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Stable properties under weakly geometrically flat maps

Daniel Barlet* and Jón Magnússon†

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ABSTRACT. In this note we show that a weakly geometrically flat map $\pi: M \rightarrow N$ between pure dimensional complex spaces has the local lifting property for cycles. From this result we also deduce that, under these hypotheses, several properties of M are transferred to N .

1 Lifting cycles locally

An equidimensional holomorphic map $\pi: M \rightarrow N$ between two pure dimensional complex spaces with $q := \dim M - \dim N$, is called **weakly geometrically flat** (or geometrically flat in the weak sense) if it is surjective and there exists an analytic family $(F_y)_{y \in N}$ of q -cycles in M such that, for any $y \in N$ we have the equality $|F_y| = \pi^{-1}(y)$.

If, moreover, for very general $y \in N$ the cycle F_y is a reduced cycle we say that π is **geometrically flat** (or geometrically flat in the strong sense if we want to avoid confusion).

REMARKS.

- (i) Weak geometric flatness is stable by base change $f: P \rightarrow N$, where f is a holomorphic map and P is a pure dimensional reduced complex space. On the contrary, this is not true, in general, for strong geometric flatness when P has at least one irreducible component whose image has empty interior in N .
- (ii) When N is normal and M, N are pure dimensional, every equidimensional holomorphic map $\pi: M \rightarrow N$ is geometrically flat (in the strong sense).

*Institut Elie Cartan, Géométrie,
Université de Lorraine, CNRS UMR 7502 and Institut Universitaire de France.

†Department of Mathematics, School of Engineering and Physical Sciences, University of Iceland.

Proposition 1.0.1 *Let $\pi: M \rightarrow N$ be a geometrically flat holomorphic map in the weak sense between reduced complex spaces of pure dimensions m and n and z_0 be a point in M . Then there exists an open connected neighborhood N_0 of $\pi(z_0)$ in N and a closed analytic subset Z of $\pi^{-1}(N_0)$ such that $\pi^{-1}(\pi(z_0)) \cap Z = \{z_0\}$ and π induces a proper and finite map $\pi_0: Z \rightarrow N_0$, which is geometrically flat in the weak sense.*

PROOF. Put $y_0 := \pi(z_0)$, $q := m - n$ and let $(F_y)_{y \in N}$ be the analytic family of fiber-cycles of π . Let $E = (U, B, j)$ be a q -scale on M , adapted to F_{y_0} and centered at z_0 . More precisely, j is a proper holomorphic embedding of an open neighborhood M_0 of z_0 in M into an open neighborhood W of the origin in \mathbb{C}^l and U, B are open relatively compact polydisks in $\mathbb{C}^q, \mathbb{C}^{l-q}$ such that $\bar{U} \times \bar{B} \subset W$ and

$$j(F_{y_0} \cap M_0) \cap (\bar{U} \times \partial B) = \emptyset.$$

Moreover, E can be chosen in such a way that $j(F_{y_0} \cap M_0) \cap (\{0\} \times \bar{B}) = \{(0, 0)\}$. Then there exists an open neighborhood N_0 of y_0 in $\pi(M_0)$ such that, for all y in N_0 , the scale E is adapted to F_y and $\deg_E(F_y) = \deg_E(F_{y_0})$.

Put $M' := M_0 \cap \pi^{-1}(N_0)$. Then $j(M') = (\pi \circ j^{-1})^{-1}(N_0)$ is an analytic subset of W which satisfies $j(M') \cap (\bar{U} \times \partial B) = \emptyset$. Hence, for all x in U , $j(M') \cap (\{x\} \times B)$ is an analytic subset of W and consequently $Z_x := j^{-1}(\{x\} \times B) \cap j(M')$ is an analytic subset of $\pi^{-1}(N_0)$. Furthermore, the induced map $\pi_x: Z_x \rightarrow N_0$ is proper, surjective and with finite fibers and we may assume that $Z_{x_0} \cap \pi^{-1}(y_0) = \{z_0\}$. For every y in N_0 , denote X_y the multigraph that F_y determines in $U \times B$ and put $A_y := j^*(X_y \cap U \times B) \cap (\{x\} \times B)$.

Then $(A_y)_{y \in N_0}$ is an analytic family of 0-cycles in Z_x such that $|A_y| = \pi_x^{-1}(y)$ and it follows that $Z := Z_0$ satisfies the required properties. \blacksquare

REMARKS.

- (i) Under the assumptions of the proposition assume that π is geometrically flat in the strong sense. Then for very general y in N_0 , the cycle F_y is reduced. So the map $\pi_0: Z_0 \rightarrow N_0$ is geometrically flat in the strong sense (after shrinking N_0 if necessary).
- (ii) Let Σ be an analytic subset of codimension r in M . Then $j(\Sigma \cap M')$ is an analytic subset of codimension at most equal to r in $j(M')$ and it follows that, for very general x in U , we have $\text{codim}_{Z_x}(\Sigma \cap Z_x) = r$ (or empty). So we may choose $x_0 \in U$ such that $Z_{x_0} \cap \Sigma$ has codimension at most equal to r in Z_{x_0} . \square

Corollary 1.0.2 *Let $\pi: M \rightarrow N$ be a geometrically flat map in the weak sense between pure dimensional complex spaces, Y be a rational p -cycle in N and y_0 be a point in $|Y|$. Then, for every x in $\pi^{-1}(y_0)$, there exists an open neighborhood N_1 of y_0 in N and a rational p -cycle X in $\pi^{-1}(N_1)$ such that $x \in |X|$ and $\pi_* X = Y \cap N_1$.*

PROOF. We may assume that Y is an irreducible analytic subset of N . Then the induced map $\pi^{-1}(Y) \rightarrow Y$ is geometrically flat in the weak sense so it is enough to consider the case where $Y = N$. Let z_0 be a fixed point in $\pi^{-1}(y_0)$.

Then, by Proposition 1.0.1, there exists an open connected neighborhood N_0 of y_0 in N and an analytic subset Z of $\pi^{-1}(N_0)$ such that π induces a proper and finite map $\pi_0: Z \rightarrow N_0$ which is geometrically flat in the weak sense and such that (set-theoretically) $\pi^{-1}(z_0) \cap Z = \{z_0\}$. As the map π_0 is both proper and open and N_0 is connected, π_0 is surjective and induces a surjective map from every irreducible component of Z_0 onto an irreducible component of N_0 . Let N'_0 be the union of those irreducible components of N_0 which do not contain y_0 . Thus $N_1 = N_0 \setminus N'_0$ is an open connected neighborhood of y_0 which has only finitely many irreducible components C_1, \dots, C_l . For every $j \in \{1, \dots, l\}$, we choose an irreducible component Z_j of $\pi^{-1}(N_1)$ such that $\pi(Z_j) = C_j$ and denote k_j the degree of the induced map $Z_j \rightarrow C_j$. Then the rational cycle

$$Y := \frac{1}{k_1}Z_1 + \dots + \frac{1}{k_l}Z_l$$

has the desired properties. ■

2 The sheaf L^\bullet

We recall here some classical equivalent definitions of L^2 -holomorphic forms on a normal complex space M .

Remember that M has **weakly rational singularities** if and only if $\omega_M^n = L_M^n$ where M has pure dimension n . Thanks to [4] (see also [5]) this implies that $\omega_M^p = L_M^p$ for each $p \in [0, n]$. If, moreover, M is Cohen-Macaulay, then M has rational singularities.

The following well known lemma gives two equivalent definitions of these (coherent) sheaves.

Lemma 2.0.1 *Let M be a normal complex space and denote $S(M)$ its singular locus. Then the following properties are equivalent for a holomorphic p -form α defined on $M \setminus S(M)$:*

1. *Let Σ is a closed analytic subset of codimension ≥ 2 in M containing the singular set $S(M)$ in M . Then for every open set M' in M and any proper holomorphic embedding $f: X \rightarrow M'$ of a p -dimensional reduced complex space such that $f(X) \not\subset \Sigma$, the integral $\int_{K \setminus f^{-1}(\Sigma)} f^*(\alpha) \wedge \overline{f^*(\alpha)}$ is finite for any compact set K in X .*
2. *There exists a desingularization $\tau: \tilde{M} \rightarrow M$ such that $\tau^*(\alpha)$ extends as a holomorphic p -form on \tilde{M} .*

PROOF. First assume that property 2. holds true and consider a proper holomorphic map $f: X \rightarrow M'$ such that $f(X) \not\subset \Sigma$. Remark first that we may assume that $\Sigma = S(M)$ since on the smooth part of M the sheaf Ω_M^p has the extension property in codimension 2. Let $\tilde{X} \rightarrow \tilde{M}'$ be the strict transform of τ by f (so $\tilde{X} := X \times_{M,strict} \tilde{M}$) and let \tilde{f} be the canonical projection. Then we have

$$\int_{pr^{-1}(K)} \tilde{f}^*(\tau^*(\alpha)) \wedge \overline{\tilde{f}^*(\tau^*(\alpha))} < +\infty$$

and this integral is equal to $\int_K f^*(\alpha) \wedge \overline{f^*(\alpha)}$ since the projection $pr: \tilde{X} \rightarrow X$ is a (proper) modification with center in $f^{-1}(S(M))$, for any compact K in X .

Conversely, if the property 1. holds true, consider a desingularization $\tau: \tilde{M} \rightarrow M$ and take a smooth generic point x_0 of an irreducible component of the exceptional set E in \tilde{M} (the preimage of the center of the modification τ). Then let $g: X \rightarrow \tilde{M}$ be a p -dimensional complex submanifold of a neighborhood of x_0 which is transversal to E^1 at the point x_0 . Then let $f := g \circ \tau$. The property 1. gives the fact that $g^*(\tau^*(\alpha))$ cannot have a pole along E , since x_0 is generic, and then $\tau^*(\alpha)$ extends holomorphically across E , using again the codimension 2 extension property of the sheaf Ω_M^p since \tilde{M} is a complex manifold. \blacksquare

REMARK. Let Z be an analytic subset of an open subset M' of M , such that $Z \cap S(M)$ is of codimension at least 2 in Z , and $j: Z \hookrightarrow M'$ be the canonical injection. Then, due to characterisation 1. above, we have a restriction map

$$j^*(L_{M'}^p) \rightarrow L_Z^p$$

extending the pull-back morphism of holomorphic p -forms, for any $p \geq 0$.

3 The Theorem

Theorem 3.0.1 *Let $\pi: M \rightarrow N$ be a geometrically flat map in the weak sense between irreducible complex spaces. Assume that M has one of the following properties:*

P.1 The complex space M is normal.

P.2 The complex space M is normal and every codimension 1 cycle in M is \mathbb{Q} -Cartier.

P.3 The complex space M is normal and every holomorphic p -form on $M \setminus S(M)$ extends as a section of the L_M^p -sheaf (so $L_M^p = \omega_M^p$).

¹Note that if E has codimension ≥ 2 the proof is complete since $\tau^*(\alpha)$ extends holomorphically across E .

P.4 The complex space M is normal and the de Rham complex $0 \rightarrow \mathbb{C} \rightarrow (\omega_M^\bullet, d^\bullet)$ is exact in degree p .

P.5 The complex space M is normal and the de Rham complex $0 \rightarrow \mathbb{C} \rightarrow (L_M^\bullet, d^\bullet)$ is locally exact in degree p on M .

Then the same property is also true for N .

PROOF. As all five properties are local we may assume, due to Proposition 1.0.1 and Remark (ii) following it, that there exists an analytic subset Z of dimension n in M satisfying the following conditions:

- The restriction $\pi|_Z$ is surjective, proper and geometrically flat in the weak sense.
- $S(M) \cap Z$ is of codimension at least 2. in Z

Denote k the degree of $\pi' := \pi|_Z$.

Let us begin by property *P.1*. Take a locally bounded meromorphic function f in an open neighborhood N' of $y_0 \in N$. Then $\pi^*(f)$ is holomorphic on $\pi^{-1}(N')$ and induces a holomorphic function g on $\pi'^{-1}(N')$. Then the trace of g by the induced map $\pi'^{-1}(N') \rightarrow N'$ is holomorphic and equal to kf .

Proof of *P.2* for N . Assume that X is a cycle of codimension 1 in N and y a point in N . Then $\pi^*(X)$ is a cycle of codimension 1 in M and we may find an open neighborhood U in M of $\pi^{-1}(y) \cap Z$ and a holomorphic function f defining the cycle $\pi^*(X)$ in U . As π' is a proper map, there exists an open neighborhood N' of y such that $\pi'^{-1}(N') \subset U \cap Z$. Put $Z' := \pi'^{-1}(N')$ and denote $\pi'' : Z' \rightarrow N'$ the restriction of π' . Then $f|_{Z'}$ defines the cycle $(\pi'')^*(X)$. Since the norm of the function $f|_{Z'}$ is meromorphic and locally bounded on N' , which is normal, it is holomorphic on N' and defines the cycle $k(X \cap N')$. Hence X is locally \mathbb{Q} -Cartier in N .

To prove property *P.3* for N , consider a p -form α on $N \setminus S(N)$. Then $\pi^*(\alpha)$ extends as a global section of the sheaf L_M^p . Since Z meets $S(M)$ in codimension at least 2 the form $\pi'^*(\alpha)$ is holomorphic on $Z \setminus S(M)$ and is a section of L_Z^p due to the characterization 1. in Lemma 2.0.1. Then, since the map π' is a proper map with finite fibers, using again the characterization 1 above with $\Sigma = \pi(S(M) \cap Z) \cap S(N)$ we conclude that α is a section of L_N^p because the normality of M implies the normality of N (see P.1), and $S(N)$ has codimension at least 2 in N . ■

To prove property *P.4* for N we take an open neighborhood N' of a point y in N and we consider $\alpha \in \Gamma(N', \omega_N^p)$ such that $d\alpha = 0$. Then the restriction of α to $N' \setminus S(N)$ is a closed holomorphic p -form. Then $\pi^*(\alpha)$ is also a closed holomorphic p -form on $\pi^{-1}(N' \setminus S(N))$. Since $\pi^{-1}(S(N))$ and $S(M)$ have codimension at least equal to 2, $\pi^*(\alpha)$ extends as a d -closed section β of ω_M^p on $\pi^{-1}(N')$. Then we may

find an open neighborhood U in M of $\pi^{-1}(y) \cap Z$ (which is a finite set) and a section $\beta \in \Gamma(U, \omega_M^{p-1})$ such that $d\beta = \pi^*(\alpha)$ on U . Then β is a $(p-1)$ -holomorphic form on $U \setminus S(M)$ and then induces a $(p-1)$ -holomorphic form on $Z \setminus (Z \cap S(M))$. Since $(Z \cap S(M))$ has codimension at least 2 in Z we obtain a section on Z of ω_Z^{p-1} (the sheaf ω_Z^\bullet has the extension property in codimension ≥ 2) and we have $d\beta|_Z = \pi^*(\alpha)|_Z$ since the sheaf ω_Z^\bullet has no torsion. The direct image by π' gives a section $\gamma := \pi'_*(\beta|_Z)$ on N' which satisfies $d\gamma = k\alpha$ where k is the degree of π' . This proves property $P.4$ for N .

The proof of $P.5$ is analogous. ■

REMARK. In the paper [4] it is proved (see Theorem 1.4²), generalizing the result of [5], that $\omega_M^n = L_M^n$ for a normal complex space implies $\omega_M^p = L_M^p$ for any $p \in [0, n]$. An interesting characterization of L^p is also given in [4] (see Theorem 1.1): A section of ω^p is in L^p if for any $\beta \in \Omega_M^{n-p}$ and any $\gamma \in \Omega_M^{n-p-1}$ the forms $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ are in L^n . □

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²In fact they prove more: if $\omega_M^k = L_M^k$ for some $k \leq n$ then $\omega_M^p = L_M^p$ for any $p \in [0, k]$.