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# About enveloping algebras of direct sums.

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## Abstract

We solve the PBW-like problem of normal ordering for enveloping algebras of direct sums.

## 1 A question about the enveloping algebra of a direct sum.

This question is imported from [13]. It is linked to this one [14] in the case of semi-direct products.

Let us consider a Lie  $\mathbf{k}$ -algebra ( $\mathbf{k}$  is a commutative ring) written as a (module) direct sum of two of its subalgebras

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad (\oplus = \oplus_{\mathbf{k}\text{-mod}})$$

and the linear maps associated to this direct sum decomposition

$$\mathfrak{g}_i \begin{array}{c} \xrightarrow{j_i} \\ \xleftarrow{p_i} \end{array} \mathfrak{g} \quad \text{such that } j_1 p_1 + j_2 p_2 = Id_{\mathfrak{g}} \quad (1)$$

( $Id_{\mathfrak{g}}$  is a sum of two orthogonal projectors, remark that only  $j_i$ 's are Lie morphisms in general). We get, at once, the maps  $\mathcal{U}(j_i)$  through the universal algebra functor  $\mathcal{U}$  (see below section 2 “Universal Constructions”) as follows.

$$\begin{array}{ccc} \mathfrak{g}_i & \xrightarrow{j_i} & \mathfrak{g} \\ \sigma_i \downarrow & & \downarrow \sigma \\ \mathcal{U}(\mathfrak{g}_i) & \xrightarrow{\mathcal{U}(j_i)} & \mathcal{U}(\mathfrak{g}) \end{array} \quad (2)$$

allowing us to multiply members of  $\mathcal{U}(\mathfrak{g}_1) \otimes_{\mathbf{k}} \mathcal{U}(\mathfrak{g}_2)$  within  $\mathcal{U}(\mathfrak{g})$  by the composite map

$$\mu_{state} = \mu \circ (\mathcal{U}(j_1) \otimes_{\mathbf{k}} \mathcal{U}(j_2)) : \mathcal{U}(\mathfrak{g}_1) \otimes_{\mathbf{k}} \mathcal{U}(\mathfrak{g}_2) \rightarrow \mathcal{U}(\mathfrak{g}) \quad (3)$$

where  $\mu$  is the multiplication of  $\mathcal{U}(\mathfrak{g})$ .

One can check, using generators, that  $\mu_{state}$  is surjective (and, in many usual cases [13] bijective).

**Question:** Is the property that  $\mu_{state}$  is bijective, true in the general case ?

**Remarks 1.** 1. Unless explicitly stated, all tensor products will be understood over  $\mathbf{k}$ .

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2. All states below are elements of spaces of the form  $M = \mathcal{A}_1 \otimes \mathcal{A}_2$  (see Eq. 10) where, for  $i = 1, 2$ ,  $(\mathcal{A}_i, *_i)$  is a  $\mathbf{k}$ -AAU (i.e. Associative Algebra with Unit). Such a space  $M$  is naturally a  $\mathcal{A}_1 - \mathcal{A}_2$  bimodule ( $\mathcal{A}_1$  module by multiplication on the left and  $\mathcal{A}_2$  module by multiplication on the right). To avoid confusion (as tensors may appear inside  $\mathcal{A}_1$ ), the separating tensor of  $M = \mathcal{A}_1 \otimes \mathcal{A}_2$  will, from time to time, be noted bold and oversized.

**Example 1.** 1. In the remarks and examples below, for any  $\mathbb{Z}$ -algebra  $\mathcal{A}$  and  $q \in \mathbb{N}_{\geq 2}$ , we will note  $(\mathcal{A})_q$  the quotient  $\mathbb{Z}$ -algebra  $\mathcal{A}/q \cdot \mathcal{A}$  (in all cases  $q \cdot \mathcal{A}$  is an ideal). For example, the situation  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  where no factor is an ideal is frequent for Lie algebras admitting a triangular decomposition<sup>1</sup>  $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$  for example

$$\mathfrak{sl}(n, \mathbb{Z}) = T_+(n, \mathbb{Z}) \oplus D(n, \mathbb{Z}) \oplus T_-(n, \mathbb{Z})$$

and one can create an example without any basis with  $\mathbf{k} = \mathbb{Z}$ ,  $q \in \mathbb{N}_{\geq 2}$  with

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sl}(n, \mathbb{Z}) = T_+(n, \mathbb{Z}) \oplus D(n, \mathbb{Z}) \oplus T_-(n, \mathbb{Z}) \text{ then} \\ (\mathfrak{g})_q &= T_+(n, \mathbb{Z})_q \oplus D(n, \mathbb{Z})_q \oplus (T_-(n, \mathbb{Z}))_q \text{ and, if one needs two factors,} \\ (\mathfrak{g})_q &= (T_+(n, \mathbb{Z}) \oplus D(n, \mathbb{Z}))_q \oplus (T_-(n, \mathbb{Z}))_q \end{aligned} \quad (4)$$

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We thank Darij Grinberg, from Drexel University and Jim Humphreys (through MathOverflow) for fruitful interactions and their interest for this question.

## 2 Universal Constructions

### 2.1 General principle.

In this subsection, we introduce the combinatorial (free) objects that we will use throughout the manuscript and the notion of enveloping algebra of a Lie algebra. These objects (call them  $G(X)$ ) together with a map  $j_X : X \rightarrow G(X)$  are all solutions of universal problems. We will recall the definition, notation and terminology about these free<sup>2</sup> objects below (cf. in general Bourbaki [1] Ch IV §3 or [9] and, in particular, [2] Ch I §7.1 and Lothaire [8] Prop 1.1.1 for words and the free monoids, Bourbaki [3] Ch II §2.2 Prop 1 and Reutenauer [11] Thm 0.4 for free Lie algebras and Bourbaki [2] for enveloping algebras i.e. towards the free associative algebras with unit and Dinh Vu Nguyen's thesis [5] for all these matters), but here, we state the general principle.

The scheme is the same for all categories considered in the following list ( $\mathbf{k}$  being a fixed ring).

$$\mathbf{Mon, Grp, k-AAU, k-Lie} \quad (5)$$

All objects of these categories can be considered as sets, we then have a natural “forgetful” functor  $F$  such that,  $\mathcal{A}$  being an object (of one of these categories),  $F(\mathcal{A})$  is the set underlying the structure  $\mathcal{A}$ . As well, any  $\mathcal{A} \in \mathbf{k-AAU}$  can be considered as a Lie algebra with the bracket  $[x, y] := xy - yx$

We are now in the position of stating the universal problem leading to left-adjoint of a functor  $F$ .

**Universal problem (w.r.t.  $F$ , naive version<sup>3</sup>).** —

For any set  $X$  ( $\mathcal{C}$  being one of the categories as above) does there exist a pair  $(j_X, G(X))$  ( $G(X)$  being an object of  $\mathcal{C}$  and  $j_X : X \rightarrow G(X)$  an heteromorphism) such that:

For any map  $f : X \rightarrow \mathcal{A}$  (heteromorphism), there exists a unique  $\hat{f} \in \text{Hom}_{\mathcal{C}}(G(X), \mathcal{A})$  such that  $f = F(\hat{f}) \circ j_X$ .

<sup>1</sup>As Kac-Moody algebras [10], have also a look at [7] Ch 1 Exercise 1.8 (local Lie algebras).

<sup>2</sup>Or freely generated in the case of enveloping algebras.

<sup>3</sup>See the theory of Heteromorphisms[12]

**Remarks 2.** *i) It might happen that  $G$  be not defined everywhere as shows the case with  $\mathcal{C} = \mathbf{FinSet}$ ,  $F$  being the inclusion functor (i.e.  $F(X) = X$  for every finite set and  $F(f) = f$  for every set-theoretical map between finite sets).*

*However a solution of the universal problem (6), for all  $X$ , provides a free functor  $G : \mathbf{Set} \rightarrow \mathcal{C}$ ,  $X \mapsto G(X)$  which is left-adjoint to the forgetful functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ . The reader must be aware that, in general, the notion of “forgetful functor” (here constructed from algebraic structures and sets) is informal.*

$$\begin{array}{ccc}
 \mathbf{Set} & \xleftarrow{\quad F \quad} & \mathcal{C} \\
 X & \xrightarrow{\quad f \quad} & \mathcal{A} \\
 & \searrow_{j_X} & \uparrow \hat{f} \\
 & & G(X).
 \end{array} \tag{6}$$

*ii) We recall here that the universal enveloping algebra of a Lie  $\mathbf{k}$ -algebra  $\mathfrak{g}$  is a pair  $(\sigma, \mathcal{U}(\mathfrak{g}))$ , where  $\mathcal{U}(\mathfrak{g})$  is an object in  $\mathbf{k-AAU}$  and  $\sigma : \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$  is a morphism in  $\mathbf{k-Lie}$ , which is a solution of the following universal problem:*

$$\begin{array}{ccc}
 \mathbf{k-Lie} & \xleftarrow{\quad F \quad} & \mathbf{k-AAU} \\
 \mathfrak{g} & \xrightarrow{\quad f \quad} & \mathcal{A} \\
 & \searrow_{\sigma} & \uparrow \hat{f} \\
 & & \mathcal{U}(\mathfrak{g}).
 \end{array} \tag{7}$$

*From this arises that there exists the universal enveloping functor*

$$\mathcal{U} : \mathbf{k-Lie} \rightarrow \mathbf{k-AAU}, \quad \mathfrak{g} \mapsto (\sigma, \mathcal{U}(\mathfrak{g})) \tag{8}$$

*which is a left-adjoint to the Lie-ation functor  $F$ .*

**Notations 1.** *In the following, we will use notations as above and also*

1. *Identity of  $\mathfrak{g}_i$  (resp. of  $\mathcal{U}(\mathfrak{g}_i)$ ) will be noted, for short,  $I_i$  (resp.  $I_{\mathcal{U}_i}$ )*
2. *The maps  $\sigma_i : \mathfrak{g}_i \mapsto \mathcal{U}(\mathfrak{g}_i)$  (resp. the map  $\sigma : \mathfrak{g} \mapsto \mathcal{U}(\mathfrak{g})$ ),*
3. *The maps deduced by universal constructions, as in the preamble, the maps  $\mathcal{U}(j_i)$*

$$\begin{array}{ccc}
 \mathfrak{g}_i & \xrightarrow{j_i} & \mathfrak{g} \\
 \sigma_i \downarrow & & \downarrow \sigma \\
 \mathcal{U}(\mathfrak{g}_i) & \xrightarrow{\mathcal{U}(j_i)} & \mathcal{U}(\mathfrak{g})
 \end{array} \tag{9}$$

*and  $\psi : T(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$  (resp.  $\psi_i : T(\mathfrak{g}_i) \rightarrow \mathcal{U}(\mathfrak{g}_i)$ ) the natural (quotient) maps (see Bourbaki [3] Ch I §2.7 p17).*

4. *The chaining of domains and maps involved is as follows*

$$\begin{array}{ccccccc}
 T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{\mathcal{U}_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\mathcal{U}(j_1) \otimes \mathcal{U}(j_2)} & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) & \xrightarrow{\mu} & \mathcal{U}(\mathfrak{g}) \\
 \underbrace{\hspace{10em}} & & \underbrace{\hspace{10em}} & & \underbrace{\hspace{10em}} & & \uparrow
 \end{array} \tag{10}$$

*The upper long arrow being  $\mu_{state} := \mu_{state}^U = \mu \circ (\mathcal{U}(j_1) \otimes \mathcal{U}(j_2))$ , the lower long (longer) arrow being  $\mu_{state}^T = \mu_{state}^U \circ (\psi_1 \otimes I_{\mathcal{U}_2})$ . We will now construct the normal form calculator and, from it, deduce a section  $s$  of  $\mu_{state}^U$  which will turn out to be bijective. Knowing already that  $\mu_{state}^U$  is surjective, it will be sufficient to establish that*

$$s \circ \mu_{state}^U = Id_{\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)}$$

### 3 Step-by-Step construction of a normal form calculator

Having remarked that the domain and codomain of  $\mu_{state} (= \mu_{state}^U)$  are  $\mathcal{U}(\mathfrak{g}_1) - \mathcal{U}(\mathfrak{g}_2)$  modules ( $\mathcal{U}(\mathfrak{g}_1)$ – by multiplication on the left and  $-\mathcal{U}(\mathfrak{g}_2)$  by multiplication on the right for the domain and through  $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$  for the codomain) our strategy will be to construct a  $-\mathcal{U}(\mathfrak{g}_2)$  section of  $\mu$  (this linearity will help us to make the construction, at first defined on  $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ , pass to quotients). We observe now, in all cases when  $\mu_{state}$  is one-to-one, there is an action on the left ( $g *_{\mathcal{U}}$ ) of  $\mathcal{U}(\mathfrak{g})$  on the space of states  $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$  provided by transport of structure [17] as follows

$$g *_{\mathcal{U}} (m_1 \otimes m_2) = \mu_{state}^{-1} (g \cdot \mu_{state} (m_1 \otimes m_2)) \quad (11)$$

Now, we will construct this action in the general case by passing to quotients a similar action on  $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$  denoted by  $g *_T -$ . This compatibility (to be proved) is illustrated by the following diagram

$$\begin{array}{ccc} T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_2} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \\ \downarrow g *_T - & & \downarrow g *_{\mathcal{U}} - \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_2} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (12)$$

where  $\psi_1 : T(\mathfrak{g}_1) \rightarrow \mathcal{U}(\mathfrak{g}_1)$  is the natural (quotient) map and  $I_2 = Id_{\mathcal{U}(\mathfrak{g}_2)}$ . We will proceed in four steps

1. Construction
2. Compatibility with  $\equiv_{\psi_1}$
3. Action  $*_{\mathcal{U}}$  as a Lie action.
4. Section and isomorphism

#### 3.1 Construction of the actions $g *$

Let us recall that  $\mathfrak{g}$  is a Lie algebra split ( $\mathbf{k}$ -module decomposition) as follows

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad (\text{here } \oplus = \oplus_{\mathbf{k}\text{-mod}}) \quad (13)$$

Let  $j_i, p_i$  be the corresponding embeddings and projectors (see also the end of paragraph 2.1). In addition, we will note  $\psi_1$  the morphism of  $\mathbf{k}$ -AAU  $\psi_1 : T(\mathfrak{g}_1) \rightarrow \mathcal{U}(\mathfrak{g}_1)$  obtained by multiplication of factors and  $\mathcal{U}(j_2)$ , the natural morphism  $\mathcal{U}(j_2) : \mathcal{U}(\mathfrak{g}_2) \rightarrow \mathcal{U}(\mathfrak{g})$  as defined above (see diagram 9) We now have the following

**Theorem 1.** *With the notations as above,*

*i) there exists a unique linear map*

$$\Phi : \mathfrak{g} \otimes T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2). \quad (14)$$

*(in the sequel,  $\Phi(g \otimes t \otimes m)$  will be alternatively noted  $g *_T (t \otimes m)$ ) such that*

$$\left\{ \begin{array}{l} g *_T (1_{T(\mathfrak{g}_1)} \otimes m) = p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g).m \text{ for all } (g, m) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_2) \\ g *_T (x \otimes t \otimes m) = [g, j_1(x)] *_T (t \otimes m) + x \otimes (g *_T (t \otimes m)) \\ \text{for all } (g, x, t, m) \in \mathfrak{g} \times \mathfrak{g}_1 \times T(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2) \end{array} \right. \quad (15)$$

*(Nota : For the sake of clarity, we have used the blue tensor product as explained in Remark 1.2.)*

*ii) This map is filtered in the following sense*

$$\Phi(\mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)) \subset T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \quad (16)$$

iii) It is compatible with

a) The  $\mathcal{U}(\mathfrak{g}_2)$  right module structure of  $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$  as

$$\begin{aligned} & \text{for } (g, t, m) \in \mathfrak{g} \times T(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2) \text{ one has} \\ & g *_T (t \otimes m) = (g *_T (t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})) \cdot m \end{aligned} \quad (17)$$

b) Multiplication of factors. Let

$$\mu_{state}^T = \mu \circ (\mathcal{U}(j_1) \otimes \mathcal{U}(j_2)) \circ (\psi_1 \otimes I_2) \quad (18)$$

(see Notation (1) and Eq. (3)) as, for all  $(g, t, m) \in \mathfrak{g} \times T(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$ , we have

$$\mu_{state}^T(g *_T (t \otimes m)) = \sigma(g) \cdot \mu_{state}^T(t \otimes m) = \sigma(g) \cdot \mu_{state}^T(t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) \cdot \mathcal{U}(j_2)(m) \quad (19)$$

iv) There is a unique map

$$\Phi_U : \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \quad (20)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} \otimes T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{I_1 \otimes \psi_1 \otimes I_{U_2}} & \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \\ \Phi \downarrow & & \Phi_U \downarrow \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (21)$$

*Proof.* i) and ii) We will show, by induction on  $n$ , the following statement:

For all  $n \geq 0$ , there exists a unique linear map

$$\Phi_n : \mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \quad (22)$$

noted  $g *_T^{(n)}(t \otimes m) := \Phi_n(g \otimes t \otimes m)$  such that,

$$\left\{ \begin{array}{l} g *_T^{(n)}(1_{T(\mathfrak{g}_1)} \otimes m) = p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g) \cdot m \text{ for all } (g, m) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_2) \\ g *_T^{(n)}(x \otimes t \otimes m) = [g, j_1(x)] *_T^{(n-1)}(t \otimes m) + x \otimes (g *_T^{(n-1)}(t \otimes m)) \\ \text{for all } (g, x, t, m) \in \mathfrak{g} \times \mathfrak{g}_1 \times T_{\leq n-1}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2) \end{array} \right. \quad (23)$$

For  $n = 0$ ,  $\Phi_0$  is clearly uniquely defined by

$$\Phi_0(g \otimes \lambda \cdot 1_{T(\mathfrak{g}_1)} \otimes m) := \lambda \cdot (p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g) \cdot m).$$

We now suppose  $(\Phi_j)_{0 \leq j \leq n}$  to be uniquely defined by (23) and show the same for some<sup>4</sup>

$$\begin{aligned} & \Phi_{n+1} : \mathfrak{g} \otimes T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T_{\leq n+2}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \text{ with} \\ & g *_T^{(n+1)}(t \otimes m) := \Phi_{n+1}(g \otimes t \otimes m) \end{aligned} \quad (24)$$

Remarking that

$$\mathfrak{g} \otimes T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) = \mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \oplus \mathfrak{g} \otimes T_{n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$$

we define  $\Phi_{n+1}$  as coinciding with  $\Phi_n$  on the sector  $\mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ .

Now for

$$(g, x, t, m) \in \mathfrak{g} \times \mathfrak{g}_1 \times T_n(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2),$$

we observe that

$$(g, x, t, m) \mapsto [g, x] *_T(t \otimes m) + x \otimes (g *_T(t \otimes m))$$

is  $\mathbf{k}$ -quadrilinear which entails existence and unicity of a linear map

$$\check{\Phi}_{n+1} : \mathfrak{g} \otimes (\mathfrak{g}_1 \otimes T_n(\mathfrak{g}_1)) \otimes \mathcal{U}(\mathfrak{g}_2) = \mathfrak{g} \otimes T_{n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T_{\leq n+2}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$$

This allows us to set  $\Phi_{n+1} = \Phi_n \oplus \check{\Phi}_{n+1}$  which is uniquely defined due to (23)<sup>5</sup>.

<sup>4</sup>which will turn out to be unique.

<sup>5</sup>In fact,  $\Phi$  is the inductive limit of the sequence  $\Phi_n$ .

iii) a) Again, by induction.

b) Compatibility with  $\mu_{state}^T$ .

Again, we prove this by induction on  $n$  on the property that, for all  $(g, t, m) \in \mathfrak{g} \times T_n(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$ , we have (19).

For  $n = 0$ , it suffices to remark that

$$\begin{aligned} \mu_{state}^T(g *_{T}(1_{T(\mathfrak{g}_1)} \otimes m)) &= \mu_{state}^T(p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g).m) = \\ \mu_{state}^T(p_1(g) \otimes m) + \mu_{state}^T(1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g).m) &= \sigma(j_1 p_1(g) + j_2 p_2(g)).\mathcal{U}(j_2)(m) = \\ \sigma(g).m &= \sigma(g).\mu_{state}^T(1_{T(\mathfrak{g}_1)} \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}).\mathcal{U}(j_2)(m) \end{aligned} \quad (25)$$

For  $n \geq 1$  we prove (19) by induction using linear generators of  $T_n(\mathfrak{g}_1)$  i.e. the family  $(x \otimes t)_{x \in \mathfrak{g}_1 \times T_{n-1}(\mathfrak{g}_1)}$  then

$$\mu_{state}^T(g *_{T}((x \otimes t) \otimes m)) = \mu_{state}^T([g, j_1(x)] *_{T}(t \otimes m)) + \mu_{state}^T(x \otimes (g *_{T}(t \otimes m))) \quad (26)$$

$$\sigma([g, j_1(x)])\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) + \mathcal{U}(j_1)\sigma_1(x).\sigma(g).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (27)$$

$$\sigma([g, j_1(x)])\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) + \sigma j_1(x).\sigma(g).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (28)$$

$$(\sigma(g).\sigma j_1(x) - \sigma j_1(x).\sigma(g))\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) + \quad (29)$$

$$\sigma j_1(x).\sigma(g).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (30)$$

$$\sigma(g).\sigma j_1(x).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \sigma(g).\mathcal{U}(j_1)\psi_1(x).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (31)$$

$$\sigma(g).\mathcal{U}(j_1)(\psi_1(x).\psi_1(t)).\mathcal{U}(j_2)(m) = \sigma(g).\mathcal{U}(j_1)\psi_1(x \otimes t).\mathcal{U}(j_2)(m) = \quad (32)$$

$$\sigma(g).\mu_{state}^T((x \otimes t) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}).\mathcal{U}(j_2)(m) \quad (33)$$

iv) We first construct  $g *_{U}(t \otimes m)$  for tensors of the type  $t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$  i.e. we construct the restriction of  $\Phi_U$  on  $\mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$  and prove that the following diagram commutes

$$\begin{array}{ccc} T(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} \\ g *_{T} \downarrow & & g *_{U} \downarrow \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (34)$$

and use the following lemma

**Lemma 1.** *Let  $\mathcal{A}_i, \mathcal{B}, i = 1..2$  be  $\mathbf{k}$ -AAU and  $s : \mathcal{A}_1 \rightarrow \mathcal{A}_2, \epsilon : \mathcal{B} \rightarrow \mathbf{k}$  be morphisms (of  $\mathbf{k}$ -AAU). Then*

i)  $\mathcal{A}_i \rightarrow \mathcal{A}_i \otimes \mathcal{B}$  defined by  $x \mapsto x \otimes 1_{\mathcal{B}}$  is injective (the image of it will be noted  $\mathcal{A}_i \otimes 1_{\mathcal{B}}$ ).

ii) The kernel of  $s \otimes Id_{\mathcal{B}}$  is  $\ker(s) \otimes 1_{\mathcal{B}}$ .

*Proof.* Left to the reader. □

**End of the proof of (iv).** —

We complete the proof of diagram (34). As  $\psi_1$  is surjective, so is  $\psi_1 \otimes I_{U_2}$  (even its restriction i.e. from  $T(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$  to  $\mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$ , let us call  $\xi_1$  this restriction) so that the diagram (34), in fact, becomes

$$\begin{array}{ccc} T(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} & \xrightarrow{\xi_1} & \mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} \\ g *_{T} \downarrow & & g *_{U} \downarrow \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (35)$$

From Lemma 1, the kernel of  $\sigma_1$  is the module generated, for  $(p, x, y) \in T(\mathfrak{g}_1) \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_1$  by the family of tensors ( $s$  is omitted in the indexation because it will not vary throughout the proof

$$E(p, x, y) := (p \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) - (p \otimes y \otimes x \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) - (p \otimes [x, y] \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})$$

then, the existence (and unicity) of  $g *_{U}$  — amounts to prove that, for  $(p, x, y) \in T(\mathfrak{g}_1) \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_1$ , we have  $g *_{T}(E(p, x, y)) = 0$ . Let us set  $T(p, x, y) := g *_{T}((p \otimes x \otimes y \otimes$

$s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$ . We proceed by cases.

First case :  $p \in T_n(\mathfrak{g}_1)$  for  $n \geq 1$ .

We check the fact for the tensors  $p = a \otimes p'$  (sufficient because these tensors generate  $T_+(\mathfrak{g}_1) = \bigoplus_{n \geq 1} T_n(\mathfrak{g}_1)$ ). Let us set  $T(p, u) := p \otimes u \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$ , we have to prove that  $g *_T (T(p, x \otimes y)) - g *_T (T(p, y \otimes x)) \equiv g *_T (T(p, [x, y]))$

where  $X \equiv Y$  stands for  $X - Y \in \ker(\psi_1 \otimes I_{U_2})$ .

By direct computation we get

$$g *_T (a \otimes p' \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) = [g, a] *_T (p' \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) + a \otimes g *_T (p' \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) \quad (36)$$

from this, we see, by induction, that all amounts to prove the fact for  $n = 0$ . Then,

Second case :  $p \in T_n(\mathfrak{g}_1)$  for  $n = 0$ .

By homogeneity, we can suppose  $p = 1_{T_n(\mathfrak{g}_1)}$ . Let us compute

$$\begin{aligned} g *_T (x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) &= \\ \underbrace{[g, x] *_T (y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})}_{T_1(x, y)} + \underbrace{x \otimes g *_T (y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})}_{T_2(x, y)} &= \\ \underbrace{[[g, x], y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})}_{T_1(x, y) = T_{11}(x, y) + T_{12}(x, y)} + \underbrace{y \otimes ([g, x] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2))))}_{T_2(x, y) = T_{21}(x, y) + T_{22}(x, y)} &= \\ + \underbrace{x \otimes ([g, y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2))))}_{T_2(x, y) = T_{21}(x, y) + T_{22}(x, y)} + \underbrace{x \otimes y \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2))))}_{T_2(x, y) = T_{21}(x, y) + T_{22}(x, y)} &= \end{aligned} \quad (37)$$

Then

$$\begin{aligned} T_{11}(x, y) - T_{11}(y, x) &= [g, [x, y]] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) \\ T_{12}(x, y) - T_{21}(y, x) &= y \otimes ([g, x] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) - y \otimes ([g, x] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) = 0 \\ T_{21}(x, y) - T_{12}(y, x) &= x \otimes ([g, y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) - x \otimes ([g, y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) = 0 \\ T_{22}(x, y) - T_{22}(y, x) &= x \otimes y \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) - y \otimes x \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) \equiv \\ [x, y] \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) & \end{aligned} \quad (38)$$

(we recall that  $X \equiv Y$  stands for  $X - Y \in \ker(\psi_1 \otimes I_{U_2})$ ). Then

$$\begin{aligned} g *_T (x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) - g *_T (y \otimes x \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) &\equiv \\ [g, [x, y]] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) + [x, y] \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) &= \\ g *_T ([x, y] \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) & \end{aligned} \quad (39)$$

then, there exists  $g *_U$  such that (34) commutes.

**End of the proof of (21).** —

We set, for  $(g, t, m) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$ ,

$$g *_U (t \otimes m) := g *_U (t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) . m \quad (40)$$

Now, we remark that  $(g, t, m) \mapsto g *_U (t \otimes m)$  is trilinear and this completes the proof.  $\square$

**Corollary 1.** For all  $(g, m_1, m_2) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$ , one has

$$\mu_{state}^U(g *_U (m_1 \otimes m_2)) = \sigma(g) . \mathcal{U}(j_1)(m_1) . \mathcal{U}(j_2)(m_2) \quad (41)$$

*Proof.* From theorem (1) (iii.b, in particular (19)) and diagram (21).  $\square$

### 3.2 $g *_U$ is a $\mathfrak{g}$ -action on $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ .

We here prove that  $g *_U$  defines a Lie  $\mathfrak{g}$ -action on  $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$  i.e. for all  $(g, h, m_1, m_2) \in \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$  we have (below  $*$  will stand for  $*_U$ )

$$g * (h * (m_1 \otimes m_2)) - h * (g * (m_1 \otimes m_2)) = [g, h] * (m_1 \otimes m_2) \quad (42)$$

Let us then set  $T(g, h) = g * (h * (m_1 \otimes m_2))$ .

We have 4 cases (which can be reduced to 3 by antisymmetry)



a)  $(g, h) \in \mathfrak{g}_1 \times \mathfrak{g}_1$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (m_1 \otimes m_2)) - h * (g * (m_1 \otimes m_2)) = \\ &= \sigma_1(g) \cdot \sigma_1(h) \cdot m_1 \otimes m_2 - \sigma_1(h) \cdot \sigma_1(g) \cdot m_1 \otimes m_2 \equiv \\ &= \sigma_1([g, h]) \cdot m_1 \otimes m_2 = [g, h] * (m_1 \otimes m_2) \end{aligned} \quad (43)$$

b)  $(g, h) \in \mathfrak{g}_2 \times \mathfrak{g}_1$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (m_1 \otimes m_2)) - h * (g * (m_1 \otimes m_2)) = \\ &= g * (\sigma_1(h) \cdot m_1 \otimes m_2) - \sigma_1(h) \cdot (g * (m_1 \otimes m_2)) = \\ &= [g, j_1(h)] * (m_1 \otimes m_2) + \sigma_1(h) \cdot (g * (m_1 \otimes m_2)) - \sigma_1(h) \cdot (g * (m_1 \otimes m_2)) = \\ &= [g, j_1(h)] * (m_1 \otimes m_2) \end{aligned} \quad (44)$$

c)  $(g, h) \in \mathfrak{g}_1 \times \mathfrak{g}_2$

Is true by antisymmetry.

d)  $(g, h) \in \mathfrak{g}_2 \times \mathfrak{g}_2$

For the computation of  $T(g, h) - T(h, g)$ , we have two cases.

d1)  $m_1 = 1_{\mathcal{U}(\mathfrak{g}_1)}$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes m_2)) - h * (g * (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes m_2)) = \\ &= 1_{\mathcal{U}(\mathfrak{g}_1)} \otimes \sigma_2(g) \cdot \sigma_2(h) \cdot m_2 - 1_{\mathcal{U}(\mathfrak{g}_1)} \otimes \sigma_2(h) \cdot \sigma_2(g) \cdot m_2 = 1_{\mathcal{U}(\mathfrak{g}_1)} \otimes \sigma_2([g, h]) \cdot m_2 = [g, h] * (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes m_2) \end{aligned} \quad (45)$$

d2)  $m_1 \in \mathcal{U}_+(\mathfrak{g}_1)$

We prove (42) by induction. Let  $m_1 \in \mathcal{U}_n(\mathfrak{g}_1)$ .

We have  $n \geq 1$  and  $\mathcal{U}_n(\mathfrak{g}_1)$  is generated by the products  $x \cdot m$  with  $x \in \mathcal{U}(\mathfrak{g}_1)$  and  $m \in \mathcal{U}_{n-1}(\mathfrak{g}_1)$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (x \cdot m \otimes m_2)) - h * (g * (x \cdot m \otimes m_2)) = \\ &= g * ([h, x] * (m \otimes m_2)) + g * (\sigma_1(x) \cdot (h * (m \otimes m_2))) \\ &\quad - h * ([g, x] * (m \otimes m_2)) - h * (\sigma_1(x) \cdot (g * (m \otimes m_2))) \\ &= \underbrace{g * ([h, x] * (m \otimes m_2))}_{T_1(g, h)} + \underbrace{[g, x] * (h * (m \otimes m_2))}_{T_2(g, h)} + \underbrace{\sigma_1(x) \cdot (g * (h * (m \otimes m_2)))}_{T_3(g, h)} \\ &\quad - \underbrace{h * ([g, x] * (m \otimes m_2))}_{T_1(h, g)} - \underbrace{[h, x] * (g * (m \otimes m_2))}_{T_2(h, g)} - \underbrace{\sigma_1(x) \cdot (h * (g * (m \otimes m_2)))}_{T_3(h, g)} \end{aligned} \quad (46)$$

Then

$$\begin{aligned} T_1(g, h) - T_2(h, g) &= [g, [h, x]] * (m \otimes m_2) \text{ by induction} \\ T_2(g, h) - T_1(h, g) &= [[g, x], h] * (m \otimes m_2) \text{ by induction} \\ T_3(g, h) - T_3(h, g) &= \sigma_1(x) \cdot ([g, h] * (m \otimes m_2)) \text{ by induction} \\ \text{Hence } T(g, h) - T(h, g) &= ([g, [h, x]] + [[g, x], h]) * (m \otimes m_2) + x \cdot ([g, h] * (m \otimes m_2)) \\ &= [[g, h], x] * (m \otimes m_2) + \sigma_1(x) \cdot ([g, h] * (m \otimes m_2)) \\ &= [g, h] * (x \cdot m \otimes m_2) \end{aligned} \quad (47)$$

We now come to the proof that  $\mu_{state}^U$  is one-to-one.

## 4 The linear map $\mu_{state}^U$ is bijective.

**Theorem 2.** *i) From the (Lie) action  $*_U$ , one deduces a unique  $\mathcal{U}(\mathfrak{g})$ -module structure on  $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$  (noted  $*_{mod}$ ) such that  $\sigma(g) *_{mod} (m_1 \otimes m_2) = g *_U (m_1 \otimes m_2)$ .*

*ii) The map  $s : m \mapsto m *_{mod} (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})$  and  $\mu_{state}^U$  are mutually inverse.*

*Proof.* i) From Theorem (1) (iv), let us note (as above)  $g *_{\mathcal{U}} -$ , the map  $m_1 \otimes m_2 \mapsto g *_{\mathcal{U}}(m_1 \otimes m_2)$ , we then get a linear map  $\varphi : \mathfrak{g} \rightarrow \text{End}(\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2))$  and, by (42), we learn that  $\varphi$  is a morphism of  $\mathbf{k}$ -Lie algebras. By universal property of  $\mathcal{U}(\mathfrak{g})$ , we get

$$\begin{array}{ccc}
 \mathbf{k}\text{-Lie} & \xleftarrow{F} & \mathbf{k}\text{-AAU} \\
 \mathfrak{g} & \xrightarrow{\varphi} & \text{End}(\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)) \\
 & \searrow \sigma & \uparrow \widehat{\varphi} \\
 & & \mathcal{U}(\mathfrak{g}).
 \end{array} \tag{48}$$

which means that, for all  $(g, m_1, m_2) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$ ,

$$\varphi(g)[m_1 \otimes m_2] = \widehat{\varphi}(\sigma(g))[m_1 \otimes m_2] \tag{49}$$

Of course, such a morphism as  $\widehat{\varphi}$  defines at once a structure of left  $\mathcal{U}(\mathfrak{g})$ -module on  $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ . Its action will be noted  $*_{mod}$  such that

$$\sigma(g) *_{mod} (m_1 \otimes m_2) := \widehat{\varphi}(\sigma(g))[m_1 \otimes m_2]$$

which completes the first point.

ii) Knowing already that  $\mu_{state}^U$  is surjective, it will be sufficient to establish that

$$s \circ \mu_{state}^U = Id_{\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)}$$

which amounts to show that for  $(g_i)_{1 \leq i \leq p}$  in  $\mathfrak{g}_1$  (resp.  $(h_i)_{1 \leq i \leq q}$  in  $\mathfrak{g}_2$ )

$$s \circ \mu_{state}^U(\sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q) \tag{50}$$

By linearity, this will prove that  $s \circ \mu_{state}^U = Id_{\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)}$ .

From (41), for  $p > 0$ , we get,

$$\begin{aligned}
 & \mu_{state}^U(\sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \\
 & \mathcal{U}(j_1)\sigma_1(g_1)\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q))
 \end{aligned} \tag{51}$$

and, remarking that  $s$  is  $\mathcal{U}(\mathfrak{g})$ -linear we have

$$\begin{aligned}
 & s \circ \mu_{state}^U(\sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \\
 & s \left( \mathcal{U}(j_1)\sigma_1(g_1)\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) \right) = \\
 & s \left( \sigma j_1(g_1)\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) \right) = \\
 & j_1(g_1) \cdot s \mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \\
 & \sigma_1(g_1) \cdot \cdots \cdot \sigma_1(g_p) \otimes \sigma_2(h_1) \cdot \cdots \cdot \sigma_2(h_q)
 \end{aligned} \tag{52}$$

the other case ( $p = 0$ ) is straightforward. Then, by induction on  $p$ , one has  $s \circ \mu_{state}^U(m_1 \otimes m_2) = m_1 \otimes m_2$  which proves the claim.

QED □

## 5 Conclusion and future

1. Quantized enveloping algebras
2. Lie superalgebras

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