Some Comparison Results for First-Order Hamilton-Jacobi Equations and Second-Order Fully Nonlinear Parabolic Equations with Ventcell Boundary Conditions

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Abstract. *In this article, we consider fully nonlinear, possibly degenerate, parabolic equations associated with Ventcell boundary conditions in bounded or unbounded, smooth domains. We first analyze the exact form of such boundary conditions in general domains in order that the notion of viscosity solutions make sense. Then we prove general comparison results under natural assumptions on the nonlinearities, assuming only that the equation is either coercive (first-order case) or strictly elliptic (second-order case) in the normal direction in a neighborhood of the boundary. Our method is inspired by the "twin blow-up method" of Forcadel-Imbert-Monneau and ideas of Lions-Souganidis which we extend to the framework of Ventcell boundary conditions.*

Key-words: Second-order elliptic and parabolic equations, Ventcell boundary conditions, comparison results, viscosity solutions.
MSC: 35D40, 35K10 35K20 35B51

1 Introduction

Introduced in 1981 by Crandall and Lions [11] (see also Crandall, Evans and Lions [9]) for first-order Hamilton-Jacobi Equations, the notion of viscosity solutions is known to be the right notion of weak solution to deal with second-order, fully nonlinear, possibly degenerate elliptic or parabolic equations. Nowadays, the basic theory can be considered as being rather complete with very general stability results, and in particular the "Half-Relaxed Limits Method" which can be powerfully used if the limit equation satisfies a *strong comparison result*, **(SCR)** for short, *i.e.* a comparison result between semicontinuous sub and supersolutions.

Such **(SCR)** not only provide the uniqueness of solutions, they are also a key tool for

obtaining their existence via the Perron's method of Ishii [15], and they exist in almost all the frameworks: whether the equations are set in the whole space or in bounded or unbounded domains, with the most classical boundary conditions (Dirichlet, State-Constraint, nonlinear Neumann boundary conditions, *etc.*) or for equations involving nonlocal terms ([6] and references therein), or equations set in a network or with discontinuities (see [5] and references therein). The reader may have a first idea of this theory by looking at the "User's guide" of Crandall, Ishii and Lions [10]; we give more references of **(SCR)** later in this introduction.

Roughly speaking, a **(SCR)** is the analog of the Maximum Principle for classical (smooth) solutions and, with few additional technical assumptions, **(SCR)** exist for any classical situation where the equation, together with the associated boundary condition formally satisfy the Maximum Principle. This last sentence is (almost) true, except for the case of *Ventcell boundary conditions*, for which no **(SCR)** was available in the literature so far; we explain why these boundary conditions create a specific difficulty later on.

The aim of this article — We provide here the very first **(SCR)** for Ventcell boundary conditions. We immediately point out that we are able to do so under reasonnable assumptions, both for the case of first-order and second-order equations. In addition, even if we consider mainly the case of equations set in an half-space, Section 4 shows that our results easily extend to the case of general regular domains via easy localization arguments and a straightforward flattening of the boundary.

The Ventcell boundary condition — Now, in order to be more specific, we consider general fully nonlinear, possibly degenerate, parabolic equation of the form

$$u_t + F(x,t,D_x u, D^2_{xx} u) = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where Ω is a bounded or unbounded domain of $\mathbb{R}^N$, the solution u is a real-valued function defined on $\overline{\Omega} \times [0,T)$, u_t, $D_x u, D^2_{xx} u$ denote its first and second-derivatives with respect to t and x respectively. Finally, $F : \overline{\Omega} \times [0,T) \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$, where $\mathcal{S}^N$ is the space of $N \times N$-symmetric matrices, is a real-valued, continuous function satisfying the ellipticity assumption

$$F(x,t,p_x,M_1) \leq F(x,t,p_x,M_2) \quad \text{if } M_1 \geq M_2, \tag{1.2}$$

for any $x \in \overline{\Omega}$, $t \in [0,T)$, $p_x \in \mathbb{R}^N$, $M_1, M_2 \in \mathcal{S}^N$, where "$\geq$" denotes the partial ordering on symmetric matrices.

In order to introduce the Ventcell boundary condition, we first consider the case when Ω is an half-space of $\mathbb{R}^N$, and to fix idea we choose

$$\Omega := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, x_N > 0\}. \tag{1.3}$$

In this context, a Ventcell boundary condition for Equation (1.1) has the form

$$-u_{x_N} + G(x', t, D_{x'}u, D^2_{x'x'}u) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.4}$$

where G satisfies similar assumptions as F, in particular an ellipticity property like (1.2). We point out that $-u_{x_N}$ is the special form, in our context, of the normal derivative of u on $\partial\Omega \times (0,T)$ and therefore (1.4) is nothing but a Neumann type boundary condition. However, this comes with an unusual dependence in the second-order tangential derivative $D^2_{x'x'}u$. This particularity is, of course, the main originality and difficulty of Ventcell boundary conditions.

The case of a general domain — If Ω is a general smooth domain[(1)], the exact form of such boundary condition and the assumptions they have to satisfy are less clear, for at least two reasons.

First, at a point x of the manifold $\partial\Omega$, it has to depend on the Hessian matrix—relatively to $\partial\Omega$—of the solution $u : \overline{\Omega} \to \mathbb{R}$ but it is well-known that the definition of such Hessian matrix on a manifold is not completely straightforward: it depends not only on $D^2_T u$, the $N \times N$-symmetric matrix corresponding to the restriction of the quadratic form $h \mapsto D^2u(x)h \cdot h$ [(2)] to $T_x\partial\Omega$, the tangent space of $\partial\Omega$ at x, but it also depends on the curvatures of $\partial\Omega$ at x.

For the time being, we just write the boundary condition as

$$\mathbb{G}(x, t, Du, D^2_T u) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.5}$$

where we recall that, if $n(x)$ denotes the outward normal to $\partial\Omega$ at x and Id is the $N \times N$ Identity matrix, $D^2_T u$ is obtained by using the projection onto $T_x\partial\Omega$, whose matrix is given by $\big(\mathrm{Id} - n(x) \otimes n(x)\big)$; hence the formula

$$D^2_T u(x) := \big(\mathrm{Id} - n(x) \otimes n(x)\big)\, D^2u(x)\, \big(\mathrm{Id} - n(x) \otimes n(x)\big)\,.$$

We refer the reader to Section 2 where we explain in an elementary way what kind of assumptions a general boundary condition like (1.5) should satisfy in order to be a "good" Ventcell boundary condition.

Of course, these restrictions are of two types: the first ones are just basic compatibility conditions in order that (1.5) is actually consistent with the Maximum Principle, and therefore that the notion of viscosity solutions makes sense. The second ones are related to comparison results and the main assumption consists in imposing that

(1) We will precise later on which type of regularity we impose.

(2) Here and throughout this article, $v_1 \cdot v_2$ stands for the standard euclidian scalar product of $v_1, v_2 \in \mathbb{R}^N$.

(1.5) can be reduced to (1.4) by (i) a suitable change of coordinates which flattens the boundary and (ii) a suitable monotonicity property in u_{x_N} after the change of coordinates to be able to write down the boundary condition as (1.4). In that way, as we explain it in Section 3, the main step in a comparison proof in a general domain is nothing but a local comparison result for (1.4).

More generally, we want to point out a key idea in this article: all the local properties for (1.1)-(1.5) are obtained from (1.1)-(1.4) since the mecanism (i)-(ii) we described above allows to reduces to this case. Now, concerning global properties such as the existence of sub and supersolutions, which are needed either for localizing the comparison proof or for Perron's method, we use only basic assumptions on $\mathbb{G}$. In fact, as this description suggests, most of the results are proved for (1.1)-(1.4).

Before coming back to the difficulties to handle such Ventcell boundary conditions and the results we are able to prove, we want to point out that, in the linear case, such boundary conditions are associated with diffusion processes with a reflection on $\partial\Omega$ as explained in N. El Karoui [13]. In modelling, these boundary conditions arise in the study of asymptotics for thin layers on the boundary; the results in this direction are either numerical ([12] and references therein) or via the Lax-Milgram Theorem ([8] and references therein).

The difficulty to handle such boundary conditions — Maybe the easiest way to explain why getting a comparison result for (1.1)-(1.4) in the viscosity solutions framework is difficult is to recall the method which is used to treat nonlinear Neumann boundary conditions, *i.e.* the case when G does not depend on $D^2_{x'x'}u$. Initiated by Lions [19] for standard linear Neumann and oblique derivatives boundary conditions, the method was then generalized under slightly different forms in the nonlinear setting (with slightly different assumptions) by Ishii [17] and Barles [4].

Of course, the difficulty comes from the condition at the boundary and the comparison proof consists in building a test-function for which the Neumann boundary condition cannot hold. With such a property, the F-inequalities necessarily hold true, both for the sub and the supersolution and, if the test-function satisfies suitable estimates, the conclusion follows.

In order to follow this strategy, a key point is that the (weak) derivatives of the sub and supersolution are nothing but derivatives of the test-function at the maximum or minimum point. Therefore, these derivatives can be directly read on the test-function and put in the equation. However, for second-order terms, any comparison proof for viscosity solutions uses the Jensen-Ishii Lemma ([18, 16]) which provides the second derivatives for the sub and supersolution in a somewhat abstract way. In particular, there is no way to build a test-function for which the boundary condition cannot hold.

How we turn around the difficulty — Let us describe here two main strategies that can be used and that we expose in this article. The simplest one in the half-space case is inspired from [5] and consist in using a "tangential regularization" in the x'-variable, at least for the subsolution and, in some other cases, both the sub and supersolution. This is why the study of the flat boundary case is more natural to begin with. Depending on the type of regularization which is doable (thanks to the properties of F and G), we can get different types of results, with different assumptions on F and G. This regularization allows to get rid of the difficulty in the x'-direction and use a standard doubling of variables in the x_N-direction.

However, this approach comes with two main connected defects: on one hand, such strategies do not seem to be able to prove results with general assumptions on F and G; on the other hand, as a consequence, the extensions to general domains require somehow unreasonnable assumptions. Anyway, we sketch in the appendix a proof using this regularization procedure which has, at least, the advantage of being very simple.

In order to obtain general results with natural assumptions on F and G, we use a combination of new arguments introduced recently by Lions and Souganidis [20, 21] and by Forcadel, Imbert and Monneau [14]. Roughly speaking, the key idea of Lions and Souganidis is to examine carefully the sub- and super-differential of the sub and supersolution respectively, at a maximum point of their difference. To do so, a blow-up argument is a key step to focus on these differentials. Then Forcadel, Imbert and Monneau improve this idea by first doubling the variables as in the classical comparison proof, and then using a "twin blow-up" argument: one on each variable (or one for the subsolution and one for the supersolution).

We extend here the strategy of Forcadel, Imbert and Monneau in order to adapt it to a second-order framework—at least for the Ventcell boundary condition but also to be able to treat the case of second-order equations. To do so, our scheme of proof in the case of (1.1)-(1.4) is the following:

(i) We use an almost classical doubling of variables method but, here, in an unusual way: it is not the main step anymore, but some kind of "preparation" to the "twin blow-up" argument. Indeed, the doubling of variables allows us to reduce to the case when the maximum points are both on the boundary—hence preparing the twin blow-up. But it also gives additionnaly some useful estimates to perform the blow-up.

(ii) The twin blow-up is done in a different way here since it has to be adapted to the Ventcell boundary condition: we use different scalings in the tangential directions (x', t) and in the normal one, *i.e.* for x_N. We perform it not only in

the equation and boundary conditions, but also in the maximum point property related to the doubling of variables, providing useful estimates.

(*iii*) Here, passing to the limit in the blow-up procedure does not allow to reduce to a one-dimensional problem, again because of the Ventcell boundary condition which mixes tangential and normal variables. However, with suitable adaptations of the Jensen-Ishii Lemma, we are able to use either the Lions-Souganidis arguments in the first-order case, and new ones in the second-order case.

In order to be able to apply this strategy, we use two specific assumptions in addition to the classical hypotheses which classically appear in such comparison results: either the equation is a first-order equation and we require a normal coercivity property, *cf.* ($\mathbf{H}_{\mathrm{NC}}$) in Section 3.1, or it is a second-order equation and we require a strong ellipticity in the normal direction, *cf.* ($\mathbf{H}_{\mathrm{NSE}}$) in Section 3.1.

A typical example that fits into the framework of this paper is the following one, posed in $\overline{\Omega} \times [0,T]$ where $\Omega := \{(x,y) \in \mathbb{R}^2 : x > 0,\ y \in \mathbb{R}\}$

$$\begin{cases} u_t - \mathrm{Tr}\left(A(x,y)D^2u\right) + b(x,y)|Du| = f(x,y) & \text{in } \Omega \times (0,T]\,, \\ -\dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} = g(y) & \text{on } \partial\Omega \times (0,T]\,, \\ u((x,y),0) = u_0(x,y) & \text{in } \Omega\,. \end{cases}$$

where we assume that $A = \sigma {}^t\sigma$ [(3)], σ, b are bounded, Lipschitz continuous functions on $\overline{\Omega}$ and f, g are bounded and continuous on $\overline{\Omega}$ and $\partial\Omega$ respectively. In order to satisfy our additional assumptions, we need that, either $A \equiv 0$ and $b(x,y) \geq \alpha > 0$ on $\overline{\Omega}$, or $A(x,y)$ is a symmetric positive matrix and, with $e_N = (1,0)$, $A(x,y)e_N \cdot e_N \geq \alpha > 0$ on $\overline{\Omega}$.

We do not know if these additional assumptions, namely ($\mathbf{H}_{\mathrm{NC}}$) and ($\mathbf{H}_{\mathrm{NSE}}$), are really necessary but (*i*) they really play a key role in our proofs of the comparison results both in the first- and second-order case; (*ii*) N. El Karoui [13] used the probabilistic analogue of ($\mathbf{H}_{\mathrm{NSE}}$) in her work; (*iii*) Proposition 3.4 in Section 3.3 shows that, if ($\mathbf{H}_{\mathrm{NSE}}$) holds then the Ventcell boundary condition is satisfied in a strong sense. In any case, one may think that ($\mathbf{H}_{\mathrm{NC}}$) or ($\mathbf{H}_{\mathrm{NSE}}$) ensures that the Ventcell boundary condition is seen in a right way.

We conclude this introduction by a remark: the approach that we use here allows to treat, as a special case, Neumann boundary condition—typically $-u_{x_N} + G(x,t,Du) = 0$. However, some of the assumptions we use in order to obtain comparison results—see ($\mathbf{H}_{\mathrm{NC}}$) and ($\mathbf{H}_{\mathrm{NSE}}$) in Section 3.1—are clearly too restrictive compared to the

[(3)] Here and below ${}^t\sigma$ denotes the transposed matrix of the matrix σ.

ones which are used in the literature on the Neumann case. But maybe some specific modification of our arguments allows not only to recover all the known results but also to improve them.

Organization — In Section 2, we define what is a "good" Ventcell boundary condition in a general, non-flat domain. Section 3 is devoted to present basic assumptions, notations and results to prepare the three next sections which are devoted to first state and then prove the comparison results. In particular, we recall how to reduce the global **(SCR)** to a local one. The statements of these results are provided in Section 4 and then we prove them in the case of first-order equations in Section 5 and in the case of second-order equations in Section 6, the proofs in these two cases being rather different even if they use similar common ingredients. Finally, in Section 7, we provide further results, we mention some open questions and we sketch simpler proofs under more restrictive assumptions.

Acknowledgements. This research was partially funded by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010 COSS.

2 The Ventcell Boundary Condition in General Domains

As we already mentioned in the introduction, contrarily to the case of classical (Dirichlet, Neumann, *etc.*) boundary conditions, the Ventcell case is particular because of the dependence in the Hessian matrix of the solution *on the boundary*. For a general boundary condition like (1.5), we have to investigate under which type of assumptions this boundary condition is consistent with the Maximum Principle, and for which a notion of viscosity solutions can exist. And to do so, we have to use the definition of an Hessian matrix on a codimension 1 manifold—which is not completely straightforward.

In this section, we have chosen to present in the simplest possible way the conditions on the function $\mathbb{G}$ in order that it yields a "good" Ventcell boundary condition. Then we show how (1.5) can be locally reduced to (1.4) by a suitable flattening of the boundary.

We argue assuming that the boundary $\partial\Omega$ is as smooth as necessary—we refer the reader to $(\mathbf{H}_\Omega)$ below for a more precise assumption concerning the regularity of the boundary. We also recall that the smoothness of $\partial\Omega$ implies that d, the distance

function to $\partial\Omega$, is smooth in a neighborhood of $\partial\Omega$, and that $Dd(x) = -n(x)$ on $\partial\Omega$; we may keep the notation $n(x)$ for $-Dd(x)$ even if x is not on the boundary.

2.1 Consistency with the maximum principle

In order to answer this first question, we adopt a viscosity solution point of view—or a Maximum Principle one—and, at least formally, we look at maximum points of $u-\phi$ where u is candidate to be a subsolution (that we assume to be smooth at first), and ϕ is a smooth test-function.

We drop the t-variable since it plays no role in the boundary condition but the reader may easily chech that t can be taken into account as any tangent variable, and so is u_t which is a tangent derivative on the boundary $\partial\Omega \times (0,T)$.

Proposition 2.1 *Let $x \in \partial\Omega$ be a local maximum point on $\overline{\Omega}$ of $y \mapsto (u-\phi)(y)$. Then the following first and second-order inequalities hold:*

$$\begin{aligned} &(i) \quad \frac{\partial u}{\partial n}(x) \geq \frac{\partial \phi}{\partial n}(x) \text{ and } Du(x) = D\phi(x) + \lambda n(x) \text{ for some } \lambda \geq 0\,, \\ &(ii) \quad D^2u(x) + \frac{\partial u}{\partial n}(x) D^2 d(x) \geq D^2\phi(x) + \frac{\partial \phi}{\partial n}(x) D^2 d(x) \text{ in } T_x\partial\Omega\,. \end{aligned} \tag{2.1}$$

Proof — If $x \in \partial\Omega$ is a local maximum point on $\overline{\Omega}$ of $u-\phi$, let us first notice that the first inequality in (i)—the normal direction one—is classical:

$$\frac{\partial (u-\phi)}{\partial n}(x) \geq 0\,.$$

For the tangential direction, we consider a smooth path $\chi : (-\eta, +\eta) \to \partial\Omega$ such that $\chi(0) = x$. Since 0 is a maximum point of $s \mapsto (u-\phi)(\chi(s))$, by differentiating it follows that $D(u-\phi)(x) \cdot \chi'(0) = 0$.

On the other hand, using that $d(\chi(s)) = 0$ and differentiating this equality at $s = 0$ implies that $Dd(x) \cdot \chi'(0) = 0$; in other words, $\tau = \chi'(0)$ belongs to $T_x\partial\Omega$. Hence, by choosing all possible paths χ as above, we deduce that, for any $\tau \in T_x\partial\Omega$, $D(u-\phi)(x) \cdot \tau = 0$. Therefore, there exists some $\lambda \in \mathbb{R}$ such that $Du(x) = D\phi(x) + \lambda n(x)$ and necessariliy $\lambda \geq 0$ from the normal inequality we recalled above, leading to (i).

We now turn to the second-order condition. Using that $h(s) := (u-\phi)(\chi(s))$ has a maximum at $s = 0$, the second-order condition yields

$$h''(0) = D^2(u-\phi)(x)\chi'(0) \cdot \chi'(0) + D(u-\phi)(x) \cdot \chi''(0) \leq 0\,. \tag{2.2}$$

Notice that $D(u-\phi)(x)\cdot\chi''(0) = \lambda n(x)\cdot\chi''(0) = -\lambda Dd(x)\cdot\chi''(0)$ and, using the second-order derivative of $d(\chi(s)) = 0$, we also have

$$D^2d(x)\chi'(0)\cdot\chi'(0) + Dd(x)\cdot\chi''(0) = 0\,.$$

Gathering these informations and denoting by τ any vector $\chi'(0) \in T_x\partial\Omega$ as above, we arrive at

$$h''(0) = D^2(u-\phi)(x)\tau\cdot\tau + \lambda D^2d(x)\tau\cdot\tau \leq 0\,.$$

Finally, since $\lambda = \frac{\partial(u-\phi)}{\partial n}$, we arrive at

$$D^2u(x) + \frac{\partial u}{\partial n}(x)D^2d(x) \geq D^2\phi(x) + \frac{\partial\phi}{\partial n}(x)D^2d(x)\,, \tag{2.3}$$

on the tangent space, which is (ii).

Q.E.D.

Consequences on $\mathbb{G}$ — In order to take into account Inequalities (2.1) in a proper way, *i.e.* in order to have

$$\mathbb{G}\big(x,t,D\phi(x),D_T^2\phi(x)\big) \leq \mathbb{G}\big(x,t,Du(x),D_T^2u(x)\big) \leq 0\,,$$

we have to require two properties on $\mathbb{G}$: on one hand, it is natural to write $\mathbb{G}$ as

$$\mathbb{G}(x,t,p,M_T) := \tilde{G}\big(x,t,p,M_T + p\cdot n(x)D^2d(x)\big)\,, \tag{2.4}$$

for any $x \in \partial\Omega$, $t \in [0,T)$, $p \in \mathbb{R}^N$ and M_T, where we recall that M_T is defined for $M \in \mathcal{S}^N$ by $M_T = (\mathrm{Id} - n(x)\otimes n(x))M(\mathrm{Id} - n(x)\otimes n(x))$. Of course, we have to assume that the function $\tilde{G}$ is elliptic in its last variable[(4)].

On the other hand, especially for (2.1)-(i), we have to assume that, for any $\lambda \geq 0$, $x \in \partial\Omega$, $t \in [0,T)$, $p \in \mathbb{R}^N$ and $M \in \mathcal{S}^N$

$$\tilde{G}\big(x,t,p+\lambda n(x),M_T + p\cdot n(x)D^2d(x)\big) - \tilde{G}\big(x,t,p,M_T + (p\cdot n(x))D^2d(x)\big) \geq 0\,.$$

Of course, these basic conditions are not even sufficient to define a nonlinear Neumann boundary condition—*i.e.* for the case where $\tilde{G}(x,t,p,M_T)$ does not depend on M_T. They have to be reinforced in order to get a "good" Ventcell boundary condition, in particular we will require the more restrictive assumption that for some $\bar{c} > 0$, and for x,t,p,M_T,λ as above,

$$\tilde{G}\big(x,t,p+\lambda n(x),M_T + p\cdot n(x)D^2d(x)\big) - \tilde{G}\big(x,t,p,M_T + p\cdot n(x)D^2d(x)\big) \geq \bar{c}\lambda. \tag{2.5}$$

In other words, under this assumption the boundary condition takes a form similar to (1.4), with constant $\bar{c}$ multiplying u_{x_N}. We refer to Section 3.1 for the exact hypotheses and more details.

[(4)] This ellipticity requirement is expected since it was expected for $\mathbb{G}$.

2.2 Reduction to a flat comparison result

Now we turn to the second question and to do so, we examine some special change of coordinates which maps $\{y_N = 0\}$ in a neighborhood of $0 \in \mathbb{R}^N$ into $\partial\Omega$. If ψ is such a diffeomorphism, we change it into

$$\Psi(y', y_N) := \psi(y', 0) + y_N Dd(\psi(y', 0)),$$

in that way, we have $d(\Psi(y', y_N)) = y_N$. Then we set

$$v(y', y_N) = u(\Psi(y', y_N))\,.$$

In the following we use the abuse of notation consisting in identifying the tangential gradients $(p_T, 0)$ with p_T, similarly we identify $D_{y'}v(y', 0)$ and $(D_{y'}v(y', 0), 0)$.

Proposition 2.2 *The derivatives of v are given by*

$$\begin{aligned}
&(i) \quad \frac{\partial v}{\partial y_N} = Du(x) \cdot Dd(x)\,,\ D_{y'}v(y', 0) = {}^tD\Psi(y', 0)D_T u(x)\,.\\
&(ii) \quad D^2_{y'y'}v(y', 0) = {}^tD\Psi(y', 0)\Big[D^2u(x) + \frac{\partial u}{\partial n}(x)D^2d(x)\Big]D\Psi(y', 0) + a(x)(D_T u(x))\,,
\end{aligned} \tag{2.6}$$

for some linear map $a(x)$ having the same regularity in x as $D^2\Psi$.

Proof — Let us compute the y'-derivatives of v for $y_N = 0$ in a direction $h = (h', 0) \in \mathbb{R}^N$. Using the notation $x = \Psi(y', y_N)$ to have simpler formulas, we get

$$\begin{aligned}
&\frac{\partial v}{\partial y_N}(y', 0) = Du(x) \cdot Dd(x)\ ,\ D_{y'}v(y', 0) \cdot h' = Du(x) \cdot D\psi(y', 0)h'\,,\\
&D^2_{y'y'}v(y', 0)h' \cdot h' = D^2u(x)D\psi(y', 0)h' \cdot D\psi(y', 0)h' + Du(x) \cdot D(D\psi(y', 0))(h', h')\,.
\end{aligned}$$

Next, applying these formulas to $y_N = d(\Psi(y', y_N))$—*i.e.* taking $u = d$—we obtain

$$\begin{aligned}
&0 = Dd(x) \cdot D\psi(y', 0)h'\,,\\
&0 = D^2d(x)D\psi(y', 0)h' \cdot D\psi(y', 0)h' + Dd(x) \cdot D(D\psi(y', 0))(h', h')\,.
\end{aligned}$$

Coming back to the first-order derivatives of $v(y', 0)$, since h' is arbitrary we deduce that $D_{y'}v(y', 0) = {}^tD\psi(y', 0)Du(x) = {}^tD\Psi(y', 0)D_T u(x)$ since ${}^tD\psi(y', 0)Dd(x) = 0$ (we use here the aforementioned abuse of notations). This yields directly (i).

Now we decompose $Du = \dfrac{\partial u}{\partial n} n(x) + D_T u$. Using that $n(x) = -Dd(x)$ we see that

$$\begin{aligned}\frac{\partial u}{\partial n}(x) n(x) \cdot D(D\psi(y',0)h') &= -\frac{\partial u}{\partial n}(x) Dd(x) \cdot D(D\psi(y',0)h') \\ &= +\frac{\partial u}{\partial n}(x) D^2 d(x) D\psi(y',0)h' \cdot D\psi(y',0)h' \,.\end{aligned}$$

Gathering everything we obtain

$$\begin{aligned}D^2_{y'y'} v(y',0)h' \cdot h' = \left[D^2 u(x) + \frac{\partial u}{\partial n}(x) D^2 d(x) \right] D\psi(y',0)h' \cdot D\psi(y',0)h' + \\ D_T u(x) \cdot D(D\psi(y',0))(h',h') \,.\end{aligned}$$

Finally, since for $y_N = 0$, we have $D\psi(y',0)h' = D\Psi(y',0)h$ and since this vector is arbitrary in $T_x \partial\Omega$, we deduce that

$$D^2_{y'y'} v(y',0) = {}^t D\Psi(y',0) \left[D^2 u(x) + \frac{\partial u}{\partial n}(x) D^2 d(x) \right] D\Psi(y',0) + a(x)(D_T u(x)),$$

where $a(x)$ acts linearly on $D_T u$ and it has the same regularity in x as $D^2\Psi$. Hence (ii) holds.

Q.E.D.

Consequences on $\mathbb{G}$ — These properties show that the "flat" Hessian matrix $D^2_{y'y'} v(y',0)$ corresponds to $D^2_T u(x) + \frac{\partial u}{\partial n}(x) D^2 d(x)$ through the change of coordinates modulo a term depending only on $D_T u(x)$, the latter corresponding to $D_{y'} v(y',0)$. Moreover, this formula can easily be inverted.

More precisely, if u is a subsolution [*resp.* super-solution] of (1.5), then v is a subsolution [*resp.* super-solution] of

$$\tilde{G}\Big(\Psi(y), t, P\big(y,t,D_y v(y,t)\big), \mathcal{M}\big(y,t,D'_y v(y,t), D^2_{y'y'} v(y,t)\big)\Big) = 0,$$

where

$$P\left(y,t,D_y v(y,t)\right) = ({}^t D\Psi)^{-1}(y) D_{y'} v(y,t) - \frac{\partial v}{\partial y_N}(y,t) n(\Psi(y)), \tag{2.7}$$

and $\mathcal{M}\big(y,t,D'_y v(y,t), D^2_{y'y'} v(y,t)\big)$ is given by

$$({}^t D\Psi)^{-1}(y) \left[D^2_{y'y'} v(y,t) - a(\Psi(y))({}^t D\Psi)^{-1}(y) D_{y'} v(y,t) \right] (D\Psi)^{-1}(y) \,,$$

with, again, the abuse of notation to identify $D^2_{y'y'} v(y,t)$ with a $N \times N$-matrix with zeros at the last line and column.

Two remarks on this admittedly complicated formula: on one hand, in order to recover the term $-\partial_{y_N} v$, one can use (2.5); this is the purpose of Lemma 2.3 below. On the other hand, the presence of the term $a(\Psi(y))({}^t D\Psi)^{-1}(y) D_{y'} v(y,t)$ perturbs the assumption we have to impose on $\tilde{G}$ to be able to use the Jensen-Ishi Lemma and justify the unusual form of ($\mathbf{H}_{\text{CONT}}$) below.

This allows to show show that a "good" Ventcell boundary condition—in the sense of Section 3.1—is locally equivalent to a "good" Ventcell boundary condition in the case of a flat boundary. Moreover, the result below proves that the boundary condition can be reduced to the form (1.4).

Lemma 2.3 *Let us assume that $\tilde{G}$ satisfies (2.5) and that $\partial\Omega$ is smooth. Then there exists a function G satisfying the same assumptions as $\tilde{G}$ such that*

$$\tilde{G}\big(\Psi(y), t, P(y,t,p'-\lambda e_N), \mathcal{M}(y,t,p',M_T)\big)$$

has the same sign as

$$-\lambda + G(y,t,p',M_T)\,.$$

As a consequence, an equation with the boundary conditions G and $\tilde{G}$ have the same subsolutions and the same supersolutions.

Proof — We first notice that we can assume that $\bar{c} = 1$ by dividing $\tilde{G}$ by $\bar{c}$. Then, if $\mathcal{D} := \mathbb{R}^{N-1} \times [0,T] \times \mathbb{R}^{N-1} \times \mathcal{S}^{N-1}$, we consider the function $f : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(X,\lambda) := \tilde{G}\big(\Psi(y), t, P(y,t,p'-\lambda e_N), \mathcal{M}(y,t,p',M_T)\big).$$

where $X = (y,t,p',M_T)$.

The property of $\tilde{G}$ implies that, for all fixed X and for all $\lambda' \geq \lambda$, we have

$$f(X,\lambda') - f(X,\lambda) \leq -(\lambda'-\lambda).$$

Hence, for all fixed X, the function $\lambda \mapsto f(X,\lambda)$ is a one-to-one function from $\mathbb{R}$ into $\mathbb{R}$ and there exists a unique $G(X)$ such that

$$f(X, G(X)) = 0,$$

and clearly $f(X,\lambda)$ has the same sign as $-\lambda + G(X)$.

For the properties of G, we just write that, if X, X' satisfy $G(X') \geq G(X)$ then, by using the above monotonicity property of f in λ and the fact that $f(X', G(X')) = f(X, G(X)) = 0$, we have

$$\begin{aligned} G(X') - G(X) &\leq f(X', G(X')) - f(X', G(X)), \\ &\leq f(X, G(X)) - f(X', G(X)). \end{aligned}$$

This inequality allows to transfer all the continuity properties of f in X to G and we trust the reader to complete the proof by using this property.

Q.E.D.

A final remark concerns the distance function which is classically used to build sub- and supersolutions. Of course, it plays this role also here; but in order to be able to do so, the form of G, namely (2.4), is essential and we point it out in the

Lemma 2.4 *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, increasing function. Then, the function $w := \psi(d)$ satisfies*

$$D_T^2 w(x) = \psi'(d(x)) D^2 d(x)\,.$$

Moreover, if (2.4) and (2.5) hold, then

$$\mathbb{G}(x,t,Dw,D_T^2 w) \leq \tilde{G}(x,t,0,0) - \bar{c}\psi'(d(x))\,.$$

Proof — A straightforward computation shows that $(Dd(x) \otimes Dd(x))_T = 0$ and $D_T^2 d(x) = D^2 d(x)$, which implies directly $D_T^2 w(x) = \psi'(d(x)) D^2 d(x)$.

Now, if $\psi' > 0$, (2.4) and (2.5) hold, then

$$\begin{aligned}
\mathbb{G}(x,t,Dw,D_T^2 w) &:= \tilde{G}\Big(x,t,\psi'(d(x))Dd(x), \psi'(d(x))D^2 d(x) + \\
&\qquad\qquad \big(\psi'(d(x))Dd(x)\cdot n(x)\big) D^2 d(x)\Big), \\
&= \tilde{G}(x,t,-\psi'(d(x))n(x),0), \\
&\leq \tilde{G}(x,t,0,0) - \bar{c}\psi'(d(x))\,.
\end{aligned}$$

In this computation, we used that $Dd(x) = -n(x)$ both for the gradient term and the $D^2 d(x)$ one, which disappears since $Dd(x)\cdot n(x) = -1$.

Q.E.D.

This property allows to consider suitable choices of ψ when building subsolutions. Of course, a similar result holds for supersolutions when $\psi' < 0$.

3 Preliminaries

In this section we first list the exact hypotheses we are going to use in the sequel: we distinguish between "basic assumptions" which, in some sense, are the keystones

of our framework and, in particular, define what is a Ventcell boundary condition; on the other hand, we have more specific assumptions which are required to obtain comparison results both in the cases when F is a first-order equation and when it is a second-order one. Then we devote several subsections to preliminary results that are used later on.

3.1 Hypotheses

We begin with the assumption on Ω which is required in order to handle a Ventcell boundary condition in a general domain, see Section 2.

($\mathbf{H}_\Omega$) — *Regularity of the domain.*
The (bounded or unbounded) domain Ω is of class $W^{4,\infty}$: there exists a bounded, $W^{4,\infty}$-function $d : \overline{\Omega} \to \mathbb{R}$ which agrees with the distance function in a neighborhood of $\partial\Omega$ and such that $d(x) > 0$ in Ω.[(5)].

We point out that, for some results, the function d being C^2, with bounded first and second derivatives, is sufficient but to simplify matter, we only use ($\mathbf{H}_\Omega$) in the paper. The $W^{4,\infty}$-regularity is justified by the change of variable we perform in Section 2: we claim that the linear map $a(x)$ has the same regularity as $D^2\Psi$ and has to be Lipschitz continuous. But Ψ is built with Dd and therefore the regularity of $D^2\Psi$ cannot be better that the one of $D^3d(x)$, hence implying the $W^{4,\infty}$-regularity.

We then proceed with the standard hypotheses on the nonlinearities that are generally needed to use the viscosity solutions' framework. To avoid repeating the same assumptions for F and G—and to point out that they are actually the same—, we introduce $H : A \times [0,T) \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ having in mind two cases

(a) $A = \Omega$, $d = N$ and $H = F$;

(b) $A = \partial\Omega$, $d = N - 1$ and $H = \mathbb{G}$.

We also use the notation $z = (x,t)$ with the usual distance $|z|^2 = |x|^2 + |t|^2$ and denote by $|||\cdot|||$ a matricial norm on $\mathcal{S}^d$.

The "basic assumptions" we mention above are

($\mathbf{H}_{\text{GEN}}$) — *General assumptions on the Hamiltonians.*

[(5)]Hence $d(x) = 0$ iff $x \in \partial\Omega$ and we recall that, if $x \in \partial\Omega$, $Dd(x) = -n(x)$ where $n(x)$ is the outward unit normal to $\partial\Omega$ at x

The nonlinearities $F, \mathbb{G}$ are continuous functions and, with the above conventions (a)-(b), we have

(i) *Lipschitz continuity.*
There exists a constant $C > 0$ such that, for any $x \in A, t \in [0,T), p_1, p_2 \in \mathbb{R}^d$ and $M_1, M_2 \in \mathcal{S}^d$

$$|H(x,t,p_1,M_1) \leq H(x,t,p_2,M_2)| \leq C\Big(|p_1 - p_2| + |M_1 - M_2|\Big).$$

(ii) *Degenerate ellipticity for the second-order case.*
For any $x \in A, t \in [0,T), p \in \mathbb{R}^d$ and $M_1, M_2 \in \mathcal{S}^d$

$$H(x,t,p,M_1) \leq H(x,t,p,M_2) \quad \text{if } M_1 \geq M_2,$$

where "$\geq$" denotes the partial ordering on symmetric matrices.

Moreover, the function $\mathbb{G} = \mathbb{G}(x,t,p,M_T)$ has the form (2.4) and

(iii) There exists a constant $\bar{c} > 0$ such that, for any $\lambda \geq 0$, $x \in \partial\Omega$, $t \in [0,T)$, $p \in \mathbb{R}^N$ and $M \in \mathcal{S}^N$

$$\tilde{G}\big((x,t,p+\lambda n(x), M_T + p\cdot n(x) D^2 d(x)\big) - \tilde{G}\big(x,t,p, M_T + p\cdot n(x) D^2 d(x)\big) \geq \bar{c}\lambda.$$

We immediately point out that it is equivalent to say that $\mathbb{G}$ or $\tilde{G}$ satisfies ($\mathbf{H}_{\text{GEN}}$)-(i)-(ii). Now, of course Assumption ($\mathbf{H}_{\text{GEN}}$) is not sufficient to prove comparison results and we introduce the following (almost classical) assumption in which $B_A(0,R)$ denotes $B(0,R) \cap A$. Of course, we still use the above conventions (a)-(b).

($\mathbf{H}_{\text{CONT}}$) — *Continuity assumption for the comparison result.*
For any $R, K > 0$ and for any function $Q : \overline{B_A(0,R)} \times [0,T] \times \mathbb{R}^d \to \mathcal{S}^d$ such that, for any $z = (x,t), \tilde{z} = (\tilde{x}, \tilde{t}) \in \overline{B_A(0,R)} \times [0,T]$, $p \in \mathbb{R}^d$

$$|||Q(z,p)||| \leq K(1+|p|)\ ,\ |||Q(z,p) - Q(\tilde{z},p)||| \leq K|z-\tilde{z}|(1+|p|),$$

there exists a modulus of continuity $\omega_{R,K}$ such that, for any $|z|, |z'| \in \overline{B_A(0,R)} \times [0,T]$, $p \in \mathbb{R}^d$ and for any $X, Y \in \mathcal{S}^d$ satisfying

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{1}{\varepsilon^2} \begin{bmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{bmatrix} + \begin{bmatrix} Q(z,p) & 0 \\ 0 & -Q(\tilde{z},p) \end{bmatrix} + \delta \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \tag{3.1}$$

for some $\varepsilon, \delta > 0$, then we have

$$H(z,p,Y) - H(\tilde{z},p,X) \leq \omega_{R,K}\Big(|z-\tilde{z}|(1+|p|) + \varepsilon^{-2}|z-\tilde{z}|^2\Big) + \omega_{R,K}(\delta). \tag{3.2}$$

As a first remark, since F can be a first-order equation, we remark that, in this case, $(\mathbf{H}_{\text{CONT}})$ reduces to

For any $R > 0$ there exists a modulus of continuity ω_R such that, for any $z = (x,t), \tilde{z} = (\tilde{x},\tilde{t}) \in \overline{B_{\overline{\Omega}}(0,R)} \times [0,T]$, $p \in \mathbb{R}^d$,

$$F(z,p) - F(\tilde{z},p) \leq \omega_R\Big(|z-\tilde{z}|(1+|p|)\Big). \tag{3.3}$$

In the classical case, the Q-term in Hypothesis $(\mathbf{H}_{\text{CONT}})$ does not exist; here it comes from the change of coordinates we perform in a neighborhood of the boundary and therefore appear only in the second-order case, *cf.* (3.3); therefore, this term is needed only in such neighborhood. In order to keep things as simple as possible, we do not try to generalize this assumption to take this remark into account.

In fact, this assumption as the classical one is satisfied by Hamilton-Jacobi-Bellman or more generally by Isaacs Equations under standard assumption, namely if F is given by

$$\sup_{\alpha}\inf_{\beta}\Big\{ -\operatorname{Tr}(a(x,t,\alpha,\beta)M) - b(x,t,\alpha,\beta)\cdot p - f(x,t,\alpha,\beta)\Big\},$$

where $a = \sigma(x,t,\alpha,\beta)\,{}^t\sigma(x,t,\alpha,\beta)$, the functions $\sigma(x,t,\alpha,\beta)$ and $b(x,t,\alpha,\beta)$ being bounded, locally Lipschitz continuous in (x,t) uniformly w.r.t. α,β and $f(x,t,\alpha,\beta)$ is continuous in (x,t) uniformly w.r.t. α,β. For $\mathbb{G}$, we may take into account nonlinearities given by similar and properly adapted formulas.

Now we introduce some specific requirements on F in the normal direction to the boundary. These are not the same according to the first or second order case. These conditions will play a crucial role and in order to get a comparison result—we refer to the book of the authors [5] for detailed explanations on the role of the normal coercivity in the first-order case. In the second-order case, the ingredient that replaces the coercivity is the normal strong ellipticity as will be clear in the comparison proof below.

$(\mathbf{H}_{\text{NC}})$ — *Normal coercivity, first-order case.*
For any $(x,t) \in \partial\Omega \times [0,T]$ there exists $r, \bar{\eta}, \bar{C} > 0$ such that, for any $(y,s) \in \overline{\Omega}$ satisfying $|y-x| + |s-t| < r$, $p \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$,

$$F(y,s,p+\lambda e_N) \geq \bar{\eta}|\lambda| - \bar{C}(1+|p|).$$

($\mathbf{H}_{\mathrm{NSE}}$) — *Normal strong ellipticity, second-order case.*
For any $(x,t) \in \partial\Omega \times (0,T)$ there exists $r, \bar{\eta}, \bar{C} > 0$ such that, for any $(y,s) \in \overline{\Omega}$ satisfying $|y-x| + |s-t| < r$, $p \in \mathbb{R}^N$, $M \in \mathcal{S}^N$ and $\lambda \in \mathbb{R}$,

$$F(y,s,p,M + \lambda e_N \otimes e_N) \leq -\bar{\eta}\lambda + \bar{C}(1 + |p| + |M|) \quad \text{if } \lambda > 0,$$

$$F(y,s,p,M + \lambda e_N \otimes e_N) \geq -\bar{\eta}\lambda - \bar{C}(1 + |p| + |M|) \quad \text{if } \lambda < 0.$$

Remark 3.1 *Let us come back on the local Lipschitz continuity in x AND t we impose in ($\mathbf{H}_{\mathrm{CONT}}$), cf. also (3.3). The reader may think that this requirement is not natural; one may just expect some continuity in t. However, in order to use efficiently ($\mathbf{H}_{\mathrm{NC}}$) in the first-order case, we need the variable t to be considered as a tangential variable x', thus imposing the same regularity on both—see the proof below. In the second-order case, though the situation is different, we still use this common regularity for some technical reason.*

We can now sum up the requirements on the equation in both the first and second order case as well as for the boundary condition for comparison results.

($\mathbf{H}_{\mathrm{COMP\text{-}1}}$) — *Assumptions on $F, \mathbb{G}$ in the first-order case.*
The nonlinearities $F, \mathbb{G}$ satisfy Assumptions ($\mathbf{H}_{\mathrm{GEN}}$), ($\mathbf{H}_{\mathrm{CONT}}$)[(6)] and the normal coercivity assumption ($\mathbf{H}_{\mathrm{NC}}$) holds for F.

($\mathbf{H}_{\mathrm{COMP\text{-}2}}$) — *Assumptions on $F, \mathbb{G}$ in the second-order case.*
The nonlinearities $F, \mathbb{G}$ satisfy Assumptions ($\mathbf{H}_{\mathrm{GEN}}$), ($\mathbf{H}_{\mathrm{CONT}}$) and the normal strong ellipticity assumption ($\mathbf{H}_{\mathrm{NSE}}$) holds for F.

These assumptions on $\mathbb{G}$ mean that the associated "flat boundary condition" $\tilde{G}$ has to satisfy first the standard second-order assumptions (ellipticity and Lipschitz continuity), but also the Neumann or Ventcell-type boundary condition already mentionned in Section 2.

[(6)] which reduces to (3.3) for F.

3.2 Global Comparison Results from Local Comparison Results

Let first us introduce a family of functions which will be used in several places, in particular to take care of the Ventcell boundary condition: for $K > 0$, we select a function $\varphi_K : [0, +\infty) \to \mathbb{R}$ satisfying

1. $\varphi_K \in C^2([0, +\infty), \mathbb{R})$, decreasing;
2. $\varphi_K(0) = 0,\ \varphi_K'(0) = -1,\ \varphi_K''(0) = -K$;
3. φ_K' has a compact support, more precisely $\mathrm{supp}(\varphi_K') = [0, 1]$.
4. In particular, φ_K is constant for $t \geq 1$ and therefore φ_K is bounded.

Now let us recall what we mean by **(SCR)** and we also define the notion of "Local Comparison Result", **(LCR)** for short.

(SCR) — *Strong (global) Comparison Result for* (1.1)-(1.5).
If $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$ is a bounded upper semicontinuous subsolution of (1.1)-(1.5), if $v : \overline{\Omega} \times [0, T) \to \mathbb{R}$ is a bounded lower semicontinuous supersolution of (1.1)-(1.5) and if $u(x, 0) \leq v(x, 0)$ in $\overline{\Omega}$, then $u(x, t) \leq v(x, t)$ in $\overline{\Omega} \times [0, T)$.

In [5], it is shown that, under suitable conditions, the proof of a **(SCR)** can be reduced to the proof of a **(LCR)**. In order to give a precise definition of a **(LCR)**, we introduce the notations

$$Q_{x,t}^{r,h} := \{(y, s) \in \overline{\Omega} \times [0, T) : |y - x| < r,\ t - h < s < t\},$$

$$\partial_p Q_{x,t}^{r,h} := \{(y, s) \in \overline{Q_{x,t}^{r,h}} : |y - x| = r\} \cup \{(y, s) \in \overline{Q_{x,t}^{r,h}} : s = t - h\}.$$

(LCR) — *Local Comparison Result for* (1.1)-(1.5).
For any $(x, t) \in \overline{\Omega} \times (0, T)$, there exists $\bar{r}, \bar{h} > 0$ such that
if $u : \overline{Q_{x,t}^{\bar{r},\bar{h}}} \to \mathbb{R}$ is a bounded upper semicontinuous subsolution of (1.1)-(1.5) in $Q_{x,t}^{r,h}$,
if $v : \overline{Q_{x,t}^{\bar{r},\bar{h}}} \to \mathbb{R}$ is a bounded lower semicontinuous supersolution of (1.1)-(1.5) in $\overline{Q_{x,t}^{\bar{r},\bar{h}}}$,
then, for any $0 < r \leq \bar{r}$ and $0 < h \leq \bar{h}$,

$$\max_{\overline{Q_{x,t}^{r,h}}} (u - v)_+ \leq \max_{\partial Q_{x,t}^{r,h}} (u - v)_+.$$

Our result is the (notice that in the result below, of course ($\mathbf{H}_{\text{GEN}}$)-(ii) is automatically satisfied if F is a first-order Hamiltonian)

Proposition 3.2 *Assume that* ($\mathbf{H}_\Omega$) *holds and that* $F, \mathbb{G}$ *satisfies* ($\mathbf{H}_{\text{GEN}}$). *Then* **(LCR)** *implies* **(SCR)**.

Proof — We slightly modify the arguments of [5] in order to take into account the Ventcell boundary condition. We denote by u and v the bounded sub and supersolution to be compared.

We first have to localize and to do so, we introduce the function

$$\chi(x,t) := (|x|^2+1)^{1/2} + k_1\varphi(d(x)) + k_2 t,$$

where $\varphi = \varphi_K$ is defined at the beginning of Section 3.2 with $K = 1$ (K is not going to play any role here). Using ($\mathbf{H}_{\text{GEN}}$)-(i) and (iii) together with Lemma 2.4, one easily shows that, by choosing k_1 large enough and then k_2 large enough, then $u_\alpha(x,t) = u(x,t) - \alpha\chi(x,t)$ is still a subsolution for (1.1)-(1.5) for any $\alpha > 0$ and $u_\alpha(x,t) \to -\infty$ when $|x| \to +\infty$ uniformly with respect to t.

The aim is to show that $u_\alpha \le v$ on $\overline{\Omega} \times [0,T)$ for any $\alpha > 0$; indeed, if this is true, we obtain the **(SCR)** by letting α tend to 0.

Because of the behavior of u_α at infinity, the maximum of $u_\alpha - v$ is achieved at some point (x,t) and we can choose t as the minimal time for which this maximum is achieved. Of course, we can assume without loss of generality that $t > 0$, otherwise we are done, and then we face two cases: either $x \in \Omega$ or $x \in \partial\Omega$.

If $x \in \Omega$, the arguments of [5] apply: we argue in $Q^{r,h}_{x,t}$ where r, h are chosen small enough in order that the **(LCR)** holds; we will also choose $h > 0$ small compared to r, its size will be precised later on. Notice that we can choose r, h such that $Q^{r,h}_{x,t}$ does not intersect $\partial\Omega \times (0,T)$ and $t - h \ge 0$.

For $K > 0$ large enough, $u^\delta_\alpha(y,s) := u_\alpha(y,s) - \delta(|y-x|^2 + K(s-t)$ is still a subsolution of (1.1) and, if $0 < h \ll r^2$, the function $|y-x|^2 + K(s-t)$ is strictly positive on the lateral boundary; indeed

$$|y-x|^2 + K(s-t) = r^2 + K(s-t) \ge r^2 - Kh > 0 \quad \text{if } h > r^2/K.$$

On the other hand, for $s = t-h$, the maximum cannot be achieved by the minimality of t and, by choosing δ small enough, we have

$$u^\delta_\alpha(x,t) - v(x,t) > \max_{|y-x|\le r} (u^\delta_\alpha(y,t-h) - v(y,t-h)).$$

Applying the **(LCR)** to u_α^δ and v and taking into account the above informations, we have

$$u_\alpha(x,t)-v(x,t) = u_\alpha^\delta(x,t)-v(x,t) \leq \max_{\partial_p Q_{x,t}^{r,h}}(u_\alpha^\delta(y,s)-v(y,s)) < \max_{\partial_p Q_{x,t}^{r,h}}(u_\alpha(y,s)-v(y,s)),$$

which yields a clear contradiction with the definition of (x,t) if $\delta > 0$ is small enough, which completes the proof in this case.

In the case when $x \in \partial\Omega$, the advantage of reducing the proof to a **(LCR)**, and therefore to a small ball around x, is that we can argue w.l.o.g. with a flat boundary, *i.e.* in the case of (1.4), *cf.* Lemma 2.3. Even if this requires a few additional arguments—in particular, the change of coordinates does not transform balls into balls; we trust the reader to be able to convince him/herself of this fact .

With this reduction, this second case is treated analogously adding an extra term to take care of the Ventcell condition, namely replacing the δ-term by

$$\delta\Big(|y-x|^2 + k\eta\varphi\big(x_N/\eta\big) + K\big(s-t\big)\Big),$$

where $k > 0$ and $\varphi = \varphi_K$ (defined at the beginning of Section 3.2) for some $K > 0$ large enough, in particular compared to k.

Using the properties of φ_K, the derivative of the φ-term is $-k$ if $d(x) = 0$, *i.e.* if $x \in \partial\Omega$. Now, for $\eta > 0$ small enough, $k\eta\varphi(x_N/\eta) = O(k\eta)$ is negative but small compared to r^2, which yields a contradiction on the lateral boundary $|y-x| = r$. On the other hand, on the boundary $s = t-h$, taking δ small enough gives the answer since, again, by the minimality of t, the maximum of $u_\alpha - v$ is strictly less than $u_\alpha^\delta(x,t) - v(x,t)$ for $s = t-h$. Again, the contradiction is obtained for $\delta > 0$ small enough, and the proof is complete.

Q.E.D.

3.3 Local Properties of the Ventcell Boundary Condition

As the title indicates it, we investigate the *local* properties of the Ventcell boundary condition and therefore we may assume without loss of generality that $\Omega = \{x_N > 0\}$ and that we are in the case of (1.1)-(1.4).

The first result concerns the "regularity" (in the sense of [5]) of sub and supersolutions in the case of Ventcell boundary conditions. To do so, we introduce the assumption implying either the normal coercivity of the nonlinearity in the case of a first-order equation or the normal strong ellipticity in the case of a second-order one.

Our result is the

Proposition 3.3 *Assume that F, G satisfy $(\mathbf{H}_{\rm GEN})$ and that F satisfies either $(\mathbf{H}_{\rm NC})$ or $(\mathbf{H}_{\rm NSE})$. Then subsolutions and supersolutions of (1.1)-(1.4) are regular on $\partial\Omega \times (0,T)$. More precisely, if u is an u.s.c. subsolution of (1.1)-(1.4) and v is a l.s.c. supersolution of (1.1)-(1.4), then for any $(x,t) \in \partial\Omega \times (0,T)$,*

$$u(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,T)}} u(y,s)\,, \quad v(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,T)}} v(y,s).$$

This proposition means that the value of u and v on the boundary are, in some sense, the limit of their interior values; there is no artificial jump on the boundary. And, of course, the same general result holds in general domains.

Proof — The arguments being similar in the sub and supersolution cases, we just give them in the subsolution one. We assume by contradiction that there exists an u.s.c. subsolution u of (1.1)-(1.4) and $(x,t) \in \partial\Omega \times (0,T)$ such that

$$u(x,t) > \limsup_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,T)}} u(y,s), \tag{3.4}$$

and the aim is to get a contradiction.

To do so, for $0 < \varepsilon \ll 1$, we introduce the function

$$(y,s) \mapsto u(y,s) - \frac{|y-x|^2}{\varepsilon^2} - \frac{|s-t|^2}{\varepsilon^2} - L\varphi_K(x_N),$$

where L is a positive constant to be chosen later on and $\varphi = \varphi_K$ is defined at the beginning of Section 3.2. We take K large enough, the size depending only on the properties of F, and being only necessary for dealing with second-order equations (recall that $\varphi_K''(0) = -K$).

For ε small enough and $L = 0$, this function has a maximum point near (x,t) and if (3.4) holds, then this maximum point is necessarily on the boundary. Moreover a property like (3.4) also holds at this maximum point and therefore it is also a maximum point of this function for any $L > 0$.

But, for fixed ε, if L is large enough, neither the inequality associated to the Ventcell boundary condition can hold, neither the one associated to the equation because of $(\mathbf{H}_{\rm NC})$ or $(\mathbf{H}_{\rm NSE})$. This gives the desired contradiction and the result.

Q.E.D.

The next result concerns the boundary condition for second-order equations which satisfy hypothesis $(\mathbf{H}_{\rm NSE})$, *i.e.* which are uniformly elliptic in the normal direction; in this case, the Ventcell boundary condition holds in a strong sense.

Proposition 3.4 *Assume that F, G satisfy $(\mathbf{H}_{\text{GEN}})$ and that F satisfies $(\mathbf{H}_{\text{NSE}})$. Then the Ventcell boundary condition is satisfied in a "strong sense" for both subsolutions and supersolutions of (1.1)-(1.4). More precisely,*

(i) if u is an u.s.c. subsolution of (1.1)-(1.4) and $(x,t) \in \partial\Omega \times (0,T)$ is a local maximum point of $u - \phi$, where ϕ is a smooth test-function then

$$-\frac{\partial \phi}{\partial x_N}(x,t) + G(x,t,D_{x'}\phi(x,t), D^2_{x'x'}\phi(x,t)) \leq 0\,.$$

(ii) if v is a l.s.c. supersolution of (1.1)-(1.4) and $(x,t) \in \partial\Omega \times (0,T)$ is a local minimum point of $v - \phi$, where ϕ is a smooth test-function then

$$-\frac{\partial \phi}{\partial x_N}(x,t) + G(x,t,D_{x'}\phi(x,t), D^2_{x'x'}\phi(x,t)) \geq 0\,.$$

Proof — We sketch the proof for the subsolution case, the supersolution one being analogous.

If $(x,t) \in \partial\Omega \times (0,T)$ is a local maximum point of $u - \phi$, it is also a local maximum point of the function

$$(y,s) \mapsto u(y,s) - \phi(y,s) - \delta x_N + L[x_N]^2,$$

for any $\delta, L > 0$. Of course, the "locality" in this property depends on δ and L. The second-derivative of the new test-function at (x,t) is now

$$D^2\phi(x,t) - Le_N \otimes e_N,$$

and, using $(\mathbf{H}_{\text{NSE}})$, it is clear that, for L large enough, the F-inequality cannot hold and therefore

$$-\frac{\partial \phi}{\partial x_N}(x,t) + \delta + G(x,t,D_{x'}\phi(x,t), D^2_{x'x'}\phi(x,t)) \leq 0.$$

Letting δ tend to 0 gives the result.

Q.E.D.

3.4 About the initial condition

A last property concerns the initial data and more precisely the points of $\partial\Omega \times \{0\}$. If (1.1)-(1.4) is associated to the initial data

$$u(x,0) = u_0(x) \quad \text{on } \overline{\Omega}, \tag{3.5}$$

where $u_0 \in C(\overline{\Omega})$, then a priori we have to use "initial data in the viscosity solutions sense" in the same way as we have "boundary conditions in the viscosity solutions sense". This is the requirement to be able to apply the half-relaxed limit method in its full powerness. It is well-known that, if u is a subsolution of (1.1)-(1.4)-(3.5) and v is a supersolution of (1.1)-(1.4)-(3.5), we have

$$u(x,0) \leq u_0(x) \leq v(x,0) \quad \text{for any } x \in \Omega. \tag{3.6}$$

But we have to show that this inequality still holds if $x \in \partial\Omega$, which is the aim of the

Proposition 3.5 *Assume that $F, \mathbb{G}$ satisfy $(\mathbf{H}_{\text{GEN}})$ and that $u_0 \in C(\overline{\Omega})$. Then (3.6) holds for any $x \in \overline{\Omega}$.*

Proof — We only prove the result for a subsolution u, the proof for a supersolution being analogous. And of course, we consider a point $x \in \partial\Omega$ for which we want to show that $u(x,0) \leq u_0(x)$.

For ε small enough and for some large enough constant $K_1 > 0$ to be chosen later on , we consider the function

$$(y,t) \mapsto u(y,t) - \frac{|y-x|^2}{\varepsilon^2} - K_1 t - \varepsilon\varphi(\frac{x_N}{\varepsilon^4}),$$

in the compact set $(\overline{B(x,1)} \cap \overline{\Omega}) \times [0,T]$ where $\varphi = \varphi_1$ defined at the beginning of Section 3.2. This function achieves its maximum at $(x_\varepsilon, t_\varepsilon)$ and, using that the $\varepsilon\varphi$-term tends to 0, classical arguments allow to show that

$$\frac{|x_\varepsilon - x|^2}{\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0.$$

In particular, for ε small enough, $x_\varepsilon \in B(x,1) \cap \overline{\Omega}$—it is not on the boundary of the ball—and we can write down viscosity subsolution inequalities. We claim that, for ε small enough and for $K_1 > 0$ large enough, we have necessarily $t_\varepsilon = 0$ and $u(x_\varepsilon, 0) \leq u_0(x_\varepsilon)$. Indeed

(*i*) If ε is small enough, the Ventcell boundry condition cannot hold since the $\varepsilon\varphi$-term has a derivative which is $+\varepsilon^{-3}$ while all the x'-derivatives at at most of order ε^{-2}.

(*ii*) On the other hand, if K_1 is large enough (of order, say, ε^{-8}), the equation cannot hold either.

Hence only the inequality associated to the initial data can hold, proving our claim. To conclude, it suffices to recall that u_0 is continuous and $u(x_\varepsilon, 0) \to u(x, 0)$ invoking again classical arguments.

Q.E.D.

Again, in the result above, of $(\mathbf{H}_{\text{GEN}})$-$(ii)$ is automatically satisfied if F is a first-order Hamiltonian.

4 Statement of the Main Comparison Results

We begin with a result in the half-space case since it is, in fact, the main result.

Theorem 4.1 *Assume that Ω is given by (1.3), that either $(\mathbf{H}_{\text{COMP-1}})$ or $(\mathbf{H}_{\text{COMP-2}})$ holds. Then the* **(LCR)** *holds for Problem (1.1)-(1.4), hence the* **(GCR)** *also holds.*

Because of the form of Assumption $(\mathbf{H}_{\text{NC}})$ or $(\mathbf{H}_{\text{NSE}})$, this result is twofold: indeed, the cases of first-order equations and of second-order equations are rather different, even if their proofs—given respectively in Sections 5 and 6—contain common features.

As we pointed out above, Assumption $(\mathbf{H}_{\text{CONT}})$—which is essential in $(\mathbf{H}_{\text{COMP-1}})$ and $(\mathbf{H}_{\text{COMP-2}})$— is nothing but the classical (3.3) in the first-order case and, in the second-order one, since Theorem 4.1 deals with a flat boundary, we can drop the Q-term in this assumption (or assume, equivalently, that it holds only for $Q \equiv 0$).

The case of general domains is just a corollary of Theorem 4.1 because of Proposition 3.2: indeed the fact that a **(GCR)** reduces to a **(LCR)** allows a local flattening of the boundary, therefore to recover the half-space case.

We formulate anyway the result.

Theorem 4.2 *Assume that $(\mathbf{H}_\Omega)$ holds, that either $(\mathbf{H}_{\text{COMP-1}})$ or $(\mathbf{H}_{\text{COMP-2}})$ holds. Then the* **(LCR)** *holds for Problem (1.1)-(1.5) hence the* **(GCR)** *also holds.*

5 Proof of (LCR) in the Half-Space Case in the First-Order Case

The aim of this section is to prove that a **(LCR)** holds for any point $(\tilde{x}, \tilde{t}) \in \overline{\Omega} \times (0, T)$ and, of course, the only difficulty is when $\tilde{x} \in \partial\Omega$, otherwise the result just follows

by a standard comparison argument if we choose $\bar{r}, \bar{h}$ small enough in order to have $Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}} \subset \Omega \times (0,T)$.

For $\tilde{x} \in \partial\Omega$, we are going to show that such a **(LCR)** holds in $\overline{Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}}$ for any $\bar{r} > 0$ and $0 < \bar{h} < \tilde{t}$. To do so, we argue by contradiction assuming that

$$\max_{\overline{Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}}} (u-v)_+ > \max_{\partial_p Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}} (u-v)_+.$$

In $\overline{Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}} \times \overline{Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}}$, we introduce the function

$$\Psi_{\varepsilon,L}(x,y,t,s) := u(x,t) - v(y,s) - \frac{|x'-y'|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} - L|x_N - y_N|,$$

where the parameters $\varepsilon > 0$ and $L > 0$ are going to be chosen small enough and large enough respectively.

This function achieves its maximum at $(\bar{x},\bar{y},\bar{t},\bar{s})$—we drop the dependence of this point in ε and L for the simplicity of notations—and with a suitable choice of ε and L (small enough and large enough respectively), we know that $(\bar{x},\bar{t}), (\bar{y},\bar{s}) \in Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}$ by our contradiction hypothesis since, by classical arguments,

$$u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) \to \max_{\overline{Q^{\bar{r},\bar{h}}_{\tilde{x},\tilde{t}}}} (u-v)_+ \quad \text{when } \varepsilon \to 0, L \to +\infty.$$

(a) *We first prove that $\bar{x}_N = \bar{y}_N = 0$ for a well-chosen constant L with respect to ε.*
Indeed, let us start by assuming that $x_N \neq y_N$. We then face two situations:
(i) if $\bar{x}_N > 0$, whether $\bar{x}_N - \bar{y}_N$ is positive or negative we may use the inside equation

$$a_\varepsilon + F(\bar{x},\bar{t},p_\varepsilon \pm Le_N) \leq -\eta,$$

where

$$a_\varepsilon := \frac{2(\bar{t}-\bar{s})}{\varepsilon^2} \quad \text{and} \quad p_\varepsilon := \frac{2(\bar{x}'-\bar{y}')}{\varepsilon^2}.$$

(ii) If $\bar{x}_N = 0$, then $|x_N - y_N| = -(x_N - y_N)$ if x_N, y_N are close enough to $\bar{x}_N, \bar{y}_N$ and the boundary condition yields

$$\min\Big(a_\varepsilon + F(\bar{x},\bar{t},p_\varepsilon + Le_N), L + G(\bar{x}',\bar{t},p_\varepsilon,\frac{1}{\varepsilon^2}\Big) \leq -\eta,$$

Now, it is clear that, for a choice of the form $L = C\varepsilon^{-2}$ with C large enough, none of these inequalities can hold and therefore $\bar{x}_N = \bar{y}_N$.

Next, let us also argue by contradiction, assuming that $\bar{x}_N = \bar{y}_N > 0$. As is well-known, we can add a term in the test-function in order that $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$ becomes a strict maximum point.

Then, regularizing the term $|x_N - y_N|$ by changing it into $(|x_N - y_N|^2 + \alpha^2)^{1/2}$ for $0 < \alpha \ll 1$, at the new maximum point $(\bar{x}_\alpha, \bar{y}_\alpha, \bar{t}_\alpha, \bar{s}_\alpha)$, we have in particular

$$a_{\varepsilon,\alpha} + F\Big(\bar{x}_\alpha, \bar{t}_\alpha, p_{\varepsilon,\alpha} + L\frac{((\bar{x}_\alpha)_N - (\bar{y}_\alpha)_N)}{(|(\bar{x}_\alpha)_N - (\bar{y}_\alpha)_N|^2 + \alpha^2)^{1/2}}\, e_N\Big) \leq 0,$$

where $a_{\varepsilon,\alpha}, p_{\varepsilon,\alpha}$ are defined in the same way as $a_\varepsilon, p_\varepsilon$ replacing $\bar{x}, \bar{y}, \bar{t}, \bar{s}$ by $\bar{x}_\alpha, \bar{y}_\alpha, \bar{t}_\alpha, \bar{s}_\alpha$. This inequality implies, using $(\mathbf{H}_{\mathrm{NC}})$, that

$$L\frac{((\bar{x}_\alpha)_N - (\bar{y}_\alpha)_N)}{(|(\bar{x}_\alpha)_N - (\bar{y}_\alpha)_N|^2 + \alpha^2)^{1/2}} = O(|p_{\varepsilon,\alpha}| + |a_{\varepsilon,\alpha}|) = o(\varepsilon^{-1}),$$

this estimate being uniform *w.r.t.* α. Notice that, in order to have the right estimate of $a_{\varepsilon,\alpha}$, we need to double the variables in the same way for both x AND t: this is where the local Lipschitz continuity in t of F, G is required, *cf.* Remark 3.1.

With this estimate, which is a key one since L is of order ε^{-2}, the classical arguments of the comparison proof for first-order Hamilton-Jacobi Equations yields a contradiction for α small enough.

The rest of the proof consists in dealing with the case $\bar{x}_N = \bar{y}_N = 0$ since we have left out the other cases.

(b) *We perform a twin blow-up à la Forcadel, Imbert and Monneau [14].*

Let us introduce the following functions:

$$\begin{aligned} u_\delta(x,t) &:= \frac{1}{\delta^2}\left(u(\bar{x}' + \delta x', \delta^2 x_N, \bar{t} + \delta t) - u(\bar{x}, \bar{t}) - \delta p_\varepsilon \cdot x' - \delta a_\varepsilon t\right), \\ v_\delta(y,s) &:= \frac{1}{\delta^2}\left(v(\bar{y}' + \delta y', \delta^2 y_N, \bar{s} + \delta s) - v(\bar{y}, \bar{s}) - \delta p_\varepsilon \cdot y' - \delta a_\varepsilon s\right). \end{aligned} \tag{5.1}$$

Notice that the Ventcell boundary condition forces us to use a different scaling in the tangent variables (x', t, y', s) and in the normal ones (x_N, y_N) and to introduce the compensating terms a_ε and p_ε, two main differences with [14].

Using the maximum property of function $\Psi_{\varepsilon,L}$ at $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$, we deduce the estimate

$$u_\delta(x,t) - v_\delta(y,s) \leq \frac{|x' - y'|^2}{\varepsilon^2} + \frac{|t - s|^2}{\varepsilon^2} + L|x_N - y_N|, \tag{5.2}$$

and $u_\delta(0,0) = v_\delta(0,0)$. This inequality shows that u_δ is bounded from above and v_δ is bounded from below.

On the other hand, they are a strict subsolution and a supersolution respectively of the following boundary problem

$$\begin{cases} \delta w_t + a_\varepsilon + F(\bar{x}' + \delta x', \delta^2 x_N, \bar{t} + \delta t, p_\varepsilon + \delta D_{x'} w + w_{x_N} e_N) = 0 & \text{in } \{x_N > 0\}, \\ -w_{x_N} + G(\bar{x}' + \delta x', \delta^2 x_N, \bar{t} + \delta t, p_\varepsilon + \delta D_{x'} w, D_{x'x'} w) = 0 & \text{on } \{x_N = 0\}. \end{cases} \tag{5.3}$$

To continue, we are going to prove that we can assume w.l.o.g that u_δ and v_δ are bounded. For u_δ, we use that, by (5.2), $(0,0)$ is a maximum point of the function

$$(x,t) \mapsto u_\delta(x,t) - \frac{|x'|^2}{\varepsilon^2} - \frac{|t|^2}{\varepsilon^2} - L x_N,$$

and, by the arguments of Proposition 2.10 in [5], there exists $\underline{\lambda} \le L$ such that

$$a_\varepsilon + F(\bar{x}', 0, \bar{t}, p_\varepsilon + \underline{\lambda} e_N) \le -\eta.$$

On the other hand, the standard subsolution inequality yields

$$\min\Big(a_\varepsilon + F(\bar{x}', 0, \bar{t}, p_\varepsilon + L e_N), -L + G(\bar{x}', 0, \bar{t}, p_\varepsilon, \varepsilon^{-2}\,\mathrm{Id})\Big) \le -\eta.$$

By standard properties of the super-differential of u_δ at $(0,0)$ (*cf.* again Proposition 2.10 in [5]) we also have

$$\min\Big(a_\varepsilon + F(\bar{x}', 0, \bar{t}, p_\varepsilon + \lambda e_N), -\lambda + G(\bar{x}', 0, \bar{t}, p_\varepsilon, \varepsilon^{-2}\,\mathrm{Id})\Big) \le -\eta,$$

for any $\lambda \in [\underline{\lambda}, +\infty)$ and therefore there exists $\bar{\lambda} \le L$ such that we have both

$$\begin{aligned} a_\varepsilon + F(\bar{x}', 0, \bar{t}, p_\varepsilon + \bar{\lambda} e_N) &\le -\eta, \\ \text{and} \quad -\bar{\lambda} + G(\bar{x}', 0, \bar{t}, p_\varepsilon, \varepsilon^{-2}\,\mathrm{Id}) &\le -\eta. \end{aligned}$$

We deduce from these inequalities that the function

$$\psi(x,t) := -1 + \bar{\lambda} x_N + \frac{|x'|^2}{\varepsilon^2} + \frac{|t|^2}{\varepsilon^2} \quad \text{(where } -1 \text{ is arbitrary)},$$

is an approximate subsolution of the problem in a neighborhood of $(0,0)$. Meaning, it is a subsolution where $\le -\eta$ is replaced by $\le -\eta + o_\delta(1)$ to take care of the terms like $\delta x'$, δt, δw_t, $\delta D_{x'} w$ in the equations.

The consequence is that $\max(\psi, u_\delta)$ is an approximate subsolution as well, which is bounded from below. A much easier but similar argument allows to bound v_δ from above by just using the coercivity of F in the w_{x_N}-direction.

(c) *Passage to the limit in both the viscosity inequalities and the maximum point property.*
Now, since the u_δ, v_δ are uniformly bounded *w.r.t.* δ, we define $\bar{u} = \limsup^* u_\delta$ and $\bar{v} = \liminf_* v_\delta$, which satisfy, on one hand, the inequality

$$\bar{u}(x,t) - \bar{v}(y,s) \leq \frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2} + L|x_N - y_N|, \tag{5.4}$$

and $\bar{u}(0,0) = \bar{v}(0,0)$.

On the other hand, if $x_N, y_N > 0$ we get

$$a_\varepsilon + F(\bar{x}, \bar{t}, p_\varepsilon + \bar{u}_{x_N} e_N) \leq -\eta < 0 \leq a_\varepsilon + F(\bar{y}, \bar{s}, p_\varepsilon + \bar{v}_{x_N} e_N),$$

while, on the boundary, we have

$$\begin{aligned} \min(a_\varepsilon + F(\bar{x}, \bar{t}, p_\varepsilon + \bar{u}_{x_N} e_N), -\bar{u}_{x_N} + G(\bar{x}', \bar{t}, p_\varepsilon, D_{x'x'}\bar{u})) \leq -\eta, \\ \max(a_\varepsilon + F(\bar{y}, \bar{s}, p_\varepsilon + \bar{v}_{x_N} e_N), -\bar{v}_{x_N} + G(\bar{y}', \bar{s}, p_\varepsilon, D_{x'x'}\bar{v})) \geq 0. \end{aligned}$$

(d) *We regularize $\bar{u}$ and $\bar{v}$ to conclude via the Lions-Souganidis approach.*

By making a tangential sup-convolution on $\bar{u}$—i.e. in variables (x', t)—and a tangential inf-convolution on $\bar{v}$, applying Lemma A.5 in [10], we can assume without loss of generality that $\bar{u}, \bar{v}$ are twice differentiable at $(0,0)$ and Lipschitz continuous in variables (x', t), uniformly in x_N. To be more precise, both are twice differentiable, not necessarily at $(0,0)$, but at least at arbitrary near points, which generates only small perturbations in the inequalities, which allows us to pass to the limit.

Moreover the coercivity assumption on F implies that $\bar{u}$ is Lipschitz continuous in x_N and, as a consequence, if there exists a superdifferential for $\bar{u}$ in the tangential direction, there exists also a global one, *i.e.* in variable (x', x_N, t). And, using the Lipschitz continuity of $\bar{u}$, the same is true for $\bar{v}$ by the maximum point property.

In the sequel, we consider only the parts of the sub and super-differentials of $\bar{u}, \bar{v}$ which are really useful, i.e. the couples corresponding to $(D^2_{x'x'}, D_{x_N})$. Mixing the above properties with the result of Lemma 15.2 in [5], there exists $(N-1)\times(N-1)$ matrices X', Y' satisfying

$$\begin{bmatrix} X' & 0 \\ 0 & -Y' \end{bmatrix} \leq \frac{1}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \tag{5.5}$$

and $\underline{\lambda}_i \leq \overline{\lambda}_i$ for $i = 1, 2$ such that

(*i*) $(X', \lambda) \in D^{2,+}\bar{u}(0,0)$ if $\lambda \geq \overline{\lambda}_1$,

(*ii*) $(Y', \lambda) \in D^{2,-}\bar{v}(0,0)$ if $\lambda \leq \underline{\lambda}_2$,

(*iii*) The Lions-Souganidis argument yields

$$\begin{aligned} a_\varepsilon + F(\bar{x}, \bar{t}, p_\varepsilon + \lambda e_N) &\leq -\eta \quad \text{if } \underline{\lambda}_1 \leq \lambda \leq \overline{\lambda}_1, \\ a_\varepsilon + F(\bar{y}, \bar{s}, p_\varepsilon + \lambda e_N) &\geq 0 \quad \text{if } \underline{\lambda}_2 \leq \lambda \leq \overline{\lambda}_2. \end{aligned}$$

(*iv*) $\underline{\lambda}_1 \leq \underline{\lambda}_2$, $\overline{\lambda}_1 \leq \overline{\lambda}_2$.

The new point in the twin blow-up argument is that the two properties in (*iii*) do not hold at the same point for F but we point out that, as long as $\underline{\lambda}_1 \leq \lambda \leq \overline{\lambda}_1$, then the coercivity assumption on F implies that $\lambda = o(\varepsilon^{-1})$ and therefore $F(\bar{x}, \bar{t}, p_\varepsilon + \lambda e_N) = F(\bar{y}, \bar{s}, p_\varepsilon + \lambda e_N) + o_\varepsilon(1)$, which means that, for ε small enough, we can assume that we are at the same point.

– If $[\underline{\lambda}_1, \overline{\lambda}_1] \cap [\underline{\lambda}_2, \overline{\lambda}_2] \neq \emptyset$ then the above remark on the estimate of λ and (*iii*) give an easy contradiction.

– Otherwise $\overline{\lambda}_1 < \underline{\lambda}_2$ and we can choose $\lambda \in [\overline{\lambda}_1, \underline{\lambda}_2]$ such that

$$a_\varepsilon + F(\bar{x}, \bar{t}, p_\varepsilon + \lambda e_N) = -\eta/2 \text{ [7]}.$$

This implies that, in the boundary inequalities, only the G-condition has to be dealt with. In this situation, since we have both $(X', \lambda) \in D^{2,+}\bar{u}(0,0)$ and $(Y', \lambda) \in D^{2,-}\bar{v}(0,0)$, we can apply the viscosity inequality for the boundary condition which leads to

$$-\lambda + G(\bar{x}', \bar{t}, p_\varepsilon, X') \leq -\eta < 0 \leq -\lambda + G(\bar{y}', \bar{s}, p_\varepsilon, Y').$$

Therefore, by $(\mathbf{H}_{\text{CONT}})$ and using that $|x' - y'| = O(\varepsilon)$, there exists $c > 0$ such that for $\varepsilon > 0$ small enough

$$-\omega_R(|x' - y'| + c\varepsilon^{-1}|x' - y'|^2) \leq -\eta,$$

which yields a contradiction for small ε.

6 Proof of (LCR) in the Half-Space Case in the Second-Order Case

For second-order equations, the strategy is exactly the same and we are not going to repeat all details here.

[7] Notice that, here again, we have $\lambda = o(\varepsilon^{-1})$ by the coercivity assumption on F.

But the first step has to be done differently since, in the first-order case, we reduce to the case when the maximum point satisfies $x_N = y_N = 0$ by a combination of normal coercivity and use of Ventcell boundary condition. Here, on the contrary, we only use the normal ellipticity of F.

We start by assuming that

$$M := \max_{\overline{Q^{\bar r,\bar h}_{\tilde x,\tilde t}}} (u-v)_+ > \max_{\partial_p Q^{\bar r,\bar h}_{\tilde x,\tilde t}} (u-v)_+ ,$$

and we denote by $(\bar x,\bar t) \in Q^{\bar r,\bar h}_{\tilde x,\tilde t}$ a point where M is attained.

(a) *Building a test-function to reduce to the case* $\bar x_N = \bar y_N = 0$.

For $\tau \in \mathbb{R}$, we set

$$\varphi(\tau) := \tau - \frac{\tau^2}{2},$$

and, in $\mathcal{E}_\varepsilon := \overline{Q^{\bar r,\bar h}_{\tilde x,\tilde t}} \times \overline{Q^{\bar r,\bar h}_{\tilde x,\tilde t}} \cap \{|x_N - y_N| \le \varepsilon\}$, we introduce the function

$$\Psi_{\varepsilon,L}(x,y,t,s) := u(x,t) - v(y,s) - \frac{|x'-y'|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} - L\varphi\left(\frac{|x_N-y_N|}{\varepsilon}\right),$$

where the parameters $\varepsilon > 0$ and $L > 0$ are going to be chosen small enough and large enough respectively. We denote by $(\bar x,\bar t,\bar y,\bar s)$ a point of maximum of $\Psi_{\varepsilon,L}$ in $\mathcal{E}_\varepsilon$, dropping the dependence in ε and L for simplicity of notations.

Notice that this penalization procedure is not as standard as usual and the following result replaces Step **(a)** from the first-order case

Lemma 6.1 *For $\varepsilon > 0$ small enough and $L > 0$ large enough (but independent of ε), the maximum point $(\bar x,\bar t,\bar y,\bar s)$ satisfies $\bar x_N = \bar y_N = 0$.*

Proof — We proceed in three steps as follows.

1. Notice first if L is chosen large enough—with a size depending only on u and v—, the maximum of function $\Psi_{\varepsilon,L}$ cannot be achieved on the boundary $|x_N - y_N| = \varepsilon$. Indeed, if $|\bar x_N - \bar y_N| = \varepsilon$, the value of the $L\varphi$-term is $L/2$, which implies that $\max(\Psi_{\varepsilon,L}) \to -\infty$ as $L \to +\infty$.

2. A second remark is that, if $\bar x_N \neq \bar y_N$, the φ-term becomes smooth at these points. Hence, for instance if $\bar x_N > 0$, we can use

$$(x,t) \mapsto v(\bar y,\bar s) + \frac{|x'-\bar y'|^2}{\varepsilon^2} + \frac{|t-\bar s|^2}{\varepsilon^2} + L\varphi\left(\frac{|x_N-\bar y_N|}{\varepsilon}\right)$$

as a test-function in the inside equation for u at $(\bar{x},\bar{t})$, which yields

$$a_\varepsilon + F\Big(\bar{x},\bar{t},p_\varepsilon \pm \frac{L}{\varepsilon}e_N, \frac{1}{\varepsilon^2}\,\mathrm{Id} - \frac{L}{\varepsilon^2}e_N\otimes e_N\Big) \le 0\,.$$

But this contradicts the ellipticity of F for L large enough, its size depending only on the properties of F. Similarly, we also reach a contradiction if $\bar{y}_N > 0$ by using the supersolution inequality for v, involving the $+L\varepsilon^{-2}\,e_N\otimes e_N$ term in F.

3. At this stage, we are left with proving that $\bar{x}_N = \bar{y}_N > 0$ cannot occur which is not as simple as in the first-order case. We first notice that, by usual arguments, we can assume w.l.o.g that $(\bar{x},\bar{t},\bar{y},\bar{s})$ is a strict maximum point by subtracting $|x-\bar{x}|^4 + |y-\bar{y}|^4 + |t-\bar{t}|^4 + |s-\bar{s}|^4$ to the function $\Psi_{\varepsilon,L}$—we keep the same notation for this new function.

Then, we denote by $\Psi_{\varepsilon,L,\alpha}$ the function which is the same as $\Psi_{\varepsilon,L}$ except that we replace ε by α in the φ-term, more precisely

$$\Psi_{\varepsilon,L}(x,y,t,s) := u(x,t) - v(y,s) - \frac{|x'-y'|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} - L\varphi\left(\frac{|x_N-y_N|}{\alpha}\right).$$

We first remark that, for $\alpha > \varepsilon$, as long as the maximum point of $\Psi_{\varepsilon,L,\alpha}$ in $\mathcal{E}_\varepsilon$ satisfies $x_N = y_N$, then this point is necessarily $(\bar{x},\bar{y},\bar{t},\bar{s})$. Indeed, this derives from the fact that $\Psi_{\varepsilon,L,\alpha}(x,t,y,s) = \Psi_{\varepsilon,L}(x,t,y,s)$ if $x_N = y_N$.

Next, we define $\bar{\alpha}$ as the supremum of all $\alpha \ge \varepsilon$ such that the maximum of $\Psi_{\varepsilon,L,\alpha}$ in $\mathcal{E}_\varepsilon$ is still achieved for $x_N = y_N > 0$, *i.e.* for which $(\bar{x},\bar{y},\bar{t},\bar{s})$ is still a maximum point. We face several cases:

- (i) If $\bar{\alpha} = +\infty$, we can drop the φ-term: $(\bar{x},\bar{t},\bar{y},\bar{s})$ is a maximum point of $\Psi_{\varepsilon,0}$ and the usual comparison arguments, leading to a contradiction, can be performed.
- (ii) If $\bar{\alpha} < +\infty$ we distinguish two sub-cases:
  - (ii)-(a) If $(\bar{x},\bar{y},\bar{t},\bar{s})$ is a *strict* maximum point of $\Psi_{\varepsilon,L,\bar{\alpha}}$, then, for any $\alpha > \bar{\alpha}$, there is a maximum point $(\bar{x}_\alpha,\bar{y}_\alpha,\bar{t}_\alpha,\bar{s}_\alpha)$ such that $(x_\alpha)_N \neq (y_\alpha)_N$ and the sequence $(\bar{x}_\alpha,\bar{y}_\alpha,\bar{t}_\alpha,\bar{s}_\alpha)$ converges to the strict maximum point $(\bar{x},\bar{y},\bar{t},\bar{s})$ as $\alpha \to \bar{\alpha}$. In this case, the usual comparison argument allows to conclude since the φ-term is smooth if $x_N \neq y_N$, see step 2. above.
  - (ii)-(b) If $(\bar{x},\bar{y},\bar{t},\bar{s})$ is NOT a *strict* maximum point of $\Psi_{\varepsilon,L,\bar{\alpha}}$, this means that there exists a sequence $(\bar{x}_k,\bar{y}_k,\bar{t}_k,\bar{s}_k)$ of maximum points of $\Psi_{\varepsilon,L,\bar{\alpha}}$ which converges to $(\bar{x}_{\bar{\alpha}},\bar{y}_{\bar{\alpha}},\bar{t}_{\bar{\alpha}},\bar{s}_{\bar{\alpha}})$ and such that $(x_k)_N \neq (y_k)_N$. Indeed, we cannot have $(x_k)_N = (y_k)_N$ (as $k\to\infty$) since $(\bar{x}_{\bar{\alpha}},\bar{y}_{\bar{\alpha}},\bar{t}_{\bar{\alpha}},\bar{s}_{\bar{\alpha}})$ is a *strict* maximum point of $\Psi_{\varepsilon,L,\bar{\alpha}} = \Psi_{\varepsilon,L}$ with the constraint $x_N = y_N$. And we conclude as in the previous case, by using the comparison arguments on $(\bar{x}_k,\bar{t}_k,\bar{y}_k,\bar{s}_k)$.

In any case, we reach a contradiction when the maximum point $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ satisfies $\bar{x}_N = \bar{y}_N > 0$, so that we can assume w.l.o.g. that $\Psi_{\varepsilon,L}$ has a maximum point such that $x_N = y_N = 0$

Q.E.D.

Remark 6.2 *In the proof of Lemma 6.1, even if this may not be completely crucial, we benefit from the same doubling of variables in x' and t since it simplifies matter, at least. This is where the local Lipschitz continuity in t plays a role, cf. Remark 3.1.*

(b) *The twin blow-up argument.*

After this first step, we perform the twin blow-up argument as in the first-order case, see (5.1). Of course, since F now depends on the second-derivatives, the equation inside the domain involves more terms than in (5.3), but we are not going to write them here since passing to the limit yields a simple formulation in the end—see below.

In order to reduce to the case when u_δ and v_δ are bounded as in the first-order case, we use sub- and supersolutions of the form

$$\psi^{\pm}(x,t) := \pm K_1(1 - x_N - K_2 x_N^2),$$

ψ^- being the subsolution and ψ^+ the supersolution. The K_1-constant is used to take care of the Ventcell boundary condition, while the K_2-one is used for the equation, using the ellipticity of F in the normal direction. Both constants depends on ε (but not on δ) and we consider these sub and supersolutions only in a small neignborhood of the boundary, *i.e.* for x_N small.

(c) *The limit problem.*

Using $(\mathbf{H}_{\mathrm{NSE}})$, the limit problem for $\bar{u}$ and $\bar{v}$ is now

$$-\bar{u}_{x_N x_N} \le 0 \le -\bar{v}_{x_N x_N} \quad \text{if } x_N > 0 \text{ and } y_N > 0 \text{ respectively}\,.$$

Notice that the strict subsolution property is lost in the limit here. On the boundary, we get

$$\min(-\bar{u}_{x_N x_N}, -\bar{u}_{x_N} + G(\bar{x}', \bar{t}, p_\varepsilon, D_{x'x'}\bar{u}) + \eta) \le 0,$$
$$\max(-\bar{v}_{x_N x_N}, -\bar{v}_{x_N} + G(\bar{y}', \bar{s}, p_\varepsilon, D_{x'x'}\bar{v})) \ge 0,$$

but using the uniform ellipticity in the normal direction of the equation inside the domain together with Proposition 3.4, these relaxed boudary conditions reduce to

$$-\bar{u}_{x_N} + G(\bar{x}', \bar{t}, p_\varepsilon, D_{x'x'}\bar{u}) + \eta \le 0,$$

$$-\bar{v}_{x_N} + G(\bar{y}', \bar{s}, p_\varepsilon, D_{x'x'}\bar{v}) \geq 0.$$

On the other hand, $\bar{u}, \bar{v}$ satisfy $\bar{u}(0,0) = \bar{v}(0,0) = 0$ and

$$\bar{u}(x,t) - \bar{v}(y,s) - \frac{|x'-y'|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} - \frac{L}{\varepsilon}|x_N - y_N| \leq 0.$$

The difference in the second-order case is that the Lipschitz continuity of the subsolution in a neighborhood of the boundary is not given for free and we are not sure that complete super and subdifferentials do exist when tangential ones exist.

(d) *Adapting the Ishii-Jensen Lemma.*

We argue by following closely the proof of the Ishii-Jensen Lemma on the boundary: we perform a sup-convolution in (x',t) to $\bar{u}$ and an inf-convolution in (y',s) to $\bar{v}$

$$\bar{u}_\alpha(x,t) := \sup\left(\bar{u}((z',x_N),\tau) - \frac{|x'-z'|^2}{\alpha^2} - \frac{|t-\tau|^2}{\alpha^2}\right),$$

$$\bar{v}_\alpha(y,s) := \sup\left(\bar{v}((z',y_N),\tau) - \frac{|y'-\tilde{z}'|^2}{\alpha^2} - \frac{|s-\tau|^2}{\alpha^2}\right),$$

for $0 < \alpha \ll \varepsilon$. It is worth pointing out that these sup- and inf-convolutions do not present any difficulty since the nonlinearities involved in the limit problem do not depend neither on x nor on t. Moreover, the equation/boundary condition satisfied by $\bar{u}_\alpha, \bar{v}_\alpha$ are exactly the same.

Inequality (5.4) implies that $\bar{u}_\alpha(0,0) = \bar{v}_\alpha(0,0) = 0$ and applying the sup-inf convolution to this inequality gives

$$\bar{u}_\alpha(x,t) - \bar{v}_\alpha(y,s) \leq (1+\frac{\alpha^2}{\varepsilon^2})\left(\frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2}\right) + \frac{L}{\varepsilon}|x_N - y_N|. \tag{6.1}$$

Hence $(0,0,0,0)$ is still a maximum point of

$$\bar{u}_\alpha(x,t) - \bar{v}_\alpha(y,s) - (1+\frac{\alpha^2}{\varepsilon^2})\left(\frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2}\right) - \frac{L}{\varepsilon}|x_N - y_N|.$$

We may even assume that $(0,0,0,0)$ is a *strict* maximum point of this function by adding suitable (small) terms.

Now, for $q \in \mathbb{R}^N$ close to 0, we consider the functions

$$\bar{u}_\alpha(x,t) - \bar{v}_\alpha(y,s) - (1+\frac{\alpha^2}{\varepsilon^2})\left(\frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2}\right) - L\varphi\left(\frac{|x_N - y_N|}{\varepsilon}\right) - q\cdot(x',t,y',s).$$

Arguing as in the first step—this is even easier here—, all these functions achieve their maximum at points such that $x_N = y_N = 0$. Then, by applying Lemma A.5 in [10] in the tangent variables, there exists a sequence $(q_k)_k$ of points in $(\mathbb{R}^{N-1} \times \mathbb{R})^2$ such that each function

$$\bar{u}_\alpha(x,t) - \bar{v}_\alpha(y,s) - (1+\frac{\alpha^2}{\varepsilon^2})\left(\frac{|x'-y'|^2}{\varepsilon^2} + \frac{|t-s|^2}{\varepsilon^2}\right) - L\varphi\left(\frac{|x_N - y_N|}{\varepsilon}\right) - q_k \cdot (x', t, y', s)$$

has a maximum point at $((x'_k, 0), t_k, (y'_k, 0), s_k)$ where $\bar{u}_\alpha, \bar{v}_\alpha$ are twice differentiable. At these points, we have full super and subdifferentials for $\bar{u}_\alpha, \bar{v}_\alpha$.

Because of the uniform ellipticity in x_N of the equation in the domain, the boundary condition is satisfied in a strong sense—see Proposition 3.4—and thanks to the structure of the super- and sub-differential of $\bar{u}_\alpha$ and v_α respectively, there exists $\overline{\lambda}_1^k, \underline{\lambda}_2^k$ such that

$$\begin{cases} -\lambda_1 + G\big(\bar{x}', \bar{t}, p_\varepsilon, D^2_{x'x'}\bar{u}_\alpha((x'_k,0),t_k)\big) \le -\eta \quad \text{for any } \lambda_1 \ge \overline{\lambda}_1^k\,, \\ -\lambda_2 + G\big(\bar{y}', \bar{s}, p_\varepsilon, D^2_{x'x'}\bar{v}_\alpha((y'_k,0),s_k)\big) \ge 0 \quad \text{for any } \lambda_2 \le \underline{\lambda}_2^k\,. \end{cases} \tag{6.2}$$

On the other hand, the matrices $D^2_{x'x'}\bar{u}_\alpha\big((x'_k,0),t_k\big), D^2_{x'x'}\bar{v}_\alpha\big((y'_k,0),s_k\big)$ satisfy the usual matrix inequality. Hence

$$G\big(\bar{x}', \bar{t}, p_\varepsilon, D^2_{x'x'}\bar{u}_\alpha((x'_k,0),t_k)\big) - G\big(\bar{y}', \bar{s}, p_\varepsilon, D^2_{x'x'}\bar{v}_\alpha((y'_k,0),s_k)\big) \ge o_\varepsilon(1) + o_k(1).$$

Using ($\mathbf{H}_{\text{CONT}}$), this yields

$$\eta - o_\varepsilon(1) - o_k(1) \le \overline{\lambda}_1^k - \underline{\lambda}_2^k.$$

At this point, we want to make precise our use of the parameters ε, α and k: α is chosen in order that $\alpha/\varepsilon \ll 1$ and then we can choose ε in order to have the above $o_\varepsilon(1)$ to be less that, say, $\eta/4$. Finally we choose k large enough (this choice is independent of the preceeding ones).

Using this inequality for ε small enough and k large enough, we can assume without loss of generality that $\overline{\lambda}_1^k - \underline{\lambda}_2^k \ge \eta/2$.

(e) *Getting a contradiction.*

Let us first notice that, after examining carefully the sub and superdifferential in the normal direction around $(0,0)$, $\overline{\lambda}_1^k$ is a minimal element in the superdifferential of $\bar{u}_\alpha$ while $\underline{\lambda}_2^k$ is a maximal element in the subdifferential of $\bar{v}_\alpha$. We will use this information below.

Then, we notice that the functions $x_N \mapsto \bar{u}_\alpha(x', x_N, t)$ are convex for any x', t close to $(0,0)$ and, in the same way, the functions $x_N \mapsto \bar{v}_\alpha(x', x_N, t)$ are concave for any

x', t close to $(0,0)$. Hence, since these functions are bounded, they are locally Lipschitz continuous and their derivatives (defined almost everywhere) are non-decreasing and non-increasing respectively.

On the other hand, the functions $x_N \mapsto \bar{u}_\alpha(x', x_N, t)$ are necessarily continuous at $x_N = 0$. Indeed, if $\bar{u}_\alpha(x', x_N, t) \to \gamma(x', t) < \bar{u}_\alpha(x', 0, t)$ for some x', t, then the same property holds in a small neighborhood of $(x', 0, t)$ and this implies that, in this neighborhood, any $\lambda \in \mathbb{R}$ is in the superdifferential of $\bar{u}_\alpha$ when it is not empty. This is clearly in contradiction with the Ventcell boundary condition. An other argument consists in using Proposition 3.3.

Now we claim that we the following inequality cannot hold for any x', t in a neighborhood of $((x'_k, 0), t_k)$:

$$\lim_{x_N \to 0} \frac{\partial \bar{u}_\alpha(x', x_N, t)}{\partial x_N} \leq \overline{\lambda}_1^k - \eta/8,$$

Indeed otherwise we would have a contradiction with the definition of $\overline{\lambda}_1^k$ and its minimality. In the same way,

$$\lim_{x_N \to 0} \frac{\partial \bar{v}_\alpha(x', x_N, t)}{\partial x_N} \geq \overline{\lambda}_2^k + \eta/8$$

cannot hold in a neighborhood of $((y'_k, 0), s_k)$.

Hence, by using suitable sequences converging to $((x'_k, 0), t_k)$ and $((y'_k, 0), s_k)$ respectively, we have

$$\begin{cases} \bar{u}_\alpha((x'_k, x_N), t_k) - \bar{u}_\alpha((x'_k, 0), t_k) \geq (\overline{\lambda}_1^k - \eta/8) x_N, \\ \bar{v}_\alpha((y'_k, x_N), s_k) - \bar{v}_\alpha((y'_k, 0), s_k) \leq (\overline{\lambda}_2^k + \eta/8) x_N, \end{cases}$$

leading to

$$\begin{aligned} \Big[\bar{u}_\alpha((x'_k, x_N), t_k) - \bar{v}_\alpha((y'_k, x_N), s_k)\Big] - \Big[\bar{u}_\alpha((x'_k, 0), t_k) - \bar{v}_\alpha((y'_k, 0), s_k)\Big] \\ \geq \Big(\overline{\lambda}_1^k - \overline{\lambda}_2^k - \frac{\eta}{4}\Big) x_N \geq \frac{\eta}{4} x_N. \end{aligned}$$

Letting $k \to \infty$, this yields

$$\bar{u}_\alpha(0, x_N, 0) - \bar{v}_\alpha(0, x_N, 0) \geq \frac{\eta}{4} x_N,$$

but (5.4) implies $\bar{u}_\alpha(0, x_N, 0) - \bar{v}_\alpha(0, x_N, 0) \leq 0$, and we reach a contradiction which ends the proof.

7 Further Results and Open Questions

In this last section we gather some comments, open questions and other results concerning Problem (1.1)-(1.5).

Existence via Perron's method

We first provide an *existence result* for Problem (1.1)-(1.5) associated to the initial condition (3.5) and to do so we use the assumption

($\mathbf{H}_{\text{B-Ex}}$) — *Boundedness assumption for existence.*
The functions $x \mapsto u_0(x)$, $(x,t) \mapsto F(x,t,0,0)$ and $(x,t) \mapsto \mathbb{G}(x,t,0,0)$ are bounded and continuous on $\overline{\Omega}$, $\overline{\Omega} \times [0,T]$ and $\partial\Omega \times [0,T]$ respectively.

The result is the

Proposition 7.1 *Under the assumptions of Theorem 4.2, if* ($\mathbf{H}_{\text{B-Ex}}$) *holds, there exists a unique, bounded continuous solution to Problem* (1.1)-(1.5)-(3.5).

Proof — We just give the main arguments since the proof is based on the classical Perron's method (*cf.* Ishii [15], see also [10]).

The key point is to build sub and supersolutions of the problem and they have the form

$$u^{\pm}(x,t) := \pm k_1 t \pm k_2 \varphi(d(x)) + k_3,$$

where φ is φ_1 defined at the beginning of Section 3.2 and k_1, k_2, k_3 are constant which are chosen in the following way:

(*i*) k_2 is chosen in order to have $u^{\pm}$ satisfying the Ventcell boundary condition, *cf.* Lemma 2.4.

(*ii*) Then k_1 is chosen in order to ensure that $u^{\pm}$ are sub and supersolution of (1.1).

(*iii*) Finally k_3 is chosen in order to have

$$u^-(x,0) \leq u_0(x) \leq u^+(x) \quad \text{on } \overline{\Omega}.$$

With all these properties, one can apply Perron's method—with an initial data being understood in the viscosity sense. And the result is proved.

Q.E.D..

Including some u_t-dependence in G

It is clear that boundary conditions like

$$\mathbb{G}(x,t,u_t,Du,D^2u)=0 \quad \text{on } \partial\Omega\times(0,T),$$

where $\mathbb{G}(x,t,p_t,p,M_T)$ is an increasing function in p_t can be treated analogously, typically

$$u_t-\frac{\partial u}{\partial x_N}+G(x',t,D_{x'}u,D^2_{x'x'}u)=0 \quad \text{on } \{x_N=0\}\times(0,T).$$

The assumptions on the dependence in p_t are analogous to those made on the tangential part of p since, as it is already the case in this article, t can be seen as a tangent variable to the boundary $\partial\Omega\times(0,T)$.

The stationary case

We point out that the stationary case can be treated analogously provided that the nonlinearity of the equation is proper in the sense of [10]. We are not going to give any detail here but both the existence and comparison result hold in this framework, as the reader will certainly be able to check.

A few open questions

Via $(\mathbf{H}_{\text{CONT}})$, we assume the same regularity for $F(x,t,p,X), G(x',t,p',X')$ in x or x' and t. We have no idea if this assumption is really necessary or if one can replace it by some weaker continuity requirement for the t-variable.

In the same way, the Lipschitz continuity assumption in $(\mathbf{H}_{\text{GEN}})$-$(i)$ may be seen as natural for G or $\mathbb{G}$ as part of the requirement for a "good Ventcell boundary condition", the linear growth in p and M_T ensuring—in some sense—that the normal derivative can control them. However, this assumption seems less natural for F which, for example, may have some superlinear gradient growth, which is incompatible with $(\mathbf{H}_{\text{GEN}})$-$(i)$. We do not address this question here but it is clearly a problem to be considered.

The $C^{0,\alpha}$-regularity of solutions for $\alpha\in(0,1]$ is an interesting question which is also a prerequisite to address other problems like the large time behavior of solutions via the study of the *ergodic problem*.

Considering the methods we used to get the comparison result suggests that these regularity results should follow from similar ideas.

Two particular cases with simpler proofs

Finally, we provide some remarks on how to derive more direct proofs in some particular cases where the comparison proof can be substantially simplified. Unfortunately those cases all suppose that the problem is set in an half-space and with too strong assumptions to have any hope of extending them to general domains. In all these cases, we just sketch the simplified proof, insisting on the main points.

(a) The first particular case is when

(i) Equation (1.1) is of first-order type, satisfying ($\mathbf{H}_{\mathrm{NC}}$).

(ii) $F(x,t,p)$ and $G(x',t,p',M')$ are convex in p and (p',M') respectively, and G does not depend on x',t.

Under these assumptions, the "tangential regularization procedure" of [5] works without any difficulty: we first regularize the subsolution u by a sup-convolution in the tangent variables—*i.e.* in (x',t)—, then by a standard convolution. This allows to reduce to the case when u is Lipschitz continuous *w.r.t.* all variables and smooth in x',t (even C^∞) with all tangent derivatives being continuous *w.r.t.* all variables.

Because of this last property, u is "almost a test-function", we have just to handle the x_N-variable. Assuming that, after the standard localization procedure, $u-v$ has a strict maximum point at $(\bar{x},\bar{t})$ satisfying $\bar{x}_N=0$, it is enough to look at the function

$$u((x',x_N),t)-v((x',y_N),t)+G\Big(D_{x'}u\big((\bar{x}',0),\bar{t}\big),D^2_{x'x'}u\big((\bar{x}',0),\bar{t}\big)\Big)(x_N-y_N)$$
$$-\frac{(x_N-y_N)^2}{\varepsilon^2}+\delta(x_N+y_N),$$

i.e. with just a doubling of variables in the normal direction, following a philosophy which is very close to the usual Neumann comparison proof.

Of course, here the conclusion follows very easily but unfortunately the Assumptions (i)-(ii) above are very restrictive.

(b) The second particular case is when F has a "separated variables" structure, *i.e.* when

$$F(x,t,Du,D^2u):=F_1(x',t,D_{x'}u,D^2_{x'x'}u)+F_2(x_N,D_{x_N}u,D^2_{x_Nx_N}u).$$

The main point here is that we can still regularize the subsolution u by a sup-convolution in the tangent variables but we can also do it on the supersolution v

by inf-convolution. The advantage is that we begin the proof with sub and supersolution which are already Lipschitz continuous and semi-convex or semi-concave in the tangent variables, which simplifies slightly the arguments. But again, if we have in mind to treat a problem set in a general domain, we face very restrictive assumptions.

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