



HAL
open science

Null-controllability and Carleman estimates for non-autonomous degenerate PDEs: a climatological application

Mohammad Akil, Genni Fragnelli, Sarah Ismail

► **To cite this version:**

Mohammad Akil, Genni Fragnelli, Sarah Ismail. Null-controllability and Carleman estimates for non-autonomous degenerate PDEs: a climatological application. 2024. hal-04563560

HAL Id: hal-04563560

<https://hal.science/hal-04563560>

Preprint submitted on 29 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Null-controllability and Carleman estimates for non-autonomous degenerate PDEs: a climatological application.

Mohammad Akil ^{*1}, Genni Fragnelli ^{†2}, and Sarah Ismail ^{‡3}

¹Université Polytechnique Hauts-de-France, CÉRAMATHS/DEMAV, le Mont Houy 59313 Valenciennes Cedex 9-France.

²Department of Ecology and Biology, Tuscia University, Largo dell'Università 01100 Viterbo, Italy.

³Università degli Studi di Bari Aldo Moro-Italy, Dipartimento di Matematica, Via E. Orbona 4, 70125 Bari-Italy.

April 29, 2024

Abstract

Inspired by a Budyko-Seller model, we consider non-autonomous degenerate parabolic equations. As a first step, using Kato's Theorem we prove the well-posedness of such problems. Then, obtaining new Carleman estimates for the non-homogeneous non-autonomous adjoint problems, we deduce null-controllability for the original ones. Some linear and semilinear extensions are also considered. We conclude the paper applying the obtained controllability result to the Budyko-Seller model given in the introduction.

Keywords: Null-controllability, degenerate equations, non-autonomous equations, Carleman estimates, operators in divergence and non-divergence form.

2020 MSC: 93B05, 93B07, 35K65

1 Introduction

1.1 Climatological Parabolic Problem

Global warming - the gradual heating of Earth's surface, oceans and atmosphere - is one of the most concerning problems facing all living beings. Warmer temperatures mainly affect Earth's polar regions and mountain glaciers. Since 1979, the Arctic has warmed nearly four times faster than the rest of the planet [44]. This warming reduces critical ice habitat and disrupts the flow of the jet stream, creating more unpredictable weather patterns around the globe.

The balance between the amount of solar radiation reflected and absorbed by Earth's surface plays an important role in regulating global temperature. To simulate the behavior and interaction of basic climate system parameters on Earth, in 1969 Budyko and Seller in [7] and [47] introduce independently a classical energy balance model that can be rewritten as the following one-dimensional nonlinear degenerate parabolic equation

$$u_t(t, x) - \rho_0 (a(x)u_x)_x = R_a(t, x, u) - R_e(t, x, u). \quad (1.1)$$

Here $u(t, x)$ is the surface temperature averaged over longitude, $a(x) = 1 - x^2$ ($x = \sin \phi \in [-1, 1]$, with ϕ being the latitude) and ρ_0 is a positive parameter. The definitions of the other functions with their interpretations in climatology are summarized in the next table:

*mohammad.akil@uphf.fr

†genni.fragnelli@unitus.it

‡sarah.ismail@uniba.it

Function	Definition and Description
R_e	<p>The energy emitted by the Earth.</p> <ul style="list-style-type: none"> • In the classical Budyko model, $R_e(t, x, u) := a + bu$, where a and b are constants (see [11]). • In the Seller model, R_e follows a Stefan-Boltzmann type law, i.e. $R_e(t, x, u) := \epsilon(u) u ^3u$, where ϵ represents the emissivity and it is assumed to be a regular, positive and bounded function (see [10], [11] and [49] for other details).
R_a	The fraction of the solar energy absorbed by the Earth, $R_a(t, x, u) := Q(t, x)\beta(u)$.
Q	<p>The incoming solar radiation flux. $Q \in C(\mathbb{R}_+ \times [-1, 1])$ and the assumption $Q(t, x) \geq 0$ allows to consider the polar night phenomena.</p> <ul style="list-style-type: none"> • When the time scale is long enough (as for instance in annual models) one may assume that the insolation function $Q = Q(x)$ is a non-negative function that does not depend on time t. • When the time scale is smaller (as in seasonal models) one uses a more realistic description according to which $Q = Q(t, x)$ is a time-periodic function (see [3] or [35]).
β	<p>The planetary coalbedo given by $\beta(u) := 1 - \alpha(u)$.</p> <ul style="list-style-type: none"> • In the model of Budyko, β is a discontinuous function taking the value β_i for $u < u_s$ and the value β_f for $u > u_s$. Here $0 < \beta_i < \beta_f$ and $u_s = -10^\circ$ is the critical value for which ice becomes white (the snow-line). • In the model of Sellers, β is a more regular function of u, which is at least Lipschitz continuous (see [11]).
α	The albedo, which is the fraction of light reflected by a surface. It is usually taken such that $\alpha(u) \in (0, 1)$.

Table 1: Description of functions used in the climatological model.

Due to its importance, the Budyko-Seller model (1.1) is studied from different points of view. For example, in [3], the author studies the existence of periodic solutions for a modified version of (1.1), using the method of subsolutions and supersolutions. In [35], the authors consider the Budyko-Seller model with mushy region, proving the existence and uniqueness of the initial value problem. The periodic problem is also considered studying the attractivity of periodic solutions; the asymptotic behavior of the model as the width of mushy region tends to zero is also studied. We underline that in [35], ρ_0 is not a parameter but a function that depends on x . In [49], J. Tort and J. Vancostenoble study some inverse problem issue that consists in recovering the so-called insolation function; they solve the uniqueness question, providing some strong stability results. In [11], the authors consider equation (1.1) with a memory term. In particular, they provide existence and regularity results, and obtain uniqueness and stability estimates that are useful for the determination of the insolation function in (1.1) with memory. Very recently, in [10], the authors consider a two-layer energy balance model that permits vertical exchanges between a surface layer and the atmosphere. The evolution equations of the surface temperature and the atmospheric temperature are coupled by the emission of infrared radiation by one level (that emission being partly captured by the other layer) and the effect of all non-radiative vertical exchanges of energy. They study the dependence of the equilibrium points with respect to the involved parameters, and they prove, in particular, that the surface temperature increases monotonically with respect to a parameter that denotes the absorptivity of the atmosphere. Other questions, such as well-posedness, uniqueness, asymptotic behavior, bifurcation, free boundary, numerical approximation are studied, for example, in [4], [19],

[20], [21], [22], [23], [27], [28], [29], [36], [38], [39], [40] (see also the references therein).

1.2 Novelty and Main Results

Starting from (1.1) up to other applications such as the Prandtl equations (see [43]), the Feller semi-groups (see [24], [25]) and the Wright-Fischer model (see [5], [30], [31], [34], [46]), null-controllability for parabolic degenerate problems is getting more intriguing. For our model of interest (1.1), null-controllability means to drive the system to a desired equilibrium climate state in finite time. The pioneering papers in degenerate problems are [1], where the authors consider a general function $a(x)$, $x \in [0, 1]$, such that $a(0) = 0$ and $a(x) > 0$ for all $x \in (0, 1]$, and [12] where $a(x) = x^\alpha$, $\alpha \in (0, 2)$ and $x \in [0, 1]$. In both papers, the authors consider an operator in divergence form; we recall also [13], [14], [15], [17], [48], [50], [51], where other arguments are considered. For degenerate equations in non-divergence form we refer to [8] or to [9]; for interior degeneracy let's recall [6], [16], [32] or [33]. In all the previous papers, the equation is autonomous, i.e. the diffusion operator does not depend on the time. Actually, in [40] a very particular case of a non-autonomous climatological model is considered and, via a fixed point method, the author proves the existence of a periodic solution. However, the equation studied in [40] is non-degenerate. Thus, in this paper, we will consider for the first time a *non-autonomous degenerate* parabolic equation in divergence or in non-divergence form. Taking into account [1] and [8], we will study null-controllability for the problem under consideration via new Carleman estimates for the associated non-autonomous adjoint system. To this aim, we give the following definition for a general function $g : [A, B] \rightarrow \mathbb{R}_+$ (assuming $A < B$) that degenerates at a boundary point.

Definition 1.1. *Assume $A < B$. We say that a function $g : [A, B] \rightarrow \mathbb{R}_+$ is*

- *weakly degenerate at $x_0 \in \{A, B\}$, (WD) at x_0 , for shortness, if*

$$\begin{aligned} (i) \quad & g \in C([A, B]) \cap C^1([A, B] \setminus \{x_0\}), \quad g > 0 \text{ in } [A, B] \setminus \{x_0\}, \quad g(x_0) = 0, \\ (ii) \quad & \exists K_{x_0} \in [0, 1) \text{ such that } (x - x_0)g'(x) \leq K_{x_0}g(x), \quad \forall x \in [A, B] \end{aligned} \quad (\text{WD})$$

(for example $g(x) = x^K$, with $[A, B] = [0, 1]$, $x_0 = 0$ and $K_{x_0} \in (0, 1)$);

- *strongly degenerate at $x_0 \in \{A, B\}$, (SD) at x_0 , for shortness, if*

$$\begin{aligned} (i) \quad & g \in C^1([A, B]), \quad g > 0 \text{ in } [A, B] \setminus \{x_0\}, \quad g(x_0) = 0, \\ (ii) \quad & \exists K_{x_0} \in [1, 2) \text{ such that } (x - x_0)g'(x) \leq K_{x_0}g(x), \quad \forall x \in [A, B] \end{aligned} \quad (\text{SD})$$

(for example $g(x) = x^K$, with $[A, B] = [0, 1]$, $x_0 = 0$ and $K_{x_0} \in [1, 2)$).

In the following, just for simplicity, we assume $[A, B] = [0, 1]$. We will study the controllability for the following non-autonomous degenerate problems

$$\begin{cases} u_t - \mathcal{A}_i(t)u = f(t, x)\chi_\omega(x), & (t, x) \in Q_T, \\ u(t, y_0) = 0, & t \in (0, T), \\ B_i u(t, x_0) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \quad (\text{P}_i)$$

$i = 1, 2$, where $Q_T := (0, T) \times (0, 1)$, $\omega \subset \subset (0, 1)$, $y_0 \in \{0, 1\} \setminus \{x_0\}$ and $B_i u(t, x_0) = 0$ are suitable boundary conditions related to the operators \mathcal{A}_i , $i = 1, 2$. In particular

$$\mathcal{A}_i(t)u := \begin{cases} b(t)a(x)u_{xx}, & i = 1, \\ b(t)(a(x)u_x)_x, & i = 2, \end{cases}$$

where $b \in W^{1,\infty}(0, T)$ is a strictly positive function, a is (WD) or (SD) at $x_0 \in \{0, 1\}$ and

$$B_i u(t, x_0) = \begin{cases} u(t, x_0) = 0, & i = 1, \\ \begin{cases} u(t, x_0) = 0, & \text{if } a \text{ is (WD) at } x_0 \in \{0, 1\}, \\ \lim_{x \rightarrow x_0} (au_x)(t, x_0) = 0, & \text{if } a \text{ is (SD) at } x_0 \in \{0, 1\}, \end{cases} & i = 2. \end{cases}$$

Using new Carleman estimates for the non-homogeneous adjoint problem associated to (P_i) , our aim is to

- find a function $f \in L^2(0, T; H)$ such that

$$u(T, x) = 0, \quad \forall x \in [0, 1];$$

- find a constant $C > 0$ such that

$$\|f\|_{L^2(0, T; H)} \leq C \|u_0\|_H.$$

Here, H is a suitable Hilbert space that depends on the operator \mathcal{A}_i .

We underline that, while in [1] only the case $a(0) = 0$ is considered, in this paper a degenerates at 0 or at 1 or a degenerates at the same time at 0 and at 1, as in [8] or [9]. On the other hand, here we prove Carleman estimates and, hence, null-controllability under weaker assumptions on the function a with respect to the ones in [8]. Hence, the results proved in this paper improve the ones in [1] or in [8]. Moreover, as observed in [8], we cannot deduce null-controllability for the problem in non-divergence form from the one in the divergence form without adding additional assumptions on the function a . For this reason, it is important to prove new Carleman estimates for the problem in non-divergence form independently of the ones in divergence form. We also underline that, as for the problem in the autonomous case, the requirement $K_{x_0} < 2$ of Definition 1.1 is essential. Indeed, as proved in [1] or [8] or [33], if $K_{x_0} \geq 2$, the problem is not null-controllable.

The paper is organized as follows: in Section 2, we introduce the Hilbert spaces needed to treat the problems and we prove some Hardy-Poincaré inequalities that are important in the divergence case when a is (SD) at 1. In Section 3, we study the well-posedness of (P_i) via the Kato Theorem. In Section 4, we consider the non-homogeneous adjoint problems associated to (P_i) proving new Carleman estimates; thanks to them we obtain the observability inequalities and, hence, null-controllability for (P_i) in a standard way. In Section 5, we use the results proved in the previous section to obtain null-controllability when a degenerates at the same time at 0 and at 1, or if we consider a linear or a semilinear extension. In Section 6, we apply the obtained null-controllability result to (1.1) in a particular case. The last section is the Appendix where some proofs are given.

Throughout this paper, (\cdot) denotes the derivative of a function depending only on the real space variable x ; (\cdot) denotes the derivative of a function depending only on the real time variable t ; y_x^2 or y_{xx}^2 means $(y_x)^2$ or $(y_{xx})^2$, respectively. Moreover, we will denote by $\|\cdot\|_{\infty, 1} := \|\cdot\|_{L^\infty(0, 1)}$ and by $\|\cdot\|_{\infty, T} := \|\cdot\|_{L^\infty(0, T)}$.

2 Preliminaries

In this section, we present the Hilbert spaces needed to treat (P_i) , $i = 1, 2$. As a first step, we recall the next results relevant to the subsequent analysis: if a is (WD) or (SD) at $x_0 \in \{0, 1\}$, then

- if $x_0 = 0$, the function

$$x \rightarrow \frac{x^\gamma}{a} \tag{2.1}$$

is non-decreasing for all $\gamma \geq K_0$ and

$$\lim_{x \rightarrow 0} \frac{x^\gamma}{a} = 0, \tag{2.2}$$

for all $\gamma > K_0$;

- if $x_0 = 1$, the function

$$x \rightarrow \frac{(1-x)^\gamma}{a} \tag{2.3}$$

is non-increasing for all $\gamma \geq K_1$ and

$$\lim_{x \rightarrow 1} \frac{(1-x)^\gamma}{a} = 0, \tag{2.4}$$

for all $\gamma > K_1$.

Here K_0 and K_1 are the constants appearing in Definition 1.1.

2.1 Hilbert spaces in the non-divergence case

One of the main differences between the systems (P_i) with $i = 1$ (in non-divergence form) and (P_i) with $i = 2$ (in divergence form) is that the natural space to study the first problem is not $L^2(0, 1)$, as for the second one, but the weighted space

$$L^2_{\frac{1}{a}}(0, 1) := \left\{ u \in L^2(0, 1) \mid \|u\|_{L^2_{\frac{1}{a}}(0, 1)}^2 < \infty \right\},$$

where

$$\|u\|_{L^2_{\frac{1}{a}}(0, 1)}^2 := \int_0^1 \frac{u^2}{a} dx \quad \text{and} \quad \langle u, v \rangle_{\frac{1}{a}} := \int_0^1 \frac{uv}{a} dx, \quad \forall u, v \in L^2_{\frac{1}{a}}(0, 1). \quad (2.5)$$

As in [8] or [9], we also consider the following Hilbert spaces

$$H^1_{\frac{1}{a}}(0, 1) := L^2_{\frac{1}{a}}(0, 1) \cap H^1_0(0, 1), \quad \langle u, v \rangle_{1, \frac{1}{a}} := \langle u, v \rangle_{\frac{1}{a}} + \langle u', v' \rangle_{L^2(0, 1)}, \quad (2.6)$$

for all $\forall u, v \in H^1_{\frac{1}{a}}(0, 1)$, and

$$H^2_{\frac{1}{a}}(0, 1) := \left\{ u \in H^1_{\frac{1}{a}}(0, 1) \mid au'' \in L^2_{\frac{1}{a}}(0, 1) \right\}, \quad \langle u, v \rangle_{2, \frac{1}{a}} := \langle u, v \rangle_{1, \frac{1}{a}} + \langle \sqrt{a}u'', \sqrt{a}v'' \rangle_{L^2(0, 1)}, \quad (2.7)$$

for all $u, v \in H^2_{\frac{1}{a}}(0, 1)$. Clearly, the previous inner products induce the following norms

$$\|u\|_{1, \frac{1}{a}}^2 := \|u\|_{L^2_{\frac{1}{a}}(0, 1)}^2 + \int_0^1 (u')^2 dx \quad \text{and} \quad \|u\|_{2, \frac{1}{a}}^2 := \|u\|_{1, \frac{1}{a}}^2 + \int_0^1 a(u'')^2 dx,$$

respectively. Observe that in [8, Corollary 2.7] and in [8, Lemma 2.1] it is proved that the spaces $H^1_{\frac{1}{a}}(0, 1)$ and $H^1_0(0, 1)$ coincide and the following Gauss-Green formula holds

$$\int_0^1 u'' v dx = - \int_0^1 u' v' dx, \quad \forall (u, v) \in H^2_{\frac{1}{a}}(0, 1) \times H^1_{\frac{1}{a}}(0, 1). \quad (2.8)$$

Moreover, the operator $(A_1, D(A_1))$, where $A_1 u := au''$ with

$$D(A_1) := H^2_{\frac{1}{a}}(0, 1), \quad (\text{ND})$$

is m-dissipative and self-adjoint in $L^2_{\frac{1}{a}}(0, 1)$ (see [8, Theorem 2.3]). Thus, by [18, Corollary 3.20], the operator $(A_1, D(A_1))$ is densely defined and generates a contraction semigroup.

2.2 Hilbert spaces in the divergence case

Following [1], for the system in divergence form, we take the following weighted Hilbert spaces. In the **weakly degenerate** case, we consider

$$H^1_a(0, 1) := \{u \in L^2(0, 1) \mid u \text{ absolutely continuous in } [0, 1], \sqrt{a}u' \in L^2(0, 1) \text{ and } u(1) = u(0) = 0\}$$

and

$$H^2_a(0, 1) := \{u \in H^1_a(0, 1) \mid au' \in H^1(0, 1)\}.$$

On the other hand, in the **strongly degenerate** case, we consider

$$H^1_a(0, 1) := \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } [0, 1] \setminus \{x_0\}, \sqrt{a}u' \in L^2(0, 1) \\ \text{and } u(y_0) = 0, \text{ where } y_0 \in \{0, 1\} \setminus \{x_0\}\}$$

and

$$H^2_a(0, 1) := \{u \in H^1_a(0, 1) \mid au' \in H^1(0, 1)\} \\ = \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } [0, 1] \setminus \{x_0\}, au \in H^1_0(0, 1), \\ au' \in H^1(0, 1), u(y_0) = 0, \text{ where } y_0 \in \{0, 1\} \setminus \{x_0\}, \text{ and } (au')(x_0) = 0\}.$$

In both cases, we consider inner products and norms given by

$$\langle u, v \rangle_{1,a} := \langle u, v \rangle_{L^2(0,1)} + \langle \sqrt{a}u', \sqrt{a}v' \rangle_{L^2(0,1)}, \quad \|u\|_{1,a}^2 := \|u\|_{L^2(0,1)}^2 + \|\sqrt{a}u'\|_{L^2(0,1)}^2,$$

for all $u, v \in H_a^1(0, 1)$, and

$$\langle u, v \rangle_{2,a} := \langle u, v \rangle_{1,a} + \langle (au) ', (av)' \rangle_{L^2(0,1)}, \quad \|u\|_{2,a}^2 := \|u\|_{1,a}^2 + \|(au)' \|_{L^2(0,1)}^2,$$

for all $u, v \in H_a^2(0, 1)$. Also, in this case, the operator $(A_2, D(A_2))$, where $A_2u := (au)'$ with

$$D(A_2) := H_a^2(0, 1), \tag{D}$$

is m-dissipative and self-adjoint in $L^2(0, 1)$ (see [1, Proposition 2.5]); thus, $(A_2, D(A_2))$ is densely defined and generates a contraction semigroup.

Moreover, in the divergence case, the following Hardy-Poincar inequalities will play a crucial role if a degenerates at 1.

Proposition 2.1. (see [1, Proposition 2.1] for the case $a(0) = 0$) Take a function $a : [0, 1] \rightarrow \mathbb{R}$ so that $a \in C([0, 1])$, $a(1) = 0$ and $a > 0$ on $[0, 1)$.

Case 1: If a is such that there exists $\gamma \in (0, 1)$ such that the function

$$x \rightarrow \frac{a(x)}{(1-x)^\gamma} \text{ is non-decreasing in } [0, 1),$$

then, for any function w , locally absolutely continuous on $[0, 1)$, continuous at 1 and satisfying

$$w(1) = 0, \text{ and } \int_0^1 a(x)|w'(x)|^2 dx < +\infty,$$

the following inequality holds

$$\int_0^1 \frac{a(x)}{(1-x)^2} w^2(x) dx \leq \frac{4}{(1-\gamma)^2} \int_0^1 a(x)|w'(x)|^2 dx. \tag{2.9}$$

Case 2: If a is such that there exists $\gamma \in (1, 2)$ such that the function

$$x \rightarrow \frac{a(x)}{(1-x)^\gamma} \text{ is non-increasing in } [0, 1),$$

then, for any function w , locally absolutely continuous on $[0, 1)$ satisfying

$$w(0) = 0, \text{ and } \int_0^1 a(x)|w'(x)|^2 dx < +\infty,$$

the inequality (2.9) holds.

Proof. The computations are similar to the ones in [1], so we sketch them. As a first step, consider the first case and fix $\beta \in (\gamma, 1)$, arbitrarily for the moment. Since $w(1) = 0$, we have $w(x) = -\int_x^1 ((1-y)^{\beta/2} w'(y))(1-y)^{-\beta/2} dy$, thus

$$\int_0^1 \frac{a(x)}{(1-x)^2} w^2(x) dx \leq \int_0^1 \frac{a(x)}{(1-x)^2} \left(\int_x^1 (1-y)^\beta |w'(y)|^2 dy \int_x^1 (1-y)^{-\beta} dy \right) dx.$$

Hence,

$$\begin{aligned} \int_0^1 \frac{a(x)}{(1-x)^2} w^2(x) dx &\leq \frac{1}{1-\beta} \int_0^1 \frac{a(x)}{(1-x)^{1+\beta}} \left(\int_x^1 (1-y)^\beta |w'(y)|^2 dy \right) dx \\ &\leq \frac{1}{1-\beta} \int_0^1 (1-y)^\beta |w'(y)|^2 \left(\int_0^y \frac{a(x)}{(1-x)^{1+\beta}} dx \right) dy. \end{aligned} \tag{2.10}$$

Now, by assumption, the function

$$x \longrightarrow \frac{a(x)}{(1-x)^\gamma} \text{ is non-decreasing in } [0, 1),$$

thus

$$\int_0^y \frac{a(x)}{(1-x)^{1+\beta}} dx \leq \frac{a(y)}{(1-y)^\gamma} \int_0^y (1-x)^{\gamma-\beta-1} dx \leq \frac{1}{\beta-\gamma} a(y)(1-y)^{-\beta}.$$

Hence,

$$\int_0^1 \frac{a(x)}{(1-x)^2} w^2(x) dx \leq \frac{1}{(1-\beta)(\beta-\gamma)} \int_0^1 |w'(y)|^2 a(y) dy.$$

We then remark that this last estimate is optimal for $\beta = \frac{\gamma+1}{2}$, which gives the desired result.

We now consider the second case. Fix $\beta \in (1, \gamma)$, arbitrarily for the moment. Again, we have

$$\begin{aligned} \int_0^1 \frac{a(x)}{(1-x)^2} w^2(x) dx &= \int_0^1 \frac{a(x)}{(1-x)^2} \left(\int_0^x ((1-y)^{\beta/2} w'(y)) (1-y)^{-\beta/2} dy \right)^2 dx \\ &\leq \int_0^1 \frac{a(x)}{(1-x)^2} \left(\int_0^x (1-y)^\beta |w'(y)|^2 dy \int_0^x (1-y)^{-\beta} dy \right) dx \\ &\leq \frac{1}{\beta-1} \int_0^1 \frac{a(x)}{(1-x)^{1+\beta}} \left(\int_0^x (1-y)^\beta |w'(y)|^2 dy \right) dx \\ &= \frac{1}{\beta-1} \int_0^1 (1-y)^\beta |w'(y)|^2 \left(\int_y^1 \frac{a(x)}{(1-x)^{1+\beta}} dx \right) dy. \end{aligned}$$

Thanks to our hypothesis,

$$x \longrightarrow \frac{a(x)}{(1-x)^\gamma} \text{ is non-increasing in } [0, 1),$$

then

$$\int_y^1 \frac{a(x)}{(1-x)^{1+\beta}} dx \leq \frac{a(y)}{(1-y)^\gamma} \int_y^1 (1-x)^{\gamma-\beta-1} dx \leq \frac{1}{\gamma-\beta} a(y)(1-y)^{-\beta}.$$

Hence,

$$\int_0^1 \frac{a(x)}{(1-x)^2} w^2(x) dx \leq \frac{1}{(\beta-1)(\gamma-\beta)} \int_0^1 a(x) |w'(x)|^2 dx$$

and we obtain the thesis. \square

Observe that in [1], the previous inequalities are proved only in the case $a(0) = 0$, thus these inequalities are new.

3 Well-posedness for the non-autonomous Cauchy problem

In order to study the well-posedness of (P_i) , we define the operator

$$\mathcal{A}_i(t) := b(t)A_i u, \quad \forall u \in D(\mathcal{A}_i(t)), \quad (3.1)$$

$i = 1, 2$, so that we can rewrite (P_i) as a non-autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = \mathcal{A}_i(t)u(t) + f(t)\chi_\omega, & t \in (0, T), \\ u(0) = u_0 \in H, \end{cases} \quad (\text{nhCP}_i)$$

$i = 1, 2$, where

$$H := \begin{cases} L^2_{\frac{1}{2}}(0, 1), & i = 1, \\ L^2(0, 1), & i = 2. \end{cases}$$

Clearly, thanks to the assumptions on b ,

$$D(\mathcal{A}_i(t)) = D(\mathcal{A}_i), \quad \forall t \in [0, T]. \quad (3.2)$$

In the following, we will consider separately the homogeneous case ($f = 0$) and the non-homogeneous one ($f \neq 0$).

3.1 The homogeneous non-autonomous Cauchy problem

As a first step, we consider the case $f = 0$, i.e.

$$\begin{cases} \dot{u}(t) = \mathcal{A}_i(t)u(t), & t \in (0, T), \\ u(0) = u_0 \in H, \end{cases} \quad (\text{CP}_i)$$

$i = 1, 2$. The following theorem holds.

Theorem 3.1. *Assume a (WD) or (SD) at $x_0 \in \{0, 1\}$ and $b \in W^{1,\infty}(0, T)$ a strictly positive function. Then, (CP_i) has a unique solution $u \in C([0, T]; H)$, $\forall u_0 \in H$. Moreover, if $u_0 \in D(\mathcal{A}_i)$, then $u \in C([0, T]; D(\mathcal{A}_i)) \cap C^1([0, T]; H) \subseteq L^2(0, T; D(\mathcal{A}_i)) \cap C([0, T]; K_i) \cap H^1(0, T; H)$, $i = 1, 2$. Here,*

$$K_i = \begin{cases} H_{\frac{1}{a}}^1(0, 1), & i = 1, \\ H_a^1(0, 1), & i = 2. \end{cases}$$

The proof of the previous Theorem is based on the next Kato's Theorem (see [41, Theorem 3.1], [42, Theorem 1.9] or [45, Theorem 2.1]).

Theorem 3.2. *Let $T > 0$ be fixed, \mathcal{H} a Hilbert space and $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset \mathcal{H} \rightarrow \mathcal{H}$ a linear operator satisfying the following hypotheses:*

$$(H_1) \quad D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \forall t \in [0, T];$$

$$(H_2) \quad D(\mathcal{A}(0)) \text{ is a dense subset of } \mathcal{H};$$

$$(H_3) \quad \forall t \in [0, T], \mathcal{A}(t) \text{ generates a strongly continuous semigroup on } \mathcal{H}, \text{ and the family } \mathcal{A} = \{\mathcal{A}(t)\}_{t \in [0, T]} \text{ is stable with stability constants } C \text{ and } p (< 0) \text{ independent of } t$$

$$\text{(i.e. the semigroup } (S_t(s))_{s \geq 0} \text{ generated by } \mathcal{A}(t) \text{ satisfies } \left\| \prod_{j=1}^k S_{t_j}(s_j)u \right\|_{\mathcal{H}} \leq C e^{ps} \|u\|_{\mathcal{H}}, \text{ for all } u \in \mathcal{H}, s_j \geq 0, 0 \leq t_1 \leq \dots \leq t_k \leq T, k \in \mathbb{N});$$

$$(H_4) \quad \frac{\partial}{\partial t} \mathcal{A} \in L_*^\infty(0, T; B(D(\mathcal{A}(0)), \mathcal{H})) \text{ which is the space of equivalent classes of essentially bounded, strongly measurable functions from } [0, T] \text{ into the set of bounded operators } B(D(\mathcal{A}(0)), \mathcal{H}).$$

Then

$$\begin{cases} \dot{u}(t) = \mathcal{A}(t)u(t), & t \in (0, T), \\ u(0) = u_0 \in \mathcal{H}, \end{cases}$$

has a unique solution $u \in C([0, T], \mathcal{H})$ for all $u_0 \in \mathcal{H}$. Moreover, if $u_0 \in D(\mathcal{A}(0))$, then $u \in C([0, T]; D(\mathcal{A}(0))) \cap C^1([0, T]; \mathcal{H})$.

In order to prove Theorem 3.1, we need to introduce another inner product in H that takes into account the term $b(t)$. Thus, for all $t \geq 0$, we consider

$$\langle u, v \rangle_t := b(t) \langle u, v \rangle_H, \quad \forall u, v \in H,$$

inducing the norm

$$\|u\|_t^2 = b(t) \|u\|_H^2, \quad \forall u \in H.$$

Observe that the norms $\|\cdot\|_t$ and $\|\cdot\|_H$ are equivalent in H since

$$m\|u\|_H^2 \leq \|u\|_t^2 \leq M\|u\|_H^2, \quad (3.3)$$

for all $u \in H$, being

$$m := \min_{t \in [0, T]} b(t) \quad \text{and} \quad M := \max_{t \in [0, T]} b(t) \quad (3.4)$$

(recall that $b \in W^{1, \infty}(0, T)$, hence $b \in C[0, T]$). Using the new norm $\|\cdot\|_t$, we consider

$$\|u\|_{1,t}^2 = \begin{cases} \|u\|_t^2 + \|u_x\|_{L^2(0,1)}^2, & \text{if } u \in H_{\frac{1}{a}}^1(0, 1), \\ \|u\|_t^2 + \|\sqrt{a}u_x\|_{L^2(0,1)}^2, & \text{if } u \in H_a^1(0, 1). \end{cases} \quad (3.5)$$

which is equivalent to $\|\cdot\|_{1, \frac{1}{a}}$ or to $\|\cdot\|_{1,a}$ by (3.4). Thanks to the norms $\|\cdot\|_t$ and $\|\cdot\|_{1,t}$, we can prove Theorem 3.1.

3.1.1 Proof of Theorem 3.1

In order to prove the well-posedness of (CP_i) , $i = 1, 2$, we need to verify that the conditions $(H_1) - (H_4)$ of Theorem 3.2 hold.

The condition (H_1) follows by (3.2). Moreover, as written before, $D(\mathcal{A}_i)$, $i = 1, 2$, is dense in H ; thus, (H_2) clearly holds in both cases since $D(\mathcal{A}_i(0)) = D(\mathcal{A}_i)$, $i = 1, 2$. The first part of (H_3) follows by the next proposition.

Proposition 3.3. *Assume a (WD) or (SD) at $x_0 \in \{0, 1\}$ and $b \in W^{1, \infty}(0, T)$ a strictly positive function. For all fixed $t \geq 0$, the operator $(\mathcal{A}_i(t), D(\mathcal{A}_i(t)))$, $i = 1, 2$, is m -dissipative and generates a strongly continuous semigroup.*

Proof. Just to fix the idea, consider $\mathcal{A}_1(t) = b(t)au_{xx}$, the computations being similar if we consider $\mathcal{A}_2(t)u = b(t)(au_x)_x$.

- $\mathcal{A}_1(t)$ is dissipative in $L_{\frac{1}{a}}^2(0, 1)$ for all $t \in [0, T]$ with respect to $\langle \cdot, \cdot \rangle_t$. Indeed, by (2.8) we have

$$\langle \mathcal{A}_1(t)u, u \rangle_t = \int_0^1 b(t) \frac{\mathcal{A}_1(t)u \cdot u}{a} dx = b^2(t) \int_0^1 u_{xx}u dx = -b^2(t) \int_0^1 u_x^2 dx \leq 0,$$

for all $u \in D(\mathcal{A}_1(t))$.

- $\mathcal{A}_1(t)$ is symmetric in $L_{\frac{1}{a}}^2(0, 1)$ for all fixed $t \in [0, T]$ with respect to $\langle \cdot, \cdot \rangle_t$:

$$\langle \mathcal{A}_1(t)u, v \rangle_t = b^2(t) \int_0^1 u_{xx}v dx = -b^2(t) \int_0^1 u_x v_x dx = b^2(t) \int_0^1 \frac{auv_{xx}}{a} dx = \langle u, \mathcal{A}_1(t)v \rangle_t,$$

for all $u, v \in D(\mathcal{A}_1(t))$.

- $I - \mathcal{A}_1(t)$ is surjective with respect to $\langle \cdot, \cdot \rangle_t$: take $f \in L_{\frac{1}{a}}^2(0, 1)$ and define

$$F(v) := \int_0^1 \frac{fv}{ab} dx, \quad \text{for all } v \in H_{\frac{1}{a}}^1(0, 1).$$

F is linear and continuous, indeed

$$\left| \int_0^1 \frac{fv}{ab} dx \right| \leq \frac{1}{m} \int_0^1 \frac{|fv|}{\sqrt{a}\sqrt{a}} dx \leq \frac{1}{m} \|f\|_{L_{\frac{1}{a}}^2(0,1)} \|v\|_{L_{\frac{1}{a}}^2(0,1)}, \quad \forall v \in H_{\frac{1}{a}}^1(0, 1).$$

Hence, $F \in (H_{\frac{1}{a}}^1(0, 1))'$. Now, define

$$\begin{aligned} L : H_{\frac{1}{a}}^1(0, 1) \times H_{\frac{1}{a}}^1(0, 1) &\longrightarrow \mathbb{R} \\ (u, z) &\longmapsto \int_0^1 \frac{uz}{ab} dx + \int_0^1 u'z' dx. \end{aligned}$$

We have

$$L(u, u) = \int_0^1 \frac{u^2}{ab} dx + \int_0^1 (u')^2 dx \geq \frac{1}{M} \int_0^1 \frac{u^2}{a} dx + \int_0^1 (u')^2 dx \geq \min \left\{ \frac{1}{M}, 1 \right\} \|u\|_{1, \frac{1}{a}}^2,$$

for all $u \in H_{\frac{1}{a}}^1(0, 1)$. Then, L is coercive. Moreover, we have

$$\begin{aligned} |L(u, z)| &\leq \frac{1}{m} \int_0^1 \frac{|uz|}{a} dx + \int_0^1 |u'z'| dx \leq \frac{1}{m} \|u\|_{L_{\frac{1}{a}}^2(0,1)} \|v\|_{L_{\frac{1}{a}}^2(0,1)} + \|u'\|_{L^2(0,1)} \|v'\|_{L^2(0,1)} \\ &\leq \max \left\{ \frac{1}{m}, 1 \right\} \|u\|_{1, \frac{1}{a}} \|z\|_{1, \frac{1}{a}}, \quad \forall u, z \in H_{\frac{1}{a}}^1(0, 1). \end{aligned}$$

Hence, L is continuous and by the Lax-Milgram Theorem, we conclude that $\exists! u \in H_{\frac{1}{a}}^1(0, 1)$ such that

$$\begin{aligned} L(u, z) &= F(z), \quad \forall z \in H_{\frac{1}{a}}^1(0, 1), \\ &\Leftrightarrow \int_0^1 \frac{uz}{ab} dx + \int_0^1 u'z' dx = \int_0^1 \frac{fz}{ab} dx, \quad \forall z \in H_{\frac{1}{a}}^1(0, 1), \\ &\Leftrightarrow \int_0^1 u'z' dx = \int_0^1 \frac{(f-u)z}{ab} dx, \quad \forall z \in H_{\frac{1}{a}}^1(0, 1). \end{aligned}$$

In particular, this equality holds for all $z \in C_c^\infty(0, 1)$, hence $u'' = \frac{(f-u)}{ab}$ a.e in $[0, 1]$ and $\mathcal{A}_1(t)u = bau'' = f - u \in L_{\frac{1}{a}}^2(0, 1)$. Thus, $u \in D(\mathcal{A}_1(t)) = H_{\frac{1}{a}}^2(0, 1)$.

As a consequence $(\mathcal{A}_1(t), D(\mathcal{A}_1(t)))$ generates a strongly continuous semigroup on $L_{\frac{1}{a}}^2(0, 1)$. Analogously for $(\mathcal{A}_2(t), D(\mathcal{A}_2(t)))$. □

Thus, the first part of (H_3) follows by Proposition 3.3. For the second part of (H_3) , it remains to prove that, for any $t \geq 0$, $\mathcal{A}_i = \{\mathcal{A}_i(t)\}_{t \in [0, T]}$ is a stable family of C_0 -semigroup generators with stability constants C and p . Indeed, by [18, Corollary 3.20], $\mathcal{A}_i(t)$ is the infinitesimal generator of a contraction semigroup on H with respect to $\|\cdot\|_t$, which means that $\|e^{s\mathcal{A}_i(t)}u\|_t \leq \|u\|_t$, $i = 1, 2$, $\forall u \in H$ and for all $s \geq 0$. Now, we have that

$$\frac{d}{dt} \|u\|_t^2 = \frac{d}{dt} \left(b(t) \|u\|_H^2 \right) = \dot{b}(t) \|u\|_H^2 \leq \frac{\|\dot{b}\|_{\infty, T}}{m} \|u\|_t^2, \quad \forall u \in H.$$

By the Gronwall inequality, one has

$$\|u\|_t^2 \leq \|u\|_s^2 e^{\int_s^t C_b d\tau} = \|u\|_s^2 e^{(t-s)C_b}, \quad \forall t \geq s \geq 0 \text{ and } \forall u \in H,$$

where $C_b = \frac{\|\dot{b}\|_{\infty, T}}{m}$. Then, we get $\|u\|_t \leq \|u\|_s e^{\frac{C_b}{2}(t-s)}$, $\forall t \geq s \geq 0$ and $\forall u \in H$. Thanks to [42, Proposition 1.1], we obtain the desired result.

Now, it remains to prove (H_4) . Since $\mathcal{A}_i(t)u = b(t)\mathcal{A}_i u$, $i = 1, 2$, we have that $\frac{\partial \mathcal{A}_i(t)}{\partial t} u = \dot{b}(t)\mathcal{A}_i u$, $\forall u \in D(\mathcal{A}_i(t))$, $i = 1, 2$. Hence,

$$\left\| \frac{\partial \mathcal{A}_i(t)}{\partial t} u \right\|_H^2 = \|\dot{b}(t)\mathcal{A}_i u\|_H^2 \leq \|\dot{b}\|_{\infty, T}^2 \|\mathcal{A}_i u\|_H^2, \quad i = 1, 2;$$

thus $\left\| \frac{\partial \mathcal{A}_i(t)}{\partial t} u \right\|_H \leq \|\dot{b}\|_{\infty, T} \|\mathcal{A}_i u\|_H$, for all $u \in D(\mathcal{A}_i(t))$, $i = 1, 2$. This shows that $\frac{\partial \mathcal{A}(t)}{\partial t} u \in L_*^\infty(0, T; B(D(\mathcal{A}(0)), L_{\frac{1}{a}}^2(0, 1)))$.

Hence, by Kato's Theorem, we conclude that $\forall u_0 \in H$, the problem (CP_i) , $i = 1, 2$, has a unique solution $u \in C([0, T]; H)$. Moreover, if $u_0 \in D(\mathcal{A}_i(t)) = D(\mathcal{A}_i)$, then $u \in C([0, T]; D(\mathcal{A}_i)) \cap C^1([0, T]; H)$, $i = 1, 2$.

3.2 The non-homogeneous non-autonomous Cauchy problem

Now, we consider the non-homogeneous case, which means $f \neq 0$.

Theorem 3.4. *Assume a (WD) or (SD) at $x_0 \in \{0, 1\}$ and $b \in W^{1,\infty}(0, T)$ a strictly positive function. If $f \in L^2(0, T; H)$ and $u_0 \in H$, then there exists a unique solution $u \in C([0, T]; H)$ of (nhCP _{i}), and the following inequality holds*

$$\sup_{t \in [0, T]} \|u(t)\|_H^2 \leq C_T \left(\|u_0\|_H^2 + \|f\|_{L^2(0, T; H)}^2 \right), \quad (3.6)$$

for a positive constant C_T . In addition, if $f \in W^{1,1}(0, T; H) \cap L^2(0, T; D(A_i))$ and $u_0 \in D(A_i)$, then $u \in C([0, T]; D(A_i)) \cap C^1([0, T]; H) \subseteq L^2(0, T; D(A_i)) \cap C([0, T]; K_i) \cap H^1(0, T; H)$, $i = 1, 2$, and we have

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|u(t)\|_{H^{\frac{1}{2}}(0, 1)}^2 \right) + \int_0^T \left(\|u_t\|_{L^{\frac{1}{2}}(0, 1)}^2 + \|au_{xx}(t)\|_{L^{\frac{1}{2}}(0, 1)}^2 \right) dt \\ & \leq C \left(\|u_0\|_{H^{\frac{1}{2}}(0, 1)}^2 + \|f\|_{L^2(0, T; L^{\frac{1}{2}}(0, 1))}^2 \right), \quad \text{if } i = 1, \end{aligned} \quad (3.7)$$

or

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|u(t)\|_{H_a^1(0, 1)}^2 \right) + \int_0^T \left(\|u_t\|_{L^2(0, 1)}^2 + \|(au_x)_x(t)\|_{L^2(0, 1)}^2 \right) dt \\ & \leq C \left(\|u_0\|_{H_a^1(0, 1)}^2 + \|f\|_{L^2(0, T; L^2(0, 1))}^2 \right), \quad \text{if } i = 2. \end{aligned}$$

We postpone the proof to the Appendix. Clearly, the previous theorem and Theorem 3.1 hold also if we consider a general interval $[A, B]$, $A < B$ and $a(A) = 0$ or $a(B) = 0$.

4 Carleman estimates, observability inequalities and null-controllability

It is known that null-controllability for a linear parabolic system is equivalent to the observability for the associated homogeneous adjoint problem (see, for example, [26] or [33]). This equivalence holds also for the considered non-autonomous problems if (3.7) holds (one can proceed as in [33, Theorem 1.3]). Thus, the idea is to prove that the homogeneous adjoint problems of (P _{i}), $i = 1, 2$, are observable via Carleman estimates. To this aim, we need to introduce the following weight function

$$\theta(t) = \frac{1}{[t(T-t)]^4}. \quad (4.1)$$

Clearly $\theta(t) \rightarrow 0$ when $t \rightarrow 0^+$ or $t \rightarrow T^-$. Moreover, there exists a constant $\rho > 0$ such that

$$|\theta^2 \dot{\theta}| \leq \rho \theta^{\frac{3}{2}}, \quad |\theta \dot{\theta}| \leq \rho \theta^3, \quad \theta^\mu \leq \rho \theta^\nu \quad (\text{for } 0 < \mu < \nu) \quad \text{and} \quad |\ddot{\theta}| \leq \rho \theta^{3/2} \leq \rho^2 \theta^3. \quad (4.2)$$

Remark 4.1. *It would be interesting to point out that a weight function may be used in calculus or engineering applications [37] including: weighting for accuracy (by giving greater weight to more accurate measurements), compensating for bias (by giving greater weight to measurements known to be less biased), and accounting for significance (in engineering applications, a weighting function may simply reflect the relative influence of various applied forces).*

As in [1], we also define the function

$$\varphi(t, x) := \theta(t) \psi(x), \quad (4.3)$$

where

$$\psi(x) := C_1 \left(\int_0^x \frac{y - x_0}{a(y)} dy - C_2 \right), \quad (4.4)$$

$x_0 \in \{0, 1\}$ is the degeneracy point, C_1 and C_2 are positive constants such that $C_2 > \frac{1}{a(1)(2-K_0)}$, if $x_0 = 0$, and $C_2 > \frac{1}{a(0)(2-K_1)}$, if $x_0 = 1$; the constant C_1 will be chosen later.

4.1 The non-homogeneous problem in non-divergence form: the case $a(0) = 0$.

In this subsection, we will prove a Carleman estimate for the non-homogeneous adjoint problem associated to (P_1) , i.e.

$$\begin{cases} v_t + b(t)a(x)v_{xx} = h, & (t, x) \in Q_T, \\ v(t, 1) = v(t, 0) = 0, & t \in (0, T). \end{cases} \quad (4.5)$$

As a consequence, we obtain an observability inequality for (4.5) when $h \equiv 0$ and, hence, null-controllability for (P_1) . For the following, we need an additional assumption on the function a . In particular, we require the next hypothesis.

Hypothesis 4.2. Assume a (WD) or (SD) at 0 and such that

$$\left(\frac{xa'}{a}\right)' \in L^\infty(0, 1). \quad (4.6)$$

Observe that (4.6) is clearly satisfied by the prototype function $a(x) = x^K$, even if $K \in (0, 1)$.

Theorem 4.3. Assume Hypothesis 4.2, $b \in W^{1,\infty}(0, T)$ a strictly positive function and $h \in L^2_{\frac{1}{a}}(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution v of (4.5) in

$$\mathcal{U} := L^2(0, T; H^2_{\frac{1}{a}}(0, 1)) \cap H^1(0, T; H^1_{\frac{1}{a}}(0, 1))$$

satisfies

$$\int_{Q_T} \left(s\theta v_x^2 b^2 + s^3 \theta^3 \left(\frac{x}{a}\right)^2 v^2 b^2 \right) e^{2s\varphi} dx dt \leq C \int_{Q_T} h^2 \frac{e^{2s\varphi}}{a} dx dt + sC \int_0^T \left[\theta v_x^2 e^{2s\varphi} b^2 \right](t, 1) dt, \quad (4.7)$$

for all $s \geq s_0$. Here $L^2_{\frac{1}{a}}(Q_T) := L^2(0, T; L^2_{\frac{1}{a}}(0, 1))$.

Observe that if $b(t) \equiv 1$, we obtain exactly the same Carleman estimate given in [8, Theorem 3.3], but here we improve the result since the assumptions on a are more general. For this reason, we rewrite the proof, even if the computations are similar to the ones of [8]. Moreover we underline that the presence of the term $b(t)$ leads to new terms that we have to estimate.

To prove Theorem 4.3, we introduce the function

$$w(t, x) := e^{s\varphi(t, x)} v(t, x), \quad (4.8)$$

with $C_1 = 1$ (C_1 is the constant that appears in (4.4)); then, using the fact that $\varphi < 0$, (4.5) can be rewritten as

$$\begin{cases} (e^{-s\varphi} w)_t + b(t)a(x)(e^{-s\varphi} w)_{xx} = h, & (t, x) \in Q_T, \\ w(t, 1) = w(t, 0) = 0, & t \in (0, T), \\ w(T, x) = w(0, x) = 0, & x \in (0, 1). \end{cases} \quad (4.9)$$

Now, defining

$$Lw := w_t + b(t)a(x)w_{xx} \quad \text{and} \quad L_s w := e^{s\varphi} L(e^{-s\varphi} w), \quad (4.10)$$

the equation in (4.9) becomes

$$L_s w = e^{s\varphi} h.$$

The next results are crucial to prove Theorem 4.3.

Proposition 4.4. Assume the Hypotheses of Theorem 4.3. Then, the operator $L_s w$ can be written as

$$L_s w = L_s^+ w + L_s^- w,$$

where L_s^+ and L_s^- denote the (formal) self-adjoint and skew-adjoint parts of L_s . In our case, we have

$$\begin{cases} L_s^+ w = baw_{xx} - s\varphi_t w + s^2 \varphi_x^2 baw, \\ L_s^- w = w_t - 2s\varphi_x baw_x - sb\varphi_{xx} w. \end{cases} \quad (4.11)$$

Proof. Computing $L(e^{-s\varphi}w)$ one has

$$\begin{aligned}
L(e^{-s\varphi}w) &= (e^{-s\varphi}w)_t + ba(e^{-s\varphi}w)_{xx} \\
&= -s\varphi_t e^{-s\varphi}w + e^{-s\varphi}w_t + ba[-s\varphi_x e^{-s\varphi}w + e^{-s\varphi}w_x]_x \\
&= -s\varphi_t e^{-s\varphi}w + e^{-s\varphi}w_t + ba[-s\varphi_{xx}e^{-s\varphi}w + (-s\varphi_x)^2 e^{-s\varphi}w - 2s\varphi_x e^{-s\varphi}w_x + e^{-s\varphi}w_{xx}] \\
&= e^{-s\varphi}[-s\varphi_t w + w_t - bas\varphi_{xx}w + ba(-s\varphi_x)^2 w - ba2s\varphi_x w_x + baw_{xx}].
\end{aligned} \tag{4.12}$$

Thus,

$$L_s w = -s\varphi_t w + w_t - s\varphi_{xx} baw + s^2 \varphi_x^2 baw - 2s\varphi_x baw_x + baw_{xx}. \tag{4.13}$$

Now, integrating by parts over Q_T , using the boundary conditions and thanks to (2.5) and (2.8), we can compute the adjoint operator L_s^* as follows

$$\begin{aligned}
\langle L_s(w), v \rangle_{L^2_{\frac{1}{a}}(Q_T)} &= \int_{Q_T} \frac{L_s(w)v}{a} dxdt \\
&= -s \int_{Q_T} \frac{\varphi_t}{a} v w dxdt + \int_{Q_T} \frac{w_t v}{a} dxdt - s \int_{Q_T} b\varphi_{xx} v w dxdt \\
&\quad + s^2 \int_{Q_T} b\varphi_x^2 v w dxdt - 2s \int_{Q_T} b\varphi_x v w_x dxdt + \int_{Q_T} b v w_{xx} dxdt \\
&= - \int_{Q_T} \frac{v_t w}{a} dxdt + \langle w, -s\varphi_t v - s\varphi_{xx} baw + s^2 \varphi_x baw \rangle_{L^2_{\frac{1}{a}}(Q)} \\
&\quad + 2s \int_{Q_T} b(\varphi_x v)_x w dxdt + \int_{Q_T} b w v_{xx} dxdt \\
&= \langle w, L_s^* v \rangle_{L^2_{\frac{1}{a}}(Q_T)},
\end{aligned} \tag{4.14}$$

where $L_s^* v := -s\varphi_t v - v_t - s\varphi_{xx} baw + s^2 \varphi_x baw + 2sb(\varphi_x v)_x a + b v_{xx} a$. Thus, we can find

$$L_s^+ w = \frac{L_s w + L_s^* w}{2} = baw_{xx} - s\varphi_t w + s^2 \varphi_x^2 baw, \tag{4.15}$$

and

$$L_s^- w = \frac{L_s w - L_s^* w}{2} = w_t - 2s\varphi_x baw_x - sb a \varphi_{xx} w. \tag{4.16}$$

□

Proposition 4.5. *Assume the Hypotheses of Theorem 4.3. Then, the following identity holds*

$$\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} = (D.T.) + (B.T.) \tag{4.17}$$

where

$$\begin{aligned}
(D.T.) &= \frac{1}{2} \int_{Q_T} \dot{b} w_x^2 dxdt + s \int_{Q_T} b^2 (a\varphi_x)_x w_x^2 dxdt + s \int_{Q_T} b^2 (a\varphi_{xx})_x w w_x dxdt \\
&\quad + s \int_{Q_T} b^2 a \varphi_{xx} w_x^2 dxdt + \frac{s}{2} \int_{Q_T} \varphi_{tt} \frac{w^2}{a} dxdt - \frac{s^2}{2} \int_{Q_T} \dot{b} \varphi_x^2 w^2 dxdt \\
&\quad - 2s^2 \int_{Q_T} b \varphi_x \varphi_{tx} w^2 dxdt + s^3 \int_{Q_T} b^2 (a' \varphi_x^3 + 2a \varphi_x^2 \varphi_{xx}) w^2 dxdt
\end{aligned} \tag{4.18}$$

and

$$(B.T.) = -s \int_0^T \theta(t) b^2(t) w_x^2(t, 1) dt. \tag{4.19}$$

Proof. Clearly,

$$\begin{aligned}
\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} &= \langle baw_{xx} - s\varphi_t w + s^2 \varphi_x^2 abw, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} \\
&= \langle baw_{xx}, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} + \langle -s\varphi_t w + s^2 \varphi_x^2 abw, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)}.
\end{aligned} \tag{4.20}$$

As in [8], integrating by parts, and using the boundary conditions given in (4.9), one has

$$\begin{aligned}
\langle baw_{xx}, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} &= -\frac{1}{2} \int_{Q_T} b(w_x^2)_t dxdt - s \int_{Q_T} b^2 a \varphi_x (w_x^2)_x dxdt - s \int_0^T b^2 a \varphi_{xx} w w_x \Big|_0^1 dt \\
&\quad + s \int_{Q_T} b^2 (a \varphi_{xx} w)_x w_x dxdt \\
&= \frac{1}{2} \int_{Q_T} \dot{b} w_x^2 dxdt - s \int_0^T b^2 a \varphi_x w_x^2 \Big|_0^1 dt + s \int_{Q_T} b^2 (a \varphi_x)_x w_x^2 dxdt \\
&\quad + s \int_{Q_T} b^2 (a \varphi_{xx})_x w w_x dxdt + s \int_{Q_T} b^2 a \varphi_{xx} w_x^2 dxdt.
\end{aligned}$$

Clearly, one has that $s \int_0^T b^2 a \varphi_x w_x^2 \Big|_0^1 dt = s \int_0^T b^2 \theta x w_x^2 \Big|_0^1 dt = s \int_0^T b^2(t) \theta(t) w_x^2(t, 1) dt$.

Moreover,

$$\begin{aligned}
\langle -s \varphi_t w + s^2 \varphi_x^2 a b w, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} &= \int_{Q_T} (-s \varphi_t w + s^2 \varphi_x^2 a b w) \frac{1}{a} (w_t - 2s b a \varphi_x w_x - s b a \varphi_{xx} w) dxdt \\
&= \frac{1}{2} \int_{Q_T} -\frac{s}{a} \varphi_t (w^2)_t dxdt + \frac{1}{2} \int_{Q_T} b s^2 \varphi_x^2 (w^2)_t dxdt - s \int_{Q_T} b (w^2)_x (-s \varphi_t \varphi_x + s^2 \varphi_x^3 a b) dxdt \\
&\quad - s \int_{Q_T} b \varphi_{xx} w^2 (-s \varphi_t + s^2 \varphi_x^2 a b) dxdt \\
&= \int_0^1 \left(-\frac{s}{2} \frac{\varphi_t}{a} + \frac{s^2}{2} \varphi_x^2 b \right) w^2 \Big|_0^T dx + \frac{s}{2} \int_{Q_T} \varphi_{tt} \frac{w^2}{a} dxdt - \frac{s^2}{2} \int_{Q_T} (\varphi_x^2 b)_t w^2 dxdt + \int_0^T b s^2 \varphi_x \varphi_t w^2 \Big|_0^1 dt \\
&\quad - s^3 \int_0^T b \varphi_x^3 a b w^2 \Big|_0^1 dt - s^2 \int_{Q_T} b w^2 (\varphi_t \varphi_x)_x dxdt + s^3 \int_{Q_T} b^2 (a \varphi_x^3)_x w^2 dxdt + s^2 \int_{Q_T} b \varphi_{xx} \varphi_t w^2 dxdt \\
&\quad - s^3 \int_{Q_T} b^2 \varphi_{xx} \varphi_x^2 a w^2 dxdt \\
&= \frac{s}{2} \int_{Q_T} \frac{\varphi_{tt} w^2}{a} dxdt - \frac{s^2}{2} \int_{Q_T} \dot{b} \varphi_x^2 w^2 dxdt - 2s^2 \int_{Q_T} b \varphi_x \varphi_{tx} w^2 dxdt + s^3 \int_{Q_T} b^2 (a' \varphi_x^3 + 2a \varphi_x^2 \varphi_{xx}) w^2 dxdt.
\end{aligned}$$

□

Proposition 4.6. *Assume the Hypotheses of Theorem 4.3. Then, there exists $s_0 > 0$ large enough and $c > 0$ such that the distributed terms in (4.22) can be estimated in the following way*

$$(D.T.) \geq c \int_{Q_T} \left(s \theta b^2 w_x^2 + s^3 \theta^3 b^2 \left(\frac{x}{a} \right)^2 w^2 \right) dxdt, \quad (4.21)$$

for all $s \geq s_0$.

Proof. Using the definition of φ and ψ , the distributed terms become

$$\begin{aligned}
(D.T.) &= \int_{Q_T} w_x^2 \left(\frac{\dot{b}}{2} + s b^2 \theta + s b^2 \theta \left(1 - \frac{x a'}{a} \right) \right) dxdt + s \int_{Q_T} b^2 \theta (a \psi'')' w w_x dxdt \\
&\quad + \frac{s}{2} \int_{Q_T} \ddot{\theta} \frac{\psi}{a} w^2 dxdt - \frac{s^2}{2} \int_{Q_T} \dot{b} \theta^2 \frac{x^2}{a^2} w^2 dxdt - 2s^2 \int_{Q_T} b \theta \dot{\theta} \frac{x^2}{a^2} w^2 dxdt \\
&\quad + s^3 \int_{Q_T} b^2 \left(a' \theta^3 \frac{x^3}{a^3} + 2\theta^3 \frac{x^2}{a^2} \left(1 - \frac{x a'}{a} \right) \right) w^2 dxdt.
\end{aligned} \quad (4.22)$$

Using the fact that $x a' \leq K_0 a$ for all $x \in [0, 1]$ (recall that K_0 is the constant that appears in

Definition 1.1 associated to the function a with $x_0 = 0$), we obtain

$$\begin{aligned}
(D.T) &\geq \int_{Q_T} w_x^2 \left(\frac{\dot{b}}{2} + s(2 - K_0)b^2\theta \right) dxdt + s \int_{Q_T} ww_x b^2\theta (a\psi'')' dxdt + \frac{s}{2} \int_{Q_T} \ddot{\theta} \frac{\psi}{a} w^2 dxdt \\
&\quad - s^2 \int_{Q_T} w^2 \left(\frac{x}{a} \right)^2 \left(\frac{\dot{b}}{2} \theta^2 + 2b\theta\dot{\theta} \right) dxdt + s^3(2 - K_0) \int_{Q_T} b^2\theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt.
\end{aligned} \tag{4.23}$$

Since, $|(a\psi'')'| = \left| \left[a \left(\frac{x}{a} \right)' \right]' \right| \leq C_3 := \left\| \left(\frac{x a'}{a} \right)' \right\|_{\infty,1}$, as in [32] and using (4.2), the Young and the Hardy inequalities, one has

$$\begin{aligned}
\left| s \int_{Q_T} b^2 ww_x \theta (a\psi'')' dxdt \right| &\leq \frac{s\varepsilon}{2} \int_{Q_T} b^2 \theta |(a\psi'')'|^2 w^2 dxdt + \frac{s}{2\varepsilon} \int_{Q_T} b^2 \theta w_x^2 dxdt \\
&\leq sA_\varepsilon \int_{Q_T} b^2 \frac{\theta w^2}{a} dxdt + \frac{s}{2\varepsilon} \int_{Q_T} b^2 \theta w_x^2 dxdt \\
&= sA_\varepsilon \int_{Q_T} \left(b^2 \frac{\theta}{\gamma} \left(\frac{x}{a} \right)^2 w^2 \right)^{\frac{1}{2}} \left(b^2 \theta \gamma \frac{w^2}{x^2} \right)^{\frac{1}{2}} dxdt + \frac{s}{2\varepsilon} \int_{Q_T} b^2 \theta w_x^2 dxdt \\
&\leq sA_\varepsilon \rho \int_{Q_T} b^2 \frac{\theta^3}{\gamma} \left(\frac{x}{a} \right)^2 w^2 dxdt + sA_\varepsilon \gamma \int_{Q_T} b^2 \theta \frac{w^2}{x^2} dxdt + \frac{s}{2\varepsilon} \int_{Q_T} b^2 \theta w_x^2 dxdt \\
&\leq \frac{sA_\varepsilon \rho}{\gamma} \int_{Q_T} b^2 \theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt + \left[4sA_\varepsilon \gamma + \frac{s}{2\varepsilon} \right] \int_{Q_T} b^2 \theta w_x^2 dxdt,
\end{aligned} \tag{4.24}$$

with $A_\varepsilon = \frac{\varepsilon}{2} C_3^2 \max_{x \in [0,1]} a$, for $\varepsilon > 0$ and $\gamma > 0$. Hence, by (4.2)

$$\begin{aligned}
(D.T) &\geq \int_{Q_T} w_x^2 \left(\frac{\dot{b}}{2} + s(2 - K_0)b^2\theta \right) dxdt - \frac{s}{2} \rho \int_{Q_T} \theta^{\frac{3}{2}} \frac{\psi}{a} w^2 dxdt \\
&\quad - s^2 \int_{Q_T} w^2 \left(\frac{x}{a} \right)^2 \left(\frac{\dot{b}}{2} \theta^2 + 2b\theta\dot{\theta} \right) dxdt + s^3(2 - K_0) \int_{Q_T} b^2 \theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt \\
&\quad - \frac{sA_\varepsilon \rho}{\gamma} \int_{Q_T} b^2 \theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt - \left[4sA_\varepsilon \gamma + \frac{s}{2\varepsilon} \right] \int_{Q_T} b^2 \theta w_x^2 dxdt.
\end{aligned} \tag{4.25}$$

Now, for $\sigma > 0$, using the Young and the Hardy inequalities, we have

$$\begin{aligned}
\left| \frac{s}{2} \rho \int_{Q_T} \theta^{\frac{3}{2}} \frac{\psi}{a} w^2 dxdt \right| &\leq \frac{s}{2} \rho B_0 \int_{Q_T} \theta^{\frac{3}{2}} \frac{w^2}{a} dxdt \\
&= \frac{s}{2} \rho B_0 \int_{Q_T} \left(\frac{1}{\sigma} \theta^2 \left(\frac{x}{a} \right)^2 w^2 \right)^{\frac{1}{2}} \left(\sigma \frac{\theta}{x^2} w^2 \right)^{\frac{1}{2}} dxdt \\
&\leq \frac{s\rho B_0}{2\sigma} \int_{Q_T} \theta^2 \left(\frac{x}{a} \right)^2 w^2 dxdt + \frac{s}{2} \rho B_0 \sigma \int_{Q_T} \frac{\theta}{x^2} w^2 dxdt \\
&\leq \frac{s}{2} B_0 \rho^2 \frac{1}{\sigma m^2} \int_{Q_T} b^2 \theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt + s\rho \sigma \frac{2B_0}{m^2} \int_0^1 b^2 \theta w_x^2 dxdt.
\end{aligned}$$

with $B_0 = \frac{1}{a(1)(2-K_0)} + C_2$. Moreover, by (3.4) and (4.2), one has

$$|\dot{b}(t)| \leq \frac{\|\dot{b}\|_{\infty,T}}{m^2} b^2(t), \quad \forall t \in [0, T], \tag{4.26}$$

thus,

$$\begin{aligned} & \left| s^2 \int_{Q_T} w^2 \left(\frac{x}{a} \right)^2 \left(\frac{\dot{b}}{2} \theta^2 + 2b\theta\dot{\theta} \right) dxdt \right| \\ & \leq \frac{s^2}{2} \frac{\|\dot{b}\|_{\infty, T}}{m^2} \rho \int_{Q_T} b^2 \theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt + 2 \frac{s^2}{m} \rho \int_{Q_T} b^2 \theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt. \end{aligned}$$

Hence, by (4.23) and the previous estimates, one has

$$\begin{aligned} (D.T) & \geq \int_{Q_T} w_x^2 \left(\frac{\dot{b}}{2} + s(2 - K_0)b^2\theta \right) dxdt - s \left(4A_\varepsilon\gamma + \frac{1}{2\varepsilon} + \frac{2B_0\sigma}{m^2}\rho \right) \int_{Q_T} b^2\theta w_x^2 dxdt \\ & \quad + \left[s^3(2 - K_0) - \frac{s}{2} \frac{\rho^2 B_0}{\sigma m^2} - \frac{s^2}{2} \frac{\|\dot{b}\|_{\infty, T}}{m^2} \rho - 2 \frac{s^2}{m} \rho - \frac{sA_\varepsilon}{\gamma} \rho \right] \int_{Q_T} b^2\theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt \end{aligned} \quad (4.27)$$

for $\varepsilon, \gamma, \sigma > 0$. Now, consider the first integral into the previous inequality, it results

$$\left| \int_{Q_T} w_x^2 \frac{\dot{b}}{2} dxdt \right| \leq \frac{\|\dot{b}\|_{\infty, T}}{2m^2} \left(\frac{T}{2} \right)^8 \int_{Q_T} b^2\theta w_x^2 dxdt.$$

Thus, for the distributed terms we obtain the following inequality

$$(D.T) \geq \left(s(2 - K_0) - \frac{\|\dot{b}\|_{\infty, T}}{2m^2} \left(\frac{T}{2} \right)^8 - \frac{s}{2\varepsilon} - 4sA_\varepsilon\gamma - s \frac{2B_0\sigma}{m^2}\rho \right) \int_{Q_T} b^2\theta w_x^2 dxdt + D, \quad (4.28)$$

for all $\varepsilon, \gamma, \sigma > 0$, where

$$D := \left(s^3(2 - K_0) - \frac{s}{2} \frac{\rho^2 B_0}{\sigma m^2} - \frac{s^2}{2} \frac{\|\dot{b}\|_{\infty, 1} + 4m}{m^2} \rho - s \frac{A_\varepsilon}{\gamma} \rho \right) \int_{Q_T} b^2\theta^3 \left(\frac{x}{a} \right)^2 w^2 dxdt.$$

Choosing, for example, $\varepsilon = \frac{2}{2 - K_0}$, $\gamma = \frac{(2 - K_0)^2}{32 C_3^2 \max_{x \in [0, 1]} a}$ and $\sigma = \frac{(2 - K_0)m^2}{32 B_0 \rho}$, one has (4.21) for all

$s \geq s_0$, with s_0 large enough and $c = \frac{2 - K_0}{2}$. For example, as s_0 one can take

$$\begin{aligned} s_0 := \max & \left\{ \frac{\|\dot{b}\|_{\infty, T}}{2m^2} \left(\frac{T}{2} \right)^8 \frac{16}{2 - K_0}; \frac{2\rho}{(2 - K_0)m^2} \left(\|\dot{b}\|_{\infty, T} + 4m \right); \right. \\ & \left. \left(\left(\frac{\rho B_0}{2\sigma m^2} + \frac{A_\varepsilon}{\gamma} \right) \frac{2m^2}{\|\dot{b}\|_{\infty, T} + 4m} \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

□

Thanks to the previous Proposition and to (4.19), one can prove the Carleman estimate given in Theorem 4.3.

Proof of Theorem 4.3. Thanks to (4.17), one has

$$(D.T.) + (B.T.) = \langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{a}}(Q_T)} \leq \frac{1}{2} \int_{Q_T} \frac{e^{2s\varphi}}{a} h^2 dxdt.$$

Hence, by Proposition 4.6 and (4.19), one has that for all $s \geq s_0$

$$\int_{Q_T} \left(s\theta b^2 w_x^2 + s^3 \theta^3 b^2 \left(\frac{x}{a} \right)^2 w^2 \right) dxdt \leq \frac{1}{c} \int_{Q_T} \frac{e^{2s\varphi}}{a} h^2 dxdt + \frac{s}{c} \int_0^T \theta(t) b^2(t) w_x^2(t, 1) dt$$

for a positive constant c . Thus, recalling the definition of $w = e^{s\varphi}v$, we have $v = we^{-s\varphi}$ and $v_x = (w_x - s\varphi_x w)e^{-s\varphi}$. Using the previous inequality, one has that there exists a positive constant C such that

$$\begin{aligned} & \int_{Q_T} sb^2\theta v_x^2 e^{2s\varphi} dxdt + \int_{Q_T} s^3b^2\theta^3 \left(\frac{x}{a}\right)^2 v^2 e^{2s\varphi} dxdt \\ & \leq C \left(\int_{Q_T} \frac{h^2 e^{2s\varphi}}{a} dxdt + s \int_0^T [b^2\theta e^{2s\varphi} v_x^2](t, 1) dt \right). \end{aligned} \quad (4.29)$$

□

Observe that if we substitute the interval $[0, 1]$ with a general interval $[A, B]$, with $a(A) = 0$, then (4.7) becomes

$$\begin{aligned} & \int_0^T \int_A^B \left(sb^2\theta v_x^2 + s^3b^2\theta^3 \left(\frac{x-A}{a}\right)^2 v^2 b^2 \right) e^{2s\varphi} dxdt \\ & \leq C \int_0^T \int_A^B \frac{h^2 e^{2s\varphi}}{a} dxdt + sC \int_0^T [b^2\theta(x-A)v_x^2 e^{2s\varphi}](t, B) dt, \end{aligned} \quad (4.30)$$

for all $s \geq s_0$. In this case, the function φ is defined as in (4.3), but

$$\psi(x) := C_1 \left(\int_A^x \frac{y-A}{a(y)} dy - C_2 \right), \quad \text{with } C_1 = 1 \text{ and } C_2 > \frac{(B-A)^{K_A}}{a(B)(2-K_A)}. \quad (4.31)$$

Now, thanks to the Carleman estimate given in Theorem 4.3, we can obtain the observability inequality for the homogeneous adjoint problem associated to (P_1) , i.e.

$$\begin{cases} v_t + b(t)a(x)v_{xx} = 0, & (t, x) \in Q_T, \\ v(t, 1) = v(t, 0) = 0, & t \in (0, T), \\ v(T, x) = v_T(x) \in L^2_{\frac{1}{a}}(0, 1). \end{cases} \quad (4.32)$$

In particular, we can prove the following observability inequality.

Proposition 4.7. *Assume Hypothesis 4.2 and let $b \in W^{1,\infty}(0, T)$ a strictly positive function. Then, $\exists C_T > 0$ such that every solution $v \in C([0, T]; L^2_{\frac{1}{a}}(0, 1))$ of (4.32) satisfies*

$$\int_0^1 v^2(0, x) \frac{b^2(0)}{a(x)} dx \leq C_T \int_0^T \int_{\omega} v^2(t, x) \frac{b^2(t)}{a(x)} dxdt. \quad (4.33)$$

The proof of the previous result is similar to the one given in [8], so we postpone it to the Appendix. Clearly, by (3.5) and (4.33), we can obtain the classical observability inequality

$$\int_0^1 v^2(0, x) \frac{1}{a(x)} dx \leq C_T \int_0^T \int_{\omega} v^2(t, x) \frac{1}{a(x)} dxdt, \quad (4.34)$$

for a positive constant C_T . Thanks to the equivalence between the observability inequality (4.34) and the null-controllability for (P_1) , one can prove the following theorem.

Theorem 4.8. *Assume Hypothesis 4.2 and let $b \in W^{1,\infty}(0, T)$ a strictly positive function. Then, given $T > 0$ and $u_0 \in L^2_{\frac{1}{a}}(0, 1)$, there exists $f \in L^2_{\frac{1}{a}}(Q_T)$ such that the solution u of (P_1) satisfies*

$$u(T, x) = 0, \quad \forall x \in [0, 1].$$

Moreover, there exists a positive constant C such that

$$\int_0^T \int_{\omega} \frac{f^2}{a} dxdt \leq C \int_0^1 \frac{u_0^2}{a} dx.$$

4.2 The non-homogeneous problem in non-divergence form: the case $a(1) = 0$.

In this subsection we will consider (4.5) where a degenerates only at 1. Proceeding as for Theorem 4.3 one can prove the next Carleman and observability inequalities.

Hypothesis 4.9. Assume a (WD) or (SD) at 1 such that

$$\left(\frac{(1-x)a'}{a}\right)' \in L^\infty(0,1). \quad (4.35)$$

Theorem 4.10. Assume Hypothesis 4.9, $b \in W^{1,\infty}(0,T)$ a strictly positive function and $h \in L^2_{\frac{1}{a}}(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution v of (4.5) in

$$\mathcal{U} := L^2(0,T; H^2_{\frac{1}{a}}(0,1)) \cap H^1(0,T; H^1_{\frac{1}{a}}(0,1))$$

satisfies

$$\int_{Q_T} \left(s\theta v_x^2 b^2 + s^3 \theta^3 \left(\frac{x-1}{a}\right)^2 v^2 b^2 \right) e^{2s\varphi} dx dt \leq C \int_{Q_T} h^2 \frac{e^{2s\varphi}}{a} dx dt + sC \int_0^T \left[\theta v_x^2 e^{2s\varphi} b^2 \right](t,0) dt, \quad (4.36)$$

for all $s \geq s_0$.

Observe that, if $b(t) \equiv 1$, we obtain exactly the same Carleman estimate given in [8, Theorem 3.4], but as before, here we improve the result since the assumptions on a are more general. Since the proof is similar to the one of Theorem 4.3, we omit it. Also in this case, if we substitute the interval $[0,1]$ with a general interval $[A,B]$, with $a(B) = 0$, then (4.36) becomes

$$\begin{aligned} & \int_0^T \int_A^B \left(s\theta v_x^2 b^2 + s^3 \theta^3 \left(\frac{x-B}{a}\right)^2 v^2 b^2 \right) e^{2s\varphi} dx dt \\ & \leq C \int_0^T \int_A^B h^2 \frac{e^{2s\varphi}}{a} dx dt + sC \int_0^T \left[\theta(B-x)v_x^2 e^{2s\varphi} b^2 \right](t,A) dt, \end{aligned} \quad (4.37)$$

for all $s \geq s_0$. In this case the function φ is defined as in (4.3), but

$$\psi(x) := C_1 \left(\int_A^x \frac{y-B}{a(y)} dy - C_2 \right), \quad \text{with } C_1 = 1 \text{ and } C_2 > \frac{(B-A)^{K_B}}{a(A)(2-K_B)}. \quad (4.38)$$

Thanks to (4.36), one can prove that Propostion 4.7 and Theorem 4.8 still hold when a degenerates only at 1.

4.3 The non-homogeneous problem in divergence form: the case $a(0) = 0$.

In this subsection, we will consider the divergence case. Since some computations are similar to the previous ones, we sketch them. As a first step, we will prove Carleman estimate for the non-homogeneous adjoint problem associated to (P_2) , i.e.

$$\begin{cases} v_t + b(t)(a(x)v_x)_x = h, & (t,x) \in Q_T, \\ v(t,1) = 0, & t \in (0,T), \\ \begin{cases} v(t,0) = 0, & \text{if } a \text{ (WD) at } 0, \\ (av_x)(t,0) = 0, & \text{if } a \text{ (SD) at } 0, \end{cases} & t \in (0,T). \end{cases} \quad (4.39)$$

For the following, we need an additional assumption on the function a . In particular, we require the next hypothesis, which is the same as in [1].

Hypothesis 4.11. Assume a (WD) or (SD) at 0, and if $K_0 \geq 1$, we require the following condition

$$\begin{cases} \exists \gamma \in (1, K_0] \text{ such that } x \rightarrow \frac{a(x)}{x^\gamma} \text{ is non-decreasing near } 0, & \text{if } K_0 > 1, \\ \exists \gamma \in (0,1) \text{ such that } x \rightarrow \frac{a(x)}{x^\gamma} \text{ is non-decreasing near } 0, & \text{if } K_0 = 1. \end{cases} \quad (4.40)$$

For (4.39), the next Carleman estimate holds.

Theorem 4.12. *Assume Hypothesis 4.11, $b \in W^{1,\infty}(0,T)$ a strictly positive function and $h \in L^2(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution v of (4.39) in*

$$\mathcal{V} := L^2(0,T; H_a^2(0,1)) \cap H^1(0,T; H_a^1(0,1))$$

satisfies

$$\int_{Q_T} \left(s\theta v_x^2 b^2 a + s^3 \theta^3 \frac{x^2}{a} v^2 b^2 \right) e^{2s\varphi} dx dt \leq C \int_{Q_T} h^2 e^{2s\varphi} dx dt + sC \int_0^T \left[\theta v_x^2 e^{2s\varphi} b^2 a \right](t,1) dt, \quad (4.41)$$

for all $s \geq s_0$. Here, θ and φ are as in (4.1) and (4.3), respectively, and $L^2(Q_T) := L^2(0,T; L^2(0,1))$.

Also in this case, if $b(t) \equiv 1$, we obtain exactly the same Carleman estimate given in [1, Theorem 3.1]. Moreover, we underline that the additional assumption on a in the strongly degenerate case is necessary to apply the Hardy-Poincaré inequality proved in [1, Proposition 2.1]. Even if the computations are similar to the ones in [1], we repeat here the proof since the presence of the term $b(t)$ leads to new terms that we have to estimate.

In order to prove Theorem 4.12, as for the non-divergence case, we define w as in (4.8). Hence, w satisfies

$$\begin{cases} (e^{-s\varphi} w)_t + b(t)(a(x)(e^{-s\varphi} w)_x)_x = h, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ \begin{cases} w(t,0) = 0, & \text{if } a \text{ is (WD) at } 0, \\ (aw_x)(0) = 0, & \text{if } a \text{ is (SD) at } 0, \end{cases} \\ w(T,x) = w(0,x) = 0, & x \in (0,1). \end{cases} \quad (4.42)$$

Moreover, setting

$$Lw := w_t + b(t)(a(x)w_x)_x \quad \text{and} \quad L_s w := e^{s\varphi} L(e^{-s\varphi} w),$$

one can rewrite the equation in (4.42) as

$$L_s w = h e^{s\varphi}.$$

In this case

$$L_s^+ w = \frac{L_s w + L_s^* w}{2} = b(aw_x)_x - s\varphi_t w + s^2 ab\varphi_x^2 w \quad (4.43)$$

and

$$L_s^- w = \frac{L_s w - L_s^* w}{2} = w_t - 2sba\varphi_x w_x - sb(a\varphi_x)_x w \quad (4.44)$$

and the following result holds.

Proposition 4.13. *Assume the Hypotheses of Theorem 4.12. Then, the following identity holds*

$$\langle L_s^+ w, L_s^- w \rangle_{L^2(Q_T)} = (D.T.) + (B.T.), \quad (4.45)$$

where

$$\begin{aligned} (D.T.) &= \frac{s}{2} \int_{Q_T} \varphi_{tt} w^2 dx dt + \frac{1}{2} \int_{Q_T} abw_x^2 dx dt - \frac{s^2}{2} \int_{Q_T} ab\varphi_x^2 w^2 dx dt \\ &+ s \int_{Q_T} ab^2(a\varphi_x)_{xx} w w_x dx dt + s \int_{Q_T} (2a\varphi_{xx} + a'\varphi_x) ab^2 w_x^2 dx dt \\ &- 2s^2 \int_{Q_T} ab\varphi_x \varphi_{xt} w^2 dx dt + s^3 \int_{Q_T} (2a\varphi_{xx} + a'\varphi_x) ab^2 \varphi_x^2 w^2 dx dt \end{aligned} \quad (4.46)$$

and

$$(B.T.) = -sC_1 a(1) \int_0^T (b^2 \theta w_x^2)(t,1) dt. \quad (4.47)$$

In particular, there exists $s_0 > 0$ large enough and $c > 0$ such that the distributed terms in (4.46) can be estimated in the following way

$$(D.T.) \geq c \left(\int_{Q_T} s^3 b^2 \theta^3 \frac{x^2}{a} w^2 dx dt + \int_{Q_T} s b^2 \theta a w_x^2 dx dt \right), \quad (4.48)$$

for all $s \geq s_0$.

Proof. As written before, some computations are similar to the ones in [1], but we repeat here for the reader's convenience since the presence of the function b leads to additional terms. It results

$$\begin{aligned} & \int_{Q_T} L_s^+ w w_t dx dt = \int_{Q_T} (b(a w_x)_x - s \varphi_t w + s^2 a b \varphi_x^2 w) w_t dx dt \\ &= \int_0^T [a b w_x w_t]_{x=0}^{x=1} dt - \frac{1}{2} \int_{Q_T} a b (w_x^2)_t dx dt - \frac{s}{2} \int_{Q_T} \varphi_t (w^2)_t dx dt + \frac{s^2}{2} \int_{Q_T} b a \varphi_x^2 (w^2)_t dx dt \quad (4.49) \\ &= \frac{1}{2} \int_{Q_T} a b w_x^2 dx dt + \frac{s}{2} \int_{Q_T} \varphi_{tt} w^2 dx dt - \frac{s^2}{2} \int_{Q_T} a b \varphi_x^2 w^2 dx dt - s^2 \int_{Q_T} a b \varphi_x \varphi_{xt} w^2 dx dt. \end{aligned}$$

In addition, we have

$$\begin{aligned} & \int_{Q_T} L_s^+ w (-2s a b \varphi_x w_x) dx dt = \int_{Q_T} (b(a w_x)_x - s \varphi_t w + s^2 a b \varphi_x^2 w) (-2s a b \varphi_x w_x) dx dt \\ &= -s \int_{Q_T} b^2 \varphi_x [(a w_x)^2]_x dx dt + s^2 \int_{Q_T} a b \varphi_t \varphi_x (w^2)_x dx dt - s^3 \int_{Q_T} a^2 b^2 \varphi_x^3 (w^2)_x dx dt \\ &= -s \int_0^T [b^2 \varphi_x (a w_x)^2](t, 1) dt - \int_0^T (-s b^2 \varphi_x (a w_x)^2 + s^2 a b \varphi_t \varphi_x w^2 - s^3 a^2 b^2 \varphi_x^3 w^2)(t, 0) dt \quad (4.50) \\ &+ s \int_{Q_T} b^2 \varphi_{xx} (a w_x)^2 dx dt + s^3 \int_{Q_T} a b^2 \varphi_x ((a \varphi_x^2)_x + \varphi_x (a \varphi_x)_x) w^2 dx dt \\ &- s^2 \int_{Q_T} b (a \varphi_t \varphi_x)_x w^2 dx dt, \end{aligned}$$

since $w(t, 1) = 0$. Analogously,

$$\begin{aligned} & \int_{Q_T} L_s^+ w (-s b (a \varphi_x)_x w) dx dt = \int_{Q_T} (b(a w_x)_x - s \varphi_t w + s^2 a b \varphi_x^2 w) (-s b (a \varphi_x)_x w) dx dt \\ &= - \int_0^T (-s a b^2 w_x w (a \varphi_x)_x)(t, 0) dt + s \int_{Q_T} a b^2 w_x ((a \varphi_x)_{xx} w + (a \varphi_x)_x w_x) dx dt \quad (4.51) \\ &+ s^2 \int_{Q_T} b (a \varphi_x)_x \varphi_t w^2 dx dt - s^3 \int_{Q_T} a b^2 \varphi_x^2 (a \varphi_x)_x w^2 dx dt. \end{aligned}$$

Now, if a is (WD) at 0, then $a w_x \in H^1(0, 1)$, $w(t, 0) = 0$, and, using the definition of φ , (2.2) and the equality $a^2 w_x^2 \varphi_x = C_1 a^2 \theta w_x \frac{x}{a}$, one has that the boundary terms in (4.49)-(4.51) reduce to

$$(B.T.) = -s C_1 a(1) \int_0^T (b^2 \theta w_x^2)(t, 1) dt. \quad (4.52)$$

Now, assume that a is (SD) at 0. Using the fact that $(a w_x)(t, 0) = 0$, the equality $a \varphi_x = \theta a \psi' = C_1 \theta x$ and [1, Proposition 2.4], one has that the boundary terms in (4.49)-(4.51) give again (4.52).

Adding (4.49)-(4.51) and taking into account (4.52), (4.45) follows immediately. Now, it remains to prove (4.48). To this aim, observe that, thanks to the choice of $\psi(x)$, one has $2a\psi''(x) + a'\psi'(x) = C_1 \frac{2a-xa'}{a}$ and $(a(x)\psi'(x))'' = 0$; hence, using the definition of φ , we rewrite the distributed term given in (4.46) as

$$\begin{aligned} (D.T.) &= \frac{s}{2} \int_{Q_T} \ddot{\theta} \psi w^2 dx dt + \frac{1}{2} \int_{Q_T} a b w_x^2 dx dt + s C_1 \int_{Q_T} b^2 \theta (2a - xa') w_x^2 dx dt \\ &+ s^3 C_1^3 \int_{Q_T} b^2 \theta^3 (2a - xa') \frac{x^2}{a^2} w^2 dx dt - s^2 C_1^2 \int_{Q_T} \left(2b\dot{\theta} + \frac{\dot{b}\theta}{2} \right) \theta \frac{x^2}{a} w^2 dx dt. \quad (4.53) \end{aligned}$$

By assumption and (4.2), one can estimate the previous terms in the following way

$$\begin{aligned}
(D.T.) &\geq -\frac{s}{2}\rho \int_{Q_T} \theta^{\frac{3}{2}} \psi w^2 dxdt + \frac{1}{2} \int_{Q_T} abw_x^2 dxdt + sC_1(2-K_0) \int_{Q_T} b^2 \theta aw_x^2 dxdt \\
&\quad + s^3 C_1^3 (2-K_0) \int_{Q_T} b^2 \theta^3 \frac{x^2}{a} w^2 dxdt - s^2 C_1^2 \rho \int_{Q_T} 2b\theta^3 \frac{x^2}{a} w^2 dxdt \\
&\quad - s^2 C_1^2 \rho \int_{Q_T} \frac{\|\dot{b}\|_{\infty, T}}{2} \theta^3 \frac{x^2}{a} w^2 dxdt.
\end{aligned}$$

Observe that

$$\frac{1}{2} \left| \int_{Q_T} abw_x^2 dxdt \right| \leq \frac{1}{2} \frac{\|\dot{b}\|_{\infty, T}}{m^2} \left(\frac{T}{2} \right)^8 \int_{Q_T} ab\theta w_x^2 dxdt. \quad (4.54)$$

On the other hand, we have

$$\begin{aligned}
\left| \frac{s}{2}\rho \int_{Q_T} \theta^{\frac{3}{2}} \psi w^2 dxdt \right| &\leq \frac{s}{2}\rho \frac{C_1}{2-K_0} \int_{Q_T} \theta^{\frac{3}{2}} \frac{x^2}{a} w^2 dxdt + \frac{s}{2}\rho C_1 C_2 \int_{Q_T} \theta^{\frac{3}{2}} w^2 dxdt \\
&\leq \frac{s}{2}\rho \frac{C_1}{(2-K_0)m^2} \left(\frac{T}{2} \right)^6 \int_{Q_T} b^2 \theta^3 \frac{x^2}{a} w^2 dxdt + \frac{s}{2}\rho C_1 C_2 \int_{Q_T} \theta^{\frac{3}{2}} w^2 dxdt,
\end{aligned} \quad (4.55)$$

where m is as in (3.4). Moreover, as in [1, Lemma 2.3], one has

$$\left| \frac{s}{2}\rho C_1 C_2 \int_{Q_T} \theta^{\frac{3}{2}} w^2 dxdt \right| \leq \frac{\sigma \tilde{C} s}{2m^2} \int_{Q_T} b^2 \theta aw_x^2 dxdt + \frac{\tilde{C} s}{2\sigma m^2} \int_{Q_T} b^2 \theta^3 \frac{x^2}{a} w^2 dxdt, \quad (4.56)$$

for positive constants σ and \tilde{C} . Hence, by (4.26),

$$\begin{aligned}
(D.T.) &\geq \left(s^3 C_1^3 (2-K_0) - \frac{s}{2}\rho \frac{C_1}{(2-K_0)m^2} \left(\frac{T}{2} \right)^6 - \frac{\tilde{C} s}{2\sigma m^2} \right) \int_{Q_T} b^2 \theta^3 \frac{x^2}{a} w^2 dxdt \\
&\quad - \left(\frac{s^2}{2} C_1^2 \rho \frac{\|\dot{b}\|_{\infty, T}}{m^2} + s^2 C_1^2 \rho \frac{2}{m} \right) \int_{Q_T} b^2 \theta^3 \frac{x^2}{a} w^2 dxdt \\
&\quad + s \left(C_1(2-K_0) - \frac{\sigma \tilde{C}}{2m^2} - \frac{\|\dot{b}\|_{\infty, T}}{2m^2} \left(\frac{T}{2} \right)^8 \right) \int_{Q_T} b^2 \theta aw_x^2 dxdt.
\end{aligned} \quad (4.57)$$

Choosing $C_1 := 2 \frac{\|\dot{b}\|_{\infty, T}}{m^2} \left(\frac{T}{2} \right)^8 \frac{1}{2-K_0}$, and $\sigma := \frac{\|\dot{b}\|_{\infty, T}}{\tilde{C}} \left(\frac{T}{2} \right)^8$, we obtain (4.48) with $c := \frac{2-K_0}{2} \max\{C_1, C_1^3\}$ for all $s \geq s_0$ and for a sufficiently large s_0 . As an example of s_0 , let $\tilde{s}_0 := \frac{\rho}{(2-K_0)C_1} \left(\frac{\|\dot{b}\|_{\infty, T}}{m^2} + \frac{4}{m} \right)$ and $\bar{s}_0 := \left(\frac{m^2}{C_1^2 \rho (\|\dot{b}\|_{\infty, T} + 4m)} \left(\frac{\rho C_1}{(2-K_0)m^2} \left(\frac{T}{2} \right)^6 + \frac{\tilde{C}}{\sigma m^2} \right) \right)^{\frac{1}{2}}$, then define $s_0 := \max\{\tilde{s}_0, \bar{s}_0\}$. \square

Using the previous Proposition, one can prove Theorem 4.12. Indeed, as for the non-divergence case, one has that for all $s \geq s_0$

$$(D.T.) + (B.T.) = \langle L_s^+ w, L_s^- w \rangle_{L^2(Q_T)} \leq \frac{1}{2} \int_{Q_T} e^{2s\varphi} h^2 dxdt,$$

i.e.

$$\int_{Q_T} \left(sb^2 \theta aw_x^2 + s^3 b^2 \theta^3 \frac{x^2}{a} w^2 \right) dxdt \leq \frac{1}{c} \int_{Q_T} e^{2s\varphi} h^2 dxdt + \frac{s}{c} a(1) \int_0^T \theta(t) b^2(t) w_x^2(t, 1) dt$$

for a positive constant c . Thus, recalling the definition of $w = e^{s\varphi}v$ and the previous inequality, one has that there exists a positive constant C such that

$$\begin{aligned} \int_{Q_T} sb^2\theta v_x^2 a e^{2s\varphi} dxdt + \int_{Q_T} s^3 b^2 \theta^3 \frac{x^2}{a} v^2 e^{2s\varphi} dxdt &\leq C \int_{Q_T} h^2 e^{2s\varphi} dxdt \\ &+ sCa(1) \int_0^T [b^2\theta e^{2s\varphi} v_x^2](t, 1) dt. \end{aligned} \quad (4.58)$$

As for the non-divergence case, if we substitute $[0, 1]$ with a general interval $[A, B]$, with $a(A) = 0$, then (4.41) becomes

$$\begin{aligned} \int_0^T \int_A^B \left(sb^2\theta v_x^2 a + s^3 b^2 \theta^3 \frac{(x-A)^2}{a} v^2 \right) e^{2s\varphi} dxdt &\leq C \int_0^T \int_A^B h^2 e^{2s\varphi} dxdt \\ &+ sC \int_0^T [b^2\theta(x-A)v_x^2 e^{2s\varphi} a](t, B) dt, \end{aligned} \quad (4.59)$$

for all $s \geq s_0$. In this case, φ is as in (4.3) with ψ defined in (4.31).

Thanks to the Carleman estimate given in Theorem 4.12, we can obtain an observability inequality for the homogeneous adjoint problem associated to (P_2) , i.e.

$$\begin{cases} v_t + b(t)(a(x)v_x)_x = 0, & (t, x) \in Q_T, \\ v(t, 1) = 0, & t \in (0, T), \\ \begin{cases} v(t, 0) = 0, & \text{if } a \text{ (WD) at } 0, \\ (av_x)(t, 0) = 0, & \text{if } a \text{ (SD) at } 0, \end{cases} & t \in (0, T), \\ v(T, x) = v_T(x) & \in L^2(0, 1). \end{cases} \quad (4.60)$$

In particular, we can prove the following observability inequality.

Proposition 4.14. *Assume Hypothesis 4.11 and let $b \in W^{1,\infty}(0, T)$ a strictly positive function. Then, $\exists C_T > 0$ such that every solution $v \in C([0, T]; L^2(0, 1))$ of (4.60) satisfies*

$$\int_0^1 v^2(0, x) b^2(0) dx \leq C_T \int_0^T \int_\omega b^2 v^2 dxdt. \quad (4.61)$$

Also in this case, thanks to (3.5), we have

$$\int_0^1 v^2(0, x) dx \leq C_T \int_0^T \int_\omega v^2 dxdt, \quad (4.62)$$

for a positive constant C_T . In order to prove the previous result, we use a technique different from the one in [1, Proposition 4.1], anyway we postpone it to Section 7 since it is similar to the proof in the non-divergence case.

Thanks to the equivalence between the observability inequality (4.62) and the null-controllability for (P_2) , one can prove the following theorem.

Theorem 4.15. *Assume Hypothesis 4.11 and let $b \in W^{1,\infty}(0, T)$ a strictly positive function. Then, given $T > 0$ and $u_0 \in L^2(0, 1)$, there exists $f \in L^2(Q_T)$ such that the solution u of (P_2) satisfies*

$$u(T, x) = 0, \quad \forall x \in [0, 1].$$

Moreover, there exists a positive constant C such that

$$\int_0^T \int_\omega f^2 dxdt \leq C \int_0^1 u_0^2 dx.$$

4.4 The non-homogeneous problem in divergence form: the case $a(1) = 0$.

In this subsection we will consider the divergence case when $a(1) = 0$. Since the computations are similar to the case $a(0) = 0$, we omit them and we give only the Carleman estimate. For the null-controllability and the observability inequality we refer to the previous subsection. Consider the non-homogeneous adjoint problem associated to (P_2) , i.e.

$$\begin{cases} v_t + b(t)(a(x)v_x)_x = h, & (t, x) \in Q_T, \\ v(t, 0) = 0, & t \in (0, T), \\ \begin{cases} v(t, 1) = 0, & \text{if } a \text{ (WD) at } 1, \\ (av_x)(t, 1) = 0, & \text{if } a \text{ (SD) at } 1, \end{cases} & t \in (0, T). \end{cases} \quad (4.63)$$

The Hypothesis 4.11 becomes the following.

Hypothesis 4.16. *Assume a (WD) or (SD) at 1, and if $K_1 \geq 1$, we require the following condition*

$$\begin{cases} \exists \gamma \in (1, K_1] \text{ such that } x \rightarrow \frac{a(x)}{(1-x)^\gamma} \text{ is non-increasing near } 1, & \text{if } K_1 > 1, \\ \exists \gamma \in (0, 1) \text{ such that } x \rightarrow \frac{a(x)}{(1-x)^\gamma} \text{ is non-increasing near } 1, & \text{if } K_1 = 1. \end{cases} \quad (4.64)$$

For (4.63), the next Carleman estimate holds.

Theorem 4.17. *Assume Hypothesis 4.16, $b \in W^{1,\infty}(0, T)$ a strictly positive function and $h \in L^2(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution v of (4.63) in*

$$\mathcal{V} := L^2(0, T; H_a^2(0, 1)) \cap H^1(0, T; H_a^1(0, 1))$$

satisfies

$$\int_{Q_T} \left(sb^2 \theta v_x^2 a + s^3 b^2 \theta^3 \frac{(x-1)^2}{a} v^2 \right) e^{2s\varphi} dx dt \leq C \int_{Q_T} h^2 e^{2s\varphi} dx dt + sC \int_0^T \left[b^2 \theta v_x^2 e^{2s\varphi} a \right] (t, 0) dt, \quad (4.65)$$

for all $s \geq s_0$. Here θ and φ are as in (4.1) and (4.3), respectively.

We underline again that the previous Carleman estimate is not proved in [1], where only the case $a(0) = 0$ is considered; thus, this result is new. Moreover, we also remark that to prove (4.65) we need the Hardy-Poincar inequalities given in Proposition 2.1.

As before, if we substitute $[0, 1]$ with a general interval $[A, B]$, with $a(B) = 0$, then (4.65) becomes

$$\begin{aligned} \int_0^T \int_A^B \left(sb^2 \theta v_x^2 a + s^3 b^2 \theta^3 \frac{(x-B)^2}{a} v^2 \right) e^{2s\varphi} dx dt &\leq C \int_0^T \int_A^B h^2 e^{2s\varphi} dx dt \\ &+ sC \int_0^T \left[b^2 \theta (B-x) v_x^2 e^{2s\varphi} a \right] (t, A) dt, \end{aligned} \quad (4.66)$$

for all $s \geq s_0$. In this case, φ is as in (4.3) with ψ defined in (4.31).

5 Some extensions

In this section, we will give some interesting extensions of the previous problem.

5.1 The case $a(0) = 0 = a(1)$.

In this subsection, we will prove null-controllability for the following system

$$\begin{cases} u_t - \mathcal{A}_i(t)u = f(t, x)\chi_\omega(x), & (t, x) \in Q_T, \\ B_i u(t, x_0) = 0, & t \in (0, T), x_0 \in \{0, 1\}, \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \quad (P_i^{\text{new}})$$

$i = 1, 2$, where Q_T , b and \mathcal{A}_i , $i = 1, 2$, are defined as before, and a degenerates at both boundary points (as prototype, we can consider, for example, $a(x) = x^k(1-x)^h$, $x \in [0, 1]$ and $k, h \in (0, 2)$). Moreover,

$$B_i u(t, x_0) = \begin{cases} u(t, x_0) = 0, & i = 1, \\ \begin{cases} u(t, x_0) = 0, & \text{if } a \text{ is (WD) at } x_0, \\ \lim_{x \rightarrow x_0} (au_x)(t, x_0) = 0, & \text{if } a \text{ is (SD) at } x_0, \end{cases} & i = 2, \end{cases}$$

where $x_0 \in \{0, 1\}$. Clearly, we have four possibilities; thus, we give the following definition.

Definition 5.1. *The function $a : [0, 1] \rightarrow \mathbb{R}$ is*

- (WWD) if a is (WD) at 0 and at 1;
- (SSD) if a is (SD) at 0 and at 1;
- (WSD) if a is (WD) at 0 and (SD) at 1;
- (SWD) if a is (SD) at 0 and (WD) at 1.

In this case, we have to define new Hilbert spaces but only for the divergence case and if a is (SSD), (WSD) or (SWD). In the non-divergence case or in the divergence case when a is (WWD), we will consider the spaces introduced in Subsection 2.1 or in Subsection 2.2, respectively.

Thus, if a is (SSD) the space $H_a^1(0, 1)$ becomes

$$H_a^1(0, 1) := \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1) \text{ and } \sqrt{a}u' \in L^2(0, 1)\};$$

if a is (SWD) we have

$$H_a^1(0, 1) := \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1], \sqrt{a}u' \in L^2(0, 1) \text{ and } u(1) = 0\};$$

and if a is (WSD) then

$$H_a^1(0, 1) := \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } [0, 1), \sqrt{a}u' \in L^2(0, 1) \text{ and } u(0) = 0\}.$$

In every case,

$$H_a^2(0, 1) := \{u \in H_a^1(0, 1) \mid au' \in H^1(0, 1)\}$$

and the norms are the same as before. If a is (WWD), (SSD), (WSD) or (SWD) and $b \in W^{1,\infty}(0, T)$ is a strictly positive function, then Theorem 3.4 still holds for (P_i^{new}). Also in this case, to prove the null-controllability for (P_i^{new}), we will prove the observability inequality for the associated homogeneous adjoint problem that holds under the following assumptions.

Hypothesis 5.2. *Assume that a is (WWD), (SSD), (WSD) or (SWD). Moreover, if $i = 1$ we assume that (4.6) and (4.35) hold. If $i = 2$ and a is (SD) at 0 then (4.11) holds; if a is (SD) at 1 then we require (4.64).*

Proposition 5.3. *Assume Hypothesis 5.2 and let $b \in W^{1,\infty}(0, T)$ a strictly positive function. Then, $\exists C_T > 0$ such that every solution $v \in C([0, T]; H)$ of (4.32) satisfies*

$$\int_0^1 v^2(0, x) \frac{b^2(0)}{a(x)} dx \leq C_T \int_0^T \int_{\omega} v^2(t, x) \frac{b^2(t)}{a(x)} dx dt, \quad (5.1)$$

if $i = 1$; and

$$\int_0^1 v^2(0, x) b^2(0) dx \leq C_T \int_0^T \int_{\omega} v^2(t, x) b^2(t) dx dt, \quad (5.2)$$

if $i = 2$.

Proof. As a first step, assume $v \in \mathcal{W}$, where

$$\mathcal{W} := \{v \text{ solution of the homogeneous adjoint problem associated to } (\mathbf{P}_{i,\text{new}}) \text{ such that } v_T \in D(\mathcal{A}_i^2(t)) = D(\mathcal{A}_i^2(0)) = D(A_i^2)\},$$

$i = 1, 2$ (see (7.13) or (7.24) for the definition of $D(\mathcal{A}_i^2(t))$).

Just to fix the idea, set $\omega := (\alpha, \beta)$, let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$ be such that $0 < \alpha < \tilde{\alpha} < \tilde{\beta} < \beta < 1$ and consider the cut-off function $\xi : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\xi(x) := \begin{cases} 1, & x \in [0, \tilde{\alpha}], \\ 0, & x \in [\tilde{\beta}, 1]. \end{cases}$$

Set $p := \xi v$ and $q := (1 - \xi)v$. Clearly, p and q satisfy

$$\begin{cases} \dot{p} + \mathcal{A}_i(t)p = h_i(t, x), & (t, x) \in (0, T) \times (0, \beta), \\ p(t, \beta) = 0, & t \in (0, T), \\ B_i p(t, 0) = 0, & t \in (0, T), \end{cases} \quad (5.3)$$

and

$$\begin{cases} \dot{q} + \mathcal{A}_i(t)q = g_i(t, x), & (t, x) \in (0, T) \times (\alpha, 1), \\ q(t, \alpha) = 0, & t \in (0, T), \\ B_i q(t, 1) = 0, & t \in (0, T), \end{cases} \quad (5.4)$$

where $g_i(t, x) := -h_i(t, x)$, $i = 1, 2$, and

$$h_i(t, x) := \begin{cases} ab(\xi_{xx}v + 2\xi_x v_x), & i = 1, \\ b((a\xi_x)_x v + 2\xi_x a v_x), & i = 2. \end{cases}$$

Clearly, one can prove that Theorem 3.4 still holds in $[0, \beta]$ or $[\alpha, 1]$ for the original problem associated to (5.3) or to (5.4); in this case H is substituted by

$$H_\beta := \begin{cases} L^2_{\frac{1}{a}}(0, \beta), & i = 1, \\ L^2(0, \beta), & i = 2, \end{cases}$$

or

$$H_\alpha := \begin{cases} L^2_{\frac{1}{a}}(\alpha, 1), & i = 1, \\ L^2(\alpha, 1), & i = 2, \end{cases}$$

for (5.3) or (5.4), respectively. Now, we will distinguish between the case $i = 1$ and $i = 2$. Assume, for the moment, $i = 1$. Set $\tilde{Q}_T := (0, T) \times (0, \beta)$, $\omega' := (\tilde{\alpha}, \tilde{\beta})$ and $\varphi_1 := \theta\psi_1$, where ψ_1 is defined as in (4.31), with $x \in [0, \beta]$. Then, by Caccioppoli's inequality (see Proposition 7.1),

$$\int_{\tilde{Q}_T} \frac{h^2}{a} e^{2s\varphi_1} dxdt \leq C \int_0^T \int_{\omega'} b^2 (v^2 + v_x^2 e^{2s\varphi_1}) dxdt \leq C \int_0^T \int_{\omega} b^2 \frac{v^2}{a} dxdt,$$

for a positive constant C . Moreover, for all $s \geq s_0$ and for $T_0 < T_1 \in [0, T]$, by Theorem 4.3 and the Hardy-Poincar inequality (see [8, Proposition 2.6]), we have

$$\begin{aligned} \int_{T_0}^{T_1} \int_0^{\tilde{\alpha}} b^2 \frac{v^2}{a} dxdt &= \int_{T_0}^{T_1} \int_0^{\tilde{\alpha}} b^2 \frac{p^2}{a} dxdt \leq C \int_{T_0}^{T_1} \int_0^\beta b^2 p_x^2 dxdt \leq C \int_0^T \int_0^\beta s_0 \theta b^2 p_x^2 e^{2s_0\varphi_1} dxdt \\ &\leq C \int_{\tilde{Q}_T} \frac{h^2}{a} e^{2s\varphi_1} dxdt \leq C \int_0^T \int_{\omega} b^2 \frac{v^2}{a} dxdt, \end{aligned}$$

for a positive constant C . Using Theorem 4.10 in $[\alpha, 1]$, with $\varphi_2 := \theta\psi_2$, ψ_2 defined as in (4.38) and $x \in [\alpha, 1]$, one has

$$\int_0^T \int_{\omega} b^2 \frac{v^2}{a} dxdt \geq C \int_{T_0}^{T_1} \int_{\tilde{\beta}}^1 b^2 \frac{q^2}{a} dxdt = C \int_{T_0}^{T_1} \int_{\tilde{\beta}}^1 b^2 \frac{v^2}{a} dxdt$$

for a positive constant C . Hence,

$$\int_0^T \int_{\omega} b^2 \frac{v^2}{a} dx dt \geq C \int_{T_0}^{T_1} \int_{(0, \bar{\alpha}) \cup (\bar{\beta}, 1)} b^2 \frac{v^2}{a} dx dt \geq C \int_{T_0}^{T_1} \int_{\omega^c} b^2 \frac{v^2}{a} dx dt,$$

where $\omega^c := (0, 1) \setminus \omega = (0, \alpha] \cup [\beta, 1)$ and C is a positive constant. Thus,

$$\begin{aligned} 2 \int_0^T \int_{\omega} b^2 \frac{v^2}{a} dx dt &\geq \int_0^T \int_{\omega} b^2 \frac{v^2}{a} dx dt + \int_{T_0}^{T_1} \int_{\omega} b^2 \frac{v^2}{a} dx dt \\ &\geq \int_{T_0}^{T_1} \int_{\omega^c} b^2 \frac{v^2}{a} dx dt + \int_{T_0}^{T_1} \int_{\omega} b^2 \frac{v^2}{a} dx dt \geq \inf\{1, C\} \int_{T_0}^{T_1} \int_0^1 b^2 \frac{v^2}{a} dx dt. \end{aligned} \quad (5.5)$$

Multiplying the equation of (4.32) by $\frac{v}{a}$, integrating over $(0, 1)$ and using the Gauss-Green formula, we have

$$\frac{d}{dt} \int_0^1 v^2(t, x) \frac{1}{a} dx = 2b \int_0^1 v_x^2 dx \geq 0, \quad \forall t \geq 0.$$

Hence the function $t \mapsto \int_0^1 \frac{1}{a} v^2(t, x) dx$ is non-decreasing in $[0, T]$ and

$$\int_0^1 v^2(0, x) \frac{b^2(0)}{a(x)} dx \leq M^2 \int_0^1 v^2(0, x) \frac{1}{a(x)} dx \leq \frac{M^2}{m^2} \int_0^1 v^2(t, x) \frac{b^2(t)}{a(x)} dx.$$

Integrating over (T_0, T_1) , by (5.5), we obtain

$$\int_0^1 v^2(0, x) \frac{b^2(0)}{a(x)} dx \leq \frac{M^2}{m^2(T_1 - T_0)} \int_{T_0}^{T_1} \int_0^1 v^2(t, x) \frac{b^2(t)}{a(x)} dx dt \leq C \int_0^T \int_{\omega} b^2 \frac{v^2}{a} dx dt,$$

and (5.1) holds. Now, we will prove (5.2). By Caccioppoli's inequality (see Proposition 7.1), which also holds in the divergence case, one has

$$\int_{\tilde{Q}_T} h^2 e^{2s\varphi_1} dx dt \leq C \int_0^T \int_{\omega'} b^2 (v^2 + v_x^2 e^{2s\varphi_1}) dx dt \leq C \int_0^T \int_{\omega} b^2 v^2 dx dt,$$

for a positive constant C ; here \tilde{Q}_T , ω' and φ_1 are as before. Now, assume a (WD) at 0. Then, by the Hardy-Poincar inequality given in [1, Proposition 2.1] and by Theorem 4.12, there exists a positive constant C such that

$$\begin{aligned} \int_{T_0}^{T_1} \int_0^{\bar{\alpha}} b^2 v^2 dx dt &\leq \frac{1}{a(1)} \int_{T_0}^{T_1} \int_0^{\bar{\alpha}} b^2 \frac{a}{x^2} p^2 dx dt \leq C \int_{T_0}^{T_1} \int_0^{\beta} b^2 a p_x^2 dx dt \\ &\leq C \int_0^T \int_0^{\beta} b^2 s_0 \theta a p_x^2 e^{2s_0 \varphi_1} dx dt \leq C \int_{\tilde{Q}_T} h^2 e^{2s\varphi_1} dx dt \leq C \int_0^T \int_{\omega} b^2 v^2 dx dt, \end{aligned}$$

for all $s \geq s_0$ and for $T_0 < T_1 \in [0, T]$. If a is (SD) at 0, take $g(x) := (ax^4)^{\frac{1}{3}}$; clearly $g(x) \leq \frac{1}{a(1)^{\frac{2}{3}}} a(x)$. Moreover, using the assumption on a , one can prove that $\frac{g(x)}{x^q}$ is non-decreasing near

0 for $q = \frac{4 + \theta}{3}$. Thus, we can apply [1, Proposition 2.1] in $[0, \beta]$ obtaining

$$\begin{aligned} \int_{T_0}^{T_1} \int_0^{\bar{\alpha}} b^2 v^2 dx dt &\leq \left(\frac{1}{a(1)} \right)^{\frac{1}{3}} \int_{T_0}^{T_1} \int_0^{\bar{\alpha}} b^2 \left(\frac{a}{x^2} \right)^{\frac{1}{3}} p^2 dx dt = \left(\frac{1}{a(1)} \right)^{\frac{1}{3}} \int_{T_0}^{T_1} \int_0^{\bar{\alpha}} b^2 \frac{g(x)}{x^2} p^2 dx dt \\ &\leq C \int_{T_0}^{T_1} \int_0^{\beta} b^2 g(x) p_x^2 dx dt \leq C \int_{T_0}^{T_1} \int_0^{\beta} b^2 a p_x^2 dx dt \\ &\leq C \int_0^T \int_0^{\beta} s_0 \theta a b^2 p_x^2 e^{2s_0 \varphi_1} dx dt \leq C \int_{\tilde{Q}_T} h^2 e^{2s\varphi_1} dx dt \leq C \int_0^T \int_{\omega} b^2 v^2 dx dt. \end{aligned}$$

In every case, we have that there exists a constant $C > 0$, such that

$$\int_{T_0}^{T_1} \int_0^{\tilde{\alpha}} b^2 v^2 dx dt \leq C \int_0^T \int_{\omega} b^2 v^2 dx dt.$$

Using Proposition 2.1 and Theorem 4.17 in $[\alpha, 1]$, with $\varphi_2 := \theta\psi_2$, ψ_2 defined as in (4.38) and $x \in [\alpha, 1]$, one has

$$\int_0^T \int_{\omega} b^2 v^2 dx dt \geq C \int_{T_0}^{T_1} \int_{\tilde{\beta}}^1 b^2 q^2 dx dt = C \int_{T_0}^{T_1} \int_{\tilde{\beta}}^1 b^2 v^2 dx dt$$

for a positive constant C . Hence, as before

$$\int_0^T \int_{\omega} b^2 v^2 dx dt \geq C \int_{T_0}^{T_1} \int_{\omega^c} b^2 v^2 dx dt,$$

and

$$\int_0^T \int_{\omega} b^2 v^2 dx dt \geq \frac{1}{2} \inf\{1, C\} \int_{T_0}^{T_1} \int_0^1 b^2 v^2 dx dt,$$

for a positive constant C . Multiplying the equation of (4.60) by v , integrating over $(0, 1)$ and using the Gauss-Green formula, we have

$$\frac{d}{dt} \int_0^1 v^2(t, x) dx = b \int_0^1 a v_x^2 dx \geq 0, \quad \forall t \geq 0.$$

Hence the function $t \mapsto \int_0^1 v^2(t, x) dx$ is non-decreasing in $[0, T]$ and

$$\int_0^1 v^2(0, x) b^2(0) dx \leq M^2 \int_0^1 v^2(0, x) dx \leq \frac{M^2}{m^2} \int_0^1 v^2(t, x) b^2(t) dx.$$

Integrating over (T_0, T_1) , we obtain

$$\int_0^1 v^2(0, x) b^2(0) dx \leq \frac{M^2}{m^2(T_1 - T_0)} \int_{T_0}^{T_1} \int_0^1 v^2(t, x) b^2(t) dx dt \leq C \int_0^T \int_{\omega} b^2 v^2 dx dt,$$

and (5.2) holds. For the rest of the proof we can proceed as in [8]. \square

Also in this case, by (3.5), (5.1) and (5.2) one has that (4.34) and (4.62) still hold if $a(0) = a(1) = 0$. Recall that in [1], the null-controllability is proved assuming that a degenerates only at 0; on the other hand in [8] the result is proved when $a(0) = a(1) = 0$ but under stronger assumptions on the function a . Thus, the previous result generalizes the one in [1] or in [8], respectively.

5.2 A linear extension

Now, we will consider the linear problem

$$\begin{cases} u_t - \mathcal{A}_i(t)u + c(t, x)u = f(t, x)\chi_w(x), & (t, x) \in Q_T, \\ u(t, y_0) = 0, & t \in (0, T), y_0 \in \{0, 1\} \setminus \{x_0\}, \\ B_i u(t, x_0) = 0, & t \in (0, T), x_0 \in \{0, 1\}, \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \quad (\text{P}_i c)$$

$i = 1, 2$, where Q_T , b , B_i , \mathcal{A}_i , $i = 1, 2$, are as before, and a degenerates at 0 or at 1. Assuming that $c \in L^\infty(Q_T)$, the operator $C(t)$ defined as

$$C(t)u := -c(t, \cdot)u$$

can be seen as a bounded perturbation of \mathcal{A}_i . Thus, working in the spaces considered above, one can prove that Theorem 3.4 still holds for $(P_i c)$. Now, consider to adjoint problem associated to $(P_i c)$

$$\begin{cases} v_t + \mathcal{A}_i(t)v - c(t, x)v = h, & (t, x) \in Q_T, \\ v(t, y_0) = 0, & t \in (0, T), y_0 \in \{0, 1\} \setminus \{x_0\}, \\ B_i v(t, x_0) = 0, & t \in (0, T), x_0 \in \{0, 1\}. \end{cases} \quad (P_i c)$$

Rewriting the equation of $(P_i c)$ as $v_t + \mathcal{A}_i(t)v = \bar{h}$, where $\bar{h} := h + cv$, one can apply the previous Carleman estimates obtaining the following results.

Theorem 5.4. *Assume Hypothesis 4.2 or 4.9, $b \in W^{1,\infty}(0, T)$ a strictly positive function and $c \in L^\infty(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution v of $(P_i c)$ in \mathcal{U} satisfies (4.7), if $a(0) = 0$ or (4.36), if $a(1) = 0$, for all $s \geq s_0$.*

Theorem 5.5. *Let $b \in W^{1,\infty}(0, T)$ a strictly positive function and $c \in L^\infty(Q_T)$.*

If Hypothesis 4.11, then there exist two positive constants C and s_0 such that every solution v of $(P_i c)$ in \mathcal{V} satisfies (4.41), for all $s \geq s_0$.

If Hypothesis 4.16, then there exist two positive constants C and s_0 such that every solution v of $(P_i c)$ in \mathcal{V} satisfies (4.65), for all $s \geq s_0$.

The proofs of Theorems 5.4 and 5.5 are the same as in [8] for the non-divergence case, or in [1] for the divergence case, so we omit it (we recall that the case $a(1) = 0$ in the divergence form is not considered in [1], but the proof is similar).

Proceeding as in the previous section one can prove that the solution of $(P_i c)$ is null-controllable, even if a degenerates at both points (for this case we can proceed as in Subsection 5.1).

5.3 A semilinear extension.

In this section, we will study the null-controllability for the semilinear degenerate problem

$$\begin{cases} \dot{u} - A_i(t)u + h(t, x, u) = f(t, x)\chi_\omega(x), & (t, x) \in Q_T, \\ u(t, y_0) = 0, & t \in (0, T), y_0 \in \{0, 1\} \setminus \{x_0\}, \\ B_i u(t, x_0) = 0, & t \in (0, T), x_0 \in \{0, 1\}, \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \quad (P_i SL)$$

$i = 1, 2$, where Q_T , b , B_i , A_i , $i = 1, 2$, are as before and a degenerates at 0 or at 1. On the function h we impose the coming assumptions.

Hypothesis 5.6. *If $i = 1$, let $h : [0, T] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be such that*

$$\forall \lambda \in \mathbb{R}, \quad (t, x) \mapsto h(t, x, \lambda) \text{ is measurable}, \quad (5.6)$$

$$\forall (t, x) \in (0, T) \times (0, 1), \quad h(t, x, 0) = 0. \quad (5.7)$$

Moreover, assume that there exists c and C such that

$$|h(t, x, \lambda) - h(t, x, \mu)| \leq c(1 + |\lambda| + |\mu|)|\lambda - \mu|$$

and

$$\forall \lambda, \mu \in \mathbb{R}, \quad \left(h(t, x, \lambda + \mu) - h(t, x, \mu) \right) \lambda \geq -C\lambda^2. \quad (5.8)$$

Hypothesis 5.7. *If $i = 2$, let $h : [0, T] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be such that (5.6), (5.7) and (5.8) hold. Moreover, suppose that there exist a non-decreasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and positive constants c, M and p with*

$$\begin{cases} p > 0 & K_{x_0} = 1, \\ p > \max \left\{ \frac{1}{2\theta}, \frac{1}{2} \left(1 - \frac{1}{\theta} \right) \right\} & K_{x_0} > 1, \end{cases} \quad (5.9)$$

such that

$$|h(t, x, \lambda) - h(t, x, \mu)| \leq \begin{cases} \varphi\left(a^p(x)(|\lambda| + |\mu|)\right)|\lambda - \mu|, & K_{x_0} \geq 1 \\ c(1 + |\lambda| + |\mu|)|\lambda - \mu|, & K_{x_0} < 1 \end{cases} \quad (5.10)$$

and

$$\forall s \in \mathbb{R}_+, \quad \varphi(s) \leq M(1 + |s|). \quad (5.11)$$

Thanks to the assumptions on h , one can prove that Theorem 3.4 still holds for $(P_i\text{SL})$. Moreover, using the compactness results of the space introduced before, the Aubin Theorem (see [2, Chapter 5]) and a fixed point method, we can obtain the analogous results of Theorem 4.8 or of Theorem 4.15 (see [1] or [8]). Thanks to Subsection 5.1, one can also obtain null-controllability for $(P_i\text{SL})$ when $a(0) = a(1) = 0$.

Clearly, these results still hold if we substitute the interval $[0, 1]$ with a general interval $[A, B]$, with $A < B$.

6 Application to the Budyko-Seller Model

The previous null-controllability result can be applied to (1.1) as soon as $h(t, x, u) := R_e(t, x, u) - R_a(t, x, u)$ satisfies Hypothesis 5.7.

Assume that $R_a(t, x, u) = Q(t, x)\beta(u)$, where $Q(t, x) \in C(\mathbb{R}_+ \times [-1, 1])$, $Q(t, x) \geq 0$ and β is a non-negative and Lipschitz continuous function, as in [3]. Moreover, we assume $R_e \in C(\mathbb{R}_+ \times [-1, 1] \times \mathbb{R})$, $R_e(t, x, u) = bu$, with $b \in \mathbb{R}$ (i.e. the classical Budyko assumption $R_e(t, x, u) = a + bu$ with $a = 0$ [11]).

Clearly, (5.6) is satisfied and, requiring $\beta(0) = 0$, we also have that (5.7) holds. Moreover, since by assumption $Q(t, x) \in C(\mathbb{R}_+ \times [-1, 1])$ and we are interested in the case $t \in [0, T]$, clearly there exists $M \geq 0$ such that $0 \leq Q(t, x) \leq M$ for all $t \in [0, T]$ and for all $x \in [-1, 1]$. Finally, since β is a Lipschitz continuous function, (5.10) follows immediately. Hence, the null-controllability holds for

$$\begin{cases} u_t(t, x) - \rho_0(t)((1 - x^2)u_x)_x - (R_a(t, x, u) - R_e(t, x, u)) = f(t, x)\chi_\omega, & (t, x) \in (0, T) \times (-1, 1), \\ B_2 u(t, \pm 1) = 0, & t \in (0, T), \\ u(0, x) = u_0(x) \in L^2(0, 1), & x \in (-1, 1), \end{cases}$$

under the previous assumptions. The cases $\beta(0) \neq 0$ and $a \neq 0$ or $R_e(t, x, u) = \epsilon(u)\|u\|^3 u$ (where $\epsilon(\cdot)$ is as in Table 1) are open problems. Actually, we think that the null-controllability for $(P_i\text{SL})$ holds under weaker assumptions on the function h . This will be the subject of a forthcoming paper.

7 Appendix

7.1 Proof of Theorem 3.4

We consider the non-homogeneous Cauchy problem (P_i) , $i = 1, 2$. Clearly, u can be written as $u = v + z$, where v and z solve the two problems

$$\begin{cases} \dot{v}(t) = \mathcal{A}_i(t)v(t), \\ v(0) = u_0 \in H, \end{cases} \quad (\text{nCP}_0)$$

$i = 1, 2$, and

$$\begin{cases} \dot{z}(t) = f(t)\chi_\omega, \\ z(0) = 0, \end{cases} \quad (\text{nCP}_f)$$

respectively. By Theorem 3.1, we know that

- $\forall u_0 \in H, \exists! v \in C([0, T]; H)$ solution of (nCP_0) .

- $\forall u_0 \in D(A_i), \exists! v \in C([0, T]; D(A_i)) \cap C^1([0, T]; H)$ solution of (nCP₀), $i = 1, 2$.

Now, we consider (nCP_f). Then,

$$z(t) = \int_0^t f(\tau) \chi_\omega d\tau. \quad (7.1)$$

If $f \in L^2(0, T; H)$, then $z \in H^1(0, T; H) \subset C([0, T]; H)$. Hence, if $f \in L^2(0, T; H)$ and $u_0 \in H$, we have that $u = v + z \in C([0, T]; H)$.

Moreover, if $f \in L^2(0, T; D(A_i)) \cap W^{1,1}(0, T; H)$, then we obtain $z \in C([0, T]; D(A_i)) \cap C^1([0, T]; H)$, $i = 1, 2$. Indeed, if $f \in W^{1,1}(0, T; H)$, then $z \in W^{2,1}(0, T; H) \subset C^1([0, T]; H)$.

Hence, if $f \in L^2(0, T; D(A_i)) \cap W^{1,1}(0, T; H)$ and $u_0 \in D(A_i)$, then $u = v + z \in C([0, T]; D(A_i)) \cap C^1([0, T]; H)$, $i = 1, 2$.

Now, we will prove the estimates given in Theorem 3.4. To fix the idea, we take $i = 1$, i.e. we consider the operator in non-divergence form $\mathcal{A}_1(t) = b(t)au_{xx}$. Take $u_0 \in D(A_1)$; as a first step we will prove (3.6). Multiplying (P_i) by $\frac{u}{a}$, integrating over $(0, 1)$ and using the Gauss-Green formula, we get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 + b(t) \|u_x(t)\|_{L^2(0,1)}^2 = \int_\omega \frac{fu}{a} dx \leq \frac{1}{2} \|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 + \frac{1}{2} \|f(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2. \quad (7.2)$$

Then, we obtain

$$\frac{d}{dt} \|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq \|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 + \|f(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2. \quad (7.3)$$

Now, integrating the above inequality over $(0, t)$, $t \in [0, T]$, we have

$$\|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq \int_0^t \|u(\tau)\|_{L^2_{\frac{1}{a}}(0,1)}^2 d\tau + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 + \|u(0)\|_{L^2_{\frac{1}{a}}(0,1)}^2. \quad (7.4)$$

By the Gronwall lemma, we have

$$\|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq \left(\|u_0\|_{L^2_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right) e^T, \quad \forall t \leq T. \quad (7.5)$$

Hence, we get

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq C_T \left(\|u_0\|_{L^2_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right). \quad (7.6)$$

Using the density of $D(A_1)$ in $L^2_{\frac{1}{a}}(0, 1)$, we obtain (3.6) when $u_0 \in L^2_{\frac{1}{a}}(0, 1)$. Now, we will prove (3.7) when $i = 1$. Again consider $u_0 \in D(A_1)$. Multiplying (P_i) by $-u_{xx}$, integrating over $(0, 1)$, and using the boundary conditions, we have

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (u_x)^2 dx + \int_0^1 b(t) au_{xx}^2 dx = - \int_\omega \frac{f}{\sqrt{a}} u_{xx} \sqrt{a} dx.$$

Hence, using the Cauchy-Schwarz inequality, we obtain

$$\frac{d}{dt} \|u_x(t)\|_{L^2(0,1)}^2 + 2m \|au_{xx}(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq \frac{1}{2\sigma} \|f(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 + \frac{\sigma}{2} \|au_{xx}(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2,$$

$\forall \sigma > 0$, where m is as in (3.4). Thus, we obtain

$$\frac{d}{dt} \|u_x(t)\|_{L^2(0,1)}^2 + \frac{4m - \sigma}{2} \|au_{xx}(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq \frac{1}{2\sigma} \|f(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2. \quad (7.7)$$

Choosing $\sigma \in (0, 4m)$ and integrating the above inequality over $(0, t)$, $t \leq T$, one has

$$\frac{2}{4m - \sigma} \|u_x(t)\|_{L^2(0,1)}^2 + \int_0^t \|au_{xx}(\tau)\|_{L^2_{\frac{1}{a}}(0,1)}^2 d\tau \leq \frac{2}{4m - \sigma} \left(\|u_x(0)\|_{L^2(0,1)}^2 + \frac{1}{2\sigma} \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right).$$

Thus,

$$\|u_x(t)\|_{L^2(0,1)}^2 \leq \max \left\{ 1, \frac{1}{2\sigma} \right\} \left(\|u_0\|_{H^1_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right), \quad (7.8)$$

for all $t \in (0, T)$, and

$$\int_0^T \|au_{xx}(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 dt \leq \frac{2}{4m - \sigma} \max \left\{ 1, \frac{1}{2\sigma} \right\} \left(\|u_0\|_{H^1_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right). \quad (7.9)$$

Clearly, (7.8) implies that

$$\sup_{t \in [0, T]} \|u_x(t)\|_{L^2(0,1)}^2 \leq \max \left\{ 1, \frac{1}{2\sigma} \right\} \left(\|u_0\|_{H^1_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right). \quad (7.10)$$

Finally, squaring (P_i) and integrating over Q_T , one has

$$\begin{aligned} \int_{Q_T} u_t^2 dxdt &= \int_{Q_T} (bau_{xx} + f\chi_\omega)^2 dxdt \leq 2 \int_{Q_T} b^2 a^2 u_{xx}^2 dxdt + 2 \int_{Q_T} a \frac{f^2}{a} dxdt \\ &\leq 2M^2 \max_{x \in [0,1]} a(x) \int_{Q_T} au_{xx}^2 dxdt + 2 \max_{x \in [0,1]} a(x) \int_{Q_T} \frac{f^2}{a} dxdt \\ &= 2 \max_{x \in [0,1]} a(x) (M^2 + 1) \left(\|au_{xx}\|_{L^2_{\frac{1}{a}}(Q_T)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right). \end{aligned} \quad (7.11)$$

By (7.9) and (7.11), we obtain the coming inequality

$$\begin{aligned} \int_0^T \|u_t(t)\|_{L^2(0,1)}^2 dt &\leq 2 \max_{x \in [0,1]} a(x) (1 + M^2) \left(\|au_{xx}\|_{L^2_{\frac{1}{a}}(Q_T)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right) \\ &\leq 2 \max_{x \in [0,1]} a(x) (1 + M^2) \left[\frac{2}{4m - \sigma} \max \left\{ 1, \frac{1}{2\sigma} \right\} \left(\|u_0\|_{H^1_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right) + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right] \\ &\leq C \left(\|u_0\|_{H^1_{\frac{1}{a}}(0,1)}^2 + \|f\|_{L^2_{\frac{1}{a}}(Q_T)}^2 \right), \quad \text{for a positive constant } C. \end{aligned} \quad (7.12)$$

Finally, by (7.6), (7.10), (7.9), and (7.12), we obtain (3.7). With similar computations one can obtain the previous estimates also in the divergence case.

7.2 Proof of Proposition 4.7

Since the proof of Proposition 4.7 is similar to the one given in [8], we sketch it. As a first step, we consider the class of functions

$$\mathcal{W}_1 := \{v \text{ solution of (4.32)} \mid v_T \in D(\mathcal{A}_1^2(t)) = D(\mathcal{A}_1^2(0)) = D(\mathcal{A}_1^2)\},$$

where

$$D(\mathcal{A}_1^2(t)) = \{u \in H^1_{\frac{1}{a}}(0,1) : \mathcal{A}_1 u \in H^2_{\frac{1}{a}}(0,1)\}. \quad (7.13)$$

Then, we have

$$\mathcal{W}_1 \subset C^1([0, T]; H^2_{\frac{1}{a}}(0,1)) \subset L^2(0, T; H^2_{\frac{1}{a}}(0,1)) \cap H^1(0, T; H^1_{\frac{1}{a}}(0,1)) \subset C([0, T]; L^2_{\frac{1}{a}}(0,1)).$$

Proposition 7.1. (*Caccioppoli's inequality*) *Let ω' and ω be open subintervals of $(0, 1)$ such that $\omega' \subset\subset \omega \subset\subset (0, 1)$. Let $s > 0$ and $\Psi(t, x) = \theta(t)\gamma(x)$, where θ is defined as before, and $\gamma \in C^1(0, 1)$ is a strictly negative function. Then, there exists a positive constant C such that every solution v of (4.32) satisfies*

$$\int_0^T \int_{\omega'} e^{2s\Psi} v_x^2 b(t) dxdt \leq C \int_0^T \int_{\omega} v^2 b(t) dxdt. \quad (7.14)$$

Proof. Let us consider a smooth function $\xi : [0, 1] \rightarrow \mathbb{R}$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \text{for all } x \in [0, 1], \\ \xi(x) = 1, & \forall x \in \omega', \\ \xi(x) = 0, & \forall x \in (0, 1) \setminus \omega. \end{cases} \quad (7.15)$$

Then, integrating by parts over Q_T , we have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \left(\int_0^1 (\xi e^{s\Psi})^2 v^2 dx \right) dt = 2s \int_{Q_T} \Psi_t (\xi e^{s\Psi})^2 v^2 dx dt - 2 \int_0^T (\xi e^{s\Psi})^2 avv_x b \Big|_0^1 dt \\ &\quad + 2 \int_{Q_T} [(\xi e^{s\Psi})^2 a]_x bvv_x dx dt + 2 \int_{Q_T} (\xi e^{s\Psi})^2 av_x^2 b dx dt. \end{aligned} \quad (7.16)$$

Thus, proceeding as in [8]

$$\begin{aligned} 2 \int_{Q_T} b \xi^2 e^{2s\Psi} av_x^2 dx dt &= -2s \int_{Q_T} \Psi_t (\xi e^{s\Psi})^2 v^2 dx dt - 2 \int_{Q_T} [\xi^2 e^{2s\Psi} a]_x bvv_x dx dt \\ &\leq -2s \int_{Q_T} \Psi_t (\xi e^{s\Psi})^2 v^2 dx dt + 4 \int_{Q_T} (\xi e^{s\Psi} \sqrt{a})_x^2 bv^2 dx dt \\ &\quad + \int_{Q_T} \xi^2 e^{2s\Psi} av_x^2 dx dt \end{aligned} \quad (7.17)$$

and, in particular,

$$\int_{Q_T} b \xi^2 e^{2s\Psi} av_x^2 dx dt \leq -2s \int_{Q_T} \Psi_t (\xi e^{s\Psi})^2 v^2 dx dt + 4 \int_{Q_T} (\xi e^{s\Psi} \sqrt{a})_x^2 bv^2 dx dt. \quad (7.18)$$

Also, we have

$$\begin{aligned} \inf_{\omega'} \{a\} \int_0^T \int_{\omega'} b e^{2s\Psi} v_x^2 dx dt &\leq \int_0^T \int_{\omega'} b \xi^2 e^{2s\Psi} av_x^2 dx dt \\ &\leq \int_{Q_T} b \xi^2 e^{2s\Psi} av_x^2 dx dt, \quad \text{and using the previous inequality, we get} \\ &\leq 4 \int_{Q_T} b [(\xi e^{s\Psi} \sqrt{a})_x]^2 v^2 dx dt - 2s \int_0^T \int_{\omega} \Psi_t \xi^2 e^{2s\Psi} v^2 dx dt. \end{aligned} \quad (7.19)$$

Hence,

$$\begin{aligned} \inf_{\omega'} \{a\} \int_0^T \int_{\omega'} b e^{2s\Psi} v_x^2 dx dt &\leq \int_0^T \int_{\omega} 4b [(\xi e^{s\Psi} \sqrt{a})_x]^2 v^2 dx dt - \frac{2s}{M} \int_0^T \int_{\omega} b \Psi_t (\xi e^{s\Psi})^2 v^2 dx dt \\ &\leq \sup_{(0,T) \times \omega} \left| 4 \left((\xi e^{s\Psi} \sqrt{a})_x \right)^2 - \frac{2s}{M} \Psi_t \xi^2 e^{2s\Psi} \right| \int_0^T \int_{\omega} bv^2 dx dt \end{aligned} \quad (7.20)$$

and the Caccioppoli's inequality (7.14) holds. \square

Proceeding as in the proof of Theorem 5.3, one can prove the following result in $[0, 1]$.

Lemma 7.2. *Assume the same conditions of Proposition 4.7. Let T_0, T_1 be such that $0 < T_0 < T_1 < T$. Then, there exists a positive constant $C = C_{T_0, T_1}$ such that every solution $v \in \mathcal{W}_1$ of (4.32) satisfies*

$$\int_{T_0}^{T_1} \int_0^1 v^2 \frac{b^2}{a} dx dt \leq C \int_0^T \int_{\omega} v^2 \frac{b^2}{a} dx dt, \quad (7.21)$$

and

$$\int_0^1 v^2(0, x) \frac{b^2(0)}{a(x)} dx \leq \frac{M^2}{m^2(T_1 - T_0)} \int_{T_0}^{T_1} \int_0^1 v^2(t, x) \frac{b^2(t)}{a(x)} dx dt, \quad (7.22)$$

where M and m are as in (3.4). In particular,

$$\int_0^1 v^2(0, x) \frac{b^2(0)}{a(x)} dx \leq \frac{M^2 C}{m^2(T_1 - T_0)} \int_0^T \int_\omega v^2(t, x) \frac{b^2(t)}{a(x)} dx dt. \quad (7.23)$$

For the rest of the proof of Proposition 4.7, we can proceed as in [8].

7.3 Proof of Proposition 4.14

As before, consider the space

$$\mathcal{W}_2 := \{v \text{ solution of (4.60)} \mid v_T \in D(\mathcal{A}_2^2(t)) = D(\mathcal{A}_2^2(0)) = D(\mathcal{A}_2^2)\},$$

where

$$D(\mathcal{A}_2^2(t)) = \{u \in H_a^1(0, 1) : \mathcal{A}_2 u \in H_a^2(0, 1)\}. \quad (7.24)$$

Then, we have

$$\mathcal{W}_2 \subset C^1([0, T]; H_a^2(0, 1)) \subset L^2(0, T; H_a^2(0, 1)) \cap H^1(0, T; H_a^1(0, 1)) \subset C([0, T]; L^2(0, 1)).$$

Proceeding as in Proposition 7.1, one can prove that (7.14) holds also in the divergence case. Moreover, the analogous of Lemma 7.2 holds.

Lemma 7.3. *Assume the same conditions of Proposition 4.14. Then, there exists a positive constant $C = C_{T_0, T_1}$ such that every solution $v \in \mathcal{W}_2$ of (4.32) satisfies*

$$\int_{T_0}^{T_1} \int_0^1 v^2 b^2 dx dt \leq C \int_0^T \int_\omega v^2 b^2 dx dt, \quad (7.25)$$

and

$$\int_0^1 v^2(0, x) b^2(0) dx \leq \frac{M^2}{m^2(T_1 - T_0)} \int_{T_0}^{T_1} \int_0^1 v^2(t, x) b^2(t) dx dt, \quad (7.26)$$

where M and m are as in (3.4). In particular,

$$\int_0^1 v^2(0, x) b^2(0) dx \leq \frac{M^2 C}{m^2(T_1 - T_0)} \int_0^T \int_\omega v^2(t, x) b^2(t) dx dt. \quad (7.27)$$

As for the non-divergence case, we can obtain Proposition 4.14.

8 Acknowledgments

G. Fragnelli and S. Ismail are members of *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)* of the Istituto Nazionale di Alta Matematica (INdAM).

G. Fragnelli is also a member of *UMI "Modellistica Socio-Epidemiologica (MSE)"* and *UMI "CliMath"*. She is partially supported by the PRIN 2022 PNR *Some Mathematical approaches to climate change and its impacts* (CUP E53D23017910001), by INdAM GNAMPA Projects *Modelli differenziali per l'evoluzione del clima e i suoi impatti* (CUP E53C22001930001) and *Analysis, control and inverse problems for evolution equations arising in climate science* (CUP E53C23001670001).

This paper is partially written during the stay of M. Akil at the University of Tuscia and at the University of Bari Aldo Moro as visiting professor supported by GNAMPA and by the Project *STEPS-STEerability and controllability of PDEs in Physical and Agricultural models*, respectively.

References

- [1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, *Carleman estimates for degenerate parabolic operators with applications to null controllability* J. Evol. Equ. **6** (2006), 161204.
- [2] S. Anița, *Analysis and Control of Age-Dependent Population Dynamics*, Mathematical Modelling: Theory and Applications **11**, Kluwer Academic Publishers, Dordrecht, 2000.
- [3] M. Badii, *Periodic solutions for Sellers type diffusive energy balance model in climatology*, Rendiconti del Seminario Matematico dell'Universit di Padova **103** (2000), 181-192.
- [4] K. Bhattacharya, M. Ghil, I. L. Vulis, *Internal variability of an energy-balance model with delayed albedo effects*, J. Atmos. Sci. **39** (1982), 17471773.
- [5] I. Boutaayamou, G. Fragnelli, *A degenerate population system: Carleman estimates and controllability*, Nonlinear Anal. **195** (2020), 111742, 129.
- [6] I. Boutaayamou, G. Fragnelli, L. Maniar, *Carleman estimates for parabolic equations with interior degeneracy and Neumann boundary conditions*, J. Anal. Math. **135** (2018), 135.
- [7] M.I. Budyko, *The effect of solar radiation variations on the climate of the Earth*, Tellus **21** (1969), 611-619.
- [8] P. Cannarsa, G. Fragnelli, D. Rocchetti, *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form*, J. Evol. Equ. **8** (2008), 583616.
- [9] P. Cannarsa, G. Fragnelli, D. Rocchetti, *Null controllability of degenerate parabolic operators with drift*, Netw. Heterog. Media **2** (2007), 695715.
- [10] P. Cannarsa, V. Lucarini, P. Martinez, C. Urbani, J. Vancostenoble, *Analysis of a two-layer energy balance model: long time behavior and greenhouse effect*, Chaos **33** (2023), Paper no. 113111, 34 pp.
- [11] P. Cannarsa, M. Malfitana, P. Martinez, *Parameter determination for energy balance models with memory*, Springer INdAM Ser., **38** (2000), Springer, Cham, 2020, 83130.
- [12] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null controllability of the degenerate heat equations*, Adv. Differential Equations **10** (2005), 153190.
- [13] P. Cannarsa, P. Martinez, J. Vancostenoble, *Global Carleman estimates for degenerate parabolic operators with applications*, Mem. Amer. Math. Soc. **239** (2016), no. 1133, ix+209 pp.
- [14] P. Cannarsa, P. Martinez, J. Vancostenoble, *The cost of controlling weakly degenerate parabolic equations by boundary controls*, Math. Control Relat. Fields **7** (2017), 171211.
- [15] P. Cannarsa, P. Martinez, J. Vancostenoble, *The cost of controlling strongly degenerate parabolic equations*, ESAIM Control Optim. Calc. Var. **26** (2020), Paper no. 2, 50 pp.
- [16] P. Cannarsa, P. Martinez, J. Vancostenoble, *Sharp estimate of the cost of controllability for a degenerate parabolic equation with interior degeneracy*, Minimax Theory Appl. **6** (2021), 251280.
- [17] P. Cannarsa, J. Tort, M. Yamamoto, *Determination of source terms in a degenerate parabolic equation*, Inverse Problems **26** (2010), 105003, 20 pp.
- [18] K. J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, Berlin, Heidelberg, 1999.
- [19] J. I. Diaz, *Mathematical analysis of some diffusive energy balance models in climatology*, Mathematics, Climate and Environment, J.I. Diaz and J.L. Lions, eds Masson (1993), 28- 56.
- [20] J. I. Diaz, *On the mathematical treatment of energy balance climate models*, NATO ASI Ser. Ser. I Glob. Environ. Change, **48**, Springer, Berlin, 1997.

- [21] J. I. Diaz, Diffusive energy balance models in climatology, Stud. Math. Appl. **31**, North-Holland, Amsterdam, 2002.
- [22] J. I. Diaz, G. Hetzer, L. Tello, *An energy balance climate model with hysteresis*, Nonlinear Analysis **64** (2006), 2053-2074.
- [23] J. I. Diaz, L. Tello, A nonlinear parabolic problem on a Riemannian manifold without boundary arising in climatology, Collectanea Mathematica, 1997.
- [24] M. Feller, *The parabolic differential equations and the associated semigroups of transformations*, Ann. of Math. **55** (1952), 468-519.
- [25] M. Feller, *Diffusion processes in one dimension*, Trans. Am. Math. Soc. **97** (1954), 131.
- [26] E. Fernandez-Cara, S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SIAM J. Control Optim. **45** (2006), 1395-1446.
- [27] G. Floridia, *Nonnegative controllability for a class of nonlinear degenerate parabolic equations with application to climate science*, Electron. J. Differential Equations (2020), Paper no. 59, 27 pp.
- [28] K. Fraedrich, *Structural and stochastic analysis of a zero-dimensional climate system*, Q. J. R. Meteorol. Soc. **104** (1978), 461-474.
- [29] K. Fraedrich, *Catastrophes and resilience of a zero-dimensional climate system with ice-albedo and greenhouse feedback*, Q. J. R. Meteorol. Soc. **105** (1979), 147-167.
- [30] G. Fragnelli, *Null controllability for a degenerate population model in divergence form via Carleman estimates*, Adv. Nonlinear Anal. **9** (2020), 1102-1129.
- [31] G. Fragnelli, *Carleman estimates and null controllability for a degenerate population model*, J. Math. Pures Appl. **115** (2018), 741-766.
- [32] G. Fragnelli, D. Mugnai, *Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations*, Mem. Amer. Math. Soc. **242** (2016), no. 1146, v+84 pp.
- [33] G. Fragnelli, D. Mugnai, *Control of degenerate and singular parabolic equations. Carleman Estimates and Observability*, BCAM SpringerBriefs, ISBN 978-3-030-69348-0, 2021.
- [34] G. Fragnelli, M. Yamamoto, *Carleman estimates and controllability for a degenerate structured population model*, Appl. Math. Optim. **84** (2021), 999-1044.
- [35] S. Ji, R. Huang, *On the Budyko-Sellers climate model with mushy region*, J. Math. Anal. Appl. **434** (2016), 581-598.
- [36] M. Ghil, *Climate stability for a Sellers-type model*, J. Atmos Sci. **33** (1976), 3-20.
- [37] K. Gröchenig, *Weight Functions in Time-Frequency Analysis*, 2006, <https://api.semanticscholar.org/CorpusID:18473520>.
- [38] G. Hetzer, *The number of stationary solutions for a one-dimensional Budyko-type climate model*, Nonlinear Analysis, Real World Applications **2** (2001), 259-272.
- [39] G. Hetzer, *Global existence for a functional reaction-diffusion problem from climate modeling*, Discrete and continuous dynamical systems **31** (2011), 660-671.
- [40] G. Hetzer, *A quasilinear functional reaction-diffusion equation from climate modeling*, Nonlinear Anal. **30** (1997), 2547-2556.
- [41] T. Kato, *Abstract differential equations and nonlinear mixed problems*, Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa; Accademia Nazionale dei Lincei, Rome, 1985.

- [42] T. Kato, Linear and Quasi-Linear Equations of Evolution of Hyperbolic Type. In *Hyperbolicity* **72** (2011) of C.I.M.E. Summer Sch. Springer, Heidelberg, 125-191.
- [43] P. Martinez, J. P- Raymond, J. Vancostenoble, *Regional null controllability for a linearized Crocco type equation*, SIAM J. Control Optim. **42** (2003), 709728.
- [44] M. Rantanen, A.Y. Karpechko, A. Lipponen, et al. *The Arctic has warmed nearly four times faster than the globe since 1979*. Commun Earth Environ **3**, 168 (2022). <https://doi.org/10.1038/s43247-022-00498-3>
- [45] D. Saba, G. Bayili, S. Nicaise, *Polynomial stabilization of the wave equation with a time varying delay term in the dynamical control*, (2023) hal-04316273.
- [46] N. Shimakura, Partial Differential Operators of elliptic type, Translations of Mathematical Monographs **99**, American Mathematical Society, Providence, RI, 1992.
- [47] W.D. Sellers, *A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System*, Journal of Applied Meteorology **8** (1969), 392-400.
- [48] J. Tort, *Determination of source terms in a degenerate parabolic equation from a locally distributed observation*, C. R. Math. Acad. Sci. Paris **348** (2010), 12871291.
- [49] J. Tort, K. Vancostenoble, *Determination of the insolation function in the nonlinear Sellers climate model*, Ann. Inst. H. Poincar Anal. Non Linéaire **29** (2012), 683713.
- [50] C. Wang, R. Du, *Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms*, SIAM J. Control Optim. **52** (2014), 14571480.
- [51] C. Wang, *Approximate controllability of a class of semilinear systems with boundary degeneracy*, J. Evol. Equ. **10** (2010), 163193.