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0. First, basic results concerning the representation of belief or plausibility functions are recalled together with the expression of Dempster's rule of combination of evidence in terms of basic probability assignments and in terms of commonality functions . Then, the normalization of belief and plausibility functions is discussed . Lastly , in case of consonant belief functions, it is stressed that Dempster's rule of combination can be interpreted as a fuzzy set intersection based on product , taking care of the normalization question.

1. Let S be a finite set : $S = \{s_1, \ldots, s_n\}$. Let $\mathcal{P}(S)$ denote the set of subsets of S . Shafer [6] has shown that any belief (or credibility) function Bel is uniquely defined through the specification of a mapping m from $\mathcal{P}(S)$ to $[0, 1]$ called 'basic probability assignment' , satisfying the conditions

$$m(\emptyset) = 0 \qquad (1)$$

and

$$\sum_{A \subseteq X} m(A) = 1 \qquad (2)$$

then , we have

$$\forall A \subseteq S , \quad Bel(A) = \sum_{B \subseteq A} m(B) \qquad (3) .$$

Conversely , m is obtained from Bel by

$$\forall A \subseteq S, \quad m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} Bel(B) \qquad (4)$$

where $|A|$ denotes the cardinality of the set A .

The set function Pl defined by duality

$$\forall A \subseteq S , \quad Pl(A) = 1 - Bel(\overline{A}) \qquad (5)$$

is called a plausibility function ($\overline{A}$ denotes the complement of A) . Note that

$$\forall A \subseteq S , \quad Pl(A) \geqslant Bel(A) .$$

We have

$$\forall A \subseteq S , \quad Pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \qquad (6)$$

[N.B.: Originally , Dempster [1] introduced plausibility and belief functions (under the names 'upper and lower probabilities') by considering a multiva_

lued mapping Γ from a probability space $(X, \mathcal{P}(X), \mu)$ to a set S , and the following subsets of X induced by Γ : $A^* = \{x \in X \ , \ \Gamma(x) \cap A \neq \emptyset\}$ and $A_* = \{x \in X \ , \Gamma(x) \neq \emptyset \ , \ \Gamma(x) \subseteq A\}$ where $A \subseteq S$; in this framework , $Pl(A) = \mu(A^*)/\mu(S^*)$ and $Bel(A) = \mu(A_*)/\mu(S_*)$ where $S^* = S_* = \{x \in X \ , \Gamma(x) \neq \emptyset\}$. Thus , Pl and Bel are completely determined by the 2^n quantities $\mu(\{x \in X, \Gamma(x) = B\})$ where B is any subset of S ; note that the subsets $\{x \in X \ , \ \Gamma(x) = B\}$ determine a partition of X , thus $\sum_{B \subseteq S} \mu(\{x \in X, \Gamma(x) = B\}) = 1$; as pointed out by Goodman [3] and Nguyen [5] , these 2^n quantities determine a random set on S . We have

$$\forall B \subseteq S \ , \ B \neq \emptyset \ , \quad m(B) = \frac{\mu(\{x \in X \ , \ \Gamma(x) = B\})}{1 - \mu(\{x \in X \ , \ \Gamma(x) = \emptyset\})} \qquad (7)$$

The set $\mathcal{F} = \{B \subseteq S \ , \ m(B) > 0\}$ may be thought as the set of possible localizations of the truth , probabilistically weighted by m . If Bel is a probability measure , $\mathcal{F}$ contains only singletons : the possible localizations of the truth are then precise. The elements of $\mathcal{F}$ are called 'focal elements' .

A so-called commonality function Q can be defined from m by

$$\forall A \subseteq S \ , \quad Q(A) = \sum_{\substack{B \subseteq S \\ A \subseteq B}} m(B) \qquad (8)$$

Note that $Q(\{s\}) = Pl(\{s\})$; $Q(\emptyset) = 1$. Bel can be expressed directly in terms of Q:

$$\forall A \subseteq S \ , \quad Bel(A) = \sum_{B \subseteq \bar{A}} (-1)^{|B|} Q(B) \qquad (9)$$

Thus , Q provides another way of specifying a belief or a plausibility function .

Given two basic probability assignments m_1 and m_2 , Dempster's rule ([1] , [6]) enables to build a basic probability assignment m representing the combination of the two evidences . We have

$$m(\emptyset) = 0 \ ; \ \forall A \subseteq S \ , A \neq \emptyset \qquad m(A) = \frac{\sum_{\substack{i,j \\ A_i \cap B_j = A}} m_1(A_i) \cdot m_2(B_j)}{1 - \sum_{\substack{i,j \\ A_i \cap B_j = \emptyset}} m_1(A_i) \cdot m_2(B_j)} \qquad (10)$$

It is easy to check that $\sum_{A \subseteq S} m(A) = 1$. Thus , Dempster 's rule corresponds to the intersection of two statistically independent random sets of the same base space , after renormalization .

N.B.: In Dempster's original framework , the combination of $((X_1, \mathcal{P}(X_1), \mu_1) \ , \Gamma_1)$ and $((X_2, \mathcal{P}(X_2), \mu_2) \ , \Gamma_2)$ yields $((X, \mathcal{P}(X) \ , \mu), \Gamma)$ defined by $X = X_1 \times X_2$, $\mu = \mu_1 \times \mu_2$ and $\Gamma(x) = \Gamma_1(x) \cap \Gamma_2(x)$.

As pointed out by Dempster [1] and Shafer [6] , the combination is more easily expressed in terms of commonality functions since we have

$$\forall A \subseteq S \ , \quad Q(A) = \frac{Q_1(A) \cdot Q_2(A)}{K} \qquad (11)$$

where K denotes the denominator of fraction (10) .

2. Zadeh [8] has questioned the validity of the normalization in (10) . It is worth noticing that the problem of normalization vanishes as soon as we

admit that a basic probability assigned to the empty set ϕ may be non-zero and we define the "intersection" of m_1 and m_2 by

$$m^*(A) = \sum_{\substack{i,j \\ A_i \cap B_j = A}} m_1(A_i).m_2(B_j) \qquad (12)$$

It is easy to check we have $\sum_{A \subseteq S} m^*(A) = 1$, even if $m_1(\phi) \neq 0$ or $m_2(\phi) \neq 0$. From now , the normalization condition of a basic probability assignment will be only given by (2) ; (1) is dropped . Keeping definition (8) unchanged , (11) becomes

$$\forall A \subseteq S , \quad Q(A) = Q_1(A) \cdot Q_2(A) \qquad (13)$$

Noticing that (3) can be written

$$\forall A \subseteq S , \quad Bel(A) = \sum_{B \subseteq S} m(B).[\inf_{s \in B} \chi_A(s)] \qquad (14)$$

where χ_A is the characteristic function of the subset A . Thus , Bel(A) appears to be the expectation that the truth lies certainly in A , computed from the set of probabilistic weights $\{ m(B) , B \subseteq S \text{ and } B \neq \phi \}$ attached to the possible localizations of the truth since $\inf \chi(s) = 1$ if $B \subseteq A$ and is zero otherwise .

According to this interpretation in terms of expectation , when $m(\phi)$ may be non-zero, (3) must be modified into

$$\forall A \subseteq S , \quad Bel(A) = \sum_{\substack{B \subseteq A \\ B \neq \phi}} m(B) \qquad (15)$$

Similarly , (6) can be written

$$\forall A \subseteq S , \quad Pl(A) = \sum_{B \subseteq S} m(B).(\sup_{s \in B} \chi_A(s)) \qquad (16)$$

Thus , Pl(A) is the expectation that the truth lies possibly in A . (6) remains unchanged , but (5) becomes

$$\forall A \subseteq S , \quad Pl(A) + Bel(\bar{A}) = 1 - m(\phi) \qquad (17)$$

However , we keep $\forall A \subseteq S$, $Pl(A) \geqslant Bel(A)$. As soon as $m(\phi) \neq 0$, Bel and Pl are no longer normalized :

$$Bel(S) < 1 \quad ; \quad Pl(S) < 1 \qquad (18)$$

We still have $Bel(\phi)$ and $Pl(\phi)$ equal to zero .
If the empty subset of S has a non-zero probabilistic weight , it means that a possible localization of the truth is outside of S , then it may seem natural that the credibility and the plausibility that the truth lies in S are strictly less than 1 ; then , (17) means that 'A non plausible' does not entail that $\bar{A}$ is completely credible .

[N.B.: (7) is modified into $\forall B \subseteq S$, $m(B) = \mu(\{ x \in X , \Gamma(x) = B \})$; i.e. $m(\phi) = 0$ if and only if the set of points in X which are not mapped to S by Γ has a measure equal to zero .

3. A basic probability assignment m is said 'consonant' if its focal elements are nested , i.e. its focal elements can be arranged in order so that each is contained in the following one : $A_1 \subseteq A_2 \subseteq \dots \subseteq A_m$ and if $A \neq A_i$, $m(A)=0$. m will be supposed consonant in this section .

It can be shown [6] that a plausibility function Pl based on a consonant assignment m (in the sense of (6)) satisfies

$$\forall A \subseteq S , A \neq \emptyset , \quad Pl(A) = \max_{s \in A} \pi(s) \qquad (19)$$

with $\pi(s) = Pl(\{s\})$. Then , Pl is nothing but a possibility measure in the sense of Zadeh [7] .

Since focal elements are nested , we can build a nested sequence (rearranging the indices if necessary) :

$$A_1 = \{s_1\} , A_2 = \{s_1, s_2\} , \dots , A_i = \{s_1, s_2, \dots s_i\} , \dots , A_n = S$$

such that $\forall A \neq A_i$, $m(A) = 0$. However , it is possible that for some i , $m(A_i) = 0$. Applying (6) , we get

$$\begin{aligned} \pi(s_1) &= m(A_1) + m(A_2) + \dots + m(A_n) \\ \pi(s_2) &= \quad\quad m(A_2) + \dots + m(A_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \pi(s_n) &= \quad\quad\quad\quad\quad\quad m(A_n) \end{aligned}$$

or more compactly

$$\forall i = 1,n , \quad \pi(s_i) = \sum_{j=i}^{n} m(A_j) \qquad (20)$$

and conversely

$$\forall i = 1,n , \quad m(A_i) = \pi(s_i) - \pi(s_{i+1}) \quad \text{with} \quad \pi(s_{n+1}) = 0 \qquad (21) ,$$

and $$m(\emptyset) = 1 - \pi(s_1) \qquad (22) .$$

Thus , $m(\emptyset) = 0$ if and only if π is normalized (i.e. $\pi(s_1) = \max_{s \in S} \pi(s) = 1$) ; m_0 is the complement to 1 of the height of the fuzzy set whose membership function is the possibility distribution π .

From (17) , (19) and (22) , we get

$$\forall A \subseteq S , A \neq \emptyset , \quad Bel(A) = \min_{s \in \bar{A}} \left[\max_{s \in S} \pi(s) - \pi(s) \right] \qquad (23)$$

It can be shown [6] that the commonality function in case of consonance is given by

$$\forall A \subseteq S , A \neq \emptyset , \quad Q(A) = \min_{s \in A} \pi(s) \qquad (24)$$

Let m_1 and m_2 be two consonant basic probability assignments whose focal elements are respectively the increasing nested sequences $A_1 \subseteq \dots \subseteq A_m$ and $B_1 \subseteq \dots \subseteq B_p$ (if $m_1(\emptyset) \neq 0$, we add $A_0 = \emptyset$; if $m_2(\emptyset) \neq 0$, we add $B_0 = \emptyset$) . The combination of m_1 and m_2 by Dempster's rule does usually yield a basic probability assignment m which is <u>not</u> consonant since m is non-zero only on the subsets $A_i \cap B_j$, i=1,m , j=1,p which are not nested generally .

However , we have from (13)

$$\forall s \in S , \quad Q(\{s\}) = Q_1(\{s\}) \cdot Q_2(\{s\}) \qquad (25)$$

i.e. since $Q_i(\{s\}) = Pl_i(\{s\}) = \pi_i(s)$, $i=1,2$, (25) expresses that $s \longmapsto Q(\{s\})$ is the membership function of a fuzzy set which is the intersection (performed using product) of the fuzzy sets whose membership functions are respectively π_1 and π_2 .

Consonance is lost and moreover $\{Q(\{s\}), s \in S\}$ does not represent all the information . Indeed the number of unknowns is generally greater than the number of equations inthe following system :

$$\begin{cases} Q(\{s\}) = \sum_{\substack{A \subseteq S \\ A \neq \{s\}}} m(A) \\ 1 = \sum_{A \subseteq S} m(A) \end{cases} \qquad (26)$$

4. The two intended purposes of this short note were

1) to question the normalization condition $m(\emptyset) = 0$ of a basic probability assignment m and to study its implications for Dempster's rule of combination especially .

2) to show in what way Dempster's rule of combination is related to fuzzy set intersection based on product . In that perspective a natural question is raised : What combination rule would be related to fuzzy set intersection based on min operation (for instance) ? Normalization of fuzzy sets and normalization of basic probability assignments appear to be clearly related .

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