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An Improved Bound for Equitable Proper Labellings

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Abstract. For every graph G with size m and no connected component isomorphic to K_2 , we prove that, for $L = (1, 1, 2, 2, \dots, \lfloor m/2 \rfloor + 2, \lfloor m/2 \rfloor + 2)$, we can assign labels of L to the edges of G in an injective way so that no two adjacent vertices of G are incident to the same sum of labels. This implies that every such graph with size m can be labelled in an equitable and proper way with labels from $\{1, \dots, \lfloor m/2 \rfloor + 2\}$, which improves on a result proved by Haslegrave, and Szabo Lyngsie and Zhong, implying this can be achieved with labels from $\{1, \dots, m\}$.

Keywords: proper labelling · equitable labelling · 1-2-3 Conjecture.

1 Introduction

Let G be a graph. For a set $S \subset \mathbb{R}$, an S -labelling $\ell : E(G) \rightarrow S$ of G is an assignment of labels from S to the edges. In case $S = \{1, \dots, k\}$ for some $k \geq 1$, we say ℓ is a k -labelling of G . For every vertex u of G , we denote by $\sigma_\ell(u)$ (or $\sigma(u)$ if there are no ambiguities) its sum by ℓ , being the sum of labels assigned to the edges incident to u , that is, $\sigma(u) = \sum_{v \in N(u)} \ell(uv)$. Now, ℓ is said *proper* if we have $\sigma(u) \neq \sigma(v)$ for every edge uv of G (that is, the resulting σ_ℓ is a proper vertex-colouring). Last, we say G is *nice* if G has no connected component isomorphic to K_2 , in which case we set $\chi_\Sigma(G)$ as the smallest $k \geq 1$ such that G admits proper k -labellings. It can be checked that, indeed, $\chi_\Sigma(G)$ is well defined if and only if G is nice.

Proper labellings and the parameter χ_Σ have been mostly investigated in the context of the so-called 1-2-3 Conjecture, raised by Karoński, Łuczak, and Thomason [5] in 2004:

Conjecture 1 (Karoński, Łuczak, Thomason [5]). If G is a nice graph, then $\chi_\Sigma(G) \leq 3$.

Several aspects behind the 1-2-3 Conjecture have been investigated in literature to date, including approaching results, algorithmic results, peculiar behaviours, and variants. Definitely the most appealing and significant result in this context is a full proof of the conjecture proposed recently by Keusch [6]. This apart, most of the most interesting related results can be found *e.g.* in the survey [9] by Seamone.

Despite Keusch’s proof of the 1-2-3 Conjecture, there are many more or less closely related open, interesting questions and problems in the field. One of these deals with a concept of equitability for proper labellings, first considered in [1] by Baudon, Piłśniak, Przybyło, Senhaji, Sopena, and Woźniak, and studied further in [2,3]. This concept revolves around the following notions. Let G be a graph, and ℓ be a labelling of G . For any $l \in \mathbb{R}$, we denote by $\text{nb}(l, \ell) \geq 0$ the number of edges of G to which label l is assigned by ℓ . We say that ℓ is *equitable* if, for any two labels l and l' assigned by ℓ , we have $|\text{nb}(l, \ell) - \text{nb}(l', \ell)| \leq 1$, or, in other words, if any two label values are assigned about the same number of times by ℓ . Assuming G is nice, we denote by $\overline{\chi}_\Sigma(G)$ the smallest $k \geq 1$ such that G admits equitable proper k -labellings.

In the very first work on the topic [1], the authors investigated the parameter $\overline{\chi}_\Sigma(G)$ for particular classes of nice graphs G , without raising a particular conjecture. It is later in [2] that the authors raised an “Equitable 1-2-3 Conjecture”, reading as follows:

Conjecture 2 (Bensmail, Fioravantes, Mc Inerney, Nisse [2]). If G is a nice graph different from K_4 , then $\overline{\chi}_\Sigma(G) \leq 3$.

Proper labellings are objects that are rather hard to comprehend in general, so, unsurprisingly, even harder to comprehend equitable proper labellings are. To date, it is known from [1,2] that

the Equitable 1-2-3 Conjecture holds for several easy, common classes of nice graphs. In [2], the authors observed that there are infinitely many graphs G with $2 = \chi_{\Sigma}(G) < \overline{\chi_{\Sigma}}(G) = 3$, and they notably proved that determining whether $\overline{\chi_{\Sigma}}(G) = 2$ holds for a given graph G with $\chi_{\Sigma}(G) = 2$ is NP-complete.

In this work, we are mainly interested in upper bounds on the parameter $\overline{\chi_{\Sigma}}$. To date, it is not even clear that there is an absolute constant $k \geq 1$ such that $\overline{\chi_{\Sigma}}(G) \leq k$ holds for every nice graph G . Actually, even establishing non-constant bounds is not that clear. Through greedy arguments, in [1] the authors observed that $\overline{\chi_{\Sigma}}(G) \leq 2^{|E(G)|}$ holds for every nice graph G , thereby establishing that it makes sense investigating upper bounds on $\overline{\chi_{\Sigma}}$, and also that the notion of nice graphs remains relevant in the context of equitable proper labellings. Later on, in [3], Bensmail, Senhaji, and Szabo Lyngsie, through the study of a combination of the 1-2-3 Conjecture and of the so-called Antimagic Labelling Conjecture³, asked whether all nice graphs G admit a proper S -labelling ℓ for $S = \{1, \dots, |E(G)|\}$ and the extra property that $\text{nb}(l, \ell) = 1$ for all $l \in S$ – the point being that, if this was true, then it would imply that $\overline{\chi_{\Sigma}}(G) \leq |E(G)|$ holds for every nice graph G . This was later proved independently by Haslegrave in [4], and by Szabo Lyngsie and Zhong in [7], through rather different approaches (probabilistic tools for the former proof, constructive ones for the latter one). This is where the investigations on Conjecture 2 stand to date, the best general upper bound on $\overline{\chi_{\Sigma}}(G)$ we know of to date being $|E(G)|$ for every nice graph G .

Our main goal in this work is to improve upon this upper bound, which we do by about a factor 2. Namely, our main result reads as follows:

Theorem 1. *If G is a nice graph, then $\overline{\chi_{\Sigma}}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor + 2$.*

Although our upper bound in Theorem 1 is still not constant, we believe the way we prove it remains of interest, as we mostly build upon the proof of Szabo Lyngsie and Zhong from [7], which we enhance with a new approach and different, dedicated arguments. Our proof of Theorem 1 can be found in Section 2. Afterwards, we finish off with a concluding discussion in Section 3, in which we explain why it might be difficult to improve upon Theorem 1 further, and come up with other, different questions and problems of independent interest for further work on the topic.

2 Proof of Theorem 1

Our proof of Theorem 1 relies mainly on a peculiar point in the definition of an equitable labelling ℓ , being that the equitability constraint (*i.e.*, that $|\text{nb}(l, \ell) - \text{nb}(l', \ell)| \leq 1$ holds) is only required to hold for pairs of label values l and l' that are actually assigned by ℓ . What we mean by that is that having $\text{nb}(l, \ell) = 0$ and $\text{nb}(l', \ell) \geq 2$ is not regarded as an objection to ℓ being equitable. Hence, if ℓ is an S -labelling for some set S of labels and $\text{nb}(l, \ell) \leq 2$ holds for all $l \in S$, then ℓ is considered equitable.

Before we get to the actual proof of Theorem 1, we need some preparation first. To make the proof more legible, and due to the ideas developed in the previous paragraph, we work with labellings in a slightly different way. Let G be a graph, and $L = (l_1, \dots, l_q)$ be a sequence of labels from \mathbb{R} (that is, a given label value may appear more than once in L). Throughout, we assume every sequence of labels is ordered increasingly, that is, $l_1 \leq \dots \leq l_q$ in the present case. For any $l \in \mathbb{R}$, we denote by $\text{nb}(l, L) \geq 0$ the number of times label l appears in L , and by $\text{mult}(L) \geq 1$ the largest value of $\text{nb}(l, L)$ over all $l \in \mathbb{R}$. Assuming $|L| \geq |E(G)|$, an L -strict-labelling ℓ of G is an assignment of labels from L to the edges of G that complies with the elements of L and their number of occurrences; that is, every label value l appearing in L must be assigned at most $\text{nb}(l, L)$ times. In some sense, L is a pool of labels in which one has to pick elements when assigning labels (so that once a label is assigned, it is no longer part of the pool). For L' being a subsequence of L , we denote by $L - L'$ the sequence obtained from L by removing all elements in L' .

As mentioned earlier, to prove Theorem 1 our goal is to show that every nice graph G admits a proper L -strict-labelling ℓ , where $L = (1, 1, 2, 2, \dots, k+1, k+1, k+2, k+2)$ and $k = \left\lfloor \frac{|E(G)|}{2} \right\rfloor$. Note

³ Raised by Hartsfield and Ringel in [8], asking whether, in general, for every graph G there is a $\{1, \dots, |E(G)|\}$ -labelling ℓ where $\text{nb}(l, \ell) = 1$ for every assigned label l , and no two vertices u and v verify $\sigma(u) = \sigma(v)$.

that $|L| \in \{|E(G)|+3, |E(G)|+4\}$ depending on whether $|E(G)|$ is odd or even. Before getting to the formal details of the proof, let us outline its rough ideas first. Essentially, we build ℓ by induction, starting initially with all edges of G being unlabelled. We then start by picking some vertex u with degree $d = d(u)$, and d labels l_{i_1}, \dots, l_{i_d} from L , before then assigning these labels l_{i_1}, \dots, l_{i_d} to the edges incident to u , and then proceeding by induction on $G - u$ and $L - (l_{i_1}, \dots, l_{i_d})$. Of course, these very general ideas suffer several issues. In particular, in $G - u$, the edges incident to u are no longer present, which means that, for any neighbour v of u in G , 1) the adjacency between u and v is present in G only, not in $G - u$, so the fact that u and v are distinguished by ℓ must rely solely on how we labelled the edges incident to u , and 2) when labelling the edges of $G - u$, we have to take into account that, for all neighbours of u in G , their eventual sums also involve labels assigned to the edges incident to u , which are not present in $G - u$.

- To deal with the latter problem above, we will actually deal with weighted graphs, where a *weighted graph* (H, c) is a graph H given together with some function $c : V(H) \rightarrow \mathbb{R}$ modelling a sum contribution for every vertex. Essentially, above, in our initial graph G all contributions will be equal to 0, modelling the fact that all edges are actually present in G , and initially unlabelled (which is similar, in terms of sums, to having all edges being assigned label 0 – any graph can actually be seen as a weighted graph with all contributions 0). Then, during the inductive step, whenever considering, in some remaining graph, a vertex u with d neighbours v_1, \dots, v_d and d labels l_{i_1}, \dots, l_{i_d} remaining from the initial L , assuming we assign label l_{i_j} to uv_j for every $j \in \{1, \dots, d\}$ before pursuing with $G - u$, we will modify the contribution (by c) of each v_j by l_{i_j} to take into account that, in the more global picture, edge uv_j is assigned label l_{i_j} , thereby contributing l_{i_j} to the full sum of v_j .
- To deal with the former problem, we need to guarantee that the edges incident to u are labelled so that, taking the sum contributions into account, no matter how the remaining edges of $G - u$ are labelled, we cannot get any conflict⁴ between u and its neighbours. To guarantee this, we will employ the notion of smallest vertex. In a weighted graph (H, c) , the *smallest possible sum* (w.r.t. a sequence L of labels) of a vertex u is $\text{small}(u) = c(u) + x$, where x is the sum of the $d(u)$ smallest elements of L . In some sense, the smallest possible sum of u is the smallest sum we can achieve for u through assigning labels of L to all the edges incident to u , taking into account the contribution $c(u)$. Now, a *smallest vertex* of (H, c) is a vertex with minimum smallest possible sum.

Note that a smallest vertex is not necessarily unique (consider *e.g.* the case of a regular graph with all contributions being 0). However, a convenient property is that if, when building an L -strict-labelling of a given weighted graph, we start by considering a smallest vertex u and assigning the smallest labels of L to the edges incident to u , and then label the other edges arbitrarily, then in most cases we are sure u cannot get involved in conflicts, as formalised in the upcoming lemma. There and further, we say a sequence $L = (l_1, l_2, \dots)$ of labels with $l_1 \leq l_2 \leq \dots$ is *shifted* if $l_1 < l_2$. Also, to be clear, in a weighted graph (H, c) with a labelling ℓ , for any vertex v when writing $\sigma_\ell(v)$ (or $\sigma(v)$) we mean the sum of labels assigned to the edges incident to v (*i.e.*, we do not take $c(v)$ into account).

Lemma 1. *Let (G, c) be a weighted graph with a smallest vertex u having d neighbours v_1, \dots, v_d , and $L = (l_1, l_2, \dots)$ be a sequence of labels with $|L| \geq |E(G)|$ and $l_1 \leq l_2 \leq \dots$, where $l_1, l_2, \dots \in \mathbb{R}^*$. If $\text{mult}(L) \leq 2$ and u is a smallest vertex of maximum degree, then, for every L -strict-labelling ℓ of (G, c) such that the edges incident to u are assigned the smallest d labels l_1, \dots, l_d of L , all conflicts involving u must also include some smallest vertex v_i where either:*

- $d(v_i) = 1$, and $\ell(uv_i) = l_1$ (if L is shifted) or $\ell(uv_i) \in \{l_1, l_2\}$ (otherwise); or
- $d(v_i) = d(u) = 2$, L is shifted, $\ell(uv_i) = l_1$, and the second edge incident to v_i is assigned the second smallest label value in L (l_2 if $l_2 < l_3$, or l_2 or l_3 if $l_2 = l_3$).

Proof. Note that, by definition, upon assigning labels l_1, \dots, l_d to the edges incident to u , since u is a smallest vertex it cannot be that u gets in conflict with an adjacent vertex that is not a smallest vertex itself. We now analyse two cases, depending on whether L is shifted or not. Assume first L is not shifted.

⁴ In a weighted graph (H, c) , two vertices u and v are *conflicting* by a labelling ℓ if $\sigma_\ell(u) + c(u) = \sigma_\ell(v) + c(v)$.

- Assume first $d(u) \geq 3$. Since L is not shifted, $l_1 = l_2 = l$, and $l_2 < l_3$ since $\text{mult}(L) \leq 2$. Consider any v_i . If $d(v_i) \geq 2$, then v_i is incident to at most one edge assigned a label with value l while two edges have been assigned this label value, meaning $\sigma(v_i) + c(v_i) > \text{small}(v_i) \geq \text{small}(u) = \sigma(u) + c(u)$ (in particular, this inequality still holds if v_i is not incident to an edge assigned a label with value l); thus, u and v_i cannot be in conflict. Now, if $d(v_i) = 1$, then, since $\text{small}(v_i) \geq \text{small}(u)$, so that we get $\sigma(v_i) + c(v_i) = \sigma(u) + c(u)$ it must be that v_i is a smallest vertex and uv_i was assigned label l_1 or l_2 .
- Assume now $d(u) = 2$. Since L is not shifted, then $l_1 = l_2 = l$ and label value l was assigned to uv_1 and uv_2 only. Focus on any $v_i \in \{v_1, v_2\}$. If $d(v_i) \geq 2$, then only one edge incident to v_i is assigned a label with value l , while L contains two occurrences of it; thus, $\sigma(v_i) + c(v_i) > \text{small}(v_i) \geq \text{small}(u) = \sigma(u) = \sigma(u) + c(u)$, and u and v_i cannot be in conflict. Now, if $d(v_i) = 1$, then, since $\sigma(u) + c(u) = \text{small}(u)$, so that $\sigma(v_i) + c(v_i) = \sigma(u) + c(u)$ it must be that v_i is a smallest vertex.
- Assume last $d(u) = 1$. Since u is a smallest vertex of maximum degree, then, since $\ell(uv_1) = l_1$, so that there is a conflict between u and v_1 it must be that v_1 is a smallest vertex, and, thus, $d(v_1) = 1$.

Assume second that L is shifted. Then $l_1 < l_2$, and, if $d \geq 3$, then $l_2 \leq l_3$ and $l_1 < l_3$ (as $\text{mult}(L) \leq 2$).

- First assume $d(u) \geq 3$. Focus on any v_i . If $d(v_i) \geq 2$, then, again, since L contains only one occurrence of label value l_1 , either v_i is not incident to the unique edge assigned label l_1 (if $\ell(uv_i) \neq l_1$) or v_i is not incident to any edge assigned label l_2 or l_3 (if $\ell(uv_i) = l_1$). Thus, in both cases $\sigma(v_i) + c(v_i) > \text{small}(v_i) \geq \text{small}(u) = \sigma(u) + c(u)$. Now, if $d(v_i) = 1$, then, since $\sigma(u) + c(u) = \text{small}(u)$, the only way to have a conflict between u and v_i is that we have $\text{small}(u) = \text{small}(v_i) = \sigma(v_i) + c(v_i)$, which requires v_i be a smallest vertex and uv_i be assigned label l_1 .
- Assume second that $d(u) = 2$. We here know that, w.l.o.g., $\ell(uv_1) = l_1$ and $\ell(uv_2) = l_2$.
 - Regarding v_1 , if $d(v_1) = 1$, then we have $\sigma(v_1) + c(v_1) = \text{small}(v_1)$, so we can only have a conflict with u if v_1 is a smallest vertex. If $d(v_1) = 2$, then, since $\text{small}(v_1) \geq \text{small}(u) = \sigma(u) + c(u)$, so that we get a conflict between u and v_1 it must be (apart from v_1 be a smallest vertex) that the second edge incident to v_1 is assigned a label with value l_2 , which, since $\ell(uv_2) = l_2$, is only possible (since $\text{mult}(L) \leq 2$) if $l_2 = l_3$ and the second edge incident to v_1 is assigned label l_3 . Now, if $d(v_1) \geq 3$, then, necessarily, either $l_2 \neq l_3$ and thus there is no edge incident to v_1 assigned a label with value l_2 (as only uv_2 verifies this), or $l_2 = l_3 = l$ and there is at most one edge incident to v_1 assigned a label with value l (possibly its second incident edge) while two edges of (G, c) are assigned a label with value l (one of which is uv_2). Thus we have $\sigma(v_1) + c(v_1) > \text{small}(v_1) \geq \text{small}(u) = \sigma(u) + c(u)$.
 - Regarding v_2 , we cannot have $\sigma(u) + c(u) = \sigma(v_2) + c(v_2)$ since L contains label l_1 , which unique label value was assigned to uv_1 only. So u and v_2 cannot be in conflict here, regardless of $d(v_2)$.
- Assume third $d(u) = 1$. Recall that, in order to have a conflict between u and v_1 , since u is a smallest vertex and $\ell(uv_1) = l_1$, it must be that v_1 is a smallest vertex; since u is a smallest vertex of maximum degree, then in that case we deduce $d(v_1) = 1$.

This concludes the proof. □

Last, we need a notion of niceness for weighted graphs. We say a weighted graph (G, c) is *nice* if it has no connected component uv isomorphic to K_2 with $c(u) = c(v)$. Note that, clearly, any weighted graph that is not nice admits no proper L -strict-labelling for any sequence L of labels.

We are now ready to prove our main result, from which Theorem 1 follows as a corollary.

Theorem 2. *Every nice weighted graph (G, c) admits a proper L -strict-labelling for every sequence L of labels with $|L| \geq |E(G)| + 3$ and $\text{mult}(L) \leq 2$.*

Proof. The proof is by induction on the order n of G . If $n = 1$, then there is nothing to prove. If $n = 2$, then either G has no edge and again there is nothing to prove, or G consists of a single edge

uv with $c(u) \neq c(v)$ (as otherwise (G, c) would not be nice), in which case, through assigning any label $l \in L$ to uv we get $\sigma(u) + c(u) = l + c(u) \neq l + c(v) = \sigma(v) + c(v)$, thus what is desired.

We now proceed with the general case. That is, we now suppose that the claim holds for all weighted graphs on up to $n - 1$ vertices, and we prove the claim for (G, c) . By the induction hypothesis, we can suppose (G, c) is connected. Indeed, if (G, c) is the disjoint union of two nice weighted graphs (G_1, c) and (G_2, c) (where, abusing the notation, c is here restricted to G_1 and G_2), then, by induction, we can get a proper L -strict-labelling ℓ_1 of (G_1, c) , thus assigning $|E(G_1)|$ labels from L ; what remains of L is then a sequence L' of labels with $|L'| \geq |E(G)| + 3 - |E(G_1)| = |E(G_2)| + 3$ with $\text{mult}(L') \leq 2$, so, by induction, we can get a proper L' -strict-labelling ℓ_2 of (G_2, c) , which, with ℓ_1 , forms what is desired for (G, c) .

From now on, we thus assume (G, c) is connected. To lead the rest of the proof, we essentially look at the vertex degrees, and show we can employ induction properly in case certain degree configurations are present. As an illustration, we first deal with a specific case that will later simplify the proof a lot.

Remark 1. The result holds if (G, c) has a vertex of degree 2 adjacent to a vertex of degree 1.

Proof of the remark. Assume v is a vertex of degree 2 of G , and let u and w be its two neighbours, where $d(u) = 1$. If G is the path uvw of length 3 (that is, $d(w) = 2$ and w , besides v , is also adjacent to a vertex x of degree 1), then $|E(G)| = 3$ and $|L| \geq |E(G)| + 3 = 6$, so, since $\text{mult}(L) \leq 2$, there are at least three pairwise distinct labels $l_{i_1}, l_{i_2}, l_{i_3}$ in L . We here assign one l_i of these labels to vw so that $c(v) + l_i \neq c(u)$ and $c(w) + l_i \neq c(x)$, which guarantees u and v , and similarly x and w , cannot eventually get in conflict regardless how we label uv and wx . Next, we assign another l_j of these labels to uv so that $L - (l_i, l_j)$ contains at least two distinct labels. It then suffices to assign any remaining label l_k to wx so that v and w are distinguished to obtain a proper L -strict-labelling of (G, c) .

Under the assumption that $d(u) = 1$, $d(v) = 2$, and G is not the path of length 3, note that $(G - v, c)$ must be nice. Also, we have $|E(G)| \geq 2$, and thus $|L| \geq |E(G)| + 3 = 5$, meaning that L must contain three pairwise distinct labels $l_{i_1}, l_{i_2}, l_{i_3}$. Let l_i be any of these three labels such that $c(u) \neq c(v) + l_i$. Set now $G' = G - v$, $L' = L - (l_i)$, and define $c' : V(G') \rightarrow \mathbb{R}$ as $c'(w) = c(w) + l_i$ and $c'(x) = c(x)$ for all $x \in V(G') \setminus \{w\}$. By the induction hypothesis, (G', c') admits a proper L' -strict-labelling ℓ' , since $\text{mult}(L') \leq 2$ and $|L'| = |L| - 1 \geq |E(G)| + 2 = |E(G')| + 4$. Now, in (G, c) , start from ℓ' , assign label l_i to vw , and last, since only uv remains to be labelled, at least four labels of L have not been assigned yet, at least two of which must have distinct values since $\text{mult}(L) \leq 2$, so we assign one l_j to uv so that, denoting ℓ the resulting labelling of (G, c) , we get $\sigma_\ell(v) + c(v) \neq \sigma_\ell(w) + c(w)$. Note that w does not get involved into a conflict since we get $\sigma_\ell(w) + c(w) = \sigma_{\ell'}(w) + c'(w)$, $\sigma_\ell(w) = \sigma_{\ell'}(w) + l_i$, and $c'(w) = c(w) + l_i$. Likewise, by the choice of l_i , we cannot have a conflict between u and v , and, by the choice of l_j , we cannot have one between v and w . Meanwhile, all vertices of G different from u , v , and w have their sums not altered by these modifications. Thus, ℓ is a proper L -strict-labelling of (G, c) . \diamond

Back to the proof of Theorem 1, we focus on u , a smallest vertex of (G, c) . Among all possible choices as u , we choose one with maximum degree $d(u)$. We consider first when $d(u)$ is small, *i.e.*, at most 2. Recall that, throughout what follows, the elements l_1, l_2, \dots of L are ordered increasingly, *i.e.*, $l_1 \leq l_2 \leq \dots$.

- Assume first that $d(u) = 1$, and let v denote the unique neighbour of u in G . If $d(v) = 1$, then actually $|V(G)| = 2$, a case we have covered already. Thus, assume $d(v) \geq 2$, and even $d(v) \geq 3$ due to Remark 1. Then note that $G' = G - u$ is necessarily nice, and thus so is (G', c') for any c' . Note also that since u is a smallest vertex of (G, c) with maximum degree, then $\text{small}(u) < \text{small}(v)$. Now set $L' = L - (l_1)$, and define $c' : V(G') \rightarrow \mathbb{R}$ as $c'(v) = c(v) + l_1$ and $c'(w) = c(w)$ for all $w \in V(G') \setminus \{v\}$. Since $|L'| = |L| - 1 \geq |E(G)| + 2 = |E(G')| + 3$, by the induction hypothesis (G', c') admits a proper L' -strict-labelling ℓ' . Now let ℓ be the L -strict-labelling of (G, c) obtained from ℓ' by assigning label l_1 to uv . As a result, note that $c'(v) + \sigma_{\ell'}(v) = c(v) + \sigma_\ell(v)$ since $\sigma_\ell(v) = \sigma_{\ell'}(v) + l_1$. Also, since we assigned label l_1 to uv , we get $c(u) + \sigma_\ell(u) = \text{small}(u) < \text{small}(v) = c(v) + \sigma_\ell(v)$, while assigning label l_1 to uv changed the sums of u and v only. Thus, ℓ is a proper L -strict-labelling of (G, c) .

- Assume now that $d(u) = 2$, and let v and w be the two neighbours of u in G . By Remark 1, we can assume $d(v), d(w) \geq 2$. Since neither v nor w can be a vertex of degree 2 adjacent to a vertex of degree 1 by Remark 1, if $G - u$ is not nice then it must be that vw is an edge, and actually that (G, c) is a weighted triangle. In this case, $|E(G)| = 3$ and $|L| \geq |E(G)| + 3 = 6$, so L contains three pairwise distinct labels $l_{i_1}, l_{i_2}, l_{i_3}$ since $\text{mult}(L) \leq 2$. Note that, w.l.o.g., we can also assume $c(u) \leq c(v) \leq c(w)$. In this case, assuming $l_{i_1} < l_{i_2} < l_{i_3}$, we assign label l_{i_1} to uv , label l_{i_2} to uw , and label l_{i_3} to vw . This guarantees $c(u) + l_{i_1} + l_{i_2} < c(v) + l_{i_1} + l_{i_3} < c(w) + l_{i_2} + l_{i_3}$. Thus, the sums of u , v , and w are distinct, and we get a proper L -strict-labelling of (G, c) .

So, now, we can assume $G - u$ is nice. We consider two cases, depending on whether L is shifted.

- If L is not shifted, then let $l = l_1 = l_2$ be the value of the two smallest labels in L (l_1 and l_2), which are thus equal. We here set $G' = G - u$, $L' = L - (l_1, l_2)$, and define $c' : V(G') \rightarrow \mathbb{R}$ as $c'(v) = c(v) + l$, $c'(w) = c(w) + l$, and $c'(x) = c(x)$ for all $x \in V(G') \setminus \{v, w\}$. Since G' is nice, so is (G', c') , so by the induction hypothesis there is a proper L' -strict-labelling ℓ' of (G', c') . We extend ℓ' to some labelling ℓ of (G, c) by setting $\ell(uv) = l_1$ and $\ell(uw) = l_2$. As a result, we get $\sigma_\ell(u) + c(u) = \text{small}(u)$. Meanwhile, note that both v and w are incident to only one edge assigned a label with value l , while this label value was assigned twice, and, recall, we have $d(v), d(w) \geq 2$. Thus, $\sigma_\ell(v) + c(v) > \text{small}(v)$ and $\sigma_\ell(w) + c(w) > \text{small}(w)$, while by our choice of u we have $\text{small}(v), \text{small}(w) \geq \text{small}(u)$. Thus u cannot be involved in conflicts (recall Lemma 1), while for all other vertices x (in $V(G') \setminus \{v, w\}$) we have $\sigma_\ell(x) + c(x) = \sigma_{\ell'}(x) + c'(x)$, implying ℓ is thus a desired proper L -strict-labelling of (G, c) .
- If L is shifted, then l_1 and l_2 are two smallest label values of L , where $l_1 < l_2$ (that is, there is only one occurrence of label value l_1 in L). Essentially we would here like to proceed just as in the previous case, considering $G' = G - u$ and $L' = L - (l_1, l_2)$, and defining $c' : V(G') \rightarrow \mathbb{R}$ as earlier, assuming we would then, say, assign label l_1 to uv and label l_2 to uw . We need to be careful however, as in some cases such arguments do not apply; precisely, in some cases we might end up, for the resulting ℓ in (G, c) , with $\sigma_\ell(v) + c(v) = \text{small}(v) = \sigma_\ell(u) + c(u)$. It can be checked, however, that these arguments apply right away when $d(v), d(w) \geq 3$. Likewise, they apply as is when $d(v) \geq 3$ and $d(w) = 2$, and, free to rename v and w , when $d(v) = 2$ and $d(w) \geq 3$. It can be checked also that we cannot have any conflict between u and v when $d(v) = 2$ and $\text{small}(v) > \text{small}(u)$, and, again free to rename v and w , when $d(w) = 2$ and $\text{small}(w) > \text{small}(u)$. This follows from arguments alike those used to prove Lemma 1: as long as we assign labels l_1 and l_2 to the edges incident to u , any conflict involving u must be with a vertex whose incident edges have been assigned the smallest label values (recall we might have $l_2 = l_3$ since $\text{mult}(L) \leq 2$, in which case l_3 is also one of the two smallest label values).

So, the very last case to consider is when $d(v) = d(w) = 2$, $\text{small}(v) = \text{small}(w) = \text{small}(u)$, and thus $c(v) = c(w) = c(u)$. If considering v instead of u does not lead to a more favourable situation, then it means that v as well is adjacent to two vertices of degree 2 being smallest vertices (one of which is u , and the other one could be w). Of course, the same goes for w . Through repeatedly considering adjacent vertices of degree 2 of (G, c) this way, either at some point we reach a smallest vertex of degree 2 whose neighbours allow for previous arguments to apply, or we determine that (G, c) is actually a weighted cycle with all vertices being of smallest possible sum $\text{small}(u)$. In this case, we can obtain a proper L -strict-labelling of (G, c) e.g. in the following way. Set $G = v_0 \dots v_{n-1} v_0$. For any two adjacent vertices v_i and v_{i+1} (where indexes are modulo n throughout), since $c(v_i) = c(v_{i+1})$, through labelling (G, c) by a labelling ℓ , to guarantee that $\sigma_\ell(v_i) + c(v_i) \neq \sigma_\ell(v_{i+1}) + c(v_{i+1})$ it suffices to guarantee that $\sigma_\ell(v_i) \neq \sigma_\ell(v_{i+1})$; and for that we need only to have $\ell(v_{i-1}v_i) \neq \ell(v_{i+1}v_{i+2})$. If G is a cycle of length at least 4, then we obtain a proper L -strict-labelling of (G, c) when considering the edges following the ordering $(v_0v_1, v_1v_2, \dots, v_{n-1}v_0)$ and assigning to them labels of L in increasing order. Indeed, this guarantees every two edges at distance 2 get assigned distinct labels, since $\text{mult}(L) \leq 2$. Now if G is a triangle $uvwu$, then we can reuse arguments we introduced at the very beginning of the current case (to deal with cases where $G - u$ is not nice).

So, we can now assume that $d(u) = d \geq 3$. Set $G' = G - u$; note that if some connected component of G' is just an edge vw , then, due to Remark 1, we deduce that uv and uw must be edges of G as well. So it might be that G' is not nice, but this must be caused by triangles attached at u in G .

Recall that l_1, \dots, l_d denote the smallest d labels of L , where $l_1 \leq \dots \leq l_d$. To build a proper L -strict-labelling ℓ of (G, c) , our goal now is, assuming (G', c') is nice for some c' , to manage to invoke the induction hypothesis just like we did before. More precisely, in general, this will be achieved as follows. Let uv_1, \dots, uv_d denote the d edges incident to u , and let ϕ be a permutation of $\{1, \dots, d\}$. An *attempt* (w.r.t. ϕ) will consist in considering $L' = L - (l_1, \dots, l_d)$ and (G', c') the weighted graph where, recall, $G' = G - u$, and we have $c' : V(G') \rightarrow \mathbb{R}$ where $c'(v_i) = c(v_i) + l_{\phi(i)}$ for every $i \in \{1, \dots, d\}$ and $c'(w) = c(w)$ for all $w \in V(G') \setminus \{v_1, \dots, v_d\}$. Assuming (G', c') is nice, we then deduce a proper L -strict-labelling ℓ of (G, c) from a proper L' -strict-labelling ℓ' of (G', c') (obtained by induction), through simply assigning label $l_{\phi(i)}$ to every edge uv_i . In many cases, there is actually a ϕ guaranteeing a successful attempt (*i.e.*, such that the eventual ℓ is proper), see below. However, there are two reasons why an attempt might fail.

- Nothing guarantees (G', c') is nice, implying the induction hypothesis cannot be invoked. However, as pointed out above, this can only occur if $G' = G - u$ contains an isolated edge $v_i v_j$ (that is, u is adjacent, in G , to two adjacent vertices of degree 2) such that $c'(v_i) = c'(v_j)$.
- To guarantee u cannot eventually get involved in conflicts by ℓ , then, by Lemma 1, since, through an attempt, we are assigning the smallest d labels of L to the edges incident to u , and $d(u) \geq 3$, we must make sure $v_{\phi(1)}$ is not a vertex of degree 1 being a smallest vertex of (G, c) . Likewise, if L is not shifted, then by definition l_2 is also the smallest label value in L (that is, $l_1 = l_2$), and so, here as well, we must make sure $v_{\phi(2)}$ is not a vertex of degree 1 being a smallest vertex of (G, c) .

In particular, if we rename v_1, \dots, v_d as w_1, \dots, w_a for some $a \geq 0$, $x_1, y_1, \dots, x_b, y_b$ for some $b \geq 0$, $x'_1, y'_1, \dots, x'_{b'}, y'_{b'}$ for some $b' \geq 0$, and z_1, \dots, z_c for some $c \geq 0$, where $a + 2b + 2b' + c = d$ and

- every w_i verifies $d(w_i) = 1$ and $\text{small}(w_i) = \text{small}(u)$;
- every pair $\{x_i, y_i\}$ forms an isolated edge $x_i y_i$ in G' and $c(x_i) < c(y_i)$;
- every pair $\{x'_i, y'_i\}$ forms an isolated edge $x'_i y'_i$ in G' and $c(x'_i) = c(y'_i)$;
- every z_i meets none of the previous conditions⁵,

then a permutation leading to a successful attempt can be obtained in most situations. In most cases, this will be done through considering another permutation ϕ' obtained from some original ϕ by *swapping* two values $\phi(i)$ and $\phi(j)$ by ϕ to get a different permutation ϕ' , which means that $\phi'(i) = \phi(j)$, $\phi'(j) = \phi(i)$, and $\phi'(k) = \phi(k)$ for all $k \in \{1, \dots, d\} \setminus \{i, j\}$, and repeating this operation as long as necessary. Throughout what follows, when dealing with any of the next configurations, we implicitly assume that none of the previous ones applies.

- $a = 0$. In this case, so that an attempt through any initial permutation ϕ is successful, we just need to guarantee, for some distinct $\alpha, \beta \in \{1, \dots, d\}$, that there is no $i \in \{1, \dots, b\}$ (if $b \geq 1$) such that $(x_i, y_i) = (v_\alpha, v_\beta)$ and $c'(x_i) = c'(y_i)$, and no $i \in \{1, \dots, b'\}$ (if $b' \geq 1$) such that $(x'_i, y'_i) = (v_\alpha, v_\beta)$ and $l_{\phi(\alpha)} = l_{\phi(\beta)}$. Note that the former configuration, since we assumed $c(v_\alpha) = c(x_i) < c(y_i) = c(v_\beta)$, is avoided through simply having $l_{\phi(\alpha)} \leq l_{\phi(\beta)}$; so, in case, through ϕ , we have $c'(v_\alpha) = c'(v_\beta)$, then we can get rid of this by simply considering the permutation ϕ' obtained from ϕ by swapping $\phi(\alpha)$ and $\phi(\beta)$.
 - Assume first $b' = 0$. If we are not done with ϕ , then there exists a pair of vertices $\{x_i, y_i\}$ such that, for some distinct $\alpha, \beta \in \{1, \dots, d\}$, we have $(x_i, y_i) = (v_\alpha, v_\beta)$ and $\phi(\alpha) > \phi(\beta)$. As explained earlier, this issue is no longer present through the permutation ϕ' obtained from ϕ by swapping $\phi(\alpha)$ and $\phi(\beta)$. After repeating this for all such conflicting pairs $\{x_i, y_i\}$, eventually we get a permutation such that the associated L -strict-labelling of (G, c) is proper.
 - Assume now $b = 0$. Again, if we are not done with ϕ , then $b' > 0$ (as otherwise the previous case would apply), and there is a pair of vertices $\{x'_i, y'_i\}$ such that for some distinct $\alpha, \beta \in \{1, \dots, d\}$, we have $(x'_i, y'_i) = (v_\alpha, v_\beta)$ and $\phi(\alpha) = \phi(\beta)$. Since $\text{mult}(L) \leq 2$ and $d(u) \geq 3$, there exists a vertex $v_\gamma \notin \{v_\alpha, v_\beta\}$ such that $\phi(\alpha) \neq \phi(\gamma)$. If v_γ does not belong to another

⁵ Note that each z_i is either a vertex of degree at least 3, a vertex of degree 2 not part of a triangle attached at u (thus not part of any pair $\{x_i, y_i\}$ or $\{x'_i, y'_i\}$), or a non-smallest vertex of degree 1.

pair $\{x'_j, y'_j\}$ (where $i \neq j$), then we can freely swap $\phi(\alpha)$ and $\phi(\gamma)$ so that $\{x'_i, y'_i\}$ is no longer a conflicting pair (while new ones are not created) by the resulting permutation. Otherwise, if v_γ belongs to some pair $\{x'_j, y'_j\}$ with $(x'_j, y'_j) = (v_{\alpha'}, v_{\beta'})$ for some $1 \leq j \leq b'$ and distinct $\alpha', \beta' \in \{1, \dots, d\}$ with $\{\alpha, \beta\} \cap \{\alpha', \beta'\} = \emptyset$, then we consider the permutation ϕ' obtained by swapping $\phi(\alpha)$ and $\phi(\alpha')$. Note that this cannot yield a ϕ' where $\phi'(\alpha) = \phi'(\beta)$ or $\phi'(\alpha') = \phi'(\beta')$ since $\text{mult}(L) \leq 2$ and $\phi(\alpha) = \phi(\beta)$. By then repeating these arguments for every conflicting pair $\{x'_k, y'_k\}$, eventually we end up with a desired permutation.

- Assume last that $b, b' \geq 1$. We start, if needed, by applying the exact same arguments as in the previous case to obtain, from ϕ , some permutation ϕ' so that for every pair $\{x'_i, y'_i\}$ such that $(x'_i, y'_i) = (v_\alpha, v_\beta)$ for some distinct $\alpha, \beta \in \{1, \dots, d\}$, we have $\phi'(\alpha) \neq \phi'(\beta)$. We then apply, if needed, the exact same arguments (on conflicting pairs $\{x_i, y_i\}$, if any) as in the first case above to reach a permutation such that the associated L -strict-labelling of (G, c) is proper.

– $b \geq 1$.

In this case, assuming w.l.o.g. that $(v_1, v_2) = (x_1, y_1)$, we consider any permutation ϕ where $\phi(1) = 1$ and $\phi(2) = 2$. This way, note that there is no v_i of degree 1 such that uv_i is assigned label l_1 or l_2 , the smallest two label values of L . By remarks above, u cannot be involved in conflicts as long as we do not swap $\phi(1)$ nor $\phi(2)$ with a $\phi(\gamma)$ such that $\gamma \in \{1, \dots, d\}$ and v_γ is a w_i . First off, in case there is a conflicting pair $(x'_i, y'_i) = (v_\alpha, v_\beta)$, then we get rid of it by swapping $\phi(1)$ and $\phi(\alpha)$. Then, in case there are more conflicting pairs of the form $\{x'_j, y'_j\}$, then we get rid of them by swapping elements as in an earlier case. So we can assume we get to some point where, by ϕ , there are no conflicting pairs of the form $\{x'_i, y'_i\}$, and there is no v_i of degree 1 such that uv_i is assigned label l_1 or l_2 . From here, for every conflicting pair $(x_i, y_i) = (v_\alpha, v_\beta)$, just as in a previous case we can get rid of it by just swapping the labels assigned to uv_α and uv_β , to eventually reach a permutation such that the associated L -strict-labelling of (G, c) is proper. In particular, note that, through swapping labels, labels l_1 and l_2 remain assigned to edges uv_i with v_i being an x_i , a y_i , an x'_i , or a y'_i .

– $b' \geq 1$ and $c \geq 1$.

Here, assume w.l.o.g. that $z_1 = v_1$, $x'_1 = v_2$, and $y'_1 = v_3$.

- If L is shifted, then recall that $l_1 < l_2$, so L contains only one label, l_1 , with smallest value. We here consider any permutation ϕ where $\phi(2) = 1$. This guarantees u cannot be involved in conflicts with v_i 's of degree 1. Likewise, this guarantees we cannot get any conflict between v_2 and v_3 . Now, if some conflicts remain, then they must involve vertices in some pair $\{x'_i, y'_i\} \neq \{x'_1, y'_1\}$ (recall we can assume $b = 0$ since the previous case does not apply), in which case, since $\{x'_1, y'_1\}$ is a pair whose vertices are not in conflict, we can again repeatedly modify ϕ by swapping elements to reach another permutation ϕ' whose associated L -strict-labelling of (G, c) is proper.
- If L is not shifted, then recall that $l_1 = l_2$. Here, we consider any permutation ϕ where $\phi(1) = 1$ and $\phi(2) = 2$. Note that this guarantees that v_2 and v_3 cannot be in conflict. To attain a permutation ϕ' from ϕ such that the associated L -strict-labelling of (G, c) is proper, we can then, if necessary, swap elements as in the last case. In particular, since labels l_1 and l_2 get assigned to edges not incident to smallest vertices of degree 1, eventually vertex u cannot be involved in any conflict.

In both cases, we are thus done as well.

– $c \geq 2$.

Assuming w.l.o.g. that $z_1 = v_1$ and $z_2 = v_2$, we are here done through considering any permutation ϕ with $\phi(1) = 1$ and $\phi(2) = 2$. This guarantees the smallest two labels of L get assigned to edges going to z_i 's, which, recall, cannot be in conflict with u by associated L -strict-labellings of (G, c) assigning the smallest d labels to edges incident to u . Again, this also guarantees u cannot be involved in conflicts at all. Recall also that, since none of the previous cases applies, we may assume $b = b' = 0$, so here we do not have to consider possible conflicting pairs $\{x_i, y_i\}$ and $\{x'_i, y'_i\}$.

– $b' \geq 2$.

Since none of the previous cases applies, we have $b = c = 0$. In this case, we can essentially be done just as in the previous one, assuming $v_1 = x'_1$, $v_2 = y'_1$, $v_3 = x'_2$, and $v_4 = y'_2$, by considering any initial permutation ϕ with $\phi(1) = 1$, $\phi(2) = 3$, $\phi(3) = 2$, and $\phi(4) = 4$. This guarantees x'_1

and y'_1 cannot be in conflict, and similarly for x'_2 and y'_2 (in particular, since $\text{mult}(L) \leq 2$, we have $l_1 < l_3$ and $l_2 < l_4$). All conflicts, if any, must now involve pairs $\{x'_i, y'_i\}$ with $i \geq 3$, and we can again get rid of any such conflict by swapping the labels assigned to ux'_i and uy'_i .

Thus, since $d(u) \geq 3$, the last situations we have to consider are when

- $a \geq 3$, and $b = b' = c = 0$;
- $a \geq 1$, $b' = 1$, and $b = c = 0$;
- $a \geq 2$, $b = b' = 0$, and $c = 1$.

In the first and second cases, note that the whole structure of (G, c) is discovered (in that all edges are incident either to u or to neighbours of u only), so, assuming we label all edges of (G, c) at once (*i.e.*, induction is not invoked), it is no longer necessary, by a proper L -strict-labelling of (G, c) , to guarantee that u is of smallest possible sum, and, thus, that the smallest d labels of L are assigned to the edges incident to u .

To conclude, we consider each of the three possible remaining cases separately.

- $a \geq 3$, and $b = b' = c = 0$.

Start from the L -strict-labelling ℓ of (G, c) where $\ell(uw_i) = l_i$ for every $i \in \{1, \dots, d\}$ (recall $d = a$ here). Since w_1 is a smallest vertex and $\ell(uw_1) = l_1$, note that we have $\sigma_\ell(u) + c(u) = l_1 + \dots + l_d + c(u) = x = l_1 + c(w_1)$, and thus $c(w_1) = x - l_1$. Likewise, if, for every $i \in \{2, \dots, d\}$, we consider the labelling obtained from ℓ by swapping l_1 and l_i , then we deduce that $c(w_i) = x - l_1$. Since $|L| \geq |E(G)| + 3$, the three largest label values $l_{d+1}, l_{d+2}, l_{d+3}$ of L have not been assigned, where $l_{d+1} \leq l_{d+2} \leq l_{d+3}$ and thus $l_{d+1} < l_{d+3}$ since $\text{mult}(L) \leq 2$, and $l_2 < l_{d+2}$ since $d \geq 3$. Now consider the L -strict-labelling ℓ' of (G, c) obtained from the initial ℓ by assigning label l_{d+2} to uw_1 and label l_{d+3} to uw_2 . As a result, note that we now have $\sigma_{\ell'}(u) + c(u) = x + l_{d+2} + l_{d+3} - l_1 - l_2$. Meanwhile, $\sigma_{\ell'}(w_1) + c(w_1) = l_{d+2} + c(w_1) = x + l_{d+2} - l_1$; since $l_{d+3} - l_2 > 1$, we thus have that u and w_1 are not in conflict. Likewise, $\sigma_{\ell'}(w_2) + c(w_2) = l_{d+3} + c(w_2) = x + l_{d+3} - l_2$; since $l_{d+2} - l_1 > 1$, again u and w_2 cannot be in conflict. Now, for every $i \in \{3, \dots, d\}$, we have $\sigma_{\ell'}(w_i) + c(w_i) = \sigma_\ell(w_i) + c(w_i) \leq l_d + c(w_i) = x + l_d - l_1$; since $l_{d+2} - l_1 > l_d - l_1$, and $l_{d+3} - l_2 > 0$, we have that u and w_i cannot be in conflict. Thus, ℓ' is proper.

- $a \geq 1$, $b' = 1$, and $b = c = 0$.

Recall that, here, u is adjacent to a vertices w_1, \dots, w_a of degree 1 where $\text{small}(u) = \text{small}(w_i)$ for every $i \in \{1, \dots, a\}$, and two vertices x'_1 and y'_1 such that $x'_1 y'_1$ is an edge and $c(x'_1) = c(y'_1)$. Also, by the choice of u we have $\text{small}(u) \leq \text{small}(x'_1), \text{small}(y'_1)$.

- If L is shifted, then $l_1 < l_2$, so we here assign labels l_1, \dots, l_d to the edges incident to u so that label l_1 is assigned to ux'_1 and label l_2 is assigned to uy'_1 (and the other labels are assigned arbitrarily), thus so that ux'_1 and uy'_1 are assigned distinct labels and the smallest label value of L is not assigned to any edge uw_i ; by then assigning any remaining label to $x'_1 y'_1$ we obtain a proper L -strict-labelling of (G, c) . In particular, u can be in conflict with neither the w_i 's nor with x'_1 and y'_1 by Lemma 1 (recall $d(u) \geq 3$), and similarly for x'_1 and y'_1 since $c(x'_1) = c(y'_1)$ and we assigned distinct labels to ux'_1 and uy'_1 (while x'_1 and y'_1 are adjacent vertices of degree 2).
- Thus consider now when L is not shifted, *i.e.*, $l_1 = l_2$, and further assume that $a \geq 2$. We start similarly as in the previous case, assigning label l_1 to ux'_1 and label l_3 to uy'_1 , where $l_1 < l_3$ since $\text{mult}(L) \leq 2$. We then assign the remaining labels l_2, l_4, \dots, l_d to the uw_i 's arbitrarily, and call ℓ the resulting labelling. Then, as in a previous case, assuming l_2 was assigned to some uw_i , we have $\sigma_\ell(u) + c(u) = l_1 + \dots + l_d + c(u) = x = l_2 + c(w_i)$, and thus $c(w_i) = x - l_2$, where recall $l_2 = l_1$. Again, free to swapping label l_2 (assigned to uw_i) and the label assigned to any other edge uw_j (with $j \neq i$), we can further assume $c(w_i) = x - l_2$ for all $i \in \{1, \dots, a\}$. Now, similarly as earlier, if $a \geq 2$, then we can replace labels l_2 and l_4 with labels l_{d+3} and l_{d+4} (since $x'_1 y'_1$ is not incident to u , note that $|L| \geq d + 4$; then, $l_3 < l_{d+3}$) to make sure u is not in conflict with the w_i 's. It then remains to label $x'_1 y'_1$. Recall that, since we assigned distinct labels (l_1 and l_3) to ux'_1 and uy'_1 and $c(x'_1) = c(y'_1)$, eventually we cannot get a conflict between x'_1 and y'_1 whatever label we assign to $x'_1 y'_1$. So it all falls down to assigning a label to $x'_1 y'_1$ so that u is in conflict with neither x'_1 nor y'_1 . Note that there are at least four labels of L that have not assigned, namely l_2, l_4, l_{d+1} , and l_{d+2} . Since

$\text{mult}(L) \leq 2$, we have $l_2 < l_4 < l_{d+2}$. So there must be a label in $\{l_2, l_4, l_{d+2}\}$ we can assign to $x'_1 y'_1$ so that u is in conflict with neither x'_1 nor y'_1 . Eventually, this process thus results in a proper L -strict-labelling of (G, c) .

- A very last case to consider is when $a = 1$ and L is not shifted (as, in the last case we just considered, it was necessary that $a \geq 2$). In that case, we start by setting $\ell(uw_1) = l_1$, $\ell(ux'_1) = l_2$, and $\ell(uy'_1) = l_3$, where, recall, $l_1 = l_2$ and $l_2 < l_3$. Since for now we necessarily have a conflict between u and w_1 (since w_1 is a smallest vertex of degree 1), we deduce $c(w_1) + l_1 = c(u) + l_1 + l_2 + l_3$. We here change the label of uy'_1 to l_6 (since $|E(G)| = 4$, recall $|L| \geq 7$), which gets rid of the conflict between u and w_1 since $l_3 < l_6$. Also, $l_2 < l_6$ so we cannot get a conflict between x'_1 and y'_1 whatever label we assign to $x'_1 y'_1$. So it remains to label $x'_1 y'_1$ so that u gets in conflict with neither x'_1 nor y'_1 , and for that any label in $\{l_3, l_4, l_5, l_7\}$ is available. Since $\text{mult}(L) \leq 2$ we deduce that l_3, l_5, l_7 are pairwise distinct. So one of these labels can be assigned to $x'_1 y'_1$ to get a proper L -strict-labelling of (G, c) .
- $a \geq 2$, $b = b' = 0$, and $c = 1$.

Recall that u is here adjacent to w_1, \dots, w_a and z_1 , where the w_i 's are smallest vertices of degree 1, and z_1 is neither a smallest vertex of degree 1 nor part of a triangle attached at u . Thus, contrarily to the previous two cases, since $c = 1$ recall that the structure of G is not quite revealed at this point (unless z_1 is a non-smallest vertex of degree 1, in which case the upcoming arguments still apply).

- If L is shifted, then recall $l_1 < l_2$. We here consider (G', c') and L' , where $G' = G - \{u, w_1, \dots, w_a\}$ (which is nice; possibly G' is empty if z_1 has degree 1 in G) and $L' = L - (l_1, \dots, l_d)$, and c' is defined as $c'(z_1) = c(z_1) + l_1$ and $c'(v) = c(v)$ for all $v \in V(G') \setminus \{z_1\}$. By the induction hypothesis, (G', c') admits a proper L' -strict-labelling ℓ' (possibly a trivial one, if G' is empty). To extend it to a proper L -strict-labelling ℓ of (G, c) , we first set $\ell(uz_1) = l_1$, so that $\sigma_\ell(z_1) + c(z_1) = \sigma_{\ell'}(z_1) + c'(z_1)$, guaranteeing z_1 , vertex u apart, cannot be involved in conflicts. It then suffices to assign, by ℓ , labels l_2, \dots, l_d to the other edges incident to u arbitrarily. This way we get $\sigma_\ell(u) = c(u) = \text{small}(u)$, so, by Lemma 1, since $l_1 < l_2$, we know u cannot be involved in conflicts. Thus, ℓ is proper.
- Last, if L is not shifted, *i.e.*, $l_1 = l_2$, then we proceed as follows. We start off similarly as in the previous case until we get ℓ . This time, however, we know there is a conflict between u and the unique w_i such that $\ell(uw_i) = l_2$. As in previous cases, this implies that if we set $x = l_1 + \dots + l_d + c(u)$, then $c(w_i) = x - l_2$, and through swapping the label assigned to uw_i and the label assigned to any uw_j with $j \neq i$, we also deduce that, actually, $c(w_j) = x - l_2$ for all $j \in \{1, \dots, a\}$. Now, assuming w.l.o.g. that $\ell(uw_1) = l_2$ and $\ell(uw_2) = l_3$ (recall that w_2 exists since $a \geq 2$), we consider the L -strict-labelling ℓ' of (G, c) obtained from ℓ by assigning label l_{d+2} to uw_1 and label l_{d+3} to uw_2 . By previous arguments, since $c(w_i) = x - l_2$ for all $i \in \{1, \dots, a\}$, we know u cannot be in conflict with none of the w_i 's (in particular, $l_3 < l_{d+2}$ since $\text{mult}(L) \leq 2$). If u is not in conflict with z_1 , then we are done. Otherwise, since $\text{mult}(L) \leq 2$, observe that it must be that $l_{d+1} + l_{d+2} \neq l_{d+2} + l_{d+3}$ (as equality would imply $l_{d+1} = l_{d+3}$); so by changing the labels assigned to uw_1 and uw_2 to l_{d+1} and l_{d+2} , respectively, we can alter the sum of u to guarantee z_1 cannot be involved in conflicts, and, again, since we altered both $\ell(uw_1)$ and $\ell(uw_2)$ (since, because $d \geq 3$, $l_2 < l_{d+1}$ and $l_3 < l_{d+2}$), also u cannot be involved in conflicts with the w_i 's. So we again end up with a desired proper L -strict-labelling of (G, c) .

This concludes the whole proof. □

As mentioned earlier, Theorem 1 now follows directly from Theorem 2.

3 Conclusion

In this work, through Theorem 1 we essentially improved the best known general upper bound on $\overline{\chi}_\Sigma$ by about a factor 2. To achieve this, we mainly considered proper labellings assigning any label value at most twice, and proved that every nice graph can actually be labelled this way.

We note that improving Theorem 2 by a bit, that is, even only to sequences of labels of size at least $|E(G)| + 1$, is not quite clear, as, analysing the cases we considered in our proof, one can

come up with examples of weighted graphs (G, c) that cannot be labelled as desired. An obvious example is *e.g.* when (G, c) is a weighted star with center u and an odd number $d \geq 3$ of leaves v_1, \dots, v_d , L is a sequence of $|E(G)| + 1$ labels with $\text{mult}(L) \leq 2$ and all label values of L appear exactly twice in L , and all vertices are smallest vertices. Indeed, as seen in the proof of Theorem 2, whatever $|L| - 1$ labels we assign to uv_1, \dots, uv_d it is not possible to have u being not involved in one conflict. It might be, however, that if we exclude such bad weighted stars, then we can improve Theorem 2 further down.

Generalising the approach we considered, one could also legitimately wonder about the more general question of labelling graphs in a proper way but so that every assigned label value is assigned to at most k edges, for some fixed $k \geq 1$. For $k = 1$, this is exactly what was considered in [3], and the best result we could hope for was proved in [4,7], being that for any sequence of labels L with $\text{mult}(L) \leq 1$ containing at least $|E(G)|$ pairwise distinct labels there is a proper L -strict-labelling of every nice graph G . What we investigated through Theorem 2 is essentially the case $k = 2$ of these considerations; although our result, being that for any sequence of labels L with $\text{mult}(L) \leq 2$ containing at least $\lfloor \frac{|E(G)|}{2} \rfloor + 2$ pairwise distinct labels there is a proper L -strict-labelling of every nice graph G , can maybe be improved, as mentioned above some pathological cases might arise when restricting L even by a bit. More generally speaking, one could wonder how these considerations behave as k grows larger.

Note that considering larger values of k this way would not bring anything new regarding Conjecture 2 and Theorem 1, as, if we are allowed to assign any label value to at least three edges, then, through a labelling ℓ , we might end up with $\text{nb}(\ell, \ell) = 1$ and $\text{nb}(\ell', \ell) = 3$ for two label values ℓ and ℓ' , which would thus not be considered equitable. Thus, towards Conjecture 2, and towards results better than Theorem 1, one has to consider other approaches. From a more general perspective, we are still far from a constant upper bound on the parameter $\overline{\chi}_\Sigma$, and it is probable that new ideas are needed.

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