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Exterior stability of Minkowski spacetime with borderline decay

Dawei Shen*

May 1, 2024

Abstract

In 1993, the global stability of Minkowski spacetime has been proven in the celebrated work of Christodoulou and Klainerman [5]. In 2003, Klainerman and Nicolò [13] revisited Minkowski stability in the exterior of an outgoing null cone. In [22], the author extended the results of [5] to minimal decay assumptions. In this paper, we prove that the exterior stability of Minkowski holds with decay which is borderline compared to the minimal decay considered in [22].

Keywords

Minkowski stability, double null foliation, borderline decay, r^p -weighted estimates.

1 Introduction

1.1 Einstein vacuum equations and the Cauchy problem

A Lorentzian 4-manifold $(\mathcal{M}, \mathbf{g})$ is called a vacuum spacetime if it solves the Einstein vacuum equations:

$$\mathbf{Ric}(\mathbf{g}) = 0 \quad \text{in } \mathcal{M}, \quad (1.1)$$

where \mathbf{Ric} denotes the Ricci tensor of the Lorentzian metric \mathbf{g} . The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a non-linear geometric coupled system of partial differential equations of order 2 for \mathbf{g} . In suitable coordinates, for example so-called wave coordinates, it can be shown that (1.1) is hyperbolic and hence admits an initial value formulation.

The corresponding initial data for the Einstein vacuum equations is given by specifying a triplet (Σ, g, k) where (Σ, g) is a Riemannian 3-manifold and k is the traceless symmetric 2-tensor on Σ satisfying the constraint equations:

$$\begin{aligned} R &= |k|^2 - (\operatorname{tr} k)^2, \\ D^j k_{ij} &= D_i(\operatorname{tr} k), \end{aligned} \quad (1.2)$$

where R denotes the scalar curvature of g , D denotes the Levi-Civita connection of g and

$$|k|^2 := g^{ad} g^{bc} k_{ab} k_{cd}, \quad \operatorname{tr} k := g^{ij} k_{ij}.$$

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In the future development $(\mathcal{M}, \mathbf{g})$ of such initial data (Σ, g, k) , $\Sigma \subset \mathcal{M}$ is a spacelike hypersurface with induced metric g and second fundamental form k .

The seminal well-posedness results for the Cauchy problem obtained in [3, 4] ensure that for any smooth Cauchy data, there exists a unique smooth maximal globally hyperbolic development $(\mathcal{M}, \mathbf{g})$ solution of Einstein equations (1.1) such that $\Sigma \subset \mathcal{M}$ and g, k are respectively the first and second fundamental forms of Σ in \mathcal{M} .

The prime example of a vacuum spacetime is Minkowski spacetime:

$$\mathcal{M} = \mathbb{R}^4, \quad \mathbf{g} = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

for which Cauchy data are given by

$$\Sigma = \mathbb{R}^3, \quad g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad k = 0.$$

In the present work, we consider the problem of the stability of Minkowski spacetime and start with reviewing the literature on this problem.

1.2 Previous works of the stability of Minkowski spacetime

In 1993, Christodoulou and Klainerman [5] proved the global stability of Minkowski for the Einstein-vacuum equations, a milestone in the domain of mathematical general relativity. In 2003, Klainerman and Nicolò [13] proved the Minkowski stability in the exterior of an outgoing cone using the double null foliation. Moreover, Klainerman and Nicolò [14] showed that under stronger asymptotic decay and regularity properties than those used in [5, 13], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties. Given that the goal of this paper is to extend the result of [13], we will state the results of [5, 13] in Section 1.3.

We now mention other proofs of Minkowski stability. In 2007, Bieri [2] gave a new proof of global stability of Minkowski requiring one less derivative and less vectorfields compared to [5]. Lindblad and Rodnianski [18, 19] gave a new proof of the stability of the Minkowski spacetime using *wave-coordinates* and showing that the Einstein equations verify the so called *weak null structure* in that gauge. Huneau [11] proved the nonlinear stability of Minkowski spacetime with a translation Killing field using generalised wave-coordinates. Using the framework of Melrose's b-analysis, Hintz and Vasy [9] reproved the stability of Minkowski space. Graf [7] proved the global nonlinear stability of Minkowski space in the context of the spacelike-characteristic Cauchy problem for Einstein vacuum equations, which together with [13] allows to reobtain [5]. Under the framework of [13] and using r^p -weighted estimates of Dafermos and Rodnianski [6], the author [21] reproved the Minkowski stability in exterior regions. More recently, Hintz [8] reproved the Minkowski stability in exterior regions by using the framework of [9]. Using r^p -weighted estimates, the author [22] extends the results of [2] to minimal decay assumption.

There are also stability results concerning Einstein's equations coupled with non trivial matter fields, see for example the introduction of [21, 22].

1.3 Minkowski Stability in [5, 13]

We recall in this section the results in [5, 13]. First, we recall the definition of a *maximal hypersurface*, which plays an important role in the statements of the main theorems in [5, 13].

Definition 1.1. An initial data (Σ, g, k) is posed on a maximal hypersurface if it satisfies

$$\operatorname{tr} k = 0. \quad (1.3)$$

In this case, we say that (Σ, g, k) is a maximal initial data set, and the constraint equations (1.2) reduce to

$$R = |k|^2, \quad \operatorname{div} k = 0, \quad \operatorname{tr} k = 0. \quad (1.4)$$

We introduce the notion of s -asymptotically flat initial data.

Definition 1.2. Given $s \geq 1$ and $q \geq 0$, we say that a data set (Σ_0, g, k) is (s, q) -asymptotically flat if there exists a coordinate system (x^1, x^2, x^3) defined outside a sufficiently large compact set such that:

- In the case of $s \geq 3$ ¹

$$g_{ij} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_{q+1}(r^{-\frac{s-1}{2}}), \quad k_{ij} = o_q(r^{-\frac{s+1}{2}}). \quad (1.5)$$

- In the case of $1 \leq s < 3$

$$g_{ij} = \delta_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \quad k_{ij} = o_q(r^{-\frac{s+1}{2}}). \quad (1.6)$$

We also introduce the following functional:

$$J_0(\Sigma_0, g, k) := \sup_{\Sigma_0} ((d_0^2 + 1)^3 |\operatorname{Ric}|^2) + \int_{\Sigma_0} \sum_{l=0}^3 (d_0^2 + 1)^{l+1} |D^l k|^2 + \int_{\Sigma_0} \sum_{l=0}^1 (d_0^2 + 1)^{l+3} |D^l B|^2,$$

where d_0 is the geodesic distance from a fixed point $O \in \Sigma_0$, and $B_{ij} := (\operatorname{curl} \widehat{R})_{ij}$ is the Bach tensor, \widehat{R} is the traceless part of Ric . Now, we can state the main theorems of [5] and [13].

Theorem 1.3 (Global stability of Minkowski space [5]). *There exists an $\varepsilon_0 > 0$ sufficiently small such that if $J_0(\Sigma_0, g, k) \leq \varepsilon_0^2$, then the initial data set (Σ_0, g, k) , $(4, 3)$ -asymptotically flat (in the sense of Definition 1.2) and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat, i.e. the Riemann curvature tensor tends to zero along any causal or spacelike geodesic. Moreover, there exists a global maximal time function t and a optical function u ² defined everywhere in an external region.*

Theorem 1.4 (Exterior stability of Minkowski [13]). *Consider an initial data set (Σ_0, g, k) , $(4, 3)$ -asymptotically flat and maximal, and assume $J_0(\Sigma_0, g, k)$ is bounded. Then, given a sufficiently large compact set $K \subset \Sigma_0$ such that $\Sigma_0 \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B}_1$, and under additional smallness assumptions, there exists a unique future development $(\mathcal{M}, \mathbf{g})$ of $\Sigma_0 \setminus K$ with the following properties:*

- $(\mathcal{M}, \mathbf{g})$ can be foliated by a double null foliation $\{C_u\}$ and $\{\underline{C}_u\}$ whose outgoing leaves C_u are complete.
- We have detailed control of all the quantities associated with the double null foliations of the spacetime, see Theorem 3.7.1 in [13].

¹The notation $f = o_l(r^{-m})$ means $\partial^\alpha f = o(r^{-m-|\alpha|})$, $|\alpha| \leq l$.

²An optical function u is a scalar function satisfying $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$.

Remark 1.5. Various choices of the parameter s in Definition 1.2 have been made in the literature. In particular.³

- $s = 4$ is used for example in [5, 7, 13].
- $s = 3 + \delta^4$ is used for example in [8, 9, 19].
- $s = 3 - \delta$ is used in [12] for the Einstein-Klein-Gordon system.
- $s \in (2, 3]$ is used in [17] for the Einstein-Klein-Gordon system.
- $s = 2$ is used in [2].
- $s \in (1, 2]$ is used in [22].

The goal of this paper is to extend the results of [13, 21] to $s = 1$ which is borderline compared to the range considered in [22]. More precisely, we prove the exterior stability of Minkowski spacetime for $(1, 3)$ -asymptotically flat initial data.

1.4 Rough version of the main theorem

In this section, we state a simple version of our main theorem. For the explicit statement, see Theorem 3.3.

Theorem 1.6 (Main Theorem (first version)). *Let an initial data set (Σ_0, g, k) which is $(1, 3)$ -asymptotically flat in the sense of Definition 1.2. Let a sufficiently large compact set $K \subset \Sigma_0$ such that $\Sigma_0 \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \bar{B}_1$. Assume that we have a smallness condition on $\Sigma_0 \setminus K$. Then, there exists a unique future development $(\mathcal{M}, \mathbf{g})$ in its future domain of dependence with the following properties:*

- $(\mathcal{M}, \mathbf{g})$ can be foliated by a double null foliation (C_u, \underline{C}_u) whose outgoing leaves C_u are complete for all $u \leq u_0$.
- We have detailed control of all the quantities associated with the double null foliations of the spacetime, see Theorem 3.3.

1.5 Structure of the paper

- In Section 2, we recall the fundamental notions and the basic equations.
- In Section 3, we present the main theorem. We then state intermediate results, and prove the main theorem. The rest of the paper focuses on the proof of these intermediary results.
- In Section 4, we make bootstrap assumptions and prove first consequences. These consequences will be used frequently in the rest of the paper.
- In Section 5, we apply r^p -weighted estimates to Bianchi equations to control the flux of curvature.
- In Section 6, we estimate the $L^p(S)$ -norms of curvature components and Ricci coefficients using the null structure equations and Bianchi equations.

³This list focuses on low values of s . For large values of s , note for example [14] valid for $s > 7$ and [21] valid for $s > 3$.

⁴Here and in the line below, δ denotes a constant satisfying $0 < \delta \ll 1$.

1.6 Acknowledgements

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2 Preliminaries

2.1 Geometry set-up

Let Σ_0 be a spacelike hypersurface. Let $K \subset \Sigma_0$ be a compact subset such that $\Sigma_0 \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_1}$ where $\overline{B_1}$ is the unit closed ball in \mathbb{R}^3 . We fix a radial foliation on the initial hypersurface $\Sigma_0 \setminus K$ by the level sets of a scalar function w . The leaves are denoted by

$$S_{(0)}(w_1) = \{p \in \Sigma_0 / w(p) = w_1\}, \quad (2.1)$$

where $w_1 \in \mathbb{R}$. We assume that

$$\partial K = \{p \in \Sigma_0 / w(p) = w_0\}, \quad K = \{p \in \Sigma_0 / w(p) \leq w_0\}, \quad (2.2)$$

where w_0 is the area radius of ∂K defined by

$$w_0 := \sqrt{\frac{|\partial K|}{4\pi}}. \quad (2.3)$$

We now construct a *double null foliation* in the future of $\Sigma_0 \setminus K$. For this, we denote T the normal vectorfield of Σ_0 oriented towards the future and N the unit vectorfield tangent to Σ_0 , oriented towards infinity and orthogonal to the leaves $S_0(w)$. We define two null vectors on $\Sigma_0 \setminus K$ by

$$L := N(w)(T + N), \quad \underline{L} := N(w)(T - N). \quad (2.4)$$

We extend the definition of L and \underline{L} to the future of $\Sigma_0 \setminus K$ by the geodesic equations:

$$\mathbf{D}_L L = 0, \quad \mathbf{D}_{\underline{L}} \underline{L} = 0. \quad (2.5)$$

We define a positive function Ω by the formula

$$\mathbf{g}(L, \underline{L}) = -\frac{1}{2\Omega^2}, \quad (2.6)$$

where Ω is called the *null lapse function*. We then define two optical functions u and \underline{u} in the future of $\Sigma_0 \setminus K$ by the *Eikonal equations*

$$L = -\text{grad } u, \quad \underline{L} = -\text{grad } \underline{u},$$

and the initial conditions

$$u|_{\Sigma_0 \setminus K} = -w, \quad \underline{u}|_{\Sigma_0 \setminus K} = w.$$

We then define the normalized null pair (e_3, e_4) by

$$e_3 = 2\underline{L}, \quad e_4 = 2\Omega^2 L. \quad (2.7)$$

The spacetime in the future of $\Sigma_0 \setminus K$ is then foliated by the level sets of u and \underline{u} respectively. We use C_u to denote the outgoing null hypersurfaces which are the level sets of u and use $\underline{C}_{\underline{u}}$ to denote the incoming null hypersurfaces which are the level sets of \underline{u} . We also denote

$$S(u, \underline{u}) := C_u \cap \underline{C}_{\underline{u}},$$

which are spacelike 2–spheres. On a given two sphere $S(u, \underline{u})$, we choose a local frame (e_1, e_2) , we call (e_1, e_2, e_3, e_4) a null frame. As a convention, throughout the paper, we use capital Latin letters A, B, C, \dots to denote an index from 1 to 2 and Greek letters $\alpha, \beta, \gamma, \dots$ to denote an index from 1 to 4, e.g. e_A denotes either e_1 or e_2 .

For any sphere $S(u, \underline{u})$ of the double null foliation (u, \underline{u}) , we denote its causal past by

$$V(u, \underline{u}) := J^-(S(u, \underline{u})) \cap D^+(\Sigma_0 \setminus K).$$

Denoting $u_0 = -w_0$ and taking $\underline{u}_* > w_0$, we define the bootstrap region by:

$$\mathcal{K} := V(u_0, \underline{u}_*).$$

By local existence theorem, the spacetime \mathcal{K} exists if \underline{u}_* is sufficiently close to w_0 . We now introduce a local coordinate system $(u, \underline{u}, \phi^A)$ in \mathcal{K} with $e_4(\phi^A) = 0$. In this coordinates system, the spacetime metric \mathbf{g} takes the following form:

$$\mathbf{g} = -2\Omega^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + \gamma_{AB}(d\phi^A - \underline{b}^A du) \otimes (d\phi^B - \underline{b}^B du), \quad (2.8)$$

where γ denotes the induced metric on $S(u, \underline{u})$ and $\underline{b} := \underline{b}^A \partial_{\phi^A}$ denotes the *null shift*. Note that

$$\Omega^2 e_3 = \partial_u + \underline{b}, \quad e_4 = \partial_{\underline{u}}.$$

We recall the null decomposition of the Ricci coefficients and curvature components of the null frame (e_1, e_2, e_3, e_4) as follows:

$$\begin{aligned} \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A e_3, e_B), & \chi_{AB} &= \mathbf{g}(\mathbf{D}_A e_4, e_B), \\ \underline{\xi}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_A), & \xi_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_A), \\ \underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \\ \underline{\eta}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_A), & \eta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_A), \\ \zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_A} e_4, e_3), \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} \alpha_{AB} &= \mathbf{R}(e_A, e_4, e_B, e_4), & \underline{\alpha}_{AB} &= \mathbf{R}(e_A, e_3, e_B, e_3), \\ \beta_A &= \frac{1}{2} \mathbf{R}(e_A, e_4, e_3, e_4), & \underline{\beta}_A &= \frac{1}{2} \mathbf{R}(e_A, e_3, e_3, e_4), \\ \rho &= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4), & \sigma &= \frac{1}{4} {}^* \mathbf{R}(e_3, e_4, e_3, e_4), \end{aligned} \quad (2.10)$$

where ${}^* \mathbf{R}$ denotes the Hodge dual of \mathbf{R} . The null second fundamental forms $\chi, \underline{\chi}$ are further decomposed in their traces $\text{tr } \chi$ and $\text{tr } \underline{\chi}$, and traceless parts $\widehat{\chi}$ and $\widehat{\underline{\chi}}$:

$$\begin{aligned} \text{tr } \chi &:= \delta^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2} \delta_{AB} \text{tr } \chi, \\ \text{tr } \underline{\chi} &:= \delta^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2} \delta_{AB} \text{tr } \underline{\chi}. \end{aligned}$$

We define the horizontal covariant operator ∇ as follows:

$$\nabla_X Y := \mathbf{D}_X Y - \frac{1}{2} \chi(X, Y) e_4 - \frac{1}{2} \underline{\chi}(X, Y) e_3.$$

We also define $\nabla_4 X$ and $\nabla_3 X$ to be the horizontal projections:

$$\begin{aligned}\nabla_4 X &:= \mathbf{D}_4 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_3) e_4 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_4 e_4) e_3, \\ \nabla_3 X &:= \mathbf{D}_3 X - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_3) e_3 - \frac{1}{2} \mathbf{g}(X, \mathbf{D}_3 e_4) e_4.\end{aligned}$$

A tensor field ψ defined on \mathcal{M} is called tangent to S if it is a priori defined on the spacetime \mathcal{M} and all the possible contractions of ψ with either e_3 or e_4 are zero. We use $\nabla_3 \psi$ and $\nabla_4 \psi$ to denote the projection to $S(u, \underline{u})$ of usual derivatives $\mathbf{D}_3 \psi$ and $\mathbf{D}_4 \psi$. As a direct consequence of (2.9), we have the following Ricci formulae:

$$\begin{aligned}\mathbf{D}_A e_3 &= \underline{\chi}_{AB} e_B + \zeta_A e_3, & \mathbf{D}_A e_4 &= \chi_{AB} e_B - \zeta_A e_4, \\ \mathbf{D}_3 e_A &= \nabla_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4, & \mathbf{D}_4 e_A &= \nabla_4 e_A + \underline{\eta}_A e_4 + \xi_A e_4, \\ \mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_B e_B, & \mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\underline{\eta}_B e_B, \\ \mathbf{D}_4 e_4 &= -2\omega e_4 + 2\underline{\xi}_B e_B, & \mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_B e_B, \\ \mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4.\end{aligned}\tag{2.11}$$

The following identities hold for the null frame (2.7):

$$\begin{aligned}\nabla \log \Omega &= \frac{1}{2}(\eta + \underline{\eta}), & \omega &= -\nabla_4(\log \Omega), & \underline{\omega} &= 0, \\ \eta &= \zeta + \nabla \log \Omega, & \underline{\eta} &= -\zeta + \nabla \log \Omega, & \xi &= \underline{\xi} = 0,\end{aligned}\tag{2.12}$$

see for example (6) in [16].

2.2 Integral formulae

Definition 2.1. Given a scalar function f on $S := S(u, \underline{u})$, we denote its average and its average free part by

$$\bar{f} := \frac{1}{|S|} \int_S f d\gamma, \quad \check{f} := f - \bar{f}.$$

Lemma 2.2. For any scalar function f , the following identities hold:

$$\begin{aligned}e_4 \left(\int_{S(u, \underline{u})} f d\gamma \right) &= \int_{S(u, \underline{u})} (e_4(f) + \text{tr } \chi f) d\gamma, \\ \Omega^2 e_3 \left(\int_{S(u, \underline{u})} f d\gamma \right) &= \int_{S(u, \underline{u})} (\Omega^2 e_3(f) + \Omega^2 \text{tr } \underline{\chi} f) d\gamma.\end{aligned}$$

Taking $f = 1$, we obtain

$$e_4(r) = \frac{\overline{\text{tr } \chi}}{2} r, \quad e_3(r) = \frac{\overline{\Omega^2 \text{tr } \underline{\chi}}}{2\Omega^2} r,$$

where r is the area radius defined by

$$r(u, \underline{u}) := \sqrt{\frac{|S(u, \underline{u})|}{4\pi}}.$$

Proof. See Lemma 2.26 in [15] or Lemma 3.1.3 in [13]. □

2.3 Hodge systems

Definition 2.3. For tensor fields defined on a 2-sphere S , we denote by $\mathfrak{s}_0 := \mathfrak{s}_0(S)$ the set of pairs of scalar functions, $\mathfrak{s}_1 := \mathfrak{s}_1(S)$ the set of 1-forms and $\mathfrak{s}_2 := \mathfrak{s}_2(S)$ the set of symmetric traceless 2-tensors.

Definition 2.4. Given $\xi \in \mathfrak{s}_1$ and $U \in \mathfrak{s}_2$, we define their Hodge dual

$${}^*\xi_A := \epsilon_{AB} \xi^B, \quad {}^*U_{AB} := \epsilon_{AC} U^C{}_B.$$

Definition 2.5. Given $\xi, \eta \in \mathfrak{s}_1$ and $U, V \in \mathfrak{s}_2$, we denote

$$\begin{aligned} \xi \cdot \eta &:= \delta^{AB} \xi_A \eta_B, & \xi \wedge \eta &:= \epsilon^{AB} \xi_A \eta_B, \\ (\xi \cdot U)_A &:= \delta^{BC} \xi_B U_{AC}, & (U \wedge V)_{AB} &:= \epsilon^{AB} U_{AC} V_{CB}, \\ (\widehat{\xi \otimes \eta})_{AB} &:= \xi_A \eta_B + \xi_B \eta_A - \delta_{AB} \xi \cdot \eta. \end{aligned}$$

Definition 2.6. For a given $\xi \in \mathfrak{s}_1$, we define the following differential operators:

$$\begin{aligned} \operatorname{div} \xi &:= \delta^{AB} \nabla_A \xi_B, \\ \operatorname{curl} \xi &:= \epsilon^{AB} \nabla_A \xi_B, \\ (\widehat{\nabla \otimes \xi})_{AB} &:= \nabla_A \xi_B + \nabla_B \xi_A - \delta_{AB} (\operatorname{div} \xi). \end{aligned}$$

Definition 2.7. We define the following Hodge type operators, as introduced in section 2.2 in [5]:

- d_1 takes \mathfrak{s}_1 into \mathfrak{s}_0 and is given by:

$$d_1 \xi := (\operatorname{div} \xi, \operatorname{curl} \xi),$$

- d_2 takes \mathfrak{s}_2 into \mathfrak{s}_1 and is given by:

$$(d_2 U)_A := \nabla^B U_{AB},$$

- d_1^* takes \mathfrak{s}_0 into \mathfrak{s}_1 and is given by:

$$d_1^*(f, f_*)_A := -\nabla_A f + \epsilon_A{}^B \nabla_B f_*,$$

- d_2^* takes \mathfrak{s}_1 into \mathfrak{s}_2 and is given by:

$$d_2^* \xi := -\frac{1}{2} \widehat{\nabla \otimes \xi}.$$

We have the following identities:

$$\begin{aligned} d_1^* d_1 &= -\Delta_1 + \mathbf{K}, & d_1 d_1^* &= -\Delta_0, \\ d_2^* d_2 &= -\frac{1}{2} \Delta_2 + \mathbf{K}, & d_2 d_2^* &= -\frac{1}{2} (\Delta_1 + \mathbf{K}). \end{aligned} \tag{2.13}$$

where \mathbf{K} denotes the Gauss curvature on S . See for example (2.2.2) in [5] for a proof of (2.13).

Definition 2.8. Let $f \in \mathfrak{s}_0$, $\xi \in \mathfrak{s}_1$ and $U \in \mathfrak{s}_2$, we define the weighted angular derivatives \mathfrak{d} :

$$\mathfrak{d}U := r d_2 U, \quad \mathfrak{d}\xi := r d_1 \xi, \quad \mathfrak{d}f := r d_1^* f.$$

We also denote for any tensor $h \in \mathfrak{s}_k$, $k = 0, 1, 2$,

$$h^{(q)} := (h, \mathfrak{d}h, \dots, \mathfrak{d}^q h), \quad q = 0, 1, 2, 3.$$

Definition 2.9. For a tensor field f on a 2-sphere S , we denote its L^p -norm:

$$|f|_{p,S} := \left(\int_S |f|^p d\gamma \right)^{\frac{1}{p}}.$$

We also introduce the following scale invariant L^p -norm:

$$\|f\|_{p,S} := |r^{-\frac{2}{p}} f|_{p,S}.$$

2.4 Null structure equations

Definition 2.10. We define the following renormalized quantity:

$$\varkappa := d_1^*(-\omega, \omega^\dagger) - \beta, \quad (2.14)$$

where ω^\dagger is defined by the solution of

$$\nabla_3 \omega^\dagger = \sigma, \quad \omega^\dagger|_{\Sigma_0 \setminus K} = 0.$$

We also define the following renormalized curvature components:

$$\widehat{\rho} := \rho - \frac{1}{2} \widehat{\chi} \cdot \widehat{\underline{\chi}}, \quad \widehat{\sigma} := \sigma - \frac{1}{2} \widehat{\chi} \wedge \widehat{\underline{\chi}}. \quad (2.15)$$

The mass aspect functions are defined as follows:

$$\mu := -\operatorname{div} \eta - \widehat{\rho}, \quad \underline{\mu} := -\operatorname{div} \underline{\eta} - \widehat{\rho}. \quad (2.16)$$

We also define the following modified mass aspect functions:

$$[\mu] := \mu + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi}, \quad [\underline{\mu}] := \underline{\mu} + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi}. \quad (2.17)$$

Proposition 2.11. We have the following null structure equations:

$$\begin{aligned} \nabla_4 \eta &= -\chi \cdot (\eta - \underline{\eta}) - \beta, \\ \nabla_3 \underline{\eta} &= -\underline{\chi} \cdot (\underline{\eta} - \eta) + \beta, \\ \nabla_4 \widehat{\chi} + \operatorname{tr} \chi \widehat{\chi} &= -\alpha, \\ \nabla_4 \operatorname{tr} \chi + \frac{1}{2} (\operatorname{tr} \chi)^2 &= -|\widehat{\chi}|^2 - 2\omega \operatorname{tr} \chi, \\ \nabla_3 \widehat{\underline{\chi}} + \operatorname{tr} \underline{\chi} \widehat{\underline{\chi}} &= -\alpha, \\ \nabla_3 \operatorname{tr} \underline{\chi} + \frac{1}{2} (\operatorname{tr} \underline{\chi})^2 &= -|\widehat{\underline{\chi}}|^2, \\ \nabla_4 \widehat{\chi} + \frac{1}{2} \operatorname{tr} \chi \widehat{\chi} &= \nabla \widehat{\otimes} \eta - \frac{1}{2} \operatorname{tr} \chi \widehat{\chi} + \eta \widehat{\otimes} \eta, \\ \nabla_3 \widehat{\underline{\chi}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \widehat{\underline{\chi}} &= \nabla \widehat{\otimes} \eta - \frac{1}{2} \operatorname{tr} \chi \widehat{\underline{\chi}} + \eta \widehat{\otimes} \eta, \\ \nabla_4 \operatorname{tr} \underline{\chi} + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} &= -2\underline{\mu} + 2|\underline{\eta}|^2, \\ \nabla_3 \operatorname{tr} \chi + \frac{1}{2} \operatorname{tr} \underline{\chi} \operatorname{tr} \chi &= -2\mu + 2|\eta|^2, \\ \nabla_3 \omega &= \rho + \frac{3}{2} |\eta - \underline{\eta}|^2 + \frac{1}{2} (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{4} |\eta + \underline{\eta}|^2. \end{aligned} \quad (2.18)$$

the Codazzi equations:

$$\begin{aligned}\operatorname{div} \widehat{\chi} &= \frac{1}{2} \nabla \operatorname{tr} \chi - \frac{1}{2} \operatorname{tr} \chi \underline{\eta} - \underline{\beta} + \underline{\eta} \cdot \widehat{\chi}, \\ \operatorname{div} \underline{\widehat{\chi}} &= \frac{1}{2} \nabla \operatorname{tr} \underline{\chi} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\eta} + \underline{\beta} - \underline{\eta} \cdot \underline{\widehat{\chi}},\end{aligned}\tag{2.19}$$

the torsion equation:

$$\operatorname{curl} \eta = -\operatorname{curl} \underline{\eta} = \widehat{\sigma},\tag{2.20}$$

and the Gauss equation:

$$\mathbf{K} + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} = -\widehat{\rho}.\tag{2.21}$$

Proof. See (3.1)–(3.5) in [16]. \square

2.5 Bianchi equations

Proposition 2.12. *We have the following Bianchi equations:*

$$\begin{aligned}\nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \alpha &= \nabla \widehat{\otimes} \beta - 3(\widehat{\chi} \rho + {}^* \widehat{\chi} \sigma) + (\zeta + 4\eta) \widehat{\otimes} \beta, \\ \nabla_4 \beta + 2 \operatorname{tr} \chi \beta &= \operatorname{div} \alpha - 2\omega \beta + \eta \cdot \alpha, \\ \nabla_3 \beta + \operatorname{tr} \underline{\chi} \beta &= \nabla \widehat{\rho} + {}^* \nabla \widehat{\sigma} + 2\widehat{\chi} \cdot \underline{\beta} + 3(\eta \rho + {}^* \eta \sigma) + \frac{1}{2} d_1^*(-\widehat{\chi} \cdot \underline{\widehat{\chi}}, \widehat{\chi} \wedge \underline{\widehat{\chi}}) \\ \nabla_4 \widehat{\rho} + \frac{3}{2} \operatorname{tr} \chi \widehat{\rho} &= \operatorname{div} \beta + (\zeta + 2\underline{\eta}) \cdot \beta - \frac{1}{2} \widehat{\chi} \cdot (\nabla \widehat{\otimes} \underline{\eta} + \underline{\eta} \widehat{\otimes} \underline{\eta}) + \frac{1}{4} \operatorname{tr} \chi |\widehat{\chi}|^2, \\ \nabla_4 \widehat{\sigma} + \frac{3}{2} \operatorname{tr} \chi \widehat{\sigma} &= -\operatorname{div} {}^* \beta - (\zeta + 2\underline{\eta}) \wedge \beta - \frac{1}{2} \widehat{\chi} \wedge (\nabla \widehat{\otimes} \underline{\eta} + \underline{\eta} \widehat{\otimes} \underline{\eta}), \\ \nabla_3 \widehat{\rho} + \frac{3}{2} \operatorname{tr} \underline{\chi} \widehat{\rho} &= -\operatorname{div} \underline{\beta} + (\zeta - 2\underline{\eta}) \cdot \underline{\beta} - \frac{1}{2} \underline{\widehat{\chi}} \cdot (\nabla \widehat{\otimes} \eta + \eta \widehat{\otimes} \eta) + \frac{1}{4} \operatorname{tr} \chi |\underline{\widehat{\chi}}|^2, \\ \nabla_3 \widehat{\sigma} + \frac{3}{2} \operatorname{tr} \underline{\chi} \widehat{\sigma} &= -\operatorname{div} {}^* \underline{\beta} + (\zeta - 2\underline{\eta}) \wedge \underline{\beta} + \frac{1}{2} \underline{\widehat{\chi}} \wedge (\nabla \widehat{\otimes} \eta + \eta \widehat{\otimes} \eta), \\ \nabla_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= -\nabla \widehat{\rho} + {}^* \nabla \widehat{\sigma} + 2\omega \underline{\beta} + 2\underline{\widehat{\chi}} \cdot \beta - 3(\eta \rho - {}^* \eta \sigma) + \frac{1}{2} d_1^*(\widehat{\chi} \cdot \underline{\widehat{\chi}}, \widehat{\chi} \wedge \underline{\widehat{\chi}}), \\ \nabla_3 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= -\operatorname{div} \underline{\alpha} + \underline{\eta} \cdot \underline{\alpha}, \\ \nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= -\nabla \widehat{\otimes} \underline{\beta} + 4\omega \underline{\alpha} - 3(\widehat{\chi} \rho - {}^* \widehat{\chi} \sigma) + (\zeta - 4\eta) \widehat{\otimes} \underline{\beta}.\end{aligned}$$

Proof. See (2.7) and (2.8) in [1]. \square

3 Main theorem

In the sequel of this paper, we always denote

$$V := V(u, \underline{u}), \quad C_u^V := C_u \cap V, \quad \underline{C}_u^V := \underline{C}_u \cap V, \quad S := S(u, \underline{u}).\tag{3.1}$$

3.1 Fundamental norms

We proceed to define our main norms on a given bootstrap spacetime $\mathcal{K} = V(u_0, \underline{u}_*)$.

3.1.1 Schematic notation Γ_g, Γ_b and Γ_w

Definition 3.1. We divide the Ricci coefficients into three parts:

$$\Gamma_g := \left\{ \text{tr } \chi - \frac{1}{r}, \text{tr } \underline{\chi} + \frac{4}{r}, \widehat{\chi}, \omega, \underline{\eta} \right\}, \quad \Gamma_b := \{ \eta, \zeta, r^{-1} \log(2\Omega) \}, \quad \Gamma_w := \{ \widehat{\chi} \}.$$

We then denote

$$\begin{aligned} \Gamma_g^{(1)} &:= (r\nabla)^{\leq 1} \Gamma_g \cup \{ r\beta, r\widehat{\rho}, r\widehat{\sigma}, r\underline{\mu}, r\underline{\varkappa} \}, \\ \Gamma_b^{(1)} &:= (r\nabla)^{\leq 1} \Gamma_b \cup \{ r\underline{\mu} \}, \quad \Gamma_w^{(1)} := (r\nabla)^{\leq 1} \Gamma_w \cup \{ r\underline{\beta} \}, \end{aligned}$$

and for $q = 2, 3$

$$\Gamma_g^{(q)} := (r\nabla)^{\leq 1} \Gamma_g^{(q-1)}, \quad \Gamma_b^{(q)} := (r\nabla)^{\leq 1} \Gamma_b^{(q-1)}, \quad \Gamma_w^{(q)} := (r\nabla)^{\leq 1} \Gamma_w^{(q-1)}.$$

Remark 3.2. The justification of Definition 3.1 has to do with the expected decay properties of the Ricci coefficients and curvature components, see Section 3.1.3 and Lemma 4.2 below.

3.1.2 \mathcal{R} norms (flux of curvature)

In the remainder of this paper, we always denote δ_0 and δ two fixed constants satisfying

$$0 < \delta_0 \ll \delta \ll 1.$$

For a tensor field h defined on C_u^V or $\underline{C}_{\underline{u}}^V$, we denote its L^2 -flux by

$$\begin{aligned} \|h\|_{2, C_u^V}^2 &:= \int_{C_u^V} |h|^2 := \int_{|u|}^u \int_{S(u, \underline{u}')} |h|^2 d\gamma d\underline{u}', \\ \|h\|_{2, \underline{C}_{\underline{u}}^V}^2 &:= \int_{\underline{C}_{\underline{u}}^V} |h|^2 := \int_{-\underline{u}}^u \int_{S(u', \underline{u})} \Omega^2 |h|^2 d\gamma du'. \end{aligned} \tag{3.2}$$

We define the norms of flux of curvature components in the bootstrap region \mathcal{K} . We denote

$$\mathcal{R} := \sum_{q=0}^2 (\mathcal{R}_q[\alpha, \beta, \widehat{\rho}, \widehat{\sigma}] + \mathcal{R}_q[\underline{\beta}] + \underline{\mathcal{R}}_q[\beta, \widehat{\rho}, \widehat{\sigma}, \underline{\beta}] + \underline{\mathcal{R}}_q[\underline{\alpha}]).$$

where for $q = 0, 1, 2$

$$\mathcal{R}_q[w] := \sup_{\mathcal{K}} \mathcal{R}_q[w](u, \underline{u}), \quad \underline{\mathcal{R}}_q[w] := \sup_{\mathcal{K}} \underline{\mathcal{R}}_q[w](u, \underline{u}).$$

It remains to define for $q = 0, 1, 2$

$$\begin{aligned} \mathcal{R}_q[\alpha, \beta, \widehat{\rho}, \widehat{\sigma}](u, \underline{u}) &:= |u|^\delta \|r^{\frac{1}{2}-\delta} (r\nabla)^q (\alpha, \beta, \widehat{\rho}, \widehat{\sigma})\|_{2, C_u^V}, \\ \underline{\mathcal{R}}_q[\beta, \widehat{\rho}, \widehat{\sigma}, \underline{\beta}](u, \underline{u}) &:= |u|^\delta \|r^{\frac{1}{2}-\delta} (r\nabla)^q (\beta, \widehat{\rho}, \widehat{\sigma}, \underline{\beta})\|_{2, \underline{C}_{\underline{u}}^V}, \\ \mathcal{R}_q[\underline{\beta}](u, \underline{u}) &:= |u|^{\frac{1}{2}+2\delta} \|r^{-2\delta} (r\nabla)^q \underline{\beta}\|_{2, C_u^V}, \\ \underline{\mathcal{R}}_q[\underline{\alpha}](u, \underline{u}) &:= |u|^{\frac{1}{2}+2\delta} \|r^{-2\delta} (r\nabla)^q \underline{\alpha}\|_{2, \underline{C}_{\underline{u}}^V}. \end{aligned}$$

3.1.3 \mathcal{O} norms ($L^p(S)$ -norms of geometric quantities)

We first define the $L^4(S)$ -norms of Γ_g , Γ_b and Γ_w in the bootstrap region \mathcal{K} . We define

$$\mathcal{O}_{[2]} := \sum_{q=0}^2 \sup_{\mathcal{K}} (\mathcal{O}_q(\Gamma_g) + \mathcal{O}_q(\Gamma_b) + \mathcal{O}_q(\Gamma_w)),$$

with

$$\begin{aligned} \mathcal{O}_0(\Gamma_g) &:= \|r\Gamma_g\|_{4,S}, & \mathcal{O}_1(\Gamma_g) &:= \|r\Gamma_g^{(1)}\|_{4,S}, & \mathcal{O}_2(\Gamma_g) &:= \|r^{1-\delta}|u|^\delta\Gamma_g^{(2)}\|_{4,S}, \\ \mathcal{O}_0(\Gamma_b) &:= \|r^{1-\delta_0}|u|^{\delta_0}\Gamma_b\|_{4,S}, & \mathcal{O}_1(\Gamma_b) &:= \|r^{1-3\delta_0}|u|^{3\delta_0}\Gamma_b^{(1)}\|_{4,S}, & \mathcal{O}_2(\Gamma_b) &:= \|r^{1-\delta-2\delta_0}|u|^{\delta+2\delta_0}\Gamma_b^{(2)}\|_{4,S}, \\ \mathcal{O}_0(\Gamma_w) &:= \|r^{1-\delta_0}|u|^{\delta_0}\Gamma_w\|_{4,S}, & \mathcal{O}_1(\Gamma_w) &:= \|r^{1-\delta}|u|^\delta\Gamma_w^{(1)}\|_{4,S}, & \mathcal{O}_2(\Gamma_w) &:= \|r^{\frac{1}{2}-\delta-\delta_0}|u|^{\frac{1}{2}+\delta+\delta_0}\Gamma_w^{(2)}\|_{4,S}. \end{aligned}$$

We then define the $H^2(S)$ -norms for the renormalized quantities introduced in Definition 2.10:

$$\mathcal{O}_3 := \mathcal{O}_2^2(\nabla \operatorname{tr} \chi) + \mathcal{O}_2^2(\nabla \operatorname{tr} \underline{\chi}) + \mathcal{O}_2^2(\varkappa) + \mathcal{O}_2^2(\mu) + \mathcal{O}_2^2(\underline{\mu}),$$

with

$$\begin{aligned} \mathcal{O}_2^2(\Gamma) &:= \sup_{\mathcal{K}} \left\| r^{2-\delta}|u|^\delta (r\nabla)^2 \Gamma \right\|_{2,S}, & \forall \Gamma \in \{ \nabla \operatorname{tr} \chi, \nabla \operatorname{tr} \underline{\chi}, \varkappa, \underline{\mu} \}, \\ \mathcal{O}_2^2(\mu) &:= \sup_{\mathcal{K}} \left\| r^{2-\delta-3\delta_0}|u|^{\delta+3\delta_0} (r\nabla)^2 \mu \right\|_{2,S}. \end{aligned}$$

We also introduce the following auxiliary norm:

$$\mathcal{O}_a := \sup_{\mathcal{K}} \left\| r^{3-\delta-\delta_0}|u|^{\delta+\delta_0} \nabla \underline{\beta} \right\|_{2,S} + \sup_{\mathcal{K}} \left\| r^{2-\delta_0}|u|^{1+\delta_0} \nabla \underline{\alpha} \right\|_{2,S}.$$

Moreover, we define

$$\mathcal{O}_\gamma := \sup_{\mathcal{K}} \frac{|\log(2\Omega)|}{\log\left(\frac{3r}{|u|}\right)} + \sup_{\mathcal{K}} \left| 1 - \frac{u-u}{2r} \right|.$$

Finally, we denote

$$\mathcal{O} := \mathcal{O}_{[2]} + \mathcal{O}_3 + \mathcal{O}_a + \mathcal{O}_\gamma.$$

3.1.4 $\mathcal{I}_{(0)}$ norm (Initial data)

We introduce the following norm on initial hypersurface Σ_0 :

$$\mathcal{I}_{(0)} := \sup_{\Sigma_0 \setminus K} \left\{ \sum_{q=0}^3 r \|\mathfrak{d}^q(\Gamma_g, \Gamma_b, \Gamma_w)\|_{2,S_{(0)}(w)} + \sum_{q=0}^2 r^2 \|\mathfrak{d}^q(\alpha, \underline{\alpha})\|_{2,S_{(0)}(w)} + \left| 1 - \frac{u-u}{2r} \right| \right\},$$

where

$$\mathfrak{d} \in \{r\nabla_3, r\nabla_4, r\nabla\}.$$

3.2 Statement of the main theorem

The goal of this paper is to prove the following theorem.

Theorem 3.3 (Main Theorem). *Consider an initial data set (Σ_0, g, k) which is $(1, 3)$ -asymptotically flat in the sense of Definition 1.2. Assume that we have on $\Sigma_0 \setminus K$:*

$$\mathcal{I}_{(0)} \leq \varepsilon_0, \quad (3.3)$$

where $\varepsilon_0 > 0$ is a constant small enough. Then, $\Sigma_0 \setminus K$ has a unique development $(\mathcal{M}, \mathbf{g})$ in its future domain of dependence with the following properties:

1. $(\mathcal{M}, \mathbf{g})$ can be foliated by a double null foliation (u, \underline{u}) . Moreover, the outgoing cones C_u are complete for all $u \leq u_0$.
2. The norms \mathcal{O} and \mathcal{R} defined in Section 3.1 satisfy

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0. \quad (3.4)$$

The proof of Theorem 3.3 is given in Section 3.4. It hinges on two basic theorems stated in Section 3.3, concerning estimates for \mathcal{O} and \mathcal{R} norms.

We choose ε_0 and ε small enough such that

$$\varepsilon \ll \delta_0 \ll \delta \ll 1, \quad \varepsilon := \varepsilon_0^{\frac{2}{3}}.$$

Here, $A \ll B$ means that $CA < B$ where C is the largest universal constant among all the constants involved in the proof via \lesssim .

3.3 Main intermediate results

Theorem 3.4. *Assume that*

$$\mathcal{I}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon. \quad (3.5)$$

Then, we have

$$\mathcal{R} \lesssim \varepsilon_0. \quad (3.6)$$

Theorem 3.4 is proved in Section 5. The proof is based on the r^p -weighted method introduced by Dafermos and Rodnianski in [6].

Theorem 3.5. *Assume that*

$$\mathcal{I}_{(0)} \leq \varepsilon_0, \quad \mathcal{O} \leq \varepsilon, \quad \mathcal{R} \lesssim \varepsilon_0. \quad (3.7)$$

Then, we have

$$\mathcal{O} \lesssim \varepsilon_0. \quad (3.8)$$

Theorem 3.5 is proved in Section 6. The proof is done by integrating the transport equations along the outgoing and incoming null cones and applying elliptic estimate on 2-spheres of the double null foliation of the spacetime \mathcal{K} .

3.4 Proof of the main theorem

We now use Theorems 3.4 and 3.5 to prove Theorem 3.3.

Definition 3.6. Let $\mathfrak{N}(\underline{u}_*)$ the set of spacetimes $\mathcal{K} = V(u_0, \underline{u}_*)$ associated with a double null foliation (u, \underline{u}) in which we have the following bounds:

$$\mathcal{O} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon. \quad (3.9)$$

Definition 3.7. We denote \mathcal{U} the set of values \underline{u}_* such that $\mathfrak{N}(\underline{u}_*) \neq \emptyset$.

The assumption $\mathcal{I}_{(0)} \leq \varepsilon_0$ and the local existence theorem imply that (3.9) holds if \underline{u}_* is sufficiently close to $|u_0|$. So, we have $\mathcal{U} \neq \emptyset$.

Define \underline{u}_* to be the supremum of the set \mathcal{U} . We want to prove $\underline{u}_* = +\infty$. We assume by contradiction that \underline{u}_* is finite. In particular we may assume $\underline{u}_* \in \mathcal{U}$. We consider the region $\mathcal{K} = V(u_0, \underline{u}_*)$. Recall that we have

$$\mathcal{I}_{(0)} \leq \varepsilon_0,$$

according to the assumption of Theorem 3.3. Applying Theorem 3.4, we obtain

$$\mathcal{R} \lesssim \varepsilon_0.$$

Then, we apply Theorem 3.5 to obtain

$$\mathcal{O} \lesssim \varepsilon_0.$$

Applying local existence results,⁵ we can extend \mathcal{K} to $\tilde{\mathcal{K}} := V(u_0, \underline{u}_* + \nu)$ for a ν sufficiently small. We denote $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{R}}$ the norms in the extended region $\tilde{\mathcal{K}}$. We have

$$\tilde{\mathcal{O}} \lesssim \varepsilon_0, \quad \tilde{\mathcal{R}} \lesssim \varepsilon_0,$$

as a consequence of continuity. We deduce that $V(u_0, \underline{u}_* + \nu)$ satisfies all the properties in Definition 3.6, and so $\mathfrak{N}(\underline{u}_* + \nu) \neq \emptyset$, which is a contradiction. Thus, we have $\underline{u}_* = +\infty$, which implies property 1 of Theorem 3.3. Moreover, we have

$$\mathcal{O} \lesssim \varepsilon_0, \quad \mathcal{R} \lesssim \varepsilon_0,$$

in the whole exterior region, which implies property 2 of Theorem 3.3. This concludes the proof of Theorem 3.3.

Remark 3.8. We have from (3.4)

$$\left\| \text{tr} \chi - \frac{1}{r} \right\|_{\infty, S} \lesssim \frac{\varepsilon_0}{r}, \quad \forall S \subseteq V(u_0, +\infty),$$

which implies for ε_0 small enough that $\text{tr} \chi > 0$. Hence, we deduce that there is no trapped surface in the exterior region $D^+(\Sigma_0 \setminus K) = V(u_0, +\infty)$.

4 Bootstrap assumptions and first consequences

In the rest of the paper, we always make the following bootstrap assumptions:

$$\mathcal{O} \leq \varepsilon, \quad \mathcal{R} \leq \varepsilon. \quad (4.1)$$

⁵See Theorem 10.2.1 in [5] and Theorem 1 in [20].

4.1 First consequences of bootstrap assumptions

In this section, we derive first consequences of (4.1) in the region $\mathcal{K} = V(u_0, \underline{u}_*)$. In the sequel, the results of this section will be used frequently without explicitly mentioning them.

Remark 4.1. According to $\mathcal{O}_\gamma \leq \varepsilon$, we have in the bootstrap region \mathcal{K}

$$\left| r - \frac{u - u}{2} \right| \lesssim \varepsilon \underline{u}. \quad (4.2)$$

Combining with $|u| \leq \underline{u}$ and $u < 0$, this yields

$$|u| \leq \underline{u} \simeq r \quad \text{in } \mathcal{K}. \quad (4.3)$$

Lemma 4.2. Under the assumption (4.1), we have the following bounds:

$$\begin{aligned} \|\Gamma_g\|_{4,S} &\lesssim \frac{\varepsilon}{r}, & \|\Gamma_g^{(1)}\|_{4,S} &\lesssim \frac{\varepsilon}{r}, & \|\Gamma_g^{(2)}\|_{4,S} &\lesssim \frac{\varepsilon}{r^{1-\delta}|u|^\delta}, \\ \|\Gamma_b\|_{4,S} &\lesssim \frac{\varepsilon}{r^{1-\delta_0}|u|^{\delta_0}}, & \|\Gamma_b^{(1)}\|_{4,S} &\lesssim \frac{\varepsilon}{r^{1-3\delta_0}|u|^{3\delta_0}}, & \|\Gamma_b^{(2)}\|_{4,S} &\lesssim \frac{\varepsilon}{r^{1-\delta-2\delta_0}|u|^{\delta+2\delta_0}}, \\ \|\Gamma_w\|_{4,S} &\lesssim \frac{\varepsilon}{r^{1-\delta_0}|u|^{\delta_0}}, & \|\Gamma_w^{(1)}\|_{4,S} &\lesssim \frac{\varepsilon}{r^{1-\delta}|u|^\delta}, & \|\Gamma_w^{(2)}\|_{4,S} &\lesssim \frac{\varepsilon}{r^{\frac{1}{2}-\delta-\delta_0}|u|^{\frac{1}{2}+\delta+\delta_0}}, \\ \|\alpha\|_{4,S} &\lesssim \frac{\varepsilon}{r^2}, & \|\alpha^{(1)}\|_{2,S} &\lesssim \frac{\varepsilon}{r^{2-\delta}|u|^\delta}, & \|\underline{\alpha}^{(1)}\|_{2,S} &\lesssim \frac{\varepsilon}{r^{1-\delta_0}|u|^{1+\delta_0}}. \end{aligned}$$

Proof. It follows directly from Definition 3.1 and (4.1). \square

Lemma 4.3. For two quantities $X_{(1)}$ and $X_{(2)}$, we denote

$$X_{(1)} \preceq X_{(2)}$$

if $X_{(1)}^{(q)}$ decays better than $X_{(2)}^{(q)}$ for $q = 0, 1, 2, 3$. Then, the following hold:

1. We have

$$\beta \preceq (\widehat{\rho}, \widehat{\sigma}) \preceq \underline{\beta} \preceq \underline{\alpha}, \quad \Gamma_g \preceq \Gamma_b \preceq \Gamma_w.$$

2. We also have

$$\Gamma_g \cdot \Gamma_g \preceq r^{-1}\Gamma_g, \quad \Gamma_g \cdot \Gamma_w \preceq r^{-1}\Gamma_w, \quad \Gamma_w \preceq \Gamma_b^{(1)}, \quad r^{-1}\Gamma_w^{(1)} \preceq \underline{\alpha}^{(0)}.$$

Proof. It follows directly from Definition 3.1, (4.1) and Lemma 4.2. \square

Remark 4.4. In the sequel, we choose the following conventions:

- Let $q \in \{0, 1, 2, 3\}$, for a quantity h satisfying the same or even better decay and regularity as $\Gamma_i^{(q)}$, for $i = g, b, w$, we write

$$h \in \Gamma_i^{(q)}, \quad i = g, b, w.$$

- For a sum of schematic notations, we write

$$X_{(1)} + X_{(2)} = X_{(2)},$$

if $X_{(1)} \preceq X_{(2)}$. For example, we write

$$\Gamma_g + \Gamma_b = \Gamma_b, \quad r^{-1}\Gamma_b \cdot \Gamma_w^{(1)} + \Gamma_b \cdot \underline{\alpha} = \Gamma_b \cdot \underline{\alpha}, \quad \Gamma_g \cdot \Gamma_w = \Gamma_g \cdot \Gamma_b^{(1)}.$$

4.2 Commutation identities

Proposition 4.5. *For any scalar function f , we have:*

$$\begin{aligned} [\nabla_4, r\nabla]f &= \Gamma_g \cdot r\nabla f, \\ [\Omega^2\nabla_3, r\nabla]f &= \Omega^2\Gamma_w \cdot r\nabla f. \end{aligned}$$

Proof. See Proposition 4.8.1 in [13] and Corollary 2.22 in [21]. \square

Proposition 4.6. *We have the following schematic commutator identities*

$$\begin{aligned} [\nabla_4, r\nabla] &= \Gamma_g \cdot r\nabla + \Gamma_g^{(1)}, \\ [\Omega^2\nabla_3, r\nabla] &= \Omega^2\Gamma_w \cdot r\nabla + \Omega^2\Gamma_w^{(1)}. \end{aligned}$$

Proof. See Lemma 7.3.3 in [5] and Proposition 2.24 in [21]. \square

4.3 Main equations in schematic form

Proposition 4.7. *We have the following schematic null structure equations:*

$$\begin{aligned} \nabla_4 \operatorname{tr} \chi + \frac{1}{2}(\operatorname{tr} \chi)^2 &= -2r^{-1}\omega + \Gamma_g \cdot \Gamma_g, \\ \nabla_3 \operatorname{tr} \underline{\chi} + \frac{1}{2}(\operatorname{tr} \underline{\chi})^2 &= \Gamma_w \cdot \Gamma_w, \\ \nabla_4 \operatorname{tr} \underline{\chi} + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} &= -2\underline{\mu} + \Gamma_g \cdot \Gamma_g, \\ \nabla_3 \operatorname{tr} \chi + \frac{1}{2} \operatorname{tr} \underline{\chi} \operatorname{tr} \chi &= -2\mu + \Gamma_b \cdot \Gamma_b, \\ \operatorname{div} \widehat{\chi} &= \frac{1}{2}\nabla \operatorname{tr} \chi - \frac{1}{2}r^{-1}\underline{\eta} - \beta + \Gamma_g \cdot \Gamma_g, \\ \operatorname{div} \widehat{\underline{\chi}} &= \frac{1}{2}\nabla \operatorname{tr} \underline{\chi} - \frac{1}{2}r^{-1}\underline{\eta} + \underline{\beta} + \Gamma_g \cdot \Gamma_w, \\ d_1 \eta &= (-\mu - \widehat{\rho}, \widehat{\sigma}), \\ d_1 \underline{\eta} &= (-\underline{\mu} - \widehat{\rho}, -\widehat{\sigma}), \\ d_1^*(-\omega, \omega^\dagger) &= \varkappa + \beta, \\ \nabla_3 \omega &= \widehat{\rho} + \Gamma_b \cdot \Gamma_b + \Gamma_g \cdot \Gamma_w. \end{aligned} \tag{4.4}$$

We also have the following transport equations for the quantities defined in Definition 2.10:

$$\begin{aligned} \nabla_3 \varkappa + \frac{1}{2} \operatorname{tr} \underline{\chi} \varkappa &= -r^{-1}\beta + r^{-1}(\Gamma_b \cdot \Gamma_b)^{(1)}, \\ \nabla_4 [\widetilde{\mu}] + \operatorname{tr} \chi [\widetilde{\mu}] &= r^{-2} \widetilde{\operatorname{tr} \chi} + r^{-1} \widetilde{\rho} + r^{-1} \Gamma_b \cdot \Gamma_b + r^{-1}(\Gamma_g \cdot \Gamma_b)^{(1)}, \\ \nabla_3 [\widetilde{\mu}] + \operatorname{tr} \underline{\chi} [\widetilde{\mu}] &= r^{-2} \widetilde{\operatorname{tr} \underline{\chi}} - r^{-1} \widetilde{\rho} + r^{-1}(\Gamma_w \cdot \Gamma_w)^{(1)}. \end{aligned} \tag{4.5}$$

Proof. Notice that (4.4) follows directly from Proposition 2.11 and Definition 3.1. See (6.19) in [16] and Lemma 4.3.2 in [13] for explicit formulae of (4.5). \square

Proposition 4.8. *We have the following schematic Bianchi equations:*

$$\begin{aligned}
\nabla_3 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\alpha} &= -2d_2^* \underline{\beta} + r^{-1} (\Gamma_g \cdot \Gamma_b)^{(1)}, \\
\nabla_4 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= d_2 \underline{\alpha} + r^{-1} (\Gamma_g \cdot \Gamma_g)^{(1)} + \Gamma_g \cdot \underline{\alpha}, \\
\nabla_3 \underline{\beta} + \operatorname{tr} \underline{\chi} \underline{\beta} &= -d_1^* (\widehat{\rho}, -\widehat{\sigma}) + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(1)}, \\
\nabla_4 (\widehat{\rho}, -\widehat{\sigma}) + \frac{3}{2} \operatorname{tr} \underline{\chi} (\widehat{\rho}, -\widehat{\sigma}) &= d_1 \underline{\beta} + r^{-1} (\Gamma_g \cdot \Gamma_g)^{(1)}, \\
\nabla_3 (\widehat{\rho}, \widehat{\sigma}) + \frac{3}{2} \operatorname{tr} \underline{\chi} (\widehat{\rho}, \widehat{\sigma}) &= -d_1 \underline{\beta} + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(1)}, \\
\nabla_4 \underline{\beta} + \operatorname{tr} \underline{\chi} \underline{\beta} &= d_1^* (\widehat{\rho}, \widehat{\sigma}) + r^{-1} (\Gamma_g \cdot \Gamma_w)^{(1)}, \\
\nabla_3 \underline{\beta} + 2 \operatorname{tr} \underline{\chi} \underline{\beta} &= -d_2 \underline{\alpha} + \Gamma_b \cdot \underline{\alpha}, \\
\nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \underline{\chi} \underline{\alpha} &= 2d_2^* \underline{\beta} + \Gamma_g \cdot \underline{\alpha} + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(1)}.
\end{aligned}$$

Proof. It follows directly from Proposition 2.12, Definition 3.1 and Remark 4.4. \square

4.4 Sobolev inequalities and L^p -estimates

Proposition 4.9. *Let F be a tensor field on S . Then, we have⁶*

$$\begin{aligned}
\|r^{\frac{3}{2}} F\|_{4,S} &\lesssim \|r^{\frac{3}{2}} F\|_{4,S(u,|u|)} + \|F\|_{2,C_u^V} + \|r \nabla F\|_{2,C_u^V} + \|r \nabla_4 F\|_{2,C_u^V}, \\
\|r|u|^{\frac{1}{2}} F\|_{4,S} &\lesssim \|r^{\frac{3}{2}} F\|_{4,S(-\underline{u},\underline{u})} + \|\Omega^{-1} F\|_{2,\underline{C}_{\underline{u}}^V} + \|\Omega^{-1} r \nabla F\|_{2,\underline{C}_{\underline{u}}^V} + |u| \|\Omega \nabla_3 F\|_{2,\underline{C}_{\underline{u}}^V}.
\end{aligned}$$

Proof. See Corollary 3.2.1.1 in [5] and Corollary 4.1.1 in [13]. \square

Proposition 4.10. *Let F be a tensor field on S . Then, we have*

$$\|F\|_{\infty,S} \lesssim \|F\|_{4,S}^{\frac{1}{2}} \|r \nabla F\|_{4,S}^{\frac{1}{2}} + \|F\|_{4,S}.$$

Proof. See Lemma 4.1.3 in [13]. \square

Proposition 4.11. *The following hold for all $p \in (1, +\infty)$:*

1. *Let $\phi \in \mathfrak{s}_0$ be a solution of $\Delta \phi = f$. Then we have*

$$\|\nabla^2 \phi\|_{p,S} + r^{-1} \|\nabla \phi\|_{p,S} + r^{-2} \|\check{\phi}\|_{p,S} \lesssim \|f\|_{p,S}.$$

2. *Let $\xi \in \mathfrak{s}_1$ be a solution of $d_1 \xi = (f, f_*)$. Then we have*

$$\|\nabla \xi\|_{p,S} + r^{-1} \|\xi\|_{p,S} \lesssim \|f\|_{p,S} + \|f_*\|_{p,S}.$$

3. *Let $U \in \mathfrak{s}_2$ be a solution of $d_2 U = f$. Then we have*

$$\|\nabla U\|_{p,S} + r^{-1} \|U\|_{p,S} \lesssim \|f\|_{p,S}.$$

Proof. See Corollary 2.3.1.1 in [5]. \square

⁶Recall that the L^2 -flux on C_u^V and $\underline{C}_{\underline{u}}^V$ has been defined in (3.2).

5 Hyperbolic estimates (proof of Theorem 3.4)

In this section, we prove Theorem 3.4 by the r^p -weighted estimate method introduced in [6] and applied to Bianchi equations in [10, 21, 22].

5.1 Schematic notation \mathbb{E}_q , \mathbb{F}_q and \mathbb{O}_ℓ

We introduce the following schematic notations as in [22].

Definition 5.1. For a quantity X , we denote

$$X \in \mathbb{E}_q,$$

if X satisfies the following estimates:

$$\|r^{\frac{q}{2}} X^{(2)}\|_{2, \underline{C}_u^V} \leq \frac{\varepsilon}{|u|^{\frac{1-q}{2}}}, \quad \|X^{(1)}\|_{4, S} \leq \frac{\varepsilon}{r^{\frac{q+3}{2}} |u|^{\frac{1-q}{2}}}.$$

Similarly, we denote

$$X \in \mathbb{F}_q,$$

if X satisfies the following estimates:

$$\|r^{\frac{q}{2}} X^{(2)}\|_{2, \underline{C}_u^V} \leq \frac{\varepsilon}{|u|^{\frac{1-q}{2}}}, \quad \|\Omega X^{(1)}\|_{4, S} \leq \frac{\varepsilon}{r^{\frac{q+2}{2}} |u|^{\frac{2-q}{2}}}.$$

Moreover, we denote

$$X \in \mathbb{O}_\ell,$$

if X satisfies the following estimate:

$$\|X^{(2)}\|_{4, S} \leq \frac{\varepsilon}{r^{\frac{\ell+1}{2}} |u|^{\frac{1-\ell}{2}}}.$$

For $\mathbb{X} \in \{\mathbb{E}, \mathbb{F}, \mathbb{O}\}$, we also denote $X \in \mathbb{X}_q^{(i)}$, for $i = 0, 1, 2$, if X satisfies the same or even better decay and regularity properties as $\mathbb{X}_q^{(i)}$.

Lemma 5.2. Under the assumption (4.1), the following hold:

1. We have

$$\begin{aligned} \beta, \widehat{\rho}, \widehat{\sigma} &\in \mathbb{E}_{1-2\delta}, & \underline{\beta} &\in \mathbb{F}_{1-2\delta}, \\ \Gamma_g &\in \mathbb{O}_{1-2\delta}, & \Gamma_b &\in \mathbb{O}_{1-2\delta-4\delta_0}, & \Gamma_w &\in \mathbb{O}_{-2\delta-2\delta_0} \cap \mathbb{O}_{1-2\delta}^{(1)}. \end{aligned} \quad (5.1)$$

2. We have⁷

$$\alpha \in \mathbb{E}_{1-2\delta}^{(1)}, \quad \alpha^{(2)} \in \mathbb{E}_{1-2\delta}^{(2)}, \quad \underline{\alpha} \in \mathbb{F}_{-2\delta_0}^{(1)}, \quad \underline{\alpha}^{(2)} \in \mathbb{F}_{-4\delta}^{(2)}. \quad (5.2)$$

3. We have

$$r^{-1}\Gamma_g^{(1)} \in \mathbb{E}_{1-2\delta-\delta_0}, \quad r^{-1}\Gamma_b^{(1)} \in \mathbb{E}_{1-2\delta-7\delta_0}, \quad r^{-1}\Gamma_w^{(1)} \in \mathbb{E}_{-4\delta} \cap \mathbb{F}_{1-2\delta}. \quad (5.3)$$

⁷We have neither $\alpha \in \mathbb{E}_{1-2\delta}$ nor $\underline{\alpha} \in \mathbb{F}_{-4\delta}$ since (4.1) does not control their $W^{1,4}(S)$ norms.

Proof. Note that (5.1) follows from $\mathcal{R} \leq \varepsilon$ and $\mathcal{O}_{[2]} \leq \varepsilon$. (5.2) follows directly from $\mathcal{R} \leq \varepsilon$ and $\mathcal{O}_a \leq \varepsilon$. Finally, (5.3) follows from $\mathcal{R} \leq \varepsilon$ and $\mathcal{O}_3 \leq \varepsilon$. \square

The following theorem provides a unified treatment of all the nonlinear error terms in curvature estimates.

Theorem 5.3. *Let $p_1, p_2, \ell \leq 1$, $p < 1$ and $q \in \mathbb{N}$. We define*

$$\mathcal{N}_p^q [\psi_{(1)}, \Gamma \cdot \psi_{(2)}] := \int_V r^p \left| \psi_{(1)}^{(q)} \cdot (\Gamma \cdot \psi_{(2)})^{(q)} \right|. \quad (5.4)$$

Then, we have the following properties:

1. *In the case $2p < p_1 + p_2 + \ell + 1$, we have*

$$\mathcal{N}_p^1 [\mathbb{E}_{p_1}^{(1)}, \mathbb{O}_\ell^{(1)} \cdot \mathbb{E}_{p_2}^{(1)}] \lesssim \frac{\varepsilon^3}{|u|^{1-p}}. \quad (5.5)$$

2. *In the case $2p < p_1 + p_2 + \ell - 1$, we have*

$$\mathcal{N}_p^1 [\mathbb{F}_{p_1}^{(1)}, \mathbb{O}_\ell^{(1)} \cdot \mathbb{F}_{p_2}^{(1)}] \lesssim \frac{\varepsilon^3}{|u|^{1-p}}. \quad (5.6)$$

Proof. See Theorem 5.8 and Lemma A.1 in [22].⁸ \square

5.2 Estimates for general Bianchi pairs

The following lemma provides the general structure of Bianchi pairs.

Lemma 5.4. *Let $k = 1, 2$ and $a_{(1)}, a_{(2)}$ real numbers. Then, we have the following properties.*

1. *If $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_k$ and $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_{k-1}$ satisfying*

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} &= -kd_k^*(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \chi \psi_{(2)} &= d_k(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (5.7)$$

Then, the pair $(\psi_{(1)}, \psi_{(2)})$ satisfies for any real number p

$$\begin{aligned} & \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + k \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\ & + \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + k \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\ & = 2kr^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2r^p \psi_{(1)} \cdot h_{(1)} + 2kr^p \psi_{(2)} \cdot h_{(2)} + r^p \Gamma_b |\psi_{(1)}|^2 + r^p \Gamma_g |\psi_{(2)}|^2. \end{aligned} \quad (5.8)$$

2. *If $\psi_{(1)}, h_{(1)} \in \mathfrak{s}_{k-1}$ and $\psi_{(2)}, h_{(2)} \in \mathfrak{s}_k$ satisfying*

$$\begin{aligned} \nabla_3(\psi_{(1)}) + a_{(1)} \operatorname{tr} \underline{\chi} \psi_{(1)} &= d_k(\psi_{(2)}) + h_{(1)}, \\ \nabla_4(\psi_{(2)}) + a_{(2)} \operatorname{tr} \chi \psi_{(2)} &= -kd_k^*(\psi_{(1)}) + h_{(2)}. \end{aligned} \quad (5.9)$$

Then, the pair $(\psi_{(1)}, \psi_{(2)})$ satisfies for any real number p

$$\begin{aligned} & k \mathbf{Div}(r^p |\psi_{(1)}|^2 e_3) + \mathbf{Div}(r^p |\psi_{(2)}|^2 e_4) \\ & + k \left(2a_{(1)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \underline{\chi} |\psi_{(1)}|^2 + \left(2a_{(2)} - 1 - \frac{p}{2}\right) r^p \operatorname{tr} \chi |\psi_{(2)}|^2 \\ & = 2r^p \operatorname{div}(\psi_{(1)} \cdot \psi_{(2)}) + 2kr^p \psi_{(1)} \cdot h_{(1)} + 2r^p \psi_{(2)} \cdot h_{(2)} + r^p \Gamma_b |\psi_{(1)}|^2 + r^p \Gamma_g |\psi_{(2)}|^2. \end{aligned} \quad (5.10)$$

⁸Remark that for $\mathbb{X} \in \{\mathbb{E}, \mathbb{F}, \mathbb{O}\}$, $\mathbb{X}_q^{(1)}$ has exactly the same decay and regularity properties as \mathbb{X}_q in [22].

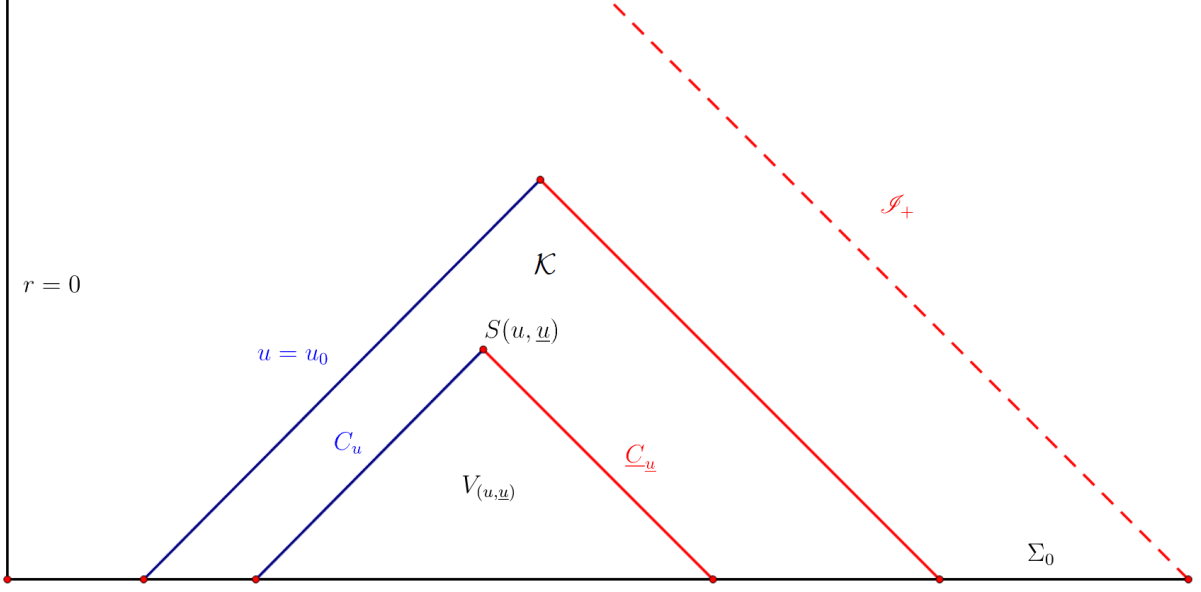


Figure 1: Domain of integration $V(u, \underline{u})$

Proof. See Lemma 4.2 in [21]. □

Remark 5.5. Note that the Bianchi equations can be written as systems of equations of the type (5.7) and (5.9). In particular

- the Bianchi pair (α, β) satisfies (5.7) with $k = 2$, $a_{(1)} = \frac{1}{2}$, $a_{(2)} = 2$,
- the Bianchi pair $(\beta, (\widehat{\rho}, -\widehat{\sigma}))$ satisfies (5.7) with $k = 1$, $a_{(1)} = 1$, $a_{(2)} = \frac{3}{2}$,
- the Bianchi pair $((\widehat{\rho}, \widehat{\sigma}), \underline{\beta})$ satisfies (5.9) with $k = 1$, $a_{(1)} = \frac{3}{2}$, $a_{(2)} = 1$,
- the Bianchi pair $(\underline{\beta}, \underline{\alpha})$ satisfies (5.9) with $k = 2$, $a_{(1)} = 2$, $a_{(2)} = \frac{1}{2}$.

The following lemma allows us to obtain $|u|$ -decay of curvature flux along $\Sigma_0 \cap V(u, \underline{u})$.

Lemma 5.6. For any $R \in \{\alpha, \beta, \widehat{\rho}, \widehat{\sigma}, \underline{\beta}, \underline{\alpha}\}$, we have the following estimate for $p < 1$:

$$\int_{\Sigma_0 \cap V(u, \underline{u})} r^p |R^{(2)}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{1-p}}.$$

Proof. We have from $\mathcal{I}_{(0)} \leq \varepsilon_0$

$$|R^{(2)}|_{2,S} \lesssim \frac{\varepsilon_0}{r}.$$

Hence, we obtain for $p < 1$

$$\int_{\Sigma_0 \cap V(u, \underline{u})} r^p |R^{(2)}|^2 \lesssim \int_{|u|}^{\infty} r^p |R^{(2)}|_{2,S}^2 dr \lesssim \int_{|u|}^{\infty} \frac{\varepsilon_0^2}{r^{2-p}} dr \lesssim \frac{\varepsilon_0^2}{|u|^{1-p}}.$$

This concludes the proof of Lemma 5.6. □

Proposition 5.7. Let $p < 1$ and let $(\psi_{(1)}, \psi_{(2)})$ be a general Bianchi pair satisfying (5.7) or (5.9) and $\psi_{(1)} \in \mathbb{E}_p^{(2)}$. Then, we have the following properties:

- In the case $2 + p - 4a_{(1)} > 0$ and $4a_{(2)} - 2 - p > 0$, we have

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(1)}|^2 + r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{1-p}} + \mathcal{N}_p^0 [\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0 [\psi_{(2)}, h_{(2)}]. \end{aligned} \quad (5.11)$$

- In the case $2 + p - 4a_{(1)} \leq 0$ and $4a_{(2)} - 2 - p > 0$, we have

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 + \int_V r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{1-p}} + \int_V r^{p-1} |\psi_{(1)}|^2 + \mathcal{N}_p^0 [\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0 [\psi_{(2)}, h_{(2)}]. \end{aligned} \quad (5.12)$$

Proof. We have from Stokes' theorem

$$\begin{aligned} \int_V \mathbf{Div} (r^p |\psi_{(1)}|^2 e_3 + r^p |\psi_{(2)}|^2 e_4) &= \int_{|u|}^u r^p |\psi_{(1)}|^2 \mathbf{g}(e_3, -e_4) du' + \int_{-u}^u r^p |\psi_{(2)}|^2 \mathbf{g}(e_4, -\Omega^2 e_3) du' \\ &+ \int_{\Sigma_0 \cap V} r^p |\psi_{(1)}|^2 \mathbf{g}(e_3, T) + r^p |\psi_{(2)}|^2 \mathbf{g}(e_4, T) \\ &= 2 \int_{C_u^V} r^p |\psi_{(1)}|^2 + 2 \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 - \int_{\Sigma_0 \cap V} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2). \end{aligned}$$

Thus, integrating (5.8) or (5.10) in $V = V(u, \underline{u})$, we obtain

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 - \left(2a_{(1)} - 1 - \frac{p}{2}\right) \int_V r^{p-1} |\psi_{(1)}|^2 + \left(2a_{(2)} - 1 - \frac{p}{2}\right) \int_V r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \int_{\Sigma_0 \cap V} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) + \mathcal{N}_p^0 [\psi_{(1)}, h_{(1)} + \Gamma_b \cdot \psi_{(1)}] + \mathcal{N}_p^0 [\psi_{(2)}, h_{(2)} + \Gamma_g \cdot \psi_{(2)}]. \end{aligned}$$

Applying Lemma 5.6, we deduce

$$\int_{\Sigma_0 \cap V} r^p (|\psi_{(1)}|^2 + |\psi_{(2)}|^2) \lesssim \frac{\varepsilon_0^2}{|u|^{1-p}}.$$

Recalling from Lemma 5.2 that $\Gamma_b \in \mathbb{O}_{1-2\delta-4\delta_0}$, we obtain from (5.5) in Theorem 5.3

$$\mathcal{N}_p^0 [\psi_{(1)}, \Gamma_b \cdot \psi_{(1)}] = \mathcal{N}_p^0 [\mathbb{E}_p^{(2)}, \mathbb{O}_{1-2\delta-4\delta_0} \cdot \mathbb{E}_p^{(2)}] \lesssim \mathcal{N}_p^2 [\mathbb{E}_p, \mathbb{O}_{1-2\delta-4\delta_0} \cdot \mathbb{E}_p] \lesssim \frac{\varepsilon^3}{|u|^{1-p}}.$$

Recalling from Lemma 4.2 that $r|\Gamma_g| \lesssim \varepsilon$, we deduce

$$\mathcal{N}_p^0 [\psi_{(2)}, \Gamma_g \cdot \psi_{(2)}] \lesssim \varepsilon \int_V r^{p-1} |\psi_{(2)}|^2.$$

Combining the above estimates, we infer

$$\begin{aligned} & \int_{C_u^V} r^p |\psi_{(1)}|^2 + \int_{\underline{C}_u^V} r^p |\psi_{(2)}|^2 - \left(2a_{(1)} - 1 - \frac{p}{2}\right) \int_V r^{p-1} |\psi_{(1)}|^2 + \left(2a_{(2)} - 1 - \frac{p}{2}\right) \int_V r^{p-1} |\psi_{(2)}|^2 \\ & \lesssim \frac{\varepsilon_0^2}{|u|^{1-p}} + \mathcal{N}_p^0 [\psi_{(1)}, h_{(1)}] + \mathcal{N}_p^0 [\psi_{(2)}, h_{(2)}] + \varepsilon \int_V r^{p-1} |\psi_{(2)}|^2. \end{aligned}$$

Recalling that $2a_{(2)} - 1 - \frac{p}{2} > 0$, we deduce that, for ε small enough, (5.11) and (5.12) hold in correspond cases. This concludes the proof of Proposition 5.7. \square

5.3 Estimates for the Bianchi pair (α, β)

Proposition 5.8. *We have the following estimate:*

$$\int_{C_u^V} r^{1-2\delta} |\alpha^{(2)}|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\beta^{(2)}|^2 + \int_V r^{-2\delta} (|\alpha^{(2)}|^2 + |\beta^{(2)}|^2) \lesssim \varepsilon_0^2. \quad (5.13)$$

Proof. We recall from Proposition 4.8 that

$$\begin{aligned} \nabla_3 \alpha + \frac{1}{2} \operatorname{tr} \underline{\chi} \alpha &= -2d_2^* \beta + r^{-1} (\Gamma_g \cdot \Gamma_b)^{(1)}, \\ \nabla_4 \beta + 2 \operatorname{tr} \chi \beta &= d_2 \alpha + r^{-1} \Gamma_g \cdot \Gamma_g^{(1)} + \Gamma_g \cdot \alpha. \end{aligned} \quad (5.14)$$

Differentiating (5.14) by \mathfrak{D}^q for $q = 0, 1, 2$ and applying Proposition 4.6, we deduce⁹

$$\begin{aligned} \nabla_3(\mathfrak{D}^q \alpha) + \frac{1}{2} \operatorname{tr} \underline{\chi}(\mathfrak{D}^q \alpha) &= -\mathfrak{D}_{(q)}^* (\mathfrak{D}^q \beta) + r^{-1} \beta^{(q-1)} + r^{-1} (\Gamma_g \cdot \Gamma_b)^{(3)} + (\Gamma_w \cdot \alpha)^{(2)}, \\ \nabla_4(\mathfrak{D}^q \beta) + 2 \operatorname{tr} \chi(\mathfrak{D}^q \beta) &= \mathfrak{D}(\mathfrak{D}^q \alpha) + r^{-1} (\Gamma_g \cdot \Gamma_g)^{(3)} + (\Gamma_g \cdot \alpha)^{(2)}, \end{aligned}$$

where $\mathfrak{D}_{(q)}^*$ is defined by

$$\mathfrak{D}_{(0)}^* := 2d_2^*, \quad \mathfrak{D}_{(1)}^* := d_1^*, \quad \mathfrak{D}_{(0)} := d_1.$$

Note that we have from Lemma 5.2

$$\begin{aligned} (\Gamma_w \cdot \alpha)^{(2)} &= (\Gamma_w^{(1)} \cdot \alpha)^{(1)} + \Gamma_w \cdot \alpha^{(2)} = (\mathbb{O}_{-2\delta-2\delta_0}^{(1)} \cdot \mathbb{E}_{1-2\delta}^{(1)})^{(1)}, \\ (\Gamma_g \cdot \alpha)^{(2)} &= (\Gamma_g^{(1)} \cdot \alpha)^{(1)} + \Gamma_g \cdot \alpha^{(2)} = (\mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta}^{(1)})^{(1)}, \\ r^{-1} (\Gamma_g \cdot \Gamma_b)^{(3)} &= (\mathbb{O}_{1-2\delta} \cdot \mathbb{E}_{1-2\delta-7\delta_0})^{(2)}, \\ r^{-1} (\Gamma_g \cdot \Gamma_g)^{(3)} &= (\mathbb{O}_{1-2\delta} \cdot \mathbb{E}_{1-2\delta-\delta_0})^{(2)}. \end{aligned}$$

Hence, we obtain by ignoring the terms which decay better

$$\begin{aligned} \nabla_3(\mathfrak{D}^q \alpha) + \frac{1}{2} \operatorname{tr} \underline{\chi}(\mathfrak{D}^q \alpha) &= -\mathfrak{D}_{(q)}^* (\mathfrak{D}^q \beta) + r^{-1} \beta^{(q-1)} + (\mathbb{O}_{-2\delta-2\delta_0}^{(1)} \cdot \mathbb{E}_{1-2\delta}^{(1)})^{(1)}, \\ \nabla_4(\mathfrak{D}^q \beta) + 2 \operatorname{tr} \chi(\mathfrak{D}^q \beta) &= \mathfrak{D}(\mathfrak{D}^q \alpha) + (\mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0})^{(1)}. \end{aligned}$$

Applying (5.11) with $\psi_{(1)} = \mathfrak{D}^q \alpha$, $\psi_{(2)} = \mathfrak{D}^q \beta$, $a_{(1)} = \frac{1}{2}$, $a_{(2)} = 2$, $p = 1 - 2\delta$ and noticing that

$$2 + p - 4a_{(1)} = 1 - 2\delta > 0, \quad 4a_{(2)} - 2 - p = 5 + 2\delta > 0, \quad (5.15)$$

we obtain from Lemma 5.2

$$\begin{aligned} &\int_{C_u^V} r^{1-2\delta} |\mathfrak{D}^q \alpha|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\mathfrak{D}^q \beta|^2 + \int_V r^{-2\delta} (|\mathfrak{D}^q \alpha|^2 + |\mathfrak{D}^q \beta|^2) \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}} + \int_V r^{-2\delta} |\mathfrak{D}^q \beta| |\mathfrak{D}^{q-1} \beta| + \mathcal{N}_{1-2\delta}^1 \left[\mathbb{E}_{1-2\delta}^{(1)}, \mathbb{O}_{-2\delta-2\delta_0}^{(1)} \cdot \mathbb{E}_{1-2\delta}^{(1)} \right]. \end{aligned} \quad (5.16)$$

The linear term on the R.H.S of (5.16) can be easily absorbed by Cauchy-Schwarz and induction. Moreover, applying Theorem 5.3 in the case $p = 1 - 2\delta$, $p_1 = 1 - 2\delta$, $p_2 = 1 - 2\delta$ and $\ell = -2\delta - 2\delta_0$, and noticing that

$$p_1 + p_2 + \ell = 2 - 6\delta - 2\delta_0 > 1 - 4\delta = 2p - 1,$$

we obtain from (5.5)

$$\mathcal{N}_{1-2\delta}^1 \left[\mathbb{E}_{1-2\delta}^{(1)}, \mathbb{O}_{-2\delta-2\delta_0}^{(1)} \cdot \mathbb{E}_{1-2\delta}^{(1)} \right] \lesssim \frac{\varepsilon^3}{|u|^{2\delta}}.$$

Injecting it into (5.16), we obtain (5.13). This concludes the proof of Proposition 5.8. \square

⁹The linear term $r^{-1} \beta^{(q-1)}$ comes from the fact that $-2d_2^* = -d_1^* d_1 + 2\mathbf{K}$, see (2.13). We also use the convention $\beta^{(-1)} = 0$.

5.4 Estimates for the Bianchi pair $(\beta, (\widehat{\rho}, -\widehat{\sigma}))$

Proposition 5.9. *We have the following estimate:*

$$\int_{C_u^V} r^{1-2\delta} |\beta^{(2)}|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |(\widehat{\rho}, \widehat{\sigma})^{(2)}|^2 + \int_V r^{-2\delta} |(\widehat{\rho}, \widehat{\sigma})^{(2)}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}}. \quad (5.17)$$

Proof. We recall from Proposition 4.8

$$\begin{aligned} \nabla_3 \beta + \text{tr} \chi \beta &= -d_1^*(\widehat{\rho}, -\widehat{\sigma}) + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}, \\ \nabla_4(\widehat{\rho}, -\widehat{\sigma}) + \frac{3}{2} \text{tr} \chi(\widehat{\rho}, -\widehat{\sigma}) &= d_1 \beta + r^{-1}(\Gamma_g \cdot \Gamma_g)^{(1)}. \end{aligned}$$

Differentiating it by $\not\partial^2$ and applying Proposition 4.6, we deduce

$$\begin{aligned} \nabla_3(\not\partial^2 \beta) + \text{tr} \chi(\not\partial^2 \beta) &= -d_1^*(\not\partial^2(\widehat{\rho}, -\widehat{\sigma})) + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(3)}, \\ \nabla_4(\not\partial^2(\widehat{\rho}, -\widehat{\sigma})) + \frac{3}{2} \text{tr} \chi(\not\partial^2(\widehat{\rho}, -\widehat{\sigma})) &= d_1(\not\partial^2 \beta) + r^{-1}(\Gamma_g \cdot \Gamma_g)^{(3)}. \end{aligned}$$

Applying Lemma 5.2, we obtain

$$\begin{aligned} \nabla_3(\not\partial^2 \beta) + \text{tr} \chi(\not\partial^2 \beta) &= -d_1^*(\not\partial^2(\widehat{\rho}, -\widehat{\sigma})) + (\mathbb{O}_{1-2\delta-4\delta_0} \cdot \mathbb{E}_{-4\delta})^{(2)}, \\ \nabla_4(\not\partial^2(\widehat{\rho}, -\widehat{\sigma})) + \frac{3}{2} \text{tr} \chi(\not\partial^2(\widehat{\rho}, -\widehat{\sigma})) &= d_1(\not\partial^2 \beta) + (\mathbb{O}_{1-2\delta} \cdot \mathbb{E}_{1-2\delta-3\delta_0})^{(2)}. \end{aligned}$$

Applying (5.12) with $\psi_{(1)} = \not\partial^2 \beta$, $\psi_{(2)} = \not\partial^2(\widehat{\rho}, -\widehat{\sigma})$, $a_{(1)} = 1$, $a_{(2)} = \frac{3}{2}$ and $p = 1 - 2\delta$, and noticing that

$$2 + p - 4a_{(1)} = -1 - 2\delta \leq 0, \quad 4a_{(2)} - 2 - p = 3 + 2\delta > 0, \quad (5.18)$$

we infer from Lemma 5.2

$$\begin{aligned} &\int_{C_u^V} r^{1-2\delta} |\not\partial^2 \beta|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\not\partial^2(\widehat{\rho}, -\widehat{\sigma})|^2 + \int_V r^{-2\delta} |\not\partial^2(\widehat{\rho}, -\widehat{\sigma})|^2 \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}} + \int_V r^{-2\delta} |\not\partial^2 \beta|^2 + \mathcal{N}_{1-2\delta}^2 [\mathbb{E}_{1-2\delta}, \mathbb{O}_{1-2\delta-4\delta_0} \cdot \mathbb{E}_{-4\delta}]. \end{aligned}$$

Applying (5.5) in Theorem 5.3, we easily deduce

$$\mathcal{N}_{1-2\delta}^2 [\mathbb{E}_{1-2\delta}, \mathbb{O}_{1-2\delta-4\delta_0} \cdot \mathbb{E}_{-4\delta}] \lesssim \frac{\varepsilon^3}{|u|^{2\delta}}.$$

Combining with (5.13), we obtain

$$\int_{C_u^V} r^{1-2\delta} |\not\partial^2 \beta|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\not\partial^2(\widehat{\rho}, -\widehat{\sigma})|^2 + \int_V r^{-2\delta} |\not\partial^2(\widehat{\rho}, -\widehat{\sigma})|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}}.$$

The estimates for $\not\partial^{\leq 1} \beta$ and $\not\partial^{\leq 1}(\widehat{\rho}, -\widehat{\sigma})$ can be deduced similarly. This concludes the proof of Proposition 5.9. \square

5.5 Estimates for the Bianchi pair $((\widehat{\rho}, \widehat{\sigma}), \beta)$

Proposition 5.10. *We have the following estimate:*

$$\int_{C_u^V} r^{1-2\delta} |(\widehat{\rho}, \widehat{\sigma})^{(2)}|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\underline{\beta}^{(2)}|^2 + \int_V r^{-2\delta} |\underline{\beta}^{(2)}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}}. \quad (5.19)$$

Proof. We recall from Proposition 4.8

$$\begin{aligned} \nabla_3(\widehat{\rho}, \widehat{\sigma}) + \frac{3}{2} \operatorname{tr} \chi(\widehat{\rho}, \widehat{\sigma}) &= -d_1 \underline{\beta} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}, \\ \nabla_4 \underline{\beta} + \operatorname{tr} \chi \underline{\beta} &= d_1^*(\widehat{\rho}, \widehat{\sigma}) + r^{-1}(\Gamma_g \cdot \Gamma_w)^{(1)}. \end{aligned}$$

Differentiating it by \mathfrak{D}^2 and applying Proposition 4.6, we deduce

$$\begin{aligned} \nabla_3(\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})) + \frac{3}{2} \operatorname{tr} \chi(\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})) &= -d_1(\mathfrak{D}^2 \underline{\beta}) + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(3)}, \\ \nabla_4(\mathfrak{D}^2 \underline{\beta}) + \operatorname{tr} \chi(\mathfrak{D}^2 \underline{\beta}) &= d_1^*(\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})) + r^{-1}(\Gamma_g \cdot \Gamma_w)^{(3)}. \end{aligned}$$

Noticing that we have from Lemma 5.2 and (2.19)

$$\begin{aligned} r^{-1}(\Gamma_g \cdot \Gamma_w)^{(3)} &= \Gamma_g \cdot \underline{\beta}^{(2)} + r^{-1}\Gamma_g^{(3)} \cdot \Gamma_w + r^{-1}\Gamma_g^{(2)} \cdot \Gamma_w^{(1)} + r^{-1}\Gamma_g^{(1)} \cdot \Gamma_w^{(2)} \\ &= \Gamma_g \cdot \underline{\beta}^{(2)} + \mathbb{E}_{1-2\delta-7\delta_0}^{(2)} \cdot \mathbb{O}_{1-2\delta}^{(1)} + \mathbb{E}_{1-2\delta-7\delta_0}^{(1)} \cdot \mathbb{O}_{1-2\delta}^{(2)} + \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(2)} \\ &= \Gamma_g \cdot \underline{\beta}^{(2)} + (\mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(1)})^{(1)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \nabla_3(\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})) + \frac{3}{2} \operatorname{tr} \chi(\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})) &= -d_1(\mathfrak{D}^2 \underline{\beta}) + (\mathbb{O}_{-2\delta-2\delta_0} \cdot \mathbb{E}_{1-2\delta-7\delta_0})^{(2)}, \\ \nabla_4(\mathfrak{D}^2 \underline{\beta}) + \operatorname{tr} \chi(\mathfrak{D}^2 \underline{\beta}) &= d_1^*(\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})) + \Gamma_g \cdot \underline{\beta}^{(2)} + (\mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(1)})^{(1)}. \end{aligned}$$

Applying (5.12) with $\psi_{(1)} = \mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})$, $\psi_{(2)} = \mathfrak{D}^2 \underline{\beta}$, $a_{(1)} = 1$, $a_{(2)} = \frac{3}{2}$ and $p = 1 - 2\delta$, we obtain

$$\begin{aligned} &\int_{C_u^V} r^{1-2\delta} |\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\mathfrak{D}^2 \underline{\beta}|^2 + \int_V r^{-2\delta} |\mathfrak{D}^2 \underline{\beta}|^2 \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}} + \int_V (r^{-2\delta} |\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})|^2 + r^{1-2\delta} |\Gamma_g| |\underline{\beta}^{(2)}|^2) + \mathcal{N}_{1-2\delta}^2 [\mathbb{E}_{1-2\delta}, \mathbb{O}_{-2\delta-2\delta_0} \cdot \mathbb{E}_{1-2\delta-7\delta_0}] \\ &\quad + \mathcal{N}_{1-2\delta}^1 [\mathbb{E}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(1)}] \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}} + \varepsilon \int_V r^{-2\delta} |\mathfrak{D}^2 \underline{\beta}|^2 + \mathcal{N}_{1-2\delta}^2 [\mathbb{E}_{1-2\delta}, \mathbb{O}_{-2\delta-2\delta_0} \cdot \mathbb{E}_{1-2\delta-7\delta_0}] + \mathcal{N}_{1-2\delta}^1 [\mathbb{E}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(1)}], \end{aligned}$$

where we used (5.17), the fact that $r|\Gamma_g| \lesssim \varepsilon$ and Proposition 4.11 in the last step. Thus, we deduce for ε small enough

$$\begin{aligned} &\int_{C_u^V} r^{1-2\delta} |\mathfrak{D}^2(\widehat{\rho}, \widehat{\sigma})|^2 + \int_{\underline{C}_u^V} r^{1-2\delta} |\mathfrak{D}^2 \underline{\beta}|^2 + \int_V r^{-2\delta} |\mathfrak{D}^2 \underline{\beta}|^2 \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{2\delta}} + \mathcal{N}_{1-2\delta}^2 [\mathbb{E}_{1-2\delta}, \mathbb{O}_{-2\delta-2\delta_0} \cdot \mathbb{E}_{1-2\delta-7\delta_0}] + \mathcal{N}_{1-2\delta}^1 [\mathbb{E}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(1)}]. \end{aligned}$$

Applying (5.5) in Theorem 5.3, we easily deduce

$$\mathcal{N}_{1-2\delta}^2 [\mathbb{E}_{1-2\delta}, \mathbb{O}_{-2\delta-2\delta_0} \cdot \mathbb{E}_{1-2\delta-7\delta_0}] + \mathcal{N}_{1-2\delta}^1 [\mathbb{E}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{E}_{1-2\delta-7\delta_0}^{(1)}] \lesssim \frac{\varepsilon^3}{|u|^{2\delta}}.$$

Combining the above estimates, this concludes the proof of Proposition 5.10. \square

5.6 Estimates for the Bianchi pair $(\underline{\beta}, \underline{\alpha})$

Proposition 5.11. *We have the following estimate:*

$$\int_{C_u^V} r^{-4\delta} |\underline{\beta}^{(2)}|^2 + \int_{\underline{C}_u^V} r^{-4\delta} |\underline{\alpha}^{(2)}|^2 + \int_V r^{-1-4\delta} |\underline{\alpha}^{(2)}|^2 \lesssim \varepsilon_0^2. \quad (5.20)$$

Proof. We have from Proposition 4.8

$$\begin{aligned} \nabla_3 \underline{\beta} + 2 \operatorname{tr} \chi \underline{\beta} &= -d_2 \underline{\alpha} + \Gamma_b \cdot \underline{\alpha}, \\ \nabla_4 \underline{\alpha} + \frac{1}{2} \operatorname{tr} \chi \underline{\alpha} &= 2d_2^* \underline{\beta} + \Gamma_g \cdot \underline{\alpha} + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(1)}. \end{aligned}$$

Differentiating it by \mathfrak{D}^2 and applying Proposition 4.6, we infer

$$\begin{aligned} \nabla_3(\mathfrak{D}^2 \underline{\beta}) + 2 \operatorname{tr} \chi(\mathfrak{D}^2 \underline{\beta}) &= -d_1^*(\mathfrak{D}^2 \underline{\alpha}) + r^{-1} (\Gamma_w \cdot \Gamma_w)^{(3)} + (\Gamma_b \cdot \underline{\alpha})^{(2)}, \\ \nabla_4(\mathfrak{D}^2 \underline{\alpha}) + \frac{1}{2} \operatorname{tr} \chi(\mathfrak{D}^2 \underline{\alpha}) &= d_1 \underline{\beta} + r^{-1} \underline{\beta}^{(1)} + (\Gamma_g \cdot \underline{\alpha})^{(2)} + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(3)}. \end{aligned}$$

Note that we have from Lemma 5.2

$$\begin{aligned} (\Gamma_b \cdot \underline{\alpha})^{(2)} &= \Gamma_b^{(2)} \cdot \underline{\alpha} + \Gamma_b^{(1)} \cdot \underline{\alpha}^{(1)} + \Gamma_b \cdot \underline{\alpha}^{(2)} = \mathbb{O}_{1-2\delta}^{(2)} \cdot \mathbb{F}_{-2\delta_0}^{(1)} + \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-4\delta}^{(2)}, \\ (\Gamma_g \cdot \underline{\alpha})^{(2)} &= \Gamma_g^{(2)} \cdot \underline{\alpha} + \Gamma_g^{(1)} \cdot \underline{\alpha}^{(1)} + \Gamma_g \cdot \underline{\alpha}^{(2)} = \mathbb{O}_{1-2\delta}^{(2)} \cdot \mathbb{F}_{-2\delta_0}^{(1)} + \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-2\delta_0}^{(2)} + \Gamma_g \cdot \underline{\alpha}^{(2)}. \end{aligned}$$

We also have from Lemma 5.2

$$r^{-1} (\Gamma_w \cdot \Gamma_w)^{(3)} = (\mathbb{O}_{-2\delta-2\delta_0} \cdot \mathbb{F}_{1-2\delta})^{(2)}, \quad r^{-1} (\Gamma_b \cdot \Gamma_w)^{(3)} = (\mathbb{O}_{1-2\delta-7\delta_0} \cdot \mathbb{F}_{1-2\delta})^{(2)}.$$

Hence, we deduce

$$\begin{aligned} \nabla_3(\mathfrak{D}^2 \underline{\beta}) + 2 \operatorname{tr} \chi(\mathfrak{D}^2 \underline{\beta}) &= -d_1^*(\mathfrak{D}^2 \underline{\alpha}) + (\mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-4\delta}^{(1)})^{(1)}, \\ \nabla_4(\mathfrak{D}^2 \underline{\alpha}) + \frac{1}{2} \operatorname{tr} \chi(\mathfrak{D}^2 \underline{\alpha}) &= d_1(\mathfrak{D}^2 \underline{\beta}) + r^{-1} \underline{\beta}^{(1)} + \Gamma_g \cdot \underline{\alpha}^{(2)} + (\mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-2\delta_0}^{(1)})^{(1)}. \end{aligned}$$

Applying (5.12) with $\psi_{(1)} = \mathfrak{D}^2 \underline{\beta}$, $\psi_{(2)} = \mathfrak{D}^2 \underline{\alpha}$, $a_{(1)} = 2$, $a_{(2)} = \frac{1}{2}$ and $p = -4\delta$, we obtain

$$\begin{aligned} &\int_{C_u^V} r^{-4\delta} |\mathfrak{D}^2 \underline{\beta}|^2 + \int_{\underline{C}_u^V} r^{-4\delta} |\mathfrak{D}^2 \underline{\alpha}|^2 + \int_V r^{-1-4\delta} |\mathfrak{D}^2 \underline{\alpha}|^2 \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{1+4\delta}} + \int_V r^{-1-4\delta} |\mathfrak{D}^2 \underline{\beta}|^2 + \int_V r^{-1-4\delta} |\mathfrak{D}^2 \underline{\beta}| |\underline{\beta}^{(1)}| + \mathcal{N}_{-4\delta}^1 [\mathbb{F}_{1-2\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-4\delta}^{(1)}] \\ &\quad + \mathcal{N}_{-4\delta}^1 [\mathbb{F}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-2\delta_0}^{(1)}] + \mathcal{N}_{-4\delta}^0 [\underline{\alpha}^{(2)}, \Gamma_g \cdot \underline{\alpha}^{(2)}] \\ &\lesssim \frac{\varepsilon_0^2}{|u|^{1+4\delta}} + \mathcal{N}_{-4\delta}^1 [\mathbb{F}_{1-2\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-4\delta}^{(1)}] + \mathcal{N}_{-4\delta}^1 [\mathbb{F}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-2\delta_0}^{(1)}] + \mathcal{N}_{-4\delta}^0 [\underline{\alpha}^{(2)}, \Gamma_g \cdot \underline{\alpha}^{(2)}], \end{aligned}$$

where we used (5.19) at the last step. Applying (5.6) in Theorem 5.3, we easily deduce

$$\mathcal{N}_{-4\delta}^1 [\mathbb{F}_{1-2\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-4\delta}^{(1)}] + \mathcal{N}_{-4\delta}^1 [\mathbb{F}_{-4\delta}^{(1)}, \mathbb{O}_{1-2\delta}^{(1)} \cdot \mathbb{F}_{-2\delta_0}^{(1)}] \lesssim \frac{\varepsilon^3}{|u|^{1+4\delta}}.$$

Moreover, we have from $r|\Gamma_g| \lesssim \varepsilon$ and Proposition 4.11

$$\mathcal{N}_{-4\delta}^0 [\underline{\alpha}^{(2)}, \Gamma_g \cdot \underline{\alpha}^{(2)}] = \int_V r^{-4\delta} |\Gamma_g| |\underline{\alpha}^{(2)}|^2 \lesssim \varepsilon \int_V r^{-1-4\delta} |\mathfrak{D}^2 \underline{\alpha}|^2.$$

Combining the above estimates, we obtain for ε small enough

$$\int_{C_u^V} r^{-4\delta} |\mathfrak{p}^2 \underline{\beta}|^2 + \int_{\underline{C}_{\underline{u}}^V} r^{-4\delta} |\mathfrak{p}^2 \underline{\alpha}|^2 + \int_V r^{-1-4\delta} |\mathfrak{p}^2 \underline{\alpha}|^2 \lesssim \frac{\varepsilon_0^2}{|u|^{1+4\delta}}.$$

Combining with Proposition 4.11, this concludes the proof of Proposition 5.11. \square

Finally, we deduce from Propositions 5.8–5.11 that

$$\mathcal{R} \lesssim \varepsilon_0,$$

this concludes the proof of Theorem 3.4.

6 Transport and $2D$ -elliptic estimates (proof of Theorem 3.5)

The goal of this section is to prove Theorem 3.5. Throughout this section, we use the following shorthand notations:

$$\int_{|u|}^u h(u, \underline{u}) := \int_{|u|}^u h(u, \underline{u}') d\underline{u}', \quad \int_{-\underline{u}}^u h(u, \underline{u}) := \int_{-\underline{u}}^u h(u', \underline{u}) du'.$$

6.1 Preliminaries

We recall the evolution lemma, which will be used repeatedly throughout this section.

Lemma 6.1. *Under the assumption $\mathcal{O} \leq \varepsilon$, the following holds:*

1. *Let U, F be k -covariant S -tangent tensor fields satisfying the outgoing evolution equation*

$$\nabla_4 U + \frac{\lambda}{2} \overline{\text{tr} \chi} U = F.$$

We have for $p \in [1, +\infty]$

$$\|r^\lambda U\|_{p,S} \lesssim \|r^\lambda U\|_{p,S(u,|u|)} + \int_{|u|}^u \|r^\lambda F\|_{p,S}.$$

2. *Let V, \underline{F} be k -covariant S -tangent tensor fields satisfying the incoming evolution equation*

$$\Omega^2 \nabla_3 V + \frac{\lambda}{2} \overline{\Omega^2 \text{tr} \underline{\chi}} V = \underline{F}.$$

We have for $p \in [1, +\infty]$

$$\|r^\lambda V\|_{p,S} \lesssim \|r^\lambda V\|_{p,S(-\underline{u},\underline{u})} + \int_{-\underline{u}}^u \|r^\lambda \underline{F}\|_{p,S}.$$

Proof. See Lemma 4.1.5 in [13] or Lemma 7 in [2]. \square

The following lemma will be useful to estimate the $L^2(S)$ -norms.

Lemma 6.2. *We have for $2\lambda > q + 1$*

$$\int_{|u|}^u r^\lambda \|\mathbb{E}_q^{(2)}\|_{2,S} \lesssim \frac{\varepsilon}{|u|^{\frac{1-q}{2}}} r^{\frac{2\lambda-1-q}{2}},$$

and for $2\lambda < q + 1$

$$\int_{|u|}^u r^\lambda \|\mathbb{E}_q^{(2)}\|_{2,S} \lesssim \varepsilon |u|^{\lambda-1}.$$

We also have for $2\lambda > -1$

$$\int_{-\underline{u}}^u |u|^\lambda \|\Omega \mathbb{F}_q^{(2)}\|_{2,S} \lesssim \frac{\varepsilon}{|u|^{\frac{1-q}{2}}} r^{\frac{2\lambda-1-q}{2}}.$$

Proof. We have for $2\lambda > q + 1$

$$\int_{|u|}^u r^\lambda \|\mathbb{E}_q^{(2)}\|_{2,S} \lesssim \left(\int_{|u|}^u r^{2\lambda-2-q} \right)^{\frac{1}{2}} \left(\int_{|u|}^u r^q |\mathbb{E}_q^{(2)}|_{2,S}^2 \right)^{\frac{1}{2}} \lesssim \frac{\varepsilon}{|u|^{\frac{1-q}{2}}} r^{\frac{2\lambda-1-q}{2}},$$

and for $2\lambda < q + 1$

$$\int_{|u|}^u r^\lambda \|\mathbb{E}_q^{(2)}\|_{2,S} \lesssim \frac{\varepsilon}{|u|^{\frac{1-q}{2}}} |u|^{\frac{2\lambda-1-q}{2}} \lesssim \varepsilon |u|^{\lambda-1}.$$

We also have for $2\lambda > -1$

$$\int_{-\underline{u}}^u |u|^\lambda \|\Omega \mathbb{F}_q^{(2)}\|_{2,S} \lesssim r^{-\frac{q+2}{2}} \left(\int_{-\underline{u}}^u |u|^{2\lambda} \right)^{\frac{1}{2}} \left(\int_{-\underline{u}}^u r^q |\Omega \mathbb{F}_q^{(2)}|_{2,S}^2 \right)^{\frac{1}{2}} \lesssim \frac{\varepsilon}{|u|^{\frac{1-q}{2}}} r^{\frac{2\lambda-1-q}{2}}.$$

This concludes the proof of Lemma 6.2. □

6.2 Estimates for curvature

Proposition 6.3. *We have the following estimates:*

$$\|(\beta, \widehat{\rho}, \widehat{\sigma})^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta} |u|^\delta}, \quad \|\underline{\beta}^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{\frac{3}{2}-\delta} |u|^{\frac{1}{2}+\delta}}.$$

Proof. It follows directly from Theorem 3.4 and Proposition 4.9. □

Proposition 6.4. *We have the following estimates:*

$$\|(\alpha, \beta, \widehat{\rho}, \widehat{\sigma})\|_{4,S} \lesssim \frac{\varepsilon_0}{r^2}, \quad \|\underline{\beta}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta} |u|^\delta}.$$

Proof. We have from Proposition 4.8

$$\nabla_3(\alpha, \beta, \widehat{\rho}, \widehat{\sigma}) = r^{-1}(\alpha, \beta, \widehat{\rho}, \widehat{\sigma}, \underline{\beta})^{(1)} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}.$$

Applying Lemma 6.1 and Proposition 6.3, we obtain

$$\begin{aligned} \|(\alpha, \beta, \widehat{\rho}, \widehat{\sigma})\|_{4,S} &\lesssim \|(\alpha, \beta, \widehat{\rho}, \widehat{\sigma})\|_{4,S(-\underline{u}, u)} + \int_{-\underline{u}}^u \Omega^2 \left(\|\underline{\beta}^{(1)}\|_{4,S} + \|(\Gamma_w \cdot \Gamma_b)^{(1)}\|_{4,S} \right) \\ &\lesssim \frac{\varepsilon_0}{r^2} + \int_{-\underline{u}}^u \frac{r^{\delta_0}}{|u|^{\delta_0}} \left(\frac{\varepsilon_0}{r^{\frac{5}{2}-\delta} |u|^{\frac{1}{2}+\delta}} + \frac{\varepsilon^2}{r^{3-\delta-2\delta_0} |u|^{\delta+2\delta_0}} \right) \\ &\lesssim \frac{\varepsilon_0}{r^2}. \end{aligned}$$

Next, we have from Proposition 4.8

$$\nabla_4 \underline{\beta} + \text{tr } \chi \underline{\beta} = r^{-1}(\widehat{\rho}, \widehat{\sigma})^{(1)} + r^{-1}(\Gamma_g \cdot \Gamma_w)^{(1)}.$$

Applying Lemma 6.1 and Proposition 6.3, we infer

$$\begin{aligned} \|r^2 \underline{\beta}\|_{4,S} &\lesssim \|r^2 \underline{\beta}\|_{4,S(u,|u|)} + \int_{|u|}^u \|r(\widehat{\rho}, \widehat{\sigma})^{(1)}\|_{4,S} + \|r(\Gamma_g \cdot \Gamma_w)^{(1)}\|_{4,S} \\ &\lesssim \varepsilon_0 + \int_{|u|}^u \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta} + \frac{\varepsilon^2}{r^{1-\delta}|u|^\delta} \\ &\lesssim \frac{\varepsilon_0}{|u|^\delta} r^\delta. \end{aligned}$$

This concludes the proof of Proposition 6.4. \square

Proposition 6.5. *We have the following estimates:*

$$\|\underline{\beta}^{(1)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta-\delta_0}|u|^{\delta+\delta_0}}, \quad \|\underline{\alpha}^{(1)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta_0}|u|^{1+\delta_0}}.$$

Proof. We have from Proposition 4.8

$$\nabla_4 \underline{\beta} + \text{tr } \chi \underline{\beta} = r^{-1}(\widehat{\rho}, \widehat{\sigma})^{(1)} + r^{-1}(\Gamma_g \cdot \Gamma_w)^{(1)}.$$

Differentiating it by $r\nabla$ and applying Proposition 4.6, we obtain

$$\nabla_4(r\nabla \underline{\beta}) + \text{tr } \chi(r\nabla \underline{\beta}) = r^{-1}(\widehat{\rho}, \widehat{\sigma})^{(2)} + r^{-1}(\Gamma_g \cdot \Gamma_w)^{(2)}.$$

Note that we have from Proposition 5.10

$$(\widehat{\rho}, \widehat{\sigma}) \in \frac{\varepsilon_0}{\varepsilon} \mathbb{E}_{1-2\delta}. \quad (6.1)$$

Applying Proposition 4.11 and Lemmas 6.1 and 6.2, we infer

$$\begin{aligned} \|r^2 \underline{\beta}^{(1)}\|_{2,S} &\lesssim \|r^2 \underline{\beta}^{(1)}\|_{2,S(u,|u|)} + \int_{|u|}^u \|r(\widehat{\rho}, \widehat{\sigma})^{(2)}\|_{2,S} + \frac{\varepsilon^2}{r^{1-\delta-\delta_0}|u|^{\delta+\delta_0}} \\ &\lesssim \varepsilon_0 + \frac{\varepsilon^2}{|u|^{\delta+\delta_0}} r^{\delta+\delta_0} + \frac{\varepsilon_0}{\varepsilon} \int_{|u|}^u r \|\mathbb{E}_{1-2\delta}^{(2)}\|_{2,S}, \\ &\lesssim \frac{\varepsilon_0}{|u|^{\delta+\delta_0}} r^{\delta+\delta_0}. \end{aligned}$$

Next, we have from Proposition 4.8

$$\nabla_4 \underline{\alpha} + \frac{1}{2} \text{tr } \chi \underline{\alpha} = r^{-1} \underline{\beta}^{(1)} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(1)}.$$

Differentiating it by $r\nabla$ and applying Proposition 4.6, we deduce

$$\nabla_4(r\nabla \underline{\alpha}) + \frac{1}{2} \text{tr } \chi(r\nabla \underline{\alpha}) = r^{-1} \underline{\beta}^{(2)} + (\Gamma_g \cdot \underline{\alpha})^{(1)} + r^{-1}(\Gamma_b \cdot \Gamma_w)^{(2)}.$$

Note that we have from Proposition 5.11

$$\underline{\beta} \in \frac{\varepsilon_0}{\varepsilon} \mathbb{E}_{-4\delta}.$$

Applying Lemmas 6.1 and 6.2, we infer

$$\begin{aligned}
\|r\underline{\alpha}^{(1)}\|_{2,S} &\lesssim \|r\underline{\alpha}^{(1)}\|_{2,S(u,|u|)} + \int_{|u|}^u \|\underline{\beta}^{(2)}\|_{2,S} + \frac{\varepsilon^2}{r^{1-\delta_0}|u|^{1+\delta_0}} + \frac{\varepsilon^2}{r^{\frac{3}{2}-\delta-\delta_0}|u|^{\frac{1}{2}+\delta+\delta_0}} \\
&\lesssim \frac{\varepsilon_0}{|u|} + \frac{\varepsilon^2}{|u|^{1+\delta_0}} r^{\delta_0} + \frac{\varepsilon^2}{|u|} + \frac{\varepsilon_0}{\varepsilon} \int_{|u|}^u \|\mathbb{E}_{-4\delta}^{(2)}\|_{2,S} \\
&\lesssim \frac{\varepsilon_0 r^{\delta_0}}{|u|^{1+\delta_0}}.
\end{aligned}$$

This concludes the proof of Proposition 6.5. \square

6.3 Estimates for \varkappa , ω and $\log(2\Omega)$

Proposition 6.6. *We have the following estimates:*

$$\|\varkappa^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^2}, \quad \|\varkappa^{(2)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta}|u|^\delta}. \quad (6.2)$$

Proof. We have from Proposition 4.7

$$\nabla_3 \varkappa + \frac{1}{2} \operatorname{tr} \underline{\chi} \varkappa = -r^{-1} \beta + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(1)}. \quad (6.3)$$

Differentiating (6.3) by $\not\partial$ and applying Proposition 4.6, we obtain

$$\nabla_3 (\not\partial \varkappa) + \frac{1}{2} \operatorname{tr} \underline{\chi} (\not\partial \varkappa) = r^{-1} \beta^{(1)} + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(2)}. \quad (6.4)$$

Applying Lemma 6.1, we deduce

$$\begin{aligned}
\|r(\not\partial \varkappa)\|_{4,S} &\lesssim \|r(\not\partial \varkappa)\|_{4,S(-\underline{u},\underline{u})} + \int_{-\underline{u}}^u \|\Omega^2 \beta^{(1)}\|_{4,S} + \|\Omega^2 (\Gamma_b \cdot \Gamma_w)^{(2)}\|_{4,S} \\
&\lesssim \frac{\varepsilon_0}{r} + \int_{-\underline{u}}^u \frac{\varepsilon_0}{r^{2-\delta-\delta_0}|u|^{\delta+\delta_0}} + \frac{\varepsilon^2}{r^{\frac{3}{2}-\delta-2\delta_0}|u|^{\frac{1}{2}+\delta+2\delta_0}} \\
&\lesssim \frac{\varepsilon_0}{r}.
\end{aligned}$$

Next, differentiating (6.4) by $\not\partial$ and applying Proposition 4.6, we infer

$$\nabla_3 (\not\partial^2 \varkappa) + \frac{1}{2} \operatorname{tr} \underline{\chi} (\not\partial^2 \varkappa) = r^{-1} \beta^{(2)} + r^{-1} (\Gamma_b \cdot \Gamma_w)^{(3)}.$$

Note that we have from Proposition 5.8 and Lemma 5.2

$$(\Omega\beta)^{(2)} \in \frac{\varepsilon_0}{\varepsilon} \mathbb{F}_{1-2\delta}^{(2)}, \quad r^{-1} \Gamma_b^{(3)}, r^{-1} \Gamma_w^{(3)} \in \mathbb{F}_{1-2\delta}^{(2)}. \quad (6.5)$$

Applying Lemmas 6.1 and 6.2, we have

$$\begin{aligned}
\|r\not\partial^2 \varkappa\|_{2,S} &\lesssim \|r\not\partial^2 \varkappa\|_{2,S(-\underline{u},\underline{u})} + \int_{-\underline{u}}^u \|\Omega^2 (\Omega\beta)^{(2)}\|_{2,S} + \|\Omega^2 (\Gamma_b \cdot \Gamma_w)^{(3)}\|_{2,S} \\
&\lesssim \frac{\varepsilon_0}{r} + \frac{\varepsilon_0}{\varepsilon} r^{\delta_0} \int_{-\underline{u}}^u |u|^{-\delta_0} \|\mathbb{F}_{1-2\delta}^{(2)}\|_{2,S} + \varepsilon r^{\frac{\delta_0+\delta}{2}} \int_{-\underline{u}}^u |u|^{-\frac{\delta+\delta_0}{2}} \|\mathbb{F}_{1-2\delta}^{(2)}\|_{2,S}, \\
&\lesssim \frac{\varepsilon_0}{r} + \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta} + \frac{\varepsilon^2}{r^{1-\delta}|u|^\delta} \\
&\lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}.
\end{aligned}$$

This concludes the proof of Proposition 6.6. \square

Proposition 6.7. *We have the following estimates:*

$$\|\omega^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r}, \quad \|\omega^{(2)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}. \quad (6.6)$$

Proof. We have from Proposition 4.7

$$\nabla_3 \omega = \widehat{\rho} + \Gamma_b \cdot \Gamma_b + \Gamma_g \cdot \Gamma_w. \quad (6.7)$$

Applying Lemma 6.1, we obtain

$$\begin{aligned} \|\omega\|_{4,S} &\lesssim \|\omega\|_{4,S(-\underline{u},\underline{u})} + \int_{-\underline{u}}^{\underline{u}} \|\Omega^2 \widehat{\rho}\|_{4,S} + \|\Omega^2 \Gamma_b \cdot \Gamma_b\|_{4,S} + \|\Omega^2 \Gamma_g \cdot \Gamma_w\|_{4,S} \\ &\lesssim \frac{\varepsilon_0}{r} + \int_{-\underline{u}}^{\underline{u}} \frac{\varepsilon_0}{r^2} + \frac{\varepsilon^2}{r^{1-3\delta_0}|u|^{3\delta_0}} + \frac{\varepsilon^2}{r^{2-\delta}|u|^\delta} \\ &\lesssim \frac{\varepsilon_0}{r}. \end{aligned}$$

Applying Propositions 4.7, 4.11 and 6.6, we deduce

$$\|\not\partial \omega\|_{4,S} \lesssim \|r(\varkappa, \beta)\|_{4,S} \lesssim \frac{\varepsilon_0}{r}, \quad \|\not\partial^2 \omega\|_{4,S} \lesssim \|r(\varkappa, \beta)^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}.$$

This concludes the proof of Proposition 6.7. \square

Proposition 6.8. *We have the following estimates:*

$$\begin{aligned} \|\log(2\Omega)\|_{\infty,S} &\lesssim \varepsilon_0 \log\left(\frac{3r}{|u|}\right), & \|\not\partial \log(2\Omega)\|_{4,S} &\lesssim \varepsilon_0 \frac{r^{\delta_0}}{|u|^{\delta_0}}, \\ \|\not\partial^2 \log(2\Omega)\|_{4,S} &\lesssim \varepsilon_0 \frac{r^\delta}{|u|^\delta}, & \|\not\partial^3 \log(2\Omega)\|_{2,S} &\lesssim \varepsilon_0 \frac{r^{\delta+\delta_0}}{|u|^{\delta+\delta_0}}. \end{aligned} \quad (6.8)$$

Proof. We recall from (2.12) that

$$\nabla_4(\log(2\Omega)) = -\omega. \quad (6.9)$$

Applying Lemma 6.1 and Propositions 6.7 and 4.10, we infer

$$\|\log(2\Omega)\|_{\infty,S} \lesssim \|\log(2\Omega)\|_{\infty,S(u,|u|)} + \int_{|u|}^{\underline{u}} \|\omega\|_{\infty,S} d\underline{u} \lesssim \varepsilon_0 + \int_{|u|}^{\underline{u}} \frac{\varepsilon_0}{r} d\underline{u} \lesssim \varepsilon_0 \log\left(\frac{3r}{|u|}\right).$$

Next, differentiating (6.9) by $\not\partial$ and applying Proposition 4.5, we obtain

$$\nabla_4(\not\partial \log(2\Omega)) = \omega^{(1)} + r\Gamma_g \cdot \Gamma_b.$$

Applying Lemma 6.1 and Proposition 6.7, we deduce

$$\|\not\partial \log(2\Omega)\|_{4,S} \lesssim \varepsilon_0 + \int_{|u|}^{\underline{u}} \frac{\varepsilon_0}{r} + \frac{\varepsilon^2}{r^{1-\delta_0}|u|^{\delta_0}} \lesssim \varepsilon_0 \frac{r^{\delta_0}}{|u|^{\delta_0}}.$$

Then, differentiating (6.9) by $\not\partial^2$ and applying Proposition 4.6, we have

$$\nabla_4(\not\partial^2 \log(2\Omega)) = \omega^{(2)} + r(\Gamma_g \cdot \Gamma_b)^{(1)}. \quad (6.10)$$

Applying Lemma 6.1 and Proposition 6.7, we obtain

$$\|\mathfrak{D}^2 \log(2\Omega)\|_{4,S} \lesssim \varepsilon_0 + \int_{|u|}^u \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta} + \frac{\varepsilon^2}{r^{1-\delta}|u|^\delta} \lesssim \varepsilon_0 \frac{r^\delta}{|u|^\delta}.$$

Finally, differentiating (6.10) by \mathfrak{D} and applying Propositions 4.5 and 4.7, we have

$$\nabla_4(\mathfrak{D}^3 \log(2\Omega)) = \omega^{(3)} + r(\Gamma_g \cdot \Gamma_b)^{(2)} = (\varkappa, \beta)^{(2)} + r(\Gamma_g \cdot \Gamma_b)^{(2)}.$$

Notice that we have from Proposition 5.9

$$\beta \in \frac{\varepsilon_0}{\varepsilon} \mathbb{E}_{1-2\delta}.$$

Applying Lemma 6.1 and Proposition 6.6, we deduce

$$\begin{aligned} \|\mathfrak{D}^3 \log(2\Omega)\|_{2,S} &\lesssim \varepsilon_0 + \int_{|u|}^u \|r(\varkappa, \beta)^{(2)}\|_{2,S} + \|r(\Gamma_g \cdot \Gamma_b)^{(2)}\|_{2,S} \\ &\lesssim \varepsilon_0 + \frac{\varepsilon_0}{\varepsilon} \int_{|u|}^u r \|\mathbb{E}_{1-2\delta}^{(2)}\|_{2,S} + \int_{|u|}^u \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta} + \frac{\varepsilon^2}{r^{1-\delta-\delta_0}|u|^{\delta+\delta_0}} \\ &\lesssim \varepsilon_0 + \frac{\varepsilon_0}{\varepsilon} \frac{\varepsilon}{|u|^\delta} r^\delta + \frac{\varepsilon_0}{|u|^\delta} r^\delta + \frac{\varepsilon^2}{|u|^{\delta+\delta_0}} r^{\delta+\delta_0}, \\ &\lesssim \frac{\varepsilon_0}{|u|^{\delta+\delta_0}} r^{\delta+\delta_0}. \end{aligned}$$

This concludes the proof of Proposition 6.8. \square

6.4 Estimates for $\widetilde{\text{tr}} \chi$ and $\widetilde{\text{tr}} \underline{\chi}$

Proposition 6.9. *We have the following estimates:*

$$\|\widetilde{\text{tr}} \chi^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r}, \quad \|\widetilde{\text{tr}} \chi^{(2)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}, \quad \|\widetilde{\text{tr}} \chi^{(3)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}. \quad (6.11)$$

Proof. We recall the following equation from Proposition 4.7:

$$\nabla_4(\text{tr } \chi) + \frac{1}{2}(\text{tr } \chi)^2 = r^{-1}\omega^{(0)} + \Gamma_g \cdot \Gamma_g. \quad (6.12)$$

Differentiating (6.12) by \mathfrak{D} and applying Proposition 4.5, we obtain

$$\nabla_4(\mathfrak{D} \text{tr } \chi) + \text{tr } \chi(\mathfrak{D} \text{tr } \chi) = r^{-1}\omega^{(1)} + \Gamma_g \cdot \Gamma_g^{(1)}. \quad (6.13)$$

Applying Lemma 6.1 and Proposition 6.6, we obtain

$$\begin{aligned} \|r^2 \mathfrak{D} \text{tr } \chi\|_{4,S} &\lesssim \|r^2 \mathfrak{D} \text{tr } \chi\|_{4,S(u,|u|)} + \int_{|u|}^u \|r\omega^{(1)}\|_{4,S} + \|r^2 \Gamma_g \cdot \Gamma_g^{(1)}\|_{4,S} \\ &\lesssim \varepsilon_0 |u| + \int_{|u|}^u \varepsilon_0 + \varepsilon^2 \\ &\lesssim \varepsilon_0 r. \end{aligned}$$

Next, differentiating (6.13) by \mathfrak{D} and applying Proposition 4.6, we infer

$$\nabla_4(\mathfrak{D}^2 \text{tr } \chi) + \text{tr } \chi(\mathfrak{D}^2 \text{tr } \chi) = r^{-1}\omega^{(2)} + (\Gamma_g \cdot \Gamma_g)^{(2)}. \quad (6.14)$$

Applying Lemma 6.1 and Proposition 6.6, we easily obtain

$$\|r^2\mathfrak{D}^2 \operatorname{tr} \chi\|_{4,S} \lesssim \|r^2\mathfrak{D}^2 \operatorname{tr} \chi\|_{4,S(u,|u|)} + \int_{|u|}^u \frac{\varepsilon_0}{|u|^\delta} r^\delta + \frac{\varepsilon^2}{|u|^\delta} r^\delta \lesssim \frac{\varepsilon_0}{|u|^\delta} r^{1+\delta}.$$

Finally, differentiating (6.14) and applying Proposition 4.5 and Proposition 4.11, we deduce

$$\nabla_4(\mathfrak{D}^3 \operatorname{tr} \chi) + \operatorname{tr} \chi(\mathfrak{D}^3 \operatorname{tr} \chi) = (\varkappa, \beta)^{(2)} + (\Gamma_g \cdot \Gamma_g)^{(3)}.$$

Notice that we have from Proposition 5.9

$$\beta^{(2)} \in \frac{\varepsilon_0}{\varepsilon} \mathbb{E}_{1-2\delta}^{(2)}, \quad r^{-1}\Gamma_g^{(3)} \in \mathbb{E}_{1-2\delta}^{(2)} + \mathbb{G}, \quad \|\mathbb{G}\|_{2,S} \lesssim \frac{\varepsilon}{r^{2-\delta}|u|^\delta}.$$

Applying Lemmas 6.1 and 6.2, we infer

$$\begin{aligned} \|r^2\mathfrak{D}^3 \operatorname{tr} \chi\|_{2,S} &\lesssim \|r\mathfrak{D}^3 \operatorname{tr} \chi\|_{2,S(u,|u|)} + \int_{|u|}^u r^2 \|\beta^{(2)}\|_{2,S} + \varepsilon r \|\Gamma_g^{(3)}\|_{2,S} + \frac{\varepsilon_0}{|u|^\delta} r^\delta \\ &\lesssim \varepsilon_0 |u| + \frac{\varepsilon_0}{|u|^\delta} r^{1+\delta} + \frac{\varepsilon_0}{\varepsilon} \int_{|u|}^u r^2 \|\mathbb{E}_{1-2\delta}^{(2)}\|_{2,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^\delta} r^{1+\delta}. \end{aligned}$$

This concludes the proof of Proposition 6.9. \square

Proposition 6.10. *We have the following estimates:*

$$\|\widetilde{\operatorname{tr} \underline{\chi}}^{(2)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r}, \quad \|\widetilde{\operatorname{tr} \underline{\chi}}^{(3)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}. \quad (6.15)$$

Proof. We have from Proposition 4.7:

$$\nabla_3 \operatorname{tr} \underline{\chi} + \frac{1}{2}(\operatorname{tr} \underline{\chi})^2 = \Gamma_w \cdot \Gamma_w. \quad (6.16)$$

Differentiating (6.16) by \mathfrak{D}^2 and applying Proposition 4.6, we obtain

$$\nabla_3(\mathfrak{D}^2 \operatorname{tr} \underline{\chi}) + \operatorname{tr} \underline{\chi}(\mathfrak{D}^2 \operatorname{tr} \underline{\chi}) = r^{-1}(\nabla \log \Omega)^{(1)} + (\Gamma_w \cdot \Gamma_w)^{(2)}. \quad (6.17)$$

Applying Lemmas 6.1 and Proposition 6.8, we infer

$$\begin{aligned} \|r^2\mathfrak{D}^2 \operatorname{tr} \underline{\chi}\|_{4,S} &\lesssim \|r^2\mathfrak{D}^2 \operatorname{tr} \underline{\chi}\|_{4,S(-u,u)} + \int_{-u}^u \|r\Omega^2(\nabla \log \Omega)^{(1)}\|_{4,S} + \|r^2\Omega^2(\Gamma_w \cdot \Gamma_w)^{(2)}\|_{4,S} \\ &\lesssim \varepsilon_0 r + \int_{-u}^u \frac{\varepsilon_0}{|u|^{2\delta_0}} r^{2\delta_0} + r^2 \frac{\varepsilon^2}{r^{\frac{3}{2}-2\delta-2\delta_0}|u|^{\frac{1}{2}+2\delta+2\delta_0}} \\ &\lesssim \varepsilon_0 r. \end{aligned}$$

Next, differentiating (6.17) by \mathfrak{D} and applying Proposition 4.5, we deduce

$$\nabla_3(\mathfrak{D}^3 \operatorname{tr} \underline{\chi}) + \operatorname{tr} \underline{\chi}(\mathfrak{D}^3 \operatorname{tr} \underline{\chi}) = r^{-1}(\nabla \log \Omega)^{(2)} + (\Gamma_w \cdot \Gamma_w)^{(3)}.$$

Applying Lemmas 6.1 and 6.2, (6.5) and Proposition 6.8, we deduce

$$\begin{aligned} \|r^2\mathfrak{D}^3 \operatorname{tr} \underline{\chi}\|_{2,S} &\lesssim \|r^2\mathfrak{D}^3 \operatorname{tr} \underline{\chi}\|_{2,S(-u,u)} + \int_{-u}^u \|r\Omega^2(\nabla \log \Omega)^{(2)}\|_{2,S} + \|r^2\Omega^2(\Gamma_w \cdot \Gamma_w)^{(3)}\|_{2,S} \\ &\lesssim \varepsilon_0 r + \int_{-u}^u \frac{\varepsilon_0 r^{\delta+\delta_0}}{|u|^{\delta+\delta_0}} + \frac{\varepsilon^2 r^2}{r^{\frac{3}{2}-2\delta-2\delta_0}|u|^{\frac{1}{2}+2\delta+2\delta_0}} + r^{3+2\delta_0} \int_{-u}^u |u|^{-2\delta_0} \|\Omega \mathbb{F}_{1-2\delta}^{(2)}\|_{2,S} \\ &\lesssim \varepsilon_0 \frac{r^{1+\delta}}{|u|^\delta}. \end{aligned}$$

This concludes the proof of Proposition 6.10. \square

6.5 Estimates for μ and $\underline{\mu}$

Proposition 6.11. *We have the following estimates:*

$$\|\mu\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{2-3\delta_0}|u|^{3\delta_0}}, \quad \|\mu^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta-2\delta_0}|u|^{\delta+2\delta_0}}, \quad \|\mu^{(2)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta-3\delta_0}|u|^{\delta+3\delta_0}}.$$

Proof. We recall from Proposition 4.7

$$\nabla_4[\widetilde{\mu}] + \text{tr } \chi[\widetilde{\mu}] = r^{-2}\widetilde{\text{tr } \chi}^{(0)} + r^{-1}\widetilde{\rho}^{(0)} + r^{-1}\Gamma_b \cdot \Gamma_b + r^{-1}(\Gamma_g \cdot \Gamma_b)^{(1)}. \quad (6.18)$$

Applying Lemma 6.1 and Propositions 6.4, 6.9 and 4.10, we infer

$$\begin{aligned} \|r^2[\widetilde{\mu}]\|_{4,S} &\lesssim \|r^2[\widetilde{\mu}]\|_{4,S(u,|u|)} + \int_{|u|}^u \|(\widetilde{\text{tr } \chi}, r\widetilde{\rho})\|_{4,S} + r\|\Gamma_b \cdot \Gamma_b\|_{4,S} + \|r(\Gamma_g \cdot \Gamma_b)^{(1)}\|_{4,S} \\ &\lesssim \varepsilon_0 + \int_{|u|}^u \frac{\varepsilon_0}{r} + \frac{\varepsilon}{r^{1-\delta_0}|u|^{\delta_0}} \frac{\varepsilon}{r^{1-\frac{\delta_0+3\delta_0}{2}}|u|^{\frac{3\delta_0+\delta_0}{2}}} + \frac{\varepsilon^2}{r^{1-3\delta_0}|u|^{3\delta_0}} \\ &\lesssim \varepsilon_0 \frac{r^{3\delta_0}}{|u|^{3\delta_0}}. \end{aligned}$$

Next, differentiating (6.18) by $\not\partial$ and applying Proposition 4.5, we deduce

$$\nabla_4(\not\partial[\widetilde{\mu}]) + \text{tr } \chi(\not\partial[\widetilde{\mu}]) = r^{-2}\widetilde{\text{tr } \chi}^{(1)} + r^{-1}\widetilde{\rho}^{(1)} + r^{-1}\Gamma_b \cdot \Gamma_b^{(1)} + r^{-1}(\Gamma_g \cdot \Gamma_b)^{(2)}. \quad (6.19)$$

Applying Lemma 6.1 and Propositions 6.4 and 6.9, we obtain

$$\begin{aligned} \|r^2\not\partial[\widetilde{\mu}]\|_{4,S} &\lesssim \|r^2\not\partial[\widetilde{\mu}]\|_{4,S(u,|u|)} + \int_{|u|}^u \|(\widetilde{\text{tr } \chi}, r\widetilde{\rho})^{(1)}\|_{4,S} + \|r(\Gamma_b \cdot \Gamma_b)^{(1)}\|_{4,S} + \|r(\Gamma_g \cdot \Gamma_b)^{(2)}\|_{4,S} \\ &\lesssim \varepsilon_0 + \int_{|u|}^u \frac{\varepsilon_0}{r} + \frac{\varepsilon_0}{r^{1-\delta}|u|^{\delta}} + \frac{\varepsilon^2}{r^{1-5\delta_0}|u|^{5\delta_0}} + \frac{\varepsilon^2}{r^{1-\delta-2\delta_0}|u|^{\delta+2\delta_0}} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\delta+2\delta_0}} r^{\delta+2\delta_0}. \end{aligned}$$

Finally, differentiating (6.19) by $\not\partial$ and applying Proposition 4.6, we obtain

$$\nabla_4(\not\partial^2[\widetilde{\mu}]) + \text{tr } \chi(\not\partial^2[\widetilde{\mu}]) = r^{-2}\widetilde{\text{tr } \chi}^{(2)} + r^{-1}\widetilde{\rho}^{(2)} + r^{-1}(\Gamma_b \cdot \Gamma_b)^{(2)} + r^{-1}(\Gamma_g \cdot \Gamma_b)^{(3)}.$$

Applying Lemmas 6.1 and 6.2, (6.1) and Propositions 5.10 and 6.9, we obtain

$$\begin{aligned} \|r^2\not\partial^2[\widetilde{\mu}]\|_{2,S} &\lesssim \|r^2\not\partial^2[\widetilde{\mu}]\|_{2,S(u,|u|)} + \int_{|u|}^u \|(\widetilde{\text{tr } \chi}, r\widetilde{\rho})^{(2)}\|_{2,S} + \|r(\Gamma_b \cdot \Gamma_b)^{(2)}\|_{2,S} + \|r(\Gamma_g \cdot \Gamma_b)^{(3)}\|_{2,S} \\ &\lesssim \varepsilon_0 + \int_{|u|}^u \frac{\varepsilon_0}{r^{1-\delta}|u|^{\delta}} + \frac{\varepsilon^2}{r^{1-\delta-3\delta_0}|u|^{\delta+3\delta_0}} + \int_{|u|}^u r\|\widetilde{\rho}^{(2)}\|_{2,S} + \|r\Gamma_b \cdot \Gamma_b^{(3)}\|_{2,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\delta+3\delta_0}} r^{\delta+3\delta_0} + \frac{\varepsilon_0}{\varepsilon} \frac{\varepsilon}{|u|^{\delta}} r^{\delta} + \frac{\varepsilon^2}{|u|^{\delta+2\delta_0}} r^{\delta+2\delta_0} \\ &\lesssim \frac{\varepsilon_0}{|u|^{\delta+3\delta_0}} r^{\delta+3\delta_0}. \end{aligned}$$

Recalling from (2.16) that

$$[\widetilde{\mu}] = \mu - \bar{\mu} + \frac{1}{4}\widetilde{\text{tr } \chi} \text{tr } \underline{\chi} = \mu + \bar{\rho} + \frac{1}{2r}\widetilde{\text{tr } \chi} - \frac{1}{2r}\text{tr } \underline{\chi} + \Gamma_g \cdot \Gamma_g,$$

we easily deduce from Propositions 6.9 and 6.10 that

$$\|\mu\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-3\delta_0}|u|^{3\delta_0}}, \quad \|\not\partial\mu\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta-2\delta_0}|u|^{\delta+2\delta_0}}, \quad \|\not\partial^2\mu\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta-3\delta_0}|u|^{\delta+3\delta_0}}.$$

This concludes the proof of Proposition 6.11. \square

Proposition 6.12. *We have the following estimates:*

$$\|\underline{\mu}^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^2}, \quad \|\underline{\mu}^{(2)}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta}|u|^\delta}.$$

Proof. We recall from Proposition 4.7

$$\nabla_3[\widetilde{\underline{\mu}}] + \text{tr} \underline{\chi}[\widetilde{\underline{\mu}}] = r^{-2} \widetilde{\text{tr} \underline{\chi}}^{(0)} + r^{-1} \widetilde{\rho}^{(0)} + r^{-1} (\Gamma_w \cdot \Gamma_w)^{(1)}. \quad (6.20)$$

Differentiating (6.20) by $\not\partial$ and applying Proposition 4.5, we deduce

$$\nabla_3[\widetilde{\underline{\mu}}]^{(1)} + \text{tr} \underline{\chi}[\widetilde{\underline{\mu}}]^{(1)} = r^{-2} \widetilde{\text{tr} \underline{\chi}}^{(1)} + r^{-1} \widetilde{\rho}^{(1)} + r^{-1} (\Gamma_w \cdot \Gamma_w)^{(2)}. \quad (6.21)$$

Applying Lemma 6.1 and Propositions 6.3 and 6.10, we infer

$$\begin{aligned} \|r^2[\widetilde{\underline{\mu}}]^{(1)}\|_{4,S} &\lesssim \|r^2[\widetilde{\underline{\mu}}]^{(1)}\|_{4,S(u,|u|)} + \int_{-\underline{u}}^u \Omega^2 \left(\|(\text{tr} \underline{\chi}, r\widehat{\rho})^{(1)}\|_{4,S} + \|r(\Gamma_w \cdot \Gamma_w)^{(2)}\|_{4,S} \right) \\ &\lesssim \varepsilon_0 + \int_{-\underline{u}}^u \frac{\varepsilon_0}{r^{1-\delta_0}|u|^{\delta_0}} + \frac{\varepsilon_0}{r^{1-\delta-2\delta_0}|u|^{\delta+2\delta_0}} + \frac{\varepsilon^2}{r^{\frac{3}{2}-\delta-2\delta_0}|u|^{\frac{1}{2}+\delta+2\delta_0}} \\ &\lesssim \varepsilon_0. \end{aligned}$$

Next, differentiating (6.21) by $\not\partial$ and applying Proposition 4.6, we have

$$\nabla_3[\not\partial^2[\widetilde{\underline{\mu}}]] + \text{tr} \underline{\chi}[\not\partial^2[\widetilde{\underline{\mu}}]] = r^{-2} \widetilde{\text{tr} \underline{\chi}}^{(2)} + r^{-1} \widetilde{\rho}^{(2)} + r^{-1} (\Gamma_w \cdot \Gamma_w)^{(3)}.$$

Applying Lemmas 6.1 and 6.2 and Propositions 5.9 and 6.10, we obtain¹⁰

$$\begin{aligned} \|r^2\not\partial^2[\widetilde{\underline{\mu}}]\|_{2,S} &\lesssim \|r^2\not\partial^2[\widetilde{\underline{\mu}}]\|_{2,S(u,|u|)} + \int_{-\underline{u}}^u \Omega^2 \left(\|(\text{tr} \underline{\chi}, r\widehat{\rho})^{(2)}\|_{2,S} + \|r(\Gamma_w \cdot \Gamma_w)^{(3)}\|_{2,S} \right) \\ &\lesssim \varepsilon_0 + \int_{-\underline{u}}^u \frac{\varepsilon_0}{r^{1-\delta-2\delta_0}|u|^{\delta+2\delta_0}} + \Omega^2 \|r\widehat{\rho}^{(2)}\|_{2,S} + \Omega^2 \|r\Gamma_w \cdot \Gamma_w^{(3)}\|_{2,S} \\ &\lesssim \varepsilon_0 + \left(\frac{\varepsilon_0}{\varepsilon} + \varepsilon \right) r^{1+2\delta_0} \int_{-\underline{u}}^u |u|^{-2\delta_0} \|\Omega \mathbb{F}_{1-2\delta}^{(2)}\|_{2,S} \\ &\lesssim \frac{\varepsilon_0}{|u|^\delta} r^\delta. \end{aligned}$$

Recalling that

$$[\widetilde{\underline{\mu}}] = \underline{\mu} - \bar{\underline{\mu}} + \frac{1}{4} \widetilde{\text{tr} \underline{\chi} \text{tr} \underline{\chi}} = \underline{\mu} + \bar{\rho} + \frac{1}{2r} \widetilde{\text{tr} \underline{\chi}} - \frac{1}{2r} \text{tr} \underline{\chi} + \Gamma_g \cdot \Gamma_g,$$

we have from Propositions 6.9 and 6.10 that

$$\|\underline{\mu}^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^2}, \quad \|\not\partial^2 \underline{\mu}\|_{2,S} \lesssim \frac{\varepsilon_0}{r^{2-\delta}|u|^\delta}.$$

This concludes the proof of Proposition 6.12. □

¹⁰Recall that we have $\widehat{\rho} \in \frac{\varepsilon_0}{\varepsilon} \mathbb{F}_{1-2\delta}$ and $r^{-1} \Gamma_w^{(3)} \in \mathbb{F}_{1-2\delta}^{(2)}$.

6.6 Estimates for η , $\underline{\eta}$, $\widehat{\chi}$ and $\widehat{\underline{\chi}}$

Proposition 6.13. *We have the following estimates:*

$$\|\eta^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-3\delta_0}|u|^{3\delta_0}}, \quad \|\eta^{(2)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta-2\delta_0}|u|^{2\delta_0}}, \quad \|\underline{\eta}^{(2)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r}. \quad (6.22)$$

We also have

$$\|\eta\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta_0}|u|^{\delta_0}}. \quad (6.23)$$

Proof. We recall from Proposition 4.7

$$d_1\eta = (-\mu - \widehat{\rho}, \widehat{\sigma}), \quad d_1\underline{\eta} = (-\underline{\mu} - \widehat{\rho}, \widehat{\sigma}).$$

Applying Propositions 4.11, we obtain for $q = 1, 2$

$$\|\eta^{(q)}\|_{4,S} \lesssim r\|(\mu, \widehat{\rho}, \widehat{\sigma})^{(q-1)}\|_{4,S}, \quad \|\underline{\eta}^{(q)}\|_{4,S} \lesssim r\|(\underline{\mu}, \widehat{\rho}, \widehat{\sigma})^{(q-1)}\|_{4,S}.$$

Combining with Propositions 6.3, 6.4, 6.11 and 6.12, we obtain (6.22) as stated.

Next, we have from Proposition 2.11

$$\nabla_4\eta + \frac{1}{2}\operatorname{tr}\chi\eta = r^{-1}\underline{\eta} - \beta + \Gamma_g \cdot \Gamma_b. \quad (6.24)$$

Applying Lemma 6.1, (6.22) and Proposition 6.4, we obtain

$$\begin{aligned} \|r\eta\|_{4,S} &\lesssim \|r\eta\|_{4,S(u,|u|)} + \int_{|u|}^u \|\underline{\eta}\|_{4,S} + r\|\beta\|_{4,S} + r\|\Gamma_g \cdot \Gamma_b\|_{4,S} \\ &\lesssim \varepsilon_0 + \int_{|u|}^u \frac{\varepsilon_0}{r} + \frac{\varepsilon^2}{r^{1-\delta_0}|u|^{\delta_0}} \\ &\lesssim \varepsilon_0 \frac{r^{\delta_0}}{|u|^{\delta_0}}, \end{aligned}$$

which implies (6.23). This concludes the proof of Proposition 6.13. \square

Proposition 6.14. *We have the following estimates:*

$$\begin{aligned} \|\widehat{\chi}^{(1)}\|_{4,S} &\lesssim \frac{\varepsilon_0}{r}, & \|\widehat{\chi}^{(2)}\|_{4,S} &\lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^{\delta}}, \\ \|\widehat{\underline{\chi}}^{(1)}\|_{4,S} &\lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^{\delta}}, & \|\widehat{\underline{\chi}}^{(2)}\|_{4,S} &\lesssim \frac{\varepsilon_0}{r^{\frac{1}{2}-\delta-\delta_0}|u|^{\frac{1}{2}+\delta+\delta_0}}. \end{aligned}$$

Proof. We recall the following Codazzi equations from Proposition 4.7

$$\begin{aligned} \operatorname{div}\widehat{\chi} &= \frac{1}{2}\nabla\operatorname{tr}\chi - r^{-1}\underline{\eta} - \beta + \Gamma_g \cdot \Gamma_g, \\ \operatorname{div}\widehat{\underline{\chi}} &= \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} - r^{-1}\underline{\eta} + \underline{\beta} + \Gamma_g \cdot \Gamma_w. \end{aligned}$$

Applying Propositions 4.11, 6.4, 6.9, 6.10 and 6.22, we obtain

$$\begin{aligned} \|\widehat{\chi}^{(1)}\|_{4,S} &\lesssim r\|(\nabla\operatorname{tr}\chi, r^{-1}\underline{\eta}, \beta)\|_{4,S} + \|r\Gamma_g \cdot \Gamma_g\|_{4,S} \lesssim \frac{\varepsilon_0}{r}, \\ \|\widehat{\underline{\chi}}^{(1)}\|_{4,S} &\lesssim r\|(\nabla\operatorname{tr}\underline{\chi}, r^{-1}\underline{\eta}, \underline{\beta})\|_{4,S} + \|r\Gamma_g \cdot \Gamma_w\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^{\delta}}. \end{aligned}$$

We also have from Propositions 4.11, 6.3, 6.9, 6.10 and 6.22

$$\begin{aligned}\|\widehat{\chi}^{(2)}\|_{4,S} &\lesssim r\|(\nabla \operatorname{tr} \chi, r^{-1}\underline{\eta}, \beta)^{(1)}\|_{4,S} + r\|(\Gamma_g \cdot \Gamma_g)^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{1-\delta}|u|^\delta}, \\ \|\underline{\widehat{\chi}}^{(2)}\|_{4,S} &\lesssim r\|(\nabla \operatorname{tr} \underline{\chi}, r^{-1}\underline{\eta}, \underline{\beta})^{(1)}\|_{4,S} + r\|(\Gamma_g \cdot \Gamma_w)^{(1)}\|_{4,S} \lesssim \frac{\varepsilon_0}{r^{\frac{1}{2}-\delta-\delta_0}|u|^{\frac{1}{2}+\delta+\delta_0}}.\end{aligned}$$

This concludes the proof of Proposition 6.14. \square

6.7 End of the proof of Theorem 3.5

Proposition 6.15. *We have the following estimates:*

$$\sup_{\mathcal{K}} r \left| \operatorname{tr} \chi - \frac{1}{r} \right| \lesssim \varepsilon_0, \quad \sup_{\mathcal{K}} r \left| \operatorname{tr} \underline{\chi} + \frac{4}{r} \right| \lesssim \varepsilon_0. \quad (6.25)$$

Proof. Applying Lemma 2.2, we obtain

$$\Omega^2 \nabla_3 \left(\frac{1}{r} \right) = -r^{-2} \Omega^2 e_3(r) = -\frac{1}{2r} \overline{\Omega^2 \operatorname{tr} \underline{\chi}} = -\frac{\Omega^2 \operatorname{tr} \underline{\chi}}{2r} + \frac{\widetilde{\Omega^2 \operatorname{tr} \underline{\chi}}}{2r}. \quad (6.26)$$

We recall from Proposition 2.11

$$\Omega^2 \nabla_3 \operatorname{tr} \chi + \frac{1}{2} \Omega^2 \operatorname{tr} \underline{\chi} \operatorname{tr} \chi = -2\Omega^2 \mu + 2|\Omega\eta|^2.$$

Hence, we deduce

$$\Omega^2 \nabla_3 \left(\operatorname{tr} \chi - \frac{1}{r} \right) + \frac{1}{2} \Omega^2 \operatorname{tr} \underline{\chi} \left(\operatorname{tr} \chi - \frac{1}{r} \right) = \Omega^2 (\mu, r^{-1} \widetilde{\operatorname{tr} \underline{\chi}})^{(0)} + \Omega^2 \Gamma_b \cdot \Gamma_b. \quad (6.27)$$

Applying Lemma 6.1, we have from Propositions 6.7 and 6.9

$$\begin{aligned}\left\| r \left(\operatorname{tr} \chi - \frac{1}{r} \right) \right\|_{\infty, S} &\lesssim \left\| r \left(\operatorname{tr} \chi - \frac{1}{r} \right) \right\|_{\infty, S(-\underline{u}, \underline{u})} + \int_{-\underline{u}}^{\underline{u}} \frac{\varepsilon_0}{r^{1-\delta_0}|u|^{\delta_0}} + \|r\Omega^2 \Gamma_b \cdot \Gamma_b\|_{\infty, S} \\ &\lesssim \varepsilon_0 + \int_{|u|}^{\underline{u}} \frac{\varepsilon_0}{r^{1-\delta_0}|u|^{\delta_0}} + \frac{\varepsilon^2}{r^{1-2\delta-4\delta_0}|u|^{2\delta+4\delta_0}} \\ &\lesssim \varepsilon_0.\end{aligned}$$

Similarly, we have from Proposition 2.11 and (6.26)

$$\nabla_3 \left(\operatorname{tr} \underline{\chi} + \frac{4}{r} \right) + \frac{1}{2} \operatorname{tr} \underline{\chi} \left(\operatorname{tr} \underline{\chi} + \frac{4}{r} \right) = r^{-1} \widetilde{\operatorname{tr} \underline{\chi}}^{(0)} + \Gamma_w \cdot \Gamma_w.$$

Applying Lemma 6.1, we infer

$$\begin{aligned}r \left\| \operatorname{tr} \underline{\chi} + \frac{4}{r} \right\|_{\infty, S} &\lesssim r \left\| \operatorname{tr} \underline{\chi} + \frac{4}{r} \right\|_{\infty, S(-\underline{u}, \underline{u})} + \int_{-\underline{u}}^{\underline{u}} \Omega^2 (\|\widetilde{\operatorname{tr} \underline{\chi}}\|_{\infty, S} + \|r\Gamma_w \cdot \Gamma_w\|_{\infty, S}) \\ &\lesssim \varepsilon_0 + \int_{-\underline{u}}^{\underline{u}} \frac{\varepsilon_0}{r^{1-\delta_0}|u|^{\delta_0}} + \frac{\varepsilon^2}{r^{1-2\delta-\delta_0}|u|^{2\delta+\delta_0}} \\ &\lesssim \varepsilon_0.\end{aligned}$$

This concludes the proof of Proposition 6.15. \square

Proposition 6.16. *We have the following estimate:*

$$\sup_{\mathcal{K}} \left| 1 - \frac{u-u}{2r} \right| \lesssim \varepsilon_0.$$

Proof. We have from Lemma 2.2 and Proposition 6.15

$$e_4 \left(r - \frac{u-u}{2} \right) = \frac{r}{2} \overline{\text{tr } \chi} - \frac{1}{2} = \frac{r}{2} \overline{\left(\text{tr } \chi - \frac{1}{r} \right)} = O(\varepsilon_0).$$

Applying Lemma 6.1, we obtain

$$\left\| r - \frac{u-u}{2} \right\|_{\infty, S} \lesssim \left\| r - \frac{u-u}{2} \right\|_{\infty, S(u, |u|)} + \int_{|u|}^u \varepsilon_0 du \lesssim \varepsilon_0 r.$$

This concludes the proof of Proposition 6.16. \square

Proposition 6.17. *We have the following estimates:*

$$\left\| \mathbf{K} - \frac{1}{r^2} \right\|_{4, S} \lesssim \frac{\varepsilon_0}{r^2}, \quad \|r \nabla \mathbf{K}\|_{4, S} \lesssim \frac{\varepsilon_0}{r^{2-\delta} |u|^\delta}.$$

Remark 6.18. *Proposition 6.17 implies that the spheres $S = S(u, \underline{u})$ are only almost round at future null infinity \mathcal{I}_+ .*

Proof. We have from (2.21) and Propositions 6.3 and 6.15

$$\begin{aligned} \left\| \mathbf{K} - \frac{1}{r^2} \right\|_{4, S} &= \left\| \mathbf{K} + \frac{1}{4} \text{tr } \chi \text{tr } \underline{\chi} - \left(\frac{1}{4} \text{tr } \chi \text{tr } \underline{\chi} + \frac{1}{r^2} \right) \right\|_{4, S} \\ &\lesssim \|\widehat{\rho}\|_{4, S} + r^{-1} \left\| \text{tr } \chi - \frac{1}{r} \right\|_{4, S} + r^{-1} \left\| \text{tr } \underline{\chi} + \frac{4}{r} \right\|_{4, S} \\ &\lesssim \frac{\varepsilon_0}{r^2}. \end{aligned}$$

Similarly, we have

$$\|r \nabla \mathbf{K}\|_{4, S} \lesssim \|r \nabla \widehat{\rho}\|_{4, S} + \|\nabla \text{tr } \chi\|_{4, S} + \|\nabla \text{tr } \underline{\chi}\|_{4, S} \lesssim \frac{\varepsilon_0}{r^{2-\delta} |u|^\delta}.$$

This concludes the proof of Proposition 6.17. \square

Combining Propositions 6.3–6.17, this concludes the proof of Theorem 3.5.

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