On non-distributive logical connectives for fuzzy sets theory

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ON NON-DISTRIBUTIVE LOGICAL CONNECTIVES FOR FUZZY SETS THEORY

by

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INTRODUCTION.

As it was proved by Bellman and Giertz (1973), it is well known that, under reasonable hypotheses (especially distributivity) the only truth-functional logical connectives for fuzzy sets are the usual Min and Max. The following and easy argument proves that distributivity and boundary conditions are essential assumptions:

$$x = F(x,1) = F(x,G(1,1)) = G(F(x,1),F(x,1)) = G(x,x),$$

F and G being, respectively, functions from $[0,1]\times[0,1]$ into $[0,1]$ generating the "meet" and the "join" by means of

$$(A\cap B)(x) = F(A(x),B(x)); \quad (A\cup B)(x) = G(A(x),B(x)),$$

for every $x\in X$ and $A,B\in \underset{\sim}{P}(X)$.

L.A. Zadeh (1965), the founder of the theory, is the first who introduces non-distributive and dual connectives by considering the couples

$$G_m(x,y) = \mathrm{Min}\{x+y,1\}\ , \quad F_m(x,y) = \mathrm{Max}\{x+y-1,0\};$$
$$G_p(x,y) = x+y-xy\ , \quad G_p(x,y) = x.y.$$

Afterwards some authors (for instance, Hamacher (1975)) have considered some classes of non-distributive connectives.

Obviously the loss of global idempotency causes the loss of the lattice structure in $\underset{\sim}{P}(X)$ when $F \neq \mathrm{Min}$ or $G \neq \mathrm{Max}$. This is the natural possibility for mantaining the lattice structure in $\underset{\sim}{P}(X)$ and, for arbitrary connectives F and G, the only sublattice is the boolean algebra of characteristic functions (classical subsets). It is also interesting to observe that the idempotents for the couples G_m,F_m and G_p,F_p are 0 and 1,and that the former verifies universally the excluded middle and non-contradiction laws. On the other hand these laws are satisfied by the couple G_p,F_p only in the case of classical sets.

In this paper it is given a general functional form for logical connectives "or" and "and" (non-distributive but associative) using additive generators. We also study the Kleene's character of the obtained logic and the connective's relation by DeMorgan laws (with adequate strong negation functions, if they exists) and give further characterizations of the Min-Max pair. Finally, in the way of controlling relative and absolute classicity of such connectives, some parameters are introduced.

1. PRELIMINARIES ON FUZZY CONNECTIVES.

Let F and G be two binary operations on $[0,1]$. In Bellman-Giertz (1973) it was showed that F = Min, G = Max are the unique solutions of the 8 conditions below:

(1) Associativity: $F(x,F(y,z)) = F(F(x,y),z)$; $G(x,G(y,z)) = G(G(x,y),z)$;
(2) Commutativity: $F(x,y) = F(y,x)$, $G(x,y) = G(y,x)$,
(3) Non-decreasing: $F(x,y) \leqslant F(x',y')$, $G(x,y) \leqslant G(x',y')$, if $x \leqslant x', y \leqslant y'$;
(4) $F(x,x) < F(x',x')$, $G(x,x) < G(x',x')$, if $x < x'$;
(5) $F(1,1) = 1$, $G(0,0) = 0$;
(6) $F(x,y) \leqslant \mathrm{Min}(x,y)$; $G(x,y) \geqslant \mathrm{Max}(x,y)$,
(7) F,G are continnous,
(8) Distributivity: $F(x,G(y,z)) = G(F(x,y),F(x,z))$,
$G(x,F(y,z)) = F(G(x,y),G(x,z))$.

Hamacher (1975) proved that conditions 1,3,4,6 and 8 are enough in order to conclude that Min-Max are the unique solutions. Thus it is clear that if we want to study fuzzy connectives, different from the classical Min-Max, it would be necessary to avoid some of the above requirements. Several authors have treated this problem (see, e.g., Dubois-Prade (1979)).

We first note that if (5) is substituted by

(5') (a) $F(x,1) = F(1,x) = x$, (b) $G(0,x) = G(x,0) = x$, for all $x \in [0,1]$.

then (3) and (5') imply (6) and (3),(5'), together with (8), given Min-Max as solutions. Thus we will exclude (6) and (8) and we can avoid also (2) and (4). More precisely, in this paper we will restrict our attention to the following classes of connectives

$$F_{ab} = \left\{ F: [a,b]^2 \longrightarrow [a,b] \,/\, F \text{ satisfies (1),(3),(5'a) and (7)} \right\},$$

$$G_{ab} = \left\{ G: [a,b]^2 \longrightarrow [a,b] \,/\, G \text{ satisfies (1),(3),(5'b) and (7)} \right\},$$

and

$$F_{ab}^{A} = \{F \in F_{ab} / \ F \text{ is Archimedean, i.e., } F(x,x) < x, \text{ for all } x \in (a,b)\},$$

$$G_{ab}^{A} = \{G \in G_{ab} / \ G \text{ is Archimedean, i.e., } G(x,x) > x, \text{ for all } x \in (a,b)\}.$$

In the sequel we will write $F = F_{01}$, $G = G_{01}$, $F_A = F_{01}^{A}$ and $G_A = G_{01}^{A}$.

These functional sets have a long distinguished history in the field of functional equations (Aczél (1966)) because they are related to the classical problem of the associativity equation. Such sets have been analyzed with detail and have become a basic tool in the theory of probabilistic metric spaces (Schweizer-Sklar (1960)) and in the theory of information (Kampé de Fériet (1969)).

From Aczél (1966), Ling (1965) we have the following important characterization of the operations in F_{ab}^{A} and G_{ab}^{A}:

Theorem 1.1 Let $H \in F_{ab}$ (resp. $H \in G_{ab}$). Then $H \in F_{ab}^{A}$ (resp. $H \in G_{ab}^{A}$) if and only if there exists a continnous and strictly decreasing function h (resp. continous and strictly increasing) from $[a,b]$ into $[0,+\infty]$, such that H is representable in the form

$$H(x,y) = h^{(-1)}(h(x)+h(y)),$$

where $h^{(-1)}$ is the pseudo-inverse of h, that is,

$$h^{(-1)}(x) = \begin{cases} b, \text{ if } 0 \leq x \leq h(b) \quad (\text{resp. } a, \text{ if } 0 \leq x \leq h(a)), \\ h^{-1}(x), \text{ if } h(b) \leq x \leq h(a), \ (\text{resp. } h^{-1}(x), \text{ if } h(a) \leq x \leq h(b)), \\ a, \text{ if } h(a) \leq x \quad (\text{resp. } b, \text{ if } h(b) \leq x), \end{cases}$$

where h^{-1} is the usual inverse of h on $[h(b),h(a)]$ (resp. $[h(a),h(b)]$).

The function h is called an additive generator of the Archimedean operation H and it is unique up to a positive constant, i.e., k.h ($k>0$) generates also H. We remark that if f generates $F \in F_A$ then $f(1) = 0$, and if g generates $G \in G_A$ then $g(0) = 0$.

We also have from Paalman de Miranda (1964) a characterization for the operations $F \in F_{ab} - F_{ab}^{A}$ and $G \in G_{ab} - G_{ab}^{A}$. If H is one of such operations, let be

$$E(H) = \{x \in [a,b] \ / \ H(x,x) = x\};$$

then $[a,b] - E(H) = \bigcup_{i \in J} (a_i,b_i)$, where $\{(a_i,b_i) \ / \ i \in J\}$ is a finite or countable collection of disjoint open intervals. Let H_i be the restriction of H to $[a_i,b_i]^2$ given by $H_i(x,y) = a_i + (b_i-a_i)H_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right)$. Then $([a_i,b_i], H_i)$ is an Archimedean semigroup and we have the representation (ordinal sum):

$$H(x,y) = \begin{cases} H_i(x,y), \text{ if } (x,y) \in [a_i,b_i]^2, \text{ for some } i \in J; \\ \text{Min}(x,y), \text{ if } (x,y) \notin \bigcup_{i\in J} [a_i,b_i]^2 \text{ and } H \in F_{ab} - F_{ab}^A ; \\ \text{Max}(x,y), \text{ if } (x,y) \notin \bigcup_{i\in J} [a_i,b_i]^2 \text{ and } H \in G_{ab} - G_{ab}^A ; \end{cases}$$

i.e., H consists of Archimedean "blocks" along the diagonal of $[a,b]^2$ and H = Min (resp. Max) outside of these blocks.

In the literature about fuzzy connectives other couples of operations different from the classical couple Min-Max have been considered. We summarize in the next table some of these operations, together with its additive generators:

$$(1.1) \quad \begin{cases} F_m(x,y) = \text{Max}(x+y-1,0) & f_m(x) = 1-x, \\ G_m(x,y) = \text{Min}(x+y,1) & g_m(x) = x; \end{cases}$$

$$(1.2) \quad \begin{cases} F_p(x,y) = x.y & f_p(x) = -\ln x, \\ G_p(x,y) = x+y-xy & g_p(x) = -\ln(1-x); \end{cases}$$

$$(1.3) \quad \begin{cases} F_0(x,y) = \dfrac{xy}{x+y-xy} & f_o(x) = \dfrac{1-x}{x}, \\ G_0(x,y) = \dfrac{x+y-2xy}{1-xy} & g_0(x) = \dfrac{x}{1-x}; \end{cases}$$

$$(1.4) \quad \begin{cases} F_a(x,y) = \dfrac{xy}{a+(1-a)(x+y-xy)} & f_a(x) = \ln \dfrac{a-(a-1)x}{x} \quad (a>0), \\ G_b(x,y) = \dfrac{(b-1)xy+x+y}{xy} & g_b(x) = \ln \dfrac{bx-1}{x} \quad (b>-1); \end{cases}$$

Examples (1.1) and (1.2) were used by Zadeh (1965) and the operations (1.3) and (1.4) were considered by Hamacher (1975). Another interesting couple introduced by Frank (1979) is given by

$$(1.5) \quad \begin{cases} F_s(x,y) = \log_s \dfrac{(s^x-1)(s^y-1)}{s-1} + 1 ; & f_s(x) = \log_s \dfrac{s-1}{s^x-1} \quad (s>0), \\ G_s(x,y) = \log_s \dfrac{(s^{1-x}-1)(s^{1-y}-1)}{s-1} + 1 ; & g_s = \log_s \dfrac{s-1}{s^{1-x}-1} \quad (s>0); \end{cases}$$

In a remarcable paper (Frank (1979)) it has been proved that (1.1), (1.2) and (1.5) are the only possible Archimedian solutions of the functional equation $F(x,y)+G(x,y) = x+y$.

2. ON STRONG NEGATIONS AND DE MORGAN'S LAWS.

The classical equality $\chi_{\overline{A}} = 1 - \chi_A$ has its natural generalization in the realm of fuzzy sets theory: $\overline{A}(x) = \overline{1-A}(x)$, where A denotes any fuzzy set. This last equality can be written as $\overline{A} = N \circ A$, where $N(x) = 1-x$ is the standard negation function on $[0,1]$. In Bellman-Giertz (1973) it was pointed out that it would be interesting to study the functions n from $[0,1]$ into $[0,1]$ with $\overline{A}(x) = n(A(x))$, satisfying only some of the classical DeMorgan's laws. We will pay attention to this problem.

To begin with we remark that the condition $\overline{\overline{A}} = A$ whenever $\overline{A}(x) = n(A(x))$ yields that the function n from $[0,1]$ into $[0,1]$ must satisfy the condition $n \circ n = j$ (where $j(x) = x$ denotes the usual identity function). Thus we will restrict our attention to the set $S([0,1])$ of strong negations, i.e.,

$$S([0,1]) = \{n:[0,1] \longrightarrow [0,1] \,/\, n \text{ is continous and strictly decreasing, } n(0) = 1, \; n(1) = 0 \text{ and } n \circ n = j\} .$$

A functional characterization of strong negations was given by one of us (Trillas (1979)) in the recent past:

Theorem 2.1 If n is any map from $[0,1]$ into $[0,1]$, then $n \in S([0,1])$ if and only if there exists a function t from $[0,1]$ into $[o,+\infty)$ such that t is increasing, $t(0) = 0$ and n admites the representation

$$n(x) = t^{-1}(t(x)+t(y)).$$

Such function is called a generator of the negation n. Note that $k.t$ $(k>0)$ generates also n. For example:

(i) the classical negation $N(x) = 1-x$ is generated by $t(x) = x$,

(ii) the round negation $n_c(x) = \sqrt{1-x^2}$ is generated by $t_c(x) = x^2$,

(iii) the Sugeno's negation $n_s(x) = \frac{1-x}{1+sx}$ $(s > -1)$ is generated by $t_s(x) = (1/s)\ln(1+sx)$.

In many uses, some "nice" additional conditions to that of strong negation are required; then, $N(x) = 1-x$ is the only possible one, e.g., N is the unique element in $S([0,1])$ satisfying the Lipschitz's inequality $|n(x)-n(y)| \leqslant k|x-y|$ $(0<k<1)$, for all $x,y \in [0,1]$.

We turn now our attention to the standard DeMorgan's law $N \circ (A \cup B) = (N \circ A) \cap (N \circ B)$.

Definition 2.1. Let $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $n \in S([0,1])$. Then F and G will be called n-duals if $F \circ (n \times n) = n \circ G$, i.e.,

$$\overline{F(n(x),n(y))} = n(G(x,y)), \text{ for all } x,y \in [0,1].$$

It is easily seen that the connectives Min-Max, as well as the examples (1.1),(1.2),(1.3) and (1.5), are examples of N-duality.

In order to built more collections of n-duals connectives we note that

Theorem 2.2. If f is an additive generator of $F \in \mathcal{F}_A$ and $n \in S([0,1])$, then $g' = f \circ n$ generates additively a function $G \in \mathcal{G}_A$ which is n-dual of F. Analogously, if $G \in \mathcal{G}_A$ is generated by g and $n \in S([0,1])$, then $f' = g \circ n$ generates an n-dual operation $F \in \mathcal{F}_A$.

We will give now a functional characterization of the n-duality in terms of the additive generators of Archimedean connectives. If $p > 0$, let $l_p(x) = p.x$,

Theorem 2.3. Let $F \in \mathcal{F}_A$ and $G \in \mathcal{G}_A$ be generated by f and g, respectively. Then F and G are n-duals, if and only if there exists a positive constant p such that

$$(*) \qquad f^{(-1)} \circ l_p \circ g = g^{(-1)} \circ l_{1/p} \circ f ,$$

and "a fortiori" n is given by the left-hand side (or by the right-hand side) of the equality (*).

Proof. The condition $n \circ F \circ (n \times n) = G$, can be written in terms of the additive generators as follows:

$$(n \circ f^{(-1)})(f(n(x))+f(n(y))) = g^{(-1)}(g(x)+g(y)),$$

but this equation states that $n \circ f$ generates G as well as g. Consequently $n \circ f$ and g must differt only in a positive constant p, i.e., $f \circ n = l_p \circ g$, and $n = f^{(-1)} \circ l_p \circ g = g^{(-1)} \circ l_{1/p} \circ f$.

Remark. We note in view of (*) that if $f(0) < +\infty$ and $g(1) < +\infty$, then (*) yields $p = f(0)/g(1)$, i.e., the constant p is unique. If f(0) (resp. g(1)) is finite and g(1) (resp. f(0)) is infinite, then there exist no constant p satisfying (*) i.e., the corresponding connectives F and G generated by f and g respectively will be never n-duals. But if $f(0) = g(1) = +\infty$, then several possibilities are avalaible, e.g., p can be unique and there will exist a unique negation n sa-

tisfying the n-duality or p is not fixed and there exists a large collection of negations stablishing the n-duality. Next examples will clarify this remark.

Example 2.1. Consider the connectives F_a and G_b given by (1.4). Then we have, considering the generators $\hat{f}_a(x) = tf_a(x)$ and $\hat{g}_b(x) = t'g_b(x)$,

$$(\hat{f}_a^{-1} \circ 1_p \circ \hat{g}_b)(x) = \frac{a(1-x)^{pt'/t}}{(bx-1)^{pt'/t} + (a-1)(1-x)^{pt'/t}} \quad , \text{ and}$$

$$(\hat{g}_b^{-1} \circ 1_{1/p} \circ \hat{f}_a)(x) = \frac{(a-(a-1)x)^{t/pt'} - x^{t/pt'}}{(a-(a-1)x)^{t/pt'} + bx^{t/pt'}} \quad ,$$

and (*) holds for $p = t/t'$. In this case the operations F_a and G_b are n_s-duals, beig n_s the Sugeno's negation $(1-x)/(1+sx)$, where $s = (b-a+1)/a$.

Example 2.2. Consider F_0 and G_{-1} given by (1.3). Then

$$(f_0^{-1} \circ 1_p \circ g_{-1})(x) = (g_{-1}^{-1} \circ 1_{1/p} \circ f_o)(x) = \frac{1-x}{1+(p-1)x} \quad ,$$

i.e., (*) holds for any $p > 0$ and the n-duality is given by anyone of the Sugeno's negation $n_p(x) = (1-x)/(1+(p-1)x)$, $p > 0$.

We study now the relation between n-duality and ordinal sums. Note first that <u>if $F \in F\text{-}F_A$ and $G \in G\text{-}G_A$ are n-duals, then necessarily $n(E(F)) = E(G)$.</u>

Theorem 2.4. Let $F \in F\text{-}F_A$ be an ordinal sum determined by a family of Archimedean semigroups $\{([a_i,b_i]),F_i) \;/\; i \in J\}$ and let $n \in S([0,1])$. Then the function $G(x,y) = n(F(n(x),n(y)))$ is a n-dual operation of F which is an ordinal sum of the Archimedean semigroups $\{([n(b_i),n(a_i)] \;, G_i) \;/\; i \in J\}$,where $G_i = n \circ F_i \circ (nxn)$.

Let $F \in F\text{-}F_A$ and $G \in G\text{-}G_A$ be ordinal sums determined by the Archimedean semigroups $\{([a_i,b_i] \;, F_i) \;/\; i \in J\}$, and $\{([c_i,d_i] \;, G_i) \;/\; i \in J\}$, respectively (both collections with identical index set). Let f_i and g_i $(i \in J)$ be additive generators of F_i and G_i, respectively. Assume the additional condition

(**) $\quad a_i < b_i \leqslant a_{i+1} < b_{i+1}$ and $c_i < d_i \leqslant c_{i-1} \leqslant d_{i-1}$

If F and G are n-duals for some $n \in S([0,1])$, then $n(E(F)) = E(G)$, $n(a_i) = d$ and $n(b_i) = c_i$, for all $i \in J$ and there exists a sequence $(p_i)_{i \in J}$ in $(0,+\infty)$ such that, for all $i \in J$

$$n(x) = c_i + (d_i - c_i) g_i^{-1} \left[p_i f_i \left(\frac{x_i - a_i}{b_i - a_i} \right) \right] \quad , \text{ for all } x \in [a_i, b_i].$$

Note that in this case n is determined only in $\bigcup_{i \in J} [a_i, b_i]$.

3. FURTHER CHARACTERIZATIONS OF THE CLASSICAL CONNECTIVES.

It is well known that Min and Max are n-duals for all $n \in S([0,1])$. We will show first that this fact characterizes these classical connectives:

Theorem 3.1. If $F \in \mathcal{F}$ and $G \in \mathcal{G}$ are n-duals for all $n \in S([0,1])$, then necessarily F = Min and G = Max.

Proof. It is sufficient to show that $G(x_o, x_o) = x_o$ for all $x_o \in (0,1)$. Let $x_o \in (0,1)$ and consider two negations $n_1, n_2 \in S([0,1])$ such that $n_1(x_o) = n_2(x_o) = x_o$ and $n_1(x) < n_2(x)$ for all $x \in [0,1] - \{0,1,x_o\}$. Then the duality conditions yields

$$n_1(G(x_o,x_o)) = F(n_1(x_o), n_1(x_o)) = F(x_o,x_o) = F(n_2(x_o), n_2(x_o)) = n_2(G(x_o,x_o)),$$

i.e., $G(x_o,x_o) \in \{0,1,x_o\}$. But $G(x_o,x_o) \geq x_o > 0$, i.e., $G(x_o,x_o)$ is either 1 or x_o. If $G(x_o,x_o) = 1$, we would have, taking the sequence of strong negations,

$$n_k(x) = \begin{cases} \dfrac{x}{1-k} + 1 \ , & \text{if } 0 \leq x \leq 1 - \dfrac{1}{k} \ , \\ (1-k)(x-1), & \text{if } 1 - \dfrac{1}{k} \leq x \leq 1, \end{cases}$$

for all $k \in N$, $0 = n_k(1) = n_k G(x_o,x_o) = F(n_k(x_o), n_k(x_o))$; consequently,

$$0 = \lim_{k \to \infty} F(n_k(x_o), n_k(x_o)) = F(\lim_{k \to \infty} n_k(x_o), \lim_{k \to \infty} n_k(x_o)) = F(1,1) = 1.$$

This contradiction implies that $G(x_o,x_o) = x_o$ and "a fortiori" G = Max and F = Min.

Now we will give an equation which characterizes at the same time Min, Max and N = 1-j.

Theorem 3.2. Let $F \in \mathcal{F}$ and $n \in S([0,1])$. Assume that $G = n \circ F \circ (n \times n)$, i.e., G is the dual operation of F via n. Then

$$F(x,y) + G(x,y) - F(x,y).G(x,y) = 1 - n(x).n(y), \tag{3.1}$$

for all $x,y \in [0,1]$, if and only if $F = \text{Min}$, $G = \text{Max}$ and $n = N$.

Proof. Substituting $y = 0$ in (3.1) we obtain $n(x) = 1-x$. Thus (3.1) can be written as follows

$$F(x,y) + G(x,y) + x.y = x + y + F(x,y).G(x,y),$$

which is equivalent ($G(x,y) = 1 - F(1-x,1-y)$) to

$$1 + xy - F(1-x,1-y) = x + y - F(x,y).F(1-x,1-y). \tag{3.2}$$

Choose any $t \in (0,1)$. The substitution $x = y = t$ on (3.2) yields

$$1 + t^2 - F(1-t,1-t) = 2t - F(t,t).F(1-t,1-t), \tag{3.3}$$

and the substitution $x = y = 1-t$ on (3.2) gives

$$1 + (1-t)^2 - F(t,t) = 2 - 2t - F(1-t,1-t).F(t,t). \tag{3.4}$$

Thus by (3.3), we have

$$F(1-t,1-t) = \frac{1 + t^2 - 2t}{1 - F(t,t)}, \tag{3.5}$$

and (3.5) together (3.4) yields:

$$F(t,t)^2 - 2tF(t,t) + t^2 = 0.$$

Whence $F(t,t) = t$ and $F = \text{Min}$. Note that this proof does not require neither continuity assumptions nor associative hypotheses.

Using a similar argument to that of theorem 3.2 it is easy to prove the following characterization.

Theorem 3.3. Let $F \in \mathcal{F}$ and let $G \in \mathcal{G}$ be its N-dual, $G(x,y) = 1 - F(1-x,1-y)$. Then

$$F(x,y).G(x,y) = x.y \text{ , for all } x,y \in [0,1]$$

if and only if $F = \text{Min}$ and $G = \text{Max}$.

Another characterization of the classical couple Min-Max, which is independent of the hypotheses on n-duality, is the following:

Theorem 3.4. Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Then

$$\left.\begin{aligned} F(x,y) + G(x,y) &= x + y \\ F(x,y).G(x,y) &= x.y \end{aligned}\right\} \text{, for all } x\in[0,1]\text{,}$$

if and only if F = Min and G = Max.

Proof. Using the result of M.J. Frank (1979) this theorem follows as a trivial corollary. We will give a short argument. Take x = y = t. Then by hypotheses we have

$$\begin{aligned} F(t,t) + G(t,t) &= 2t \\ F(t,t).G(t,t) &= t^2 \end{aligned}$$

whence (using the fact that $F(t,t) > 0$, for all $t > 0$) we have:

$$F(t,t) + \frac{t^2}{F(t,t)} = 2t,$$

and it follows $F(t,t) = t$, i.e., necessarily F = Min and a "fortiori" G = Max.

4. SOME PARAMETERS ASSOCIATED WITH NON-CLASSICAL CONNECTIVES.

In the last section we have analyzed some strong conditions about connectives which characterize the classical couple Min-Max, but there is a large family of logical conditions with more solutions. For example Kleene's inequality "$x \wedge \overline{x} \leqslant y \vee \overline{y}$, for all x,y" is satisfied universally when $\wedge$-$\vee$ is substituted by any couple of connectives.

Theorem 4.1. If $F\in F$, $G\in G$ and $n\in S([0,1])$ then Kleene's inequality $F(x,n(x)) \leqslant G(y,n(y))$ holds for all $x,y\in[0,1]$.

Proof. Let x_n be the fixed point of n. Then we have for all $x,y\in[0,1]$:

$$F(x,n(x)) \leqslant \text{Min}(x,n(x)) \leqslant x_n \leqslant \text{Max}(y,n(y)) \leqslant G(y,n(y)).$$

In view of this inequality we can consider the parameter

$$K(F;n) = \inf\left\{n(F(x,n(x)) - F(x,n(x)) \;/\; x\in[0,1]\right\},$$

for given $F\in F$ and $n\in S([0,1])$. Then it easy to prove:

(a) $0 \leqslant K(F;n) \leqslant 1$;

(b) $K(F;n) = 0$ if and only if $F(x_n,x_n) = x_n$, i.e., the fixed point x_n

of n is an idempotent element of F;

(c) If $F \in \mathcal{F}_A$ then $K(F;n) > 0$;

(d) If $F \in \mathcal{F}_A$ and F is strictly increasing, then $K(F;n) < 1$;

(e) $K(F_m;n) = 1$ if and only if $n \leq N$.

In the other hand, given $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we can introduce a different parameter in order to evaluate the "degree of classicity" of the couple F,G. Define $C: \mathcal{F} \times \mathcal{G} \to R$ by

$$C(F,G) = \int_0^1 (G(x,x)-F(x,x))dx,$$

and consider

$$C_1(F) = \int_0^1 (Min(x,x)-F(x,x))dx,$$

$$C_2(G) = \int_0^1 (G(x,x)-Max(x,x))dx.$$

C(F,G) evaluates the "degree of classicity" of (F,G) and $C_1(F), C_2(G)$ evaluate the degrees of Archimedeanity of F and G respectively. Then we can show easily the following properties:

(i) $0 \leq C(F,G) \leq 1$, $0 \leq C_1(F) \leq 1$, $0 \leq C_2(G) \leq 1$;

(ii) $C(F,G) = C_1(F) + C_2(G)$;

(iii) $C_1(F) = 0$, if and only if F = Min: $C_2(G) = 0$, if and only if G = Max;

(iv) $C(F,G) = 0$, if and only if F = Min and G = Max;

(v) If F and G are N-duals then $C_1(F) = C_2(G)$ and consequently $C(F,G) = 2C_1(F) = 2C_2(G)$.

(vi) If F and G are N-duals and F is an ordinal sum determined by a family of Archimedean semigroups $\{([a_i,b_i], F_i) / i \in J\}$, then

$$C_1(F) = C_2(G) = \sum_{i \in J} (b_i-a_i)^2 C_1(F_i),$$

$$C(F,G) = 2 \sum_{i \in J} (b_i-a_i)^2 C_1(F_i).$$

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