# Degree of truth: matching statements againt reality

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# DEGREE OF TRUTH : MATCHING STATEMENT AGAINST REALITY

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0. This short note offers some remarks which are extensions of the contents of papers [4] and [5].

1. In [4], the truth of a proposition is viewed as the conformity of what is stated with what is known about reality or what reality is supposed to be. Thus, the degree of truth of a proposition is obtained as the result of a comparison procedure which matches a representation of the contents of the proposition against a representation of the reality. In such an approach -- which sensibly departs from recently expressed points of view [6] and which is more in the spirit of [1] -- the degree of truth may depend on the representation and on the comparison procedure which are used. Moreover, if our perception of the reality or our belief of what reality is changes, the degree of truth of a given statement might be modified. Here, a PRUF-like representation [10] is supposed to be used. Therefore the comparison procedure has to deal with possibility distributions [9] (or if we prefer with fuzzy sets).

2. In [4], two independent scalar comparison indices are considered in order to estimate the agreement of a possibility $\pi_P$ with respect to another one $\pi_R$ ; these two possibility distributions are supposed to respectively represent the contents of a proposition P and our perception of the corresponding reality R modeled on the universe of discourse U attached to P. Namely

$$\mathrm{Pos}(P|R) = \sup_{u \in U} \min(\pi_P(u), \pi_R(u)) \qquad (1)$$

$$\mathrm{Nes}(P|R) = \inf_{u \in U} \max(\pi_P(u), 1 - \pi_R(u)) \qquad (2)$$

Note that $\mathrm{Nes}(P|R) = 1 - \mathrm{Pos}(\neg P|R)$ where $\neg P$ denotes the negation of P with $\pi_{\neg P} = 1 - \pi_P$. (1) is a degree of overlapping of $\pi_P$ and $\pi_R$, P and R playing symmetrical roles ; (2) is a degree of containment of $\pi_R$ in $\pi_P$ (to what extent does R entail P ?). $\mathrm{Pos}(P|R)$ is an estimate of the possibility [9] that the reality is in agreement with the contents of the proposition P taking into account the uncertainty on R. $\mathrm{Nes}(P|R)$ is the necessity (certainty)measure of the same event ; note that we are not estimating the coincidence of P and R -- the comparison procedure does not deal with P and R in a symmetrical manner -- but just the covering of R by P ; indeed P does not need to be as much precise as R to be true. If we accept some approximation in the covering of R by P, we can introduce a tolerance relation $\mu_T$ (modeling an approximate equality) in order to enlarge $\pi_P$ into $\pi_{P \circ T}(u) = \sup_{v \in U} \min(\pi_P(u), \mu_T(u,v))$ in formulae (1) and (2). The

reader is referred to[4] for a presentation of the properties of the two scalar comparison indices Pos(P|R) and Nes(P|R) and for a discussion of their possible aggregation in order to yield a degree of truth . These two comparison indices can also be used in a slightly different purpose when we want to retrieve the data which correspond to a given requirement in a data base ; then the requirement and each datum play the role of the proposition P and of a fragment of the reality R respectively ; see [2] , [3] .

3. 1 . Note that if P entails Q , i.e. $\pi_P \subseteq \pi_Q$ , we have $Pos(P|R) \leqslant Pos(Q|R)$ and $Nes(P|R) \leqslant Nes(Q|R)$ , which is in agreement with the fact that the truth of Q must not be more uncertain than the truth of P ; roughly speaking , Q must be at least as true as P .

2 . When U is a Cartesian product , for instance $U = U_1 \times U_2$ , if $\pi_P$ or $\pi_R$ are interactive fuzzy sets (see [8] ) , i.e. if they are strictly included in (rather than equal to) the Cartesian product of their projections $\pi_{P1}, \pi_{R1}$ and $\pi_{P2}, \pi_{R2}$ on $U_1$ and $U_2$ respectively , Pos(P|R) (resp. Nes(P|R) ) cannot be expressed only in terms of $Pos(P_1|R_1)$ and $Pos(P_2|R_2)$ ( resp. $Nes(P_1|R_1)$ and $Nes(P_2|R_2)$ ) . When the fuzzy sets are non-interactive (i.e. $\pi_P = \pi_{P1} \times \pi_{P2}$ , $\pi_R = \pi_{R1} \times \pi_{R2}$ where $\times$ is defined by means of 'min' operation ), we have :

$$Pos(P|R) = \min(Pos(P_1|R_1) , Pos(P_2|R_2)) \quad (3)$$
$$Nes(P|R) = \min(Nes(P_1|R_1) , Nes(P_2|R_2)) \quad (4)$$

These formulae correspond to the conjunction case where $P = P_1 \wedge P_2$ and $R = R_1 \wedge R_2$ , $P_1$ and $P_2$ , $R_1$ and $R_2$ being independent , the dual formulae corresponding to disjunction are :

$$Pos(P_1 \vee P_2 | R_1 \wedge R_2) = \max(Pos(P_1|R_1) , Pos(P_2|R_2)) \quad (5)$$
$$Nes(P_1 \vee P_2 | R_1 \wedge R_2) = \max(Nes(P_1|R_1) , Nes(P_2|R_2)) \quad (6)$$

4.1 As suggested in[4], a more sophisticated comparison index , whose value is a fuzzy set of [0, 1] and which may be used in our setting as a fuzzy degree of truth , is the compatibility (see [1] ) . The compatibility of $\pi_P$ with respect to $\pi_R$ is defined by its membership function :

$$\mu_{Cp(P|R)}(v) = \sup_{v=\pi_P(u)} \pi_R(u)$$
$$= 0 \text{ if } \pi_P^{-1}(v) = \emptyset \quad (7)$$

The fuzzy set Cp(P|R) is nothing but the fuzzy set of the possible values of the degree of membership to $\pi_P$ of an element whose apriori possible values are restricted by $\pi_R$. Cp(P|R) may be viewed as $\mu_P(R)$ , the fuzzy degree of membership in $\pi_P$ of elements more or less belonging to $\pi_R$.

As pointed out in [5] , Cp(P|R) encompasses the informations Pos(P|R) and Nes(P|R) ; see figure 1.

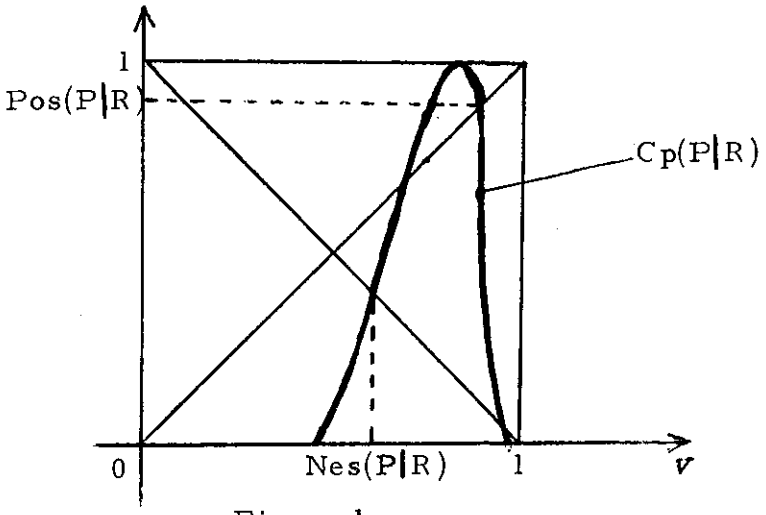


Figure 1

2. As pointed out in [5], $Cp(P|\neg R)$ ( where $\pi_{\neg R} = 1 - \pi_R$ ) is an information which cannot be deduced from $Cp(P|R)$ only, while $Cp(\neg P|R) = 1 \ominus Cp(P|R)$ where $\ominus$ denotes the extended subtraction ( see [11] ). The following table where P and R are supposed to be non-fuzzy and $\pi_R$ to be non-empty ( then the possible values of $Cp(P|R)$ are the crisp subsets of $[0, 1]$, $\{1\}$, $\{0, 1\}$ and $\{0\}$ which can be logically interpreted as 'true' (T), 'indeterminate' (I) and 'false' (F) respectively ) suggests in what way $Cp(P|\neg R)$ precises the information carried by $Cp(P|R)$:

| $Cp(P\|R)$ | | $Cp(P\|\neg R)$ | |
|---|---|---|---|
| T | $\pi_R \subseteq \pi_P$ | T | $\pi_P = U$ |
| | | I | $\pi_R \subsetneq \pi_P$ |
| | | F | $\pi_R = \pi_P$ |
| I | $\pi_R \cap \pi_P \neq \emptyset$ and $\pi_R \cap \pi_{\neg P} \neq \emptyset$ | T | $\pi_{\neg R} \subsetneq \pi_P$ |
| | | I | $\pi_{\neg R} \cap \pi_P \neq \emptyset$ & $\pi_{\neg R} \cap \pi_{\neg P} \neq \emptyset$ |
| | | F | $\pi_R \supsetneq \pi_P$ |
| F | $\pi_R \subseteq \pi_{\neg P}$ | T | $\pi_{\neg R} = \pi_P$ |
| | | I | $\pi_R \subsetneq \pi_{\neg P}$ |
| | | F | $\pi_P = \emptyset$ |

$\subsetneq$ denotes the strict inclusion . U is supposed to have at least 3 elements .

3. Generalizing (3) and (4), (5) and (6), we have (see [5]):

$$Cp(P_1 \wedge P_2 | R_1 \wedge R_2) = \widetilde{\min}(Cp(P_1|R_1), Cp(P_2|R_2)) \quad (8)$$
$$Cp(P_1 \vee P_2 | R_1 \wedge R_2) = \widetilde{\max}(Cp(P_1|R_1), Cp(P_2|R_2)) \quad (9)$$

where $\widetilde{\min}$ and $\widetilde{\max}$ denote the extended min and max operations (see [11])

4. Note that $Cp(\cdot|R)$, which is a mapping from the set of fuzzy sets of U to the set of fuzzy sets of $[0, 1]$, $\pi_R$ being a normalized possibility distribution, can be considered as a 'fuzzy-valued fuzzy measure' at least in a weak sense, since we

have the boundary conditions :
i) $Cp(\emptyset|R) = 0$ ; $Cp(U|R) = 1$
and $Cp(\cdot|R)$ is non-decreasing in the sense of $\widetilde{max}$ and $\widetilde{min}$ ( see [11] for the definitions and properties of these operations ) :
ii) if $\pi_P \subseteq \pi_Q$ (i.e. $\forall u$, $\pi_P(u) \leq \pi_Q(u)$ ), then

$$\widetilde{max}(Cp(P|R), Cp(Q|R)) \supseteq Cp(Q|R) \qquad (10)$$

(as well as $\widetilde{min}(Cp(P|R), Cp(Q|R)) \supseteq Cp(P|R)$ )

since it is clear that

$$\sup_{\{(x,y),\max(x,y)=z\}} \min\Big(\sup_{\{u,\,x=\pi_P(u)\}} \pi_R(u),\ \sup_{\{v,\,y=\pi_Q(v)\}} \pi_R(v)\Big) \geqslant \sup_{\{u,\,z=\pi_Q(u)\}} \pi_R(u)$$

However , (10) is not an equality in general .

5. Partial truth:

In face of a proposition which contains highlevel or composite predicates(i.e. predicates whose definition involves several more elementary predicates) it is customary to say that such a proposition is 'partially true' as soon as some of the elementary predicates are satisfied ( i.e. some of the restrictions of the proposition to these elementary predicates are true or at least almost true). Then the global degree of truth of the proposition may be modeled by the set of the partial degrees of truth corresponding to the elementary predicates rather than by a logical aggregation of these partial degrees by means of formulae such as (3)-(6) or (8) and (9). If we use compatibility measures to model the partial degrees of truth , the global degree of truth will then be represented by a set of fuzzy truth-values or more generally a fuzzy set of fuzzy truth-values in order to take into account that each elementary predicate has not the same importance in the definition of the composite predicate . Thus , we get a level 2 fuzzy set (see [7] )

$$\int_i \alpha_i / Cp(P_i|R_i) \qquad (11)$$

which may be reduced to an ordinary fuzzy set of the interval [0, 1]

$$\bigcup_i \alpha_i \cdot Cp(P_i|R_i) \qquad (12)$$

( rather than , for instance , $\widetilde{min}_i\ Cp(P_i|R_i)$ or $\widetilde{max}_i\ Cp(P_i|R_i)$ ) .
What we get is a possibility distribution of the degree of truth of the proposition , the value of this degree depending on the elementary predicate we consider and taking into account their relative importance $\alpha_i$ . The situation is quite similar to the multi-criteria evaluation of an item : either we aggregate the elementary evaluations or we keep the set of all elementary evaluations . The larger (in the sense of the usual fuzzy set inclusion ) the fuzzy truth-value given by (12) is , the more conflicting the partial degrees of truth are and the more ambiguous the proposition is from a truth point of view .

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