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► **To cite this version:**

G rard Henry Edmond Duchamp, Christophe Tollu, Jean-Gabriel Luque, Vu Nguyen Dinh. About enveloping algebras of direct sums. 2024. hal-04565261v1

HAL Id: hal-04565261

<https://hal.science/hal-04565261v1>

Preprint submitted on 2 May 2024 (v1), last revised 10 May 2024 (v2)

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About enveloping algebras of direct sums.

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May 1, 2024

Abstract

We solve the PBW-like problem of normal ordering for enveloping algebras of direct sums.

1 A question about the enveloping algebra of a direct sum.

This question is imported from [13]. It is linked to this one [14] in the case of semi-direct products.

Let us consider a Lie \mathbf{k} -algebra (\mathbf{k} is a commutative ring) written as a (module) direct sum of two of its subalgebras

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad (\oplus = \oplus_{\mathbf{k}\text{-mod}})$$

and the associated linear maps

$$\mathfrak{g}_i \begin{array}{c} \xrightarrow{j_i} \\ \xleftarrow{p_i} \end{array} \mathfrak{g} \quad \text{such that } j_1 p_1 + j_2 p_2 = Id_{\mathfrak{g}} \quad (1)$$

(it is a sum of two orthogonal projectors, remark that only j_i 's are Lie morphisms in general). We get, at once, the maps $\mathcal{U}(j_i)$ through the universal algebra functor \mathcal{U} (see below section 2 “Universal Constructions”) as follows.

$$\begin{array}{ccc} \mathfrak{g}_i & \xrightarrow{j_i} & \mathfrak{g} \\ \sigma_i \downarrow & & \downarrow \sigma \\ \mathcal{U}(\mathfrak{g}_i) & \xrightarrow{\mathcal{U}(j_i)} & \mathcal{U}(\mathfrak{g}) \end{array} \quad (2)$$

allowing us to multiply members of $\mathcal{U}(\mathfrak{g}_1) \otimes_{\mathbf{k}} \mathcal{U}(\mathfrak{g}_2)$ within $\mathcal{U}(\mathfrak{g})$ by the composite map

$$\mu_{state} = \mu \circ (\mathcal{U}(j_1) \otimes_{\mathbf{k}} \mathcal{U}(j_2)) : \mathcal{U}(\mathfrak{g}_1) \otimes_{\mathbf{k}} \mathcal{U}(\mathfrak{g}_2) \rightarrow \mathcal{U}(\mathfrak{g}) \quad (3)$$

where μ is the multiplication of $\mathcal{U}(\mathfrak{g})$.

One can check, using generators, that μ_{state} is surjective (and, in many usual cases [13] bijective).

Question: Is the property that μ_{state} is bijective, true in the general case ?

Remarks 1. 1. Unless explicitly stated, all tensor products will be understood over \mathbf{k} .

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2. All states below are elements of spaces of the form $M = \mathcal{A}_1 \otimes \mathcal{A}_2$ (see Eq. 10) where, for $i = 1, 2$, $(\mathcal{A}_i, *_i)$ is a \mathbf{k} -AAU. Such a space M is naturally a $\mathcal{A}_1 - \mathcal{A}_2$ bimodule (\mathcal{A}_1 module by multiplication on the left and \mathcal{A}_2 module by multiplication on the right). To avoid confusion (as tensors may appear inside \mathcal{A}_1), the separating tensor of $M = \mathcal{A}_1 \otimes \mathcal{A}_2$ will, from time to time, be noted bold and oversized.

Example 1. 1. In the remarks and examples below, for any \mathbb{Z} -algebra \mathcal{A} and $q \in \mathbb{N}_{\geq 2}$, we will note $(\mathcal{A})_q$ the quotient \mathbb{Z} -algebra $\mathcal{A}/q.\mathcal{A}$ (in all cases $q.\mathcal{A}$ is an ideal). For example, the situation $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where no factor is an ideal is frequent for Lie algebras admitting a triangular decomposition¹ $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ for example

$$\mathfrak{sl}(n, \mathbb{Z}) = T_+(n, \mathbb{Z}) \oplus D(n, \mathbb{Z}) \oplus T_-(n, \mathbb{Z})$$

and one can create an example without any basis with $\mathbf{k} = \mathbb{Z}$, $q \in \mathbb{N}_{\geq 2}$ with

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sl}(n, \mathbb{Z}) = T_+(n, \mathbb{Z}) \oplus D(n, \mathbb{Z}) \oplus T_-(n, \mathbb{Z}) \text{ then} \\ (\mathfrak{g})_q &= T_+(n, \mathbb{Z})_q \oplus D(n, \mathbb{Z})_q \oplus (T_-(n, \mathbb{Z}))_q \text{ and, if one needs two factors,} \\ (\mathfrak{g})_q &= (T_+(n, \mathbb{Z}) \oplus D(n, \mathbb{Z}))_q \oplus (T_-(n, \mathbb{Z}))_q \end{aligned} \quad (4)$$

Acknowledgements

We thank Darij Grinberg, from Drexel University and Jim Humphreys (through Math-Overflow) for fruitful interactions and their interest for this question.

2 Universal Constructions

2.1 General principle.

In this subsection, we introduce the combinatorial (free) objects that we will use throughout the manuscript. These objects (call them $G(X)$) together with a map $j_X : X \rightarrow G(X)$ are all solutions of universal problems. We will recall the definition, notation and terminology about these free objects below (cf. in general Bourbaki [1] Ch IV §3 or [9] and, in particular, [2] Ch I §7.1 and Lothaire [8] Prop 1.1.1 for words and the free monoids, Bourbaki [3] Ch II §2.2 Prop 1 and Reutenauer [11] Thm 0.4 for free Lie algebras and Bourbaki [2] for enveloping algebras i.e. towards the free associative algebras with unit and Dinh Vu Nguyen's thesis [5] for all these matters), but here, we state the general principle.

The scheme is the same for all categories considered in the following list (\mathbf{k} being a fixed ring).

$$\mathbf{Mon, Grp, k-AAU, k-Lie} \quad (5)$$

All objects of these categories can be considered as sets, we then have a natural “forgetful” functor F such that, \mathcal{A} being an object (of one of these categories), $F(\mathcal{A})$ is the set underlying the structure \mathcal{A} . We are now in the position of stating the universal problem corresponding to the forgetful functor F .

Universal problem (w.r.t. F , naive version²). —

For any set X (\mathcal{C} being one of the categories as above) does there exist a pair $(j_X, G(X))$ ($G(X)$ being an object of \mathcal{C} and $j_X : X \rightarrow G(X)$ an heteromorphism) such that:

For any map $f : X \rightarrow \mathcal{A}$ (heteromorphism), there exists a unique $\hat{f} \in \text{Hom}_{\mathcal{C}}(G(X), \mathcal{A})$ such that $f = F(\hat{f}) \circ j_X$.

Remarks 2. *i)* It might happen that G be not defined everywhere as shows the case with $\mathcal{C} = \mathbf{FinSet}$, F being the inclusion functor (i.e. $F(X) = X$ for every finite set and $F(f) = f$ for every set-theoretical map between finite sets).

However a solution of the universal problem (6), for all X , provides a free functor $G : \mathbf{Set} \rightarrow$

¹As Kac-Moody algebras [10], have also a look at [7] Ch 1 Exercise 1.8 (local Lie algebras) probably due to Vinberg [4] who heralded the notion of pre-Lie algebras.

²See the theory of Heteromorphisms[12]

$\mathcal{C}, X \mapsto G(X)$ which is left-adjoint to the forgetful functor $F : \mathcal{C} \rightarrow \mathbf{Set}$. The reader must be aware that, in general, the notion of “forgetful functor” (here constructed from algebraic structures and sets) is informal.

$$\begin{array}{ccc}
 \mathbf{Set} & \xleftarrow{\quad F \quad} & \mathcal{C} \\
 X & \xrightarrow{\quad f \quad} & \mathcal{A} \\
 & \searrow \scriptstyle j_X & \uparrow \scriptstyle \widehat{f} \\
 & & G(X).
 \end{array} \tag{6}$$

ii) We recall here that the universal enveloping algebra of a Lie \mathbf{k} -algebra \mathfrak{g} is a pair $(\sigma, \mathcal{U}(\mathfrak{g}))$, where $\mathcal{U}(\mathfrak{g})$ is an object in $\mathbf{k-AAU}$ and $\sigma : \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$ is a morphism in $\mathbf{k-Lie}$, which is a solution of the following universal problem:

$$\begin{array}{ccc}
 \mathbf{k-Lie} & \xleftarrow{\quad F \quad} & \mathbf{k-AAU} \\
 \mathfrak{g} & \xrightarrow{\quad f \quad} & \mathcal{A} \\
 & \searrow \scriptstyle \sigma & \uparrow \scriptstyle \widehat{f} \\
 & & \mathcal{U}(\mathfrak{g}).
 \end{array} \tag{7}$$

From this arises that there exists the universal enveloping functor

$$\mathcal{U} : \mathbf{k-Lie} \rightarrow \mathbf{k-AAU}, \quad \mathfrak{g} \mapsto (\sigma, \mathcal{U}(\mathfrak{g})) \tag{8}$$

which is a left-adjoint to the Lie-ation functor F .

Notations 1. In the following, we will use notations as above and also

1. Identity of \mathfrak{g}_i (resp. of $\mathcal{U}(\mathfrak{g}_i)$) will be noted, for short, I_i (resp. $I_{\mathcal{U}_i}$)
2. The maps $\sigma_i : \mathfrak{g}_i \mapsto \mathcal{U}(\mathfrak{g}_i)$ (resp. the map $\sigma : \mathfrak{g} \mapsto \mathcal{U}(\mathfrak{g})$),
3. The maps deduced by universal constructions, as in the preamble, the maps $\mathcal{U}(j_i)$

$$\begin{array}{ccc}
 \mathfrak{g}_i & \xrightarrow{\quad j_i \quad} & \mathfrak{g} \\
 \sigma_i \downarrow & & \downarrow \sigma \\
 \mathcal{U}(\mathfrak{g}_i) & \xrightarrow{\quad \mathcal{U}(j_i) \quad} & \mathcal{U}(\mathfrak{g})
 \end{array} \tag{9}$$

and $\psi : T(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$ (resp. $\psi_i : T(\mathfrak{g}_i) \rightarrow \mathcal{U}(\mathfrak{g}_i)$) the natural (quotient) maps (see Bourbaki [3] Ch I §2.7 p17).

4. The chaining of domains and maps involved is as follows

$$\begin{array}{ccccccc}
 T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{\mathcal{U}_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\mathcal{U}(j_1) \otimes \mathcal{U}(j_2)} & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) & \xrightarrow{\mu} & \mathcal{U}(\mathfrak{g}) \\
 & \searrow & & \searrow & & \nearrow & \\
 & & & & & &
 \end{array} \tag{10}$$

The upper long arrow being $\mu_{state} := \mu_{state}^U = \mu \circ (\mathcal{U}(j_1) \otimes \mathcal{U}(j_2))$, the lower long (longer) arrow being $\mu_{state}^T = \mu_{state}^U \circ (\psi_1 \otimes I_{\mathcal{U}_2})$. We will now construct the normal form calculator and, from it, deduce a section s of μ_{state}^U which will turn out to be bijective. Knowing already that μ_{state}^U is surjective, it will be sufficient to establish that

$$s \circ \mu_{state}^U = Id_{\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)}$$

3 Step-by-Step construction of a normal form calculator

Having remarked that the domain and codomain of $\mu_{state} (= \mu_{state}^U)$ are $\mathcal{U}(\mathfrak{g}_1) - \mathcal{U}(\mathfrak{g}_2)$ modules ($\mathcal{U}(\mathfrak{g}_1) -$ by multiplication on the left and $-\mathcal{U}(\mathfrak{g}_2)$ by multiplication on the right for the domain and through $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ for the codomain) our strategy will be to construct a $-\mathcal{U}(\mathfrak{g}_2)$ section of μ (this linearity will help us to make the construction, at first defined on $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$, pass to quotients). We observe now, in all cases when μ_{state} is one-to-one, there is an action on the left ($g *_{\mathcal{U}}$) of $\mathcal{U}(\mathfrak{g})$ on the space of states $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ provided by transport of structure [17] as follows

$$g *_{\mathcal{U}} (m_1 \otimes m_2) = \mu_{state}^{-1} (g \cdot \mu_{state} (m_1 \otimes m_2)) \quad (11)$$

Now, we will construct this action in the general case by passing to quotients a similar action on $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ denoted by $g *_{\mathcal{T}}$. This compatibility (to be proved) is illustrated by the following diagram

$$\begin{array}{ccc} T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_2} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \\ \downarrow g *_{\mathcal{T}} & & \downarrow g *_{\mathcal{U}} \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_2} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (12)$$

where $\psi_1 : T(\mathfrak{g}_1) \rightarrow \mathcal{U}(\mathfrak{g}_1)$ is the natural (quotient) map and $I_2 = Id_{\mathcal{U}(\mathfrak{g}_2)}$. We will proceed in four steps

1. Construction
2. Compatibility with \equiv_{ψ_1}
3. Action $*_{\mathcal{U}}$ as a Lie action.
4. Section and isomorphism

3.1 Construction of the actions $g*$

Let us recall that \mathfrak{g} is a Lie algebra split (\mathbf{k} -module decomposition) as follows

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad (\text{here } \oplus = \oplus_{\mathbf{k}\text{-mod}}) \quad (13)$$

Let j_i, p_i be the corresponding embeddings and projectors (see also the end of paragraph 2.1). In addition, we will note ψ_1 the morphism of \mathbf{k} -AAU $\psi_1 : T(\mathfrak{g}_1) \rightarrow \mathcal{U}(\mathfrak{g}_1)$ obtained by multiplication of factors and $\mathcal{U}(j_2)$, the natural morphism $\mathcal{U}(j_2) : \mathcal{U}(\mathfrak{g}_2) \rightarrow \mathcal{U}(\mathfrak{g})$ as defined above (see diagram 9) We now have the following

Theorem 1. *With the notations as above,*

i) there exists a unique linear map

$$\Phi : \mathfrak{g} \otimes T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2). \quad (14)$$

*(in the sequel, $\Phi(g \otimes t \otimes m)$ will be alternatively noted $g *_{\mathcal{T}} (t \otimes m)$) such that*

$$\left\{ \begin{array}{l} g *_{\mathcal{T}} (1_{T(\mathfrak{g}_1)} \otimes m) = p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g).m \text{ for all } (g, m) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_2) \\ g *_{\mathcal{T}} (x \otimes t \otimes m) = [g, j_1(x)] *_{\mathcal{T}} (t \otimes m) + x \otimes (g *_{\mathcal{T}} (t \otimes m)) \\ \text{for all } (g, x, t, m) \in \mathfrak{g} \times \mathfrak{g}_1 \times T(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2) \end{array} \right. \quad (15)$$

(Nota : For the sake of clarity, we have used the blue tensor product as explained in Remark 1.2.)

ii) This map is filtered in the following sense

$$\Phi(\mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)) \subset T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \quad (16)$$

iii) It is compatible with

a) The $\mathcal{U}(\mathfrak{g}_2)$ right module structure of $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ as

$$\begin{aligned} & \text{for } (g, t, m) \in \mathfrak{g} \times T(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2) \text{ one has} \\ & g *_T (t \otimes m) = (g *_T (t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})) \cdot m \end{aligned} \quad (17)$$

b) Multiplication of factors. Let

$$\mu_{state}^T = \mu \circ (\mathcal{U}(j_1) \otimes \mathcal{U}(j_2)) \circ (\psi_1 \otimes I_2) \quad (18)$$

(see Notation (1) and Eq. (3)) as, for all $(g, t, m) \in \mathfrak{g} \times T(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$, we have

$$\mu_{state}^T(g *_T (t \otimes m)) = \sigma(g) \cdot \mu_{state}^T(t \otimes m) = \sigma(g) \cdot \mu_{state}^T(t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) \cdot \mathcal{U}(j_2)(m) \quad (19)$$

iv) There is a unique map

$$\Phi_U : \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \quad (20)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} \otimes T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{I_1 \otimes \psi_1 \otimes I_{U_2}} & \mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \\ \Phi \downarrow & & \Phi_U \downarrow \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (21)$$

Proof. i) and ii) We will show, by induction on n , the following statement:

For all $n \geq 0$, there exists a unique linear map

$$\Phi_n : \mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \quad (22)$$

noted $g *_T^{(n)}(t \otimes m) := \Phi_n(g \otimes t \otimes m)$ such that,

$$\left\{ \begin{array}{l} g *_T^{(n)}(1_{T(\mathfrak{g}_1)} \otimes m) = p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g) \cdot m \text{ for all } (g, m) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_2) \\ g *_T^{(n)}(x \otimes t \otimes m) = [g, j_1(x)] *_T^{(n-1)}(t \otimes m) + x \otimes (g *_T^{(n-1)}(t \otimes m)) \\ \text{for all } (g, x, t, m) \in \mathfrak{g} \times \mathfrak{g}_1 \times T_{\leq n-1}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2) \end{array} \right. \quad (23)$$

For $n = 0$, Φ_0 is clearly uniquely defined by

$$\Phi_0(g \otimes \lambda \cdot 1_{T(\mathfrak{g}_1)} \otimes m) := \lambda \cdot (p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g) \cdot m).$$

We now suppose $(\Phi_j)_{0 \leq j \leq n}$ to be uniquely defined by (23) and show the same for some³

$$\begin{aligned} & \Phi_{n+1} : \mathfrak{g} \otimes T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T_{\leq n+2}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \text{ with} \\ & g *_T^{(n+1)}(t \otimes m) := \Phi_{n+1}(g \otimes t \otimes m) \end{aligned} \quad (24)$$

Remarking that

$$\mathfrak{g} \otimes T_{\leq n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) = \mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \oplus \mathfrak{g} \otimes T_{n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$$

we define Φ_{n+1} as coinciding with Φ_n on the sector $\mathfrak{g} \otimes T_{\leq n}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$.

Now for

$$(g, x, t, m) \in \mathfrak{g} \times \mathfrak{g}_1 \times T_n(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2),$$

we observe that

$$(g, x, t, m) \mapsto [g, x] *_T(t \otimes m) + x \otimes (g *_T(t \otimes m))$$

is \mathbf{k} -quadrilinear which entails existence and unicity of a linear map

$$\check{\Phi}_{n+1} : \mathfrak{g} \otimes (\mathfrak{g}_1 \otimes T_n(\mathfrak{g}_1)) \otimes \mathcal{U}(\mathfrak{g}_2) = \mathfrak{g} \otimes T_{n+1}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \rightarrow T_{\leq n+2}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$$

This allows us to set $\Phi_{n+1} = \Phi_n \oplus \check{\Phi}_{n+1}$ which is uniquely defined due to (23)⁴.

³which will turn out to be unique.

⁴In fact, Φ is the inductive limit of the sequence Φ_n .

iii) a) Again, by induction.

b) Compatibility with μ_{state}^T .

Again, we prove this by induction on n on the property that, for all $(g, t, m) \in \mathfrak{g} \times T_n(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$, we have (19).

For $n = 0$, it suffices to remark that

$$\begin{aligned} \mu_{state}^T(g *_{T}(1_{T(\mathfrak{g}_1)} \otimes m)) &= \mu_{state}^T(p_1(g) \otimes m + 1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g).m) = \\ \mu_{state}^T(p_1(g) \otimes m) + \mu_{state}^T(1_{T(\mathfrak{g}_1)} \otimes \sigma_2 p_2(g).m) &= \sigma(j_1 p_1(g) + j_2 p_2(g)).\mathcal{U}(j_2)(m) = \\ \sigma(g).m &= \sigma(g).\mu_{state}^T(1_{T(\mathfrak{g}_1)} \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}).\mathcal{U}(j_2)(m) \end{aligned} \quad (25)$$

For $n \geq 1$ we prove (19) by induction using linear generators of $T_n(\mathfrak{g}_1)$ i.e. the family $(x \otimes t)_{x \in \mathfrak{g}_1 \times T_{n-1}(\mathfrak{g}_1)}$ then

$$\mu_{state}^T(g *_{T}((x \otimes t) \otimes m)) = \mu_{state}^T([g, j_1(x)] *_{T}(t \otimes m)) + \mu_{state}^T(x \otimes (g *_{T}(t \otimes m))) \quad (26)$$

$$\sigma([g, j_1(x)])\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) + \mathcal{U}(j_1)\sigma_1(x).\sigma(g).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (27)$$

$$\sigma([g, j_1(x)])\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) + \sigma j_1(x).\sigma(g).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (28)$$

$$(\sigma(g).\sigma j_1(x) - \sigma j_1(x).\sigma(g))\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) + \quad (29)$$

$$\sigma j_1(x).\sigma(g).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (30)$$

$$\sigma(g).\sigma j_1(x).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \sigma(g).\mathcal{U}(j_1)\psi_1(x).\mathcal{U}(j_1)\psi_1(t).\mathcal{U}(j_2)(m) = \quad (31)$$

$$\sigma(g).\mathcal{U}(j_1)(\psi_1(x).\psi_1(t)).\mathcal{U}(j_2)(m) = \sigma(g).\mathcal{U}(j_1)\psi_1(x \otimes t).\mathcal{U}(j_2)(m) = \quad (32)$$

$$\sigma(g).\mu_{state}^T((x \otimes t) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}).\mathcal{U}(j_2)(m) \quad (33)$$

iv) We first construct $g *_{U}(t \otimes m)$ for tensors of the type $t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$ i.e. we construct the restriction of Φ_U on $\mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$ and prove that the following diagram commutes

$$\begin{array}{ccc} T(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} \\ g *_{T} \downarrow & & g *_{U} \downarrow \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (34)$$

and use the following lemma

Lemma 1. *Let $\mathcal{A}_i, \mathcal{B}, i = 1..2$ be \mathbf{k} -AAU and $s : \mathcal{A}_1 \rightarrow \mathcal{A}_2, \epsilon : \mathcal{B} \rightarrow \mathbf{k}$ be morphisms (of \mathbf{k} -AAU). Then*

i) $\mathcal{A}_i \rightarrow \mathcal{A}_i \otimes \mathcal{B}$ defined by $x \mapsto x \otimes 1_{\mathcal{B}}$ is injective (the image of it will be noted $\mathcal{A}_i \otimes 1_{\mathcal{B}}$).

ii) The kernel of $s \otimes Id_{\mathcal{B}}$ is $\ker(s) \otimes 1_{\mathcal{B}}$.

Proof. Left to the reader. □

End of the proof of (iv). —

We complete the proof of diagram (34). As ψ_1 is surjective, so is $\psi_1 \otimes I_{U_2}$ (even its restriction i.e. from $T(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$ to $\mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$, let us call ξ_1 this restriction) so that the diagram (34), in fact, becomes

$$\begin{array}{ccc} T(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} & \xrightarrow{\xi_1} & \mathcal{U}(\mathfrak{g}_1) \otimes 1_{\mathcal{U}(\mathfrak{g}_2)} \\ g *_{T} \downarrow & & g *_{U} \downarrow \\ T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) & \xrightarrow{\psi_1 \otimes I_{U_2}} & \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2) \end{array} \quad (35)$$

From Lemma 1, the kernel of σ_1 is the module generated, for $(p, x, y) \in T(\mathfrak{g}_1) \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_1$ by the family of tensors (s is omitted in the indexation because it will not vary throughout the proof

$$E(p, x, y) := (p \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) - (p \otimes y \otimes x \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) - (p \otimes [x, y] \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})$$

then, the existence (and unicity) of $g *_{U}$ — amounts to prove that, for

$(p, x, y) \in T(\mathfrak{g}_1) \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_1$, we have $g *_{T}(E(p, x, y)) = 0$. Let us set $T(p, x, y) := g *_{T}((p \otimes x \otimes y \otimes$

$s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$. We proceed by cases.

First case : $p \in T_n(\mathfrak{g}_1)$ for $n \geq 1$.

We check the fact for the tensors $p = a \otimes p'$ (sufficient because these tensors generate $T_+(\mathfrak{g}_1) = \bigoplus_{n \geq 1} T_n(\mathfrak{g}_1)$). Let us set $T(p, u) := p \otimes u \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}$, we have to prove that $g *_T (T(p, x \otimes y)) - g *_T (T(p, y \otimes x)) \equiv g *_T (T(p, [x, y]))$

where $X \equiv Y$ stands for $X - Y \in \ker(\psi_1 \otimes I_{U_2})$.

By direct computation we get

$$g *_T (a \otimes p' \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) = [g, a] *_T (p' \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) + a \otimes g *_T (p' \otimes x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) \quad (36)$$

from this, we see, by induction, that all amounts to prove the fact for $n = 0$. Then,

Second case : $p \in T_n(\mathfrak{g}_1)$ for $n = 0$.

By homogeneity, we can suppose $p = 1_{T_n(\mathfrak{g}_1)}$. Let us compute

$$\begin{aligned} g *_T (x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) &= \\ \underbrace{[g, x] *_T (y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})}_{T_1(x, y)} + \underbrace{x \otimes g *_T (y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})}_{T_2(x, y)} &= \\ \underbrace{[[g, x], y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) + y \otimes ([g, x] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))}_{T_1(x, y) = T_{11}(x, y) + T_{12}(x, y)} & \\ + \underbrace{x \otimes ([g, y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) + x \otimes y \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))}_{T_2(x, y) = T_{21}(x, y) + T_{22}(x, y)} & \end{aligned} \quad (37)$$

Then

$$\begin{aligned} T_{11}(x, y) - T_{11}(y, x) &= [g, [x, y]] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) \\ T_{12}(x, y) - T_{21}(y, x) &= y \otimes ([g, x] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) - y \otimes ([g, x] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) = 0 \\ T_{21}(x, y) - T_{12}(y, x) &= x \otimes ([g, y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) - x \otimes ([g, y] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) = 0 \\ T_{22}(x, y) - T_{22}(y, x) &= x \otimes y \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) - y \otimes x \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) \equiv \\ [x, y] \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) & \end{aligned} \quad (38)$$

(we recall that $X \equiv Y$ stands for $X - Y \in \ker(\psi_1 \otimes I_{U_2})$). Then

$$\begin{aligned} g *_T (x \otimes y \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) - g *_T (y \otimes x \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) &\equiv \\ [g, [x, y]] *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) + [x, y] \otimes (g *_T (s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)))) &= \\ g *_T ([x, y] \otimes s \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) & \end{aligned} \quad (39)$$

then, there exists $g *_U$ such that (34) commutes.

End of the proof of (21). —

We set, for $(g, t, m) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$,

$$g *_U (t \otimes m) := g *_U (t \otimes 1_{\mathcal{U}(\mathfrak{g}_2)}) . m \quad (40)$$

Now, we remark that $(g, t, m) \mapsto g *_U (t \otimes m)$ is trilinear and this completes the proof. \square

Corollary 1. For all $(g, m_1, m_2) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$, one has

$$\mu_{state}^U(g *_U (m_1 \otimes m_2)) = \sigma(g) . \mathcal{U}(j_1)(m_1) . \mathcal{U}(j_2)(m_2) \quad (41)$$

Proof. From theorem (1) (iii.b, in particular (19)) and diagram (21). \square

3.2 $g *_U$ is a \mathfrak{g} -action on $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$.

We here prove that $g *_U$ defines a Lie \mathfrak{g} -action on $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ i.e. for all $(g, h, m_1, m_2) \in \mathfrak{g} \times \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$ we have (below $*$ will stand for $*_U$)

$$g * (h * (m_1 \otimes m_2)) - h * (g * (m_1 \otimes m_2)) = [g, h] * (m_1 \otimes m_2) \quad (42)$$

Let us then set $T(g, h) = g * (h * (m_1 \otimes m_2))$.

We have 4 cases (which can be reduced to 3 by antisymmetry)

a) $(g, h) \in \mathfrak{g}_1 \times \mathfrak{g}_1$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (m_1 \otimes m_2)) - h * (g * (m_1 \otimes m_2)) = \\ &= \sigma_1(g) \cdot \sigma_1(h) \cdot m_1 \otimes m_2 - \sigma_1(h) \cdot \sigma_1(g) \cdot m_1 \otimes m_2 \equiv \\ &= \sigma_1([g, h]) \cdot m_1 \otimes m_2 = [g, h] * (m_1 \otimes m_2) \end{aligned} \quad (43)$$

b) $(g, h) \in \mathfrak{g}_2 \times \mathfrak{g}_1$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (m_1 \otimes m_2)) - h * (g * (m_1 \otimes m_2)) = \\ &= g * (\sigma_1(h) \cdot m_1 \otimes m_2) - \sigma_1(h) \cdot (g * (m_1 \otimes m_2)) = \\ &= [g, j_1(h)] * (m_1 \otimes m_2) + \sigma_1(h) \cdot (g * (m_1 \otimes m_2)) - \sigma_1(h) \cdot (g * (m_1 \otimes m_2)) = \\ &= [g, j_1(h)] * (m_1 \otimes m_2) \end{aligned} \quad (44)$$

c) $(g, h) \in \mathfrak{g}_1 \times \mathfrak{g}_2$

Is true by antisymmetry.

d) $(g, h) \in \mathfrak{g}_2 \times \mathfrak{g}_2$

For the computation of $T(g, h) - T(h, g)$, we have two cases.

d1) $m_1 = 1_{\mathcal{U}(\mathfrak{g}_1)}$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes m_2)) - h * (g * (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes m_2)) = \\ &= 1_{\mathcal{U}(\mathfrak{g}_1)} \otimes \sigma_2(g) \cdot \sigma_2(h) \cdot m_2 - 1_{\mathcal{U}(\mathfrak{g}_1)} \otimes \sigma_2(h) \cdot \sigma_2(g) \cdot m_2 = 1_{\mathcal{U}(\mathfrak{g}_1)} \otimes \sigma_2([g, h]) \cdot m_2 = [g, h] * (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes m_2) \end{aligned} \quad (45)$$

d2) $m_1 \in \mathcal{U}_+(\mathfrak{g}_1)$

We prove (42) by induction. Let $m_1 \in \mathcal{U}_n(\mathfrak{g}_1)$.

We have $n \geq 1$ and $\mathcal{U}_n(\mathfrak{g}_1)$ is generated by the products $x \cdot m$ with $x \in \mathcal{U}(\mathfrak{g}_1)$ and $m \in \mathcal{U}_{n-1}(\mathfrak{g}_1)$

$$\begin{aligned} T(g, h) - T(h, g) &= g * (h * (x \cdot m \otimes m_2)) - h * (g * (x \cdot m \otimes m_2)) = \\ &= g * ([h, x] * (m \otimes m_2)) + g * (\sigma_1(x) \cdot (h * (m \otimes m_2))) \\ &\quad - h * ([g, x] * (m \otimes m_2)) - h * (\sigma_1(x) \cdot (g * (m \otimes m_2))) \\ &= \underbrace{g * ([h, x] * (m \otimes m_2))}_{T_1(g, h)} + \underbrace{[g, x] * (h * (m \otimes m_2))}_{T_2(g, h)} + \underbrace{\sigma_1(x) \cdot (g * (h * (m \otimes m_2)))}_{T_3(g, h)} \\ &\quad - \underbrace{h * ([g, x] * (m \otimes m_2))}_{T_1(h, g)} - \underbrace{[h, x] * (g * (m \otimes m_2))}_{T_2(h, g)} - \underbrace{\sigma_1(x) \cdot (h * (g * (m \otimes m_2)))}_{T_3(h, g)} \end{aligned} \quad (46)$$

Then

$$\begin{aligned} T_1(g, h) - T_2(h, g) &= [g, [h, x]] * (m \otimes m_2) \text{ by induction} \\ T_2(g, h) - T_1(h, g) &= [[g, x], h] * (m \otimes m_2) \text{ by induction} \\ T_3(g, h) - T_3(h, g) &= \sigma_1(x) \cdot ([g, h] * (m \otimes m_2)) \text{ by induction} \\ \text{Hence } T(g, h) - T(h, g) &= ([g, [h, x]] + [[g, x], h]) * (m \otimes m_2) + x \cdot ([g, h] * (m \otimes m_2)) \\ &= [[g, h], x] * (m \otimes m_2) + \sigma_1(x) \cdot ([g, h] * (m \otimes m_2)) \\ &= [g, h] * (x \cdot m \otimes m_2) \end{aligned} \quad (47)$$

We now come to the proof that μ_{state}^U is one-to-one.

4 The linear map μ_{state}^U is bijective.

Theorem 2. *i) From the (Lie) action $*_U$, one deduces a unique $\mathcal{U}(\mathfrak{g})$ -module structure on $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ (noted $*_{mod}$) such that $\sigma(g) *__{mod} (m_1 \otimes m_2) = g *_U (m_1 \otimes m_2)$.*

*ii) The map $s : m \mapsto m *__{mod} (1_{\mathcal{U}(\mathfrak{g}_1)} \otimes 1_{\mathcal{U}(\mathfrak{g}_2)})$ and μ_{state}^U are mutually inverse.*

Proof. i) From Theorem (1) (iv), let us note (as above) $g*\mathcal{U}-$, the map $m_1 \otimes m_2 \mapsto g*\mathcal{U}(m_1 \otimes m_2)$, we then get a linear map $\varphi : \mathfrak{g} \rightarrow \text{End}(\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2))$ and, by (42), we learn that φ is a morphism of \mathbf{k} -Lie algebras. By universal property of $\mathcal{U}(\mathfrak{g})$, we get

$$\begin{array}{ccc}
 \mathbf{k}\text{-Lie} & \xleftarrow{F} & \mathbf{k}\text{-AAU} \\
 \mathfrak{g} & \xrightarrow{\varphi} & \text{End}(\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)) \\
 & \searrow \sigma & \uparrow \widehat{\varphi} \\
 & & \mathcal{U}(\mathfrak{g}).
 \end{array} \tag{48}$$

which means that, for all $(g, m_1, m_2) \in \mathfrak{g} \times \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$,

$$\varphi(g)[m_1 \otimes m_2] = \widehat{\varphi}(\sigma(g))[m_1 \otimes m_2] \tag{49}$$

Of course, such a morphism as $\widehat{\varphi}$ defines at once a structure of left $\mathcal{U}(\mathfrak{g})$ -module on $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$. Its action will be noted $*_{mod}$ such that

$$\sigma(g) *_{mod} (m_1 \otimes m_2) := \widehat{\varphi}(\sigma(g))[m_1 \otimes m_2]$$

which completes the first point.

ii) Knowing already that μ_{state}^U is surjective, it will be sufficient to establish that

$$s \circ \mu_{state}^U = Id_{\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)}$$

which amounts to show that for $(g_i)_{1 \leq i \leq p}$ in \mathfrak{g}_1 (resp. $(h_i)_{1 \leq i \leq q}$ in \mathfrak{g}_2)

$$s \circ \mu_{state}^U(\sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q) \tag{50}$$

By linearity, this will prove that $s \circ \mu_{state}^U = Id_{\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)}$.

From (41), for $p > 0$, we get,

$$\begin{aligned}
 & \mu_{state}^U(\sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \\
 & \mathcal{U}(j_1)\sigma_1(g_1)\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q))
 \end{aligned} \tag{51}$$

and, remarking that s is $\mathcal{U}(\mathfrak{g})$ -linear we have

$$\begin{aligned}
 & s \circ \mu_{state}^U(\sigma_1(g_1) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \\
 & s\left(\mathcal{U}(j_1)\sigma_1(g_1)\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q))\right) = \\
 & s\left(\sigma j_1(g_1)\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q))\right) = \\
 & j_1(g_1) \cdot s\mu_{state}^U(\sigma_2(g_2) \cdots \sigma_1(g_p) \otimes \sigma_2(h_1) \cdots \sigma_2(h_q)) = \\
 & \sigma_1(g_1) \cdot \cdots \cdot \sigma_1(g_p) \otimes \sigma_2(h_1) \cdot \cdots \cdot \sigma_2(h_q)
 \end{aligned} \tag{52}$$

the other case ($p = 0$) is straightforward. Then, by induction on p , one has $s \circ \mu_{state}^U(m_1 \otimes m_2) = m_1 \otimes m_2$ which proves the claim.

QED □

5 Conclusion and future

1. Quantized enveloping algebras
2. Lie superalgebras

References

- [1] N. Bourbaki, *Theory of Sets*, Springer-Verlag Berlin Heidelberg New York; (2nd printing 1989).
- [2] N. BOURBAKI, *Algebra I (Chapters 1-3)*, Springer 1989.
- [3] N. Bourbaki, *Lie groups and Lie algebras, Chapters 1-3*, Springer-Verlag; (1989).
- [4] Cartier P., *Vinberg algebras, Lie groups and combinatorics*, Quanta of Maths, Clay mathematics proceedings; vol. 11 (January 2011)
- [5] Dinh (Vu Nguyen), *Combinatorics of Lazard Elimination and Interactions*, <https://hal.science/tel-04367964> and (more updated) <https://www-lipn.univ-paris13.fr/~duchamp/Theses/Vu/>
- [6] D. Grinberg and V. Reiner, *Hopf algebras in Combinatorics*, ArXiv version, <http://www.arxiv.org/abs/1409.8356v7> `arXiv:1409.8356v7`.
See also, for the free algebra of NonCommutative polynomials, Exercise 1.6.8 p31, there <http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf> (More frequently updated version).
- [7] Kac V., *Infinite dimensional Lie algebras*
- [8] M. Lothaire, *Combinatorics on words*, Cambridge University Press (2003).
- [9] S. Mac Lane, *Categories for the Working Mathematician*, second edition, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York; (1998).
- [10] Robert V. Moody, Arturo Pianzola, *Lie Algebras with Triangular Decompositions*, ISBN: 978-0-471-63304-4 April 1995
- [11] C. Reutenauer, *Free Lie Algebras*, Université du Québec à Montréal, Clarendon Press, Oxford; (1993).
- [12] Heteromorphism, <https://ncatlab.org/nlab/show/heteromorphism>
- [13] About Enveloping Algebras of direct sums <https://mathoverflow.net/questions/300851>
- [14] Could we define the semi direct product of two universal enveloping algebras ? <https://mathoverflow.net/questions/142623>
- [15] Are semi-direct products categorical colimits, <https://mathoverflow.net/questions/96078>
- [16] Tropical semiring, https://en.wikipedia.org/wiki/Tropical_semiring
- [17] Transport of structure, https://en.wikipedia.org/wiki/Transport_of_structure

6 WHBD (What Has Been Done and some comments)

1. **22/04/24.** — (GHED from DG)
2. In (2), the top arrow should be j_i , not j_1 .
In (2), there is a) parenthesis too much in $U(g_i)$. → Done
3. p2 Remark 1 item 2 is not quite clear, but I assume it only concerns some examples?
Yes, it is only an “example generator” to show that the “all rings” requirement is not isolated,
Changed [The situation] for [For example, the situation]
4. Page 3:
Notations 1 item 1: Extra) parenthesis at the beginning, missing) at the end.
Notations 1 item 4: Extra) parenthesis in the diagram.
Notations 1 item 4: Below the diagram, $U(J_2)$ should be $U(j_2)$.
Done x 3
5. Page 4: (15): On the second line of (15), when you say “ $x \otimes (g *_T (t \otimes m))$ ”, do you mean “ $(x \otimes 1)(g *_T (t \otimes m))$ ”?
Here $T(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ is viewed as a $T(\mathfrak{g}_1) - \mathcal{U}(\mathfrak{g}_2)$ bimodule i.e. $T(\mathfrak{g})$ - module on the left and $\mathcal{U}(\mathfrak{g}_2)$ module on the right, then, if $g *_T (t \otimes m) = \sum_i t_i \otimes m_i$, we have
$$x \otimes (g *_T (t \otimes m)) = \sum_i (x \otimes t_i) \otimes m_i$$

I will put a word on this bimodule structure.
6. **22/04/24.** —
7. Added a second item in Remark 1
8. Created the example generator under environment “Example”
9. Put a Nota after the first item of Th. 1
10. **24/04/24.** —
11. Added bibliography and pointed to [7] Ch 1 Ex 8 (probably due to Vinberg, check **todo** Kac V. G. [1968 B] in this book)
12. **29/04/24.** —
13. Moved Darij from co-authorship to acknowledgements
14. Changed [As tensors may appear .../... and oversized.] (in the beginning) for [To avoid confusion .../... and oversized.]

7 Related quest!ons

1. Infinite fold tensor product of universal enveloping algebras
<https://mathoverflow.net/questions/334887>
2. Infinite tensor product of states
<https://mathoverflow.net/questions/161169>
3. Math of QM: 10.1 Motivation for Infinite Tensor Product of Hilbert Spaces (Roland Speicher)
<https://www.youtube.com/watch?v=2yRvGLh1SUQ>
4. Infinite tensor product as infinite coproduct in the category of R algebras
<https://math.stackexchange.com/questions/2054068>