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A NEW DEFINITION OF THE FUZZY CARDINALITY OF FINITE FUZZY SETS PRESERVING THE CLASSICAL ADDITIVITY PROPERTY

Didier DUBOIS

CERT/DERA, B.P. 4025, 31055 TOULOUSE Cedex

Background

A previous note /2/ had investigated the definition of the fuzzy cardinality of a finite fuzzy set, as proposed by Zadeh /6/. It was noticed that the classical additivity property of cardinality is no longer valid under such a definition.

More precisely, let U be a finite universe, A be a fuzzy set on U; its cardinality $|A|$ is defined according to Zadeh as a fuzzy set of the set of natural integers $\mathbb{N}$, such that:

$$\forall n \in \mathbb{N},\ \mu_{|A|}(n) = \sup \{ \alpha \in]0,1] \mid |A_\alpha| = n \} \qquad (1)$$

where $A_\alpha = \{ u \in U \mid \mu_A(u) \geqslant \alpha \}$ is the α-cut of A.

Denoting $\oplus$ the addition extended to fuzzy integers by means of the sup-min extension principle, the following equality generally does not hold:

$$|A| \oplus |B| = |A \cup B| \oplus |A \cap B| \qquad (2)$$

where A and B are fuzzy sets on U.

If $\widehat{|A|}$ denotes the fuzzy set of $\mathbb{N}$ defined by means of the following equalities:

$$\forall n \in \mathbb{N},\ \mu_{\widehat{|A|}}(n) = \sup \{ \alpha \in]0,1] \mid |A_\alpha| \geqslant n \} = \sup_{x \geqslant n} \mu_{|A|}(x) \qquad (3)$$

then :

$$\widehat{|A|} \oplus \widehat{|B|} = \widehat{|A \cap B|} \oplus \widehat{|A \cup B|} = \widehat{|A| \oplus |B|} \qquad (4)$$

N. Blanchard /1/ suggested using (3) as a definition of the fuzzy cardinality, so as to avoid the lack of convexity of $|A|$ in $\mathbb{N}$.

However, if A is a crisp subset of U, with cardinality $n_A \in \mathbb{N}$, it is easy to check that

$$\forall n \in \mathbb{N},\ 0 \leqslant n \leqslant n_A \Rightarrow \mu_{\widehat{|A|}}(n) = 1$$

i.e. $\widehat{|A|}$ is not the usual cardinality but the interval (of integers) $[0, n_A] \subseteq \mathbb{N}$.

This remark prevents (3) from being considered any longer.

N.B.: $\widehat{|A|}$ is erroneously called "enveloppe convexe" in /2/.

A new representation of fuzzy sets

A suitable definition of fuzzy cardinality requires an extension of the familiar representation of fuzzy sets in terms of α-cuts. Such an extension has been proposed in a recent paper /3/ and was motivated by some counter-intuitive behavior of a definition of Yager /5/, regarding fuzzy probabilities of fuzzy events. This anomalie stemmed from the use of α-cuts in the definition. It could be suppressed through representing a fuzzy set A by a set of crisp sets each of which is rated by a scalar value. Namely, A is represented by:

$$\left\{ (S, c_A(S)) \mid S \subseteq U \ , \ c_A(S) \in]0,1] \right\}$$

where

$$\begin{aligned} c_A(S) &= \inf_{u \in S} \mu_A(u) && \text{if S includes } A_1 \text{ (the 1-cut or "peak" of A)} \\ &= 0 && \text{otherwise} \end{aligned} \qquad (5)$$

$c_A(S)$ is clearly a worst-case matching rate between S and A. It is simple to check that:

$$\forall u \ , \ \mu_A(u) = \sup_{S \atop u \in S} c_A(S) \qquad (6)$$

(6) generalizes the classical identity:

$$\mu_A(u) = \sup \left\{ \alpha \mid u \in A_\alpha \right\} \qquad (7)$$

since $c_A(A_\alpha) = \alpha$ and $c_A(S) < \alpha \ , \ \forall S \supset A_\alpha$.

N.B.: For all subsets S containing A_1 , c_A coincides with the "commonality" function of the possibility distribution μ_A (see Prade's paper in this issue).

A new definition of fuzzy cardinality

Let us now define the fuzzy cardinality of the fuzzy set A denoted $\|A\|$, by its membership function:

$$\forall n \in \mathbb{N} \ , \ \mu_{\|A\|}(n) = \sup_{\substack{S \supseteq A_1 \\ |S| = n}} \inf_{u \in S} \mu_A(u)$$

i.e. $\mu_{\|A\|}(n) = \sup \left\{ c_A(S) \mid |S| = n \right\}$

The following propositions or equalities are then valid:

-) If A is a crisp set with cardinality n_A , then $\|A\| = |A| = \{n_A\}$

-) $\forall\, n < |A_1|$, $\mu_{\|A\|}(n) = 0$

-) If $n = |A_\alpha|$ then $\mu_{\|A\|}(n) = \alpha = \mu_{|A|}(n)$

-) If $\exists\, u,v \in A_\alpha$ s.t. $\mu_A(u) = \mu_A(v) = \alpha$ and if $n = |A_\alpha|$

then $\mu_{\|A\|}(n-1) = \alpha$ (while $\mu_{|A|}(n-1) = 0$)

-) $\|A\|$ is convex on $\mathbb{N}$ and its membership function is non-increasing on the set of integers $\{ n \mid n \geq |A_1| \}$; i.e. $\left(\mu_{\|A\|}(n)\right)_n$ is a non-increasing sequence for $n \geq |A_1|$

-) $\|A\| = \widehat{|A|} \cap [0, |A_1| - 1]$

-) $\|A\|$ is the convex hull (Lowèn /4/) of $|A|$

-) $\|\cdot\|$ satisfies the additivity rule :

$$\|A\| \oplus \|B\| = \|A \cap B\| \oplus \|A \cup B\|$$

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