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# New developments and explicit results for the thermal constriction resistance of a circular contact under mixed axisymmetric boundary conditions

Najib Laraqi(\*), Jean-Gabriel Bauzin

Université Paris Nanterre, Laboratoire Thermique Interfaces Environnement, LTIE  
50 Rue de Sèvres, 92410, Ville d'Avray, France

(\*) Corresponding author: [nlaraqi@parisnanterre.fr](mailto:nlaraqi@parisnanterre.fr), [nlaraqi@gmail.com](mailto:nlaraqi@gmail.com)

## Abstract

This article presents a study of the thermal constriction resistance developed in a semi-infinite medium subjected to a circular heat source with mixed boundary conditions. The problem is considered axisymmetric and stationary. Two cases of mixed boundary conditions are considered. The contact area can be subjected to a non-uniform axisymmetric heat flux density while the remainder of the surface is isothermal, or the contact is subjected to a non-uniform axisymmetric surface temperature while the remainder of the surface is adiabatic. The thermal constriction resistance is determined analytically. It takes a simple form and can be used for various problems. To offer a wide choice of applications, a general distribution law for the heat flux density or the temperature of the contact area is considered. Verifications of the proposed solutions are carried out by comparison with some analytical results available in the literature. An analysis of the results according to the kind of boundary conditions and the curvature of the axisymmetric source is also performed.

**Key words:** Thermal constriction resistance, analytical solution, mixed boundary conditions, singularities

## Nomenclature

$a$	Radius of the contact area, m	$\phi$	Total heat flux, W
$F$	Hypergeometric function	$\Gamma$	Gamma function
$J_\nu$	Bessel function of the first kind of order $\nu$		Subscript
$P_n$	Legendre polynomial of order $n$	$0$	Uniform boundary condition
$(r,z)$	Cylindrical coordinate, (m,m)	$c$	Contact
$R_c$	Thermal constriction resistance, $K.W^{-1}$	$c,av$	Contact-average
$T$	Temperature rise, K	$c-c$	Contact-concave
$T_{ref}$	Reference temperature, K	$c-h$	Contact-homogeneous
	Greek symbols		Superscript
$\lambda$	Thermal conductivity, $W.m^{-1}.K^{-1}$	$(F)$	Heat flux density condition
$\mu$	Geometric parameter to define the distribution of the imposed function	$(T)$	Temperature condition
$\varphi$	Heat flux density, $W.m^{-2}$	$*$	Dimensionless quantity
		$\tilde{X}$	Hankel transform of $X$

## I. Introduction

The works devoted to the study of thermal constriction resistance are very numerous and have been the subject of several developments since the 1950s. It would be utopian to want to cite them all. We only mention a few examples here. A synthesis on the behavior of the thermal constriction resistance both at the theoretical and experimental cases has been performed [1]. The effect of the geometry of the contact on the evolution of the thermal constriction resistance has been investigated [2]. A theoretical study on the effect of the randomness of microcontacts on the behavior of thermal constriction resistance has been developed [3].

This article deals with another aspect, little studied in the literature and more complex. This is the case of mixed boundary conditions. This problem is encountered in several scientific fields, not only in heat transfer area (such as solid-solid interface, control, characterization ...), but also in mechanics (studies of stresses and cracks), electrostatics (such as the theory of potentials) and electrochemistry (such as charge transfer for electrodes). All these problems are governed by similar equations.

To our knowledge, only two special cases are well known in the literature. They are given in [4, pp. 216-217] for a circular contact at uniform temperature while the remainder of the surface is adiabatic or circular contact at uniform heat flux density while the remainder of the surface is isothermal.

Some other works have been devoted to the transient problem with mixed boundary conditions by considering the contact between two solid media through a circular contact [5] or equally spaced strips contacts [6]. The transient problem with mixed boundary conditions involving a half-space with an isothermal disk has been investigated by approximant techniques [7,8].

The effect of the relative size of a circular contact receiving a uniform heat flux density while the remainder of the surface is partially or completely at zero reference temperature has been investigated for a finite insulated cylinder [9]. The authors used the technique of truncated series to obtain an analytical solution involving series. Another technique using scale analysis has been developed to obtain simple and accurate correlations for an axisymmetric problem involving three mixed boundary conditions [10].

The physical equations of the problems involving mixed boundary conditions come up against the difficulty of the presence of singularities in the vicinity of the edges which separate these boundary conditions. They are often described by the techniques of dual integrals equations [11-12] or by conformal transforms [13] for axisymmetric cases.

The resolution of this kind of problem by numerical methods is very complex. Indeed, the heat flux density in the vicinity of the singularities tends towards infinity. It then becomes necessary to adopt a very thin mesh in the vicinity of this zone.

In this proposed study the problems are reduced to an integral equation of the first kind which is presented under the form of an Abel's integral equation.

Analytical developments are presented for this kind of problem. It is shown, from an original way, that the thermal constriction resistance can be expressed under an exact and a simple form to use. The developed solutions are then used to analyze the behavior of the thermal constriction resistance as a function on the one hand of the mixed boundary conditions imposed and on the other hand as a function of the curvature of the heat source.

## II. Governing equations

We consider a semi-infinite medium (Figure 1) subjected to a circular heat source imposing an axisymmetric temperature while the remainder of the surface is adiabatic (Figure 1- (T)) or dissipating an axisymmetric heat flux density while the remainder of the surface is at zero reference temperature ((Figure 1- (F)).

These two configurations are governed by the following equations

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} = 0, \quad [0 < r < \infty, 0 < z < \infty] \quad (1)$$

$$\left( \frac{\partial T}{\partial r} \right)_{r=0} = 0, \quad [0 < z < \infty] \quad (2)$$

$$T(\infty, z) = 0, \quad [0 < z < \infty] \quad (3)$$

$$T(r, \infty) = 0, \quad [0 < r < \infty] \quad (4)$$

Case of the temperature imposed on the contact area (T)

$$\begin{cases} T(r, 0) = T_c(r) & [r < a] \\ \left( \frac{\partial T}{\partial z} \right)_{z=0} = 0 & [r > a] \end{cases} \quad (5)$$

Case of the heat flux density imposed on the contact area (F)

$$\begin{cases} -\lambda \left( \frac{\partial T}{\partial z} \right)_{r, z=0} = \varphi_c(r) & [r < a] \\ T(r, 0) = 0 & [r > a] \end{cases} \quad (6)$$

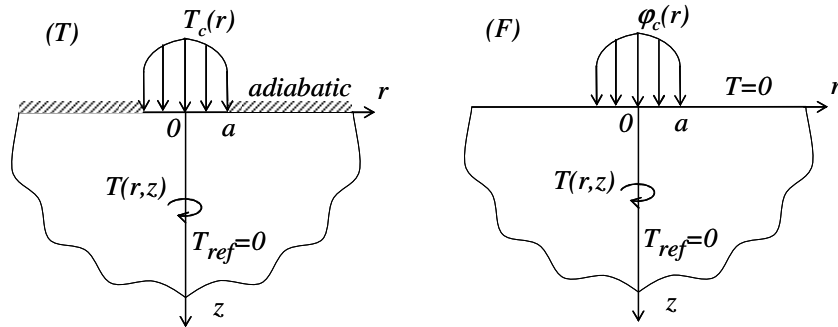


Figure 1: Studied problems with mixed boundary conditions: (T) imposed temperature on the contact area, (F) imposed heat flux density on the contact area

### III. analytical developments

Taking into account the cylindrical geometry of the two studied configurations and the boundary conditions in the radial direction (2) and (3), it is judicious to use the infinite Hankel transformation [4, p. 458] such as

$$\tilde{T} = \int_{r=0}^{\infty} r T(r, z) J_0(\beta r) dr \quad (7)$$

The transformation of eq. (1) leads to a simple differential equation

$$\frac{\partial^2 \tilde{T}}{\partial z^2} - \beta^2 \tilde{T} = 0, \quad [0 < z < \infty] \quad (1a)$$

Taking into account the boundary condition in z-direction (4), the solution of eq. (1a) is then

$$\tilde{T} = A e^{-\beta z} \quad (8)$$

The constant A will be determined according the boundary conditions (5) and (6).

### III.1- Solution for the contact at an axisymmetric non-uniform temperature (T)

This problem is governed by eqs. (1-4) and the surface mixed boundary conditions (5). In this case, the temperature of the contact area is known and we can therefore calculate its average value. The goal is to determine the total heat flux across the contact area.

The Hankel transform (7) of the heat flux density (unknown) over the contact area is

$$\tilde{\varphi}_c = -\lambda \left( \frac{\partial \tilde{T}}{\partial z} \right)_{z=0} = \lambda \beta A = \int_{\rho=0}^a \rho \varphi_c(\rho) J_0(\beta \rho) d\rho \quad (9)$$

In eq. (9) we have taken into account part of the boundary condition (5), which consists in considering the adiabatic part of the surface for ( $r > a$ ). We deduce the constant A under the form

$$A = \frac{1}{\lambda \beta} \int_{\rho=0}^a \rho \varphi_c(\rho) J_0(\beta \rho) d\rho \quad (10)$$

We then use the inverse infinite Hankel transform [4, p. 458] defined by

$$T(r, z) = \int_{\beta=0}^{\infty} \beta \tilde{T} J_0(\beta r) d\beta \quad (11)$$

By replacing (9) and (10) into (11), we obtain the surface temperature under the following form

$$T(r, 0) = \frac{1}{\lambda} \int_{\beta=0}^{\infty} J_0(\beta r) d\beta \int_{\rho=0}^a \rho \varphi_c(\rho) J_0(\beta \rho) d\rho \quad (12)$$

We now use the second part of the boundary condition (5) for which we have  $T(r < a, 0) = T_c(r)$  (known). For this, we use the relation (12) for ( $r < a$ ). We also reverse the order of integration such as

$$\int_{\rho=0}^a \rho \varphi_c(\rho) d\rho \int_{\beta=0}^{\infty} J_0(\beta \rho) J_0(\beta r) d\beta = \lambda T_c(r) \quad [0 < r < a] \quad (13)$$

This is an integral equation of the first kind.

We multiply to the left and to the right of eq. (13) by  $r/\sqrt{x^2-r^2}$  and we integrate with respect  $r$  between 0 and  $x$ . Knowing that [14, 6.567-13]

$$\int_{r=0}^x \frac{rJ_0(\beta r) dr}{\sqrt{x^2-r^2}} = \frac{\sin(\beta x)}{\beta} \quad (13a)$$

eq. (13) then becomes

$$\int_{\rho=0}^a \rho \varphi_c(\rho) d\rho \int_{\beta=0}^{\infty} \frac{J_0(\beta \rho) \sin(\beta x) d\beta}{\beta} = \lambda \int_{r=0}^x \frac{rT_c(r) dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (14)$$

We now derive the eq. (14) with respect to  $x$  to obtain the following equation

$$\int_{\rho=0}^a \rho \varphi_c(\rho) d\rho \int_{\beta=0}^{\infty} J_0(\beta \rho) \cos(\beta x) d\beta = \lambda \frac{d}{dx} \int_{r=0}^x \frac{rT_c(r) dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (15)$$

The integral following  $\beta$  is given by [14, 6.671-2]

$$\int_{\beta=0}^{\infty} \cos(\beta x) J_0(\beta \rho) d\beta = \begin{cases} 0 & [\rho \leq x] \\ \frac{1}{\sqrt{\rho^2-x^2}} & [\rho > x] \end{cases} \quad (16)$$

Eq. (15) will take the following form

$$\int_{\rho=x}^a \frac{\rho \varphi_c(\rho) d\rho}{\sqrt{\rho^2-x^2}} = \lambda \frac{d}{dx} \int_0^x \frac{rT_c(r) dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (17)$$

It is the Abel's integral equation, whose solution is written [15, p.35]

$$\varphi_c(\rho) = -\frac{2\lambda}{\pi} \frac{1}{\rho} \frac{d}{d\rho} \int_{\rho}^a \frac{x dx}{\sqrt{x^2-\rho^2}} \frac{d}{dx} \int_0^x \frac{rT_c(r) dr}{\sqrt{x^2-r^2}} \quad (18)$$

We have to determine the total heat flux through the contact area. It follows from (18)

$$\phi_c = 2\pi \int_0^a \rho \varphi_c(\rho) d\rho = -\frac{2\lambda}{\pi} 2\pi \left[ \int_{\rho}^a \frac{x dx}{\sqrt{x^2-\rho^2}} dx \left\{ \frac{d}{dx} \int_0^x \frac{rT_c(r) dr}{\sqrt{x^2-r^2}} \right\} \right]_{\rho=0}^{\rho=a} \quad (19)$$

We finally get

$$\phi_c = 4\lambda \int_0^a dx \left\{ \frac{d}{dx} \int_0^x \frac{rT_c(r) dr}{\sqrt{x^2 - r^2}} \right\} = 4\lambda \left[ \int_0^x \frac{rT_c(r) dr}{\sqrt{x^2 - r^2}} \right]_{x=0}^{x=a} = 4\lambda \int_0^a \frac{rT_c(r) dr}{\sqrt{a^2 - r^2}} \quad (20)$$

To implement the analytical solution (20) we propose a distribution law of the contact temperature under the following form.

$$T_c(r) = T_0 \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^\mu \quad (21)$$

The parameter  $\mu$  allows, depending on its value, to study several shapes of temperature distribution that can be flat, and more or less rounded. The greater the value of  $\mu$ , the pointed the curve will be, such as a Gaussian-like shape.

The value of  $\mu$  must be positive or zero for the case of imposed temperature. Negative values of  $\mu$  lead to infinite temperatures, which is not physical.

We get the total heat flux using relation (20) under the form

$$\phi_c = \frac{4\lambda a T_0}{2\mu + 1} [\mu \geq 0] \quad (22)$$

The average temperature of the contact area is deduced from (21) as follows

$$T_{c,av} = \frac{T_0}{\mu + 1} [\mu \geq 0] \quad (23)$$

With eqs. (22) and (23) we determine a simple expression of the thermal constriction resistance

$$R_c^{*(T)} = \lambda a R_c^{(T)} = \lambda a \frac{T_{c,av} - T_{ref}}{\phi_c} = \frac{2\mu + 1}{4(\mu + 1)} [\mu \geq 0] \quad (24)$$

Here the reference temperature is  $T_{ref} = 0$ .

### III.2- Solution for the contact at an axisymmetric non-uniform heat flux density (F)

This problem is governed by eqs. (1-4) and the surface mixed boundary conditions (6). The goal this time is to determine the average temperature of the contact area. We also apply the infinite Hankel transform (7) to equations (1-6) as previously. Taking into account the part of the boundary condition (6) which imposes a zero surface temperature for  $(r > a)$ , equations (7) and (8) lead to

$$\tilde{T}(z) = e^{-\beta z} \int_{\rho=0}^a \rho T_c(\rho) J_0(\beta \rho) d\rho \quad (25)$$

The inverse Hankel transform allows us to deduce the expression for the temperature under the following form

$$T(r, z) = \int_{\beta=0}^{\infty} \beta J_0(\beta r) e^{-\beta z} d\beta \int_{\rho=0}^a \rho T_c(\rho) J_0(\beta \rho) d\rho \quad (26)$$

The surface heat flux density is deduced from (26)

$$\varphi(r) = -\lambda \left( \frac{\partial T}{\partial z} \right)_{z=0} = \lambda \int_{\beta=0}^{\infty} \beta^2 J_0(\beta r) d\beta \int_{\rho=0}^a \rho T_c(\rho) J_0(\beta \rho) d\rho \quad (27)$$

Now using the second part of the boundary condition (6) for which the heat flux density on the contact area is known, eq. (27) leads to

$$\int_{\beta=0}^{\infty} \beta^2 J_0(\beta r) d\beta \int_{\rho=0}^a \rho T_c(\rho) J_0(\beta \rho) d\rho = \frac{\varphi_c(r)}{\lambda} \quad [0 < r < a] \quad (28)$$

This is also an integral equation of the first kind.

One proceeds first to the integration by parts of the integral with respect to  $\rho$ . We thus have:

$$\int_{\rho=0}^a \rho T_c(\rho) J_0(\beta \rho) d\rho = \frac{a}{\beta} J_1(\beta a) T_c(a) - \frac{1}{\beta} \int_{\rho=0}^a \rho T_c'(\rho) J_1(\beta \rho) d\rho \quad (29)$$

Where  $T_c'(\rho)$  is the derivative of  $T_c(\rho)$  with respect to  $\rho$ .

By replacing (29) in (28) and permuting the order of the integrations, we obtain

$$a T_c(a) \int_{\beta=0}^{\infty} \beta J_1(\beta a) J_0(\beta r) d\beta - \int_{\rho=0}^a \rho T_c'(\rho) d\rho \int_{\beta=0}^{\infty} \beta J_1(\beta \rho) J_0(\beta r) d\beta = \frac{\varphi_c(r)}{\lambda} \quad [0 < r < a] \quad (30)$$

As in the previous case, we multiply to the left and to the right of eq. (30) by  $r / \sqrt{x^2 - r^2}$  and we integrate with respect to  $r$  between 0 and  $x$ . Eq. (30) then becomes

$$a T_c(a) \int_{\beta=0}^{\infty} J_1(\beta a) \sin(\beta x) d\beta - \int_{\rho=0}^a \rho T_c'(\rho) d\rho \int_{\beta=0}^{\infty} J_1(\beta \rho) \sin(\beta x) d\beta = \frac{1}{\lambda} \int_{r=0}^x \frac{r \varphi_c(r) dr}{\sqrt{x^2 - r^2}} \quad [0 < x < a] \quad (31)$$

We also have [14, 6.671-1]

$$\int_{\beta=0}^{\infty} J_1(\beta\rho)\sin(\beta x)d\beta = \begin{cases} \frac{x}{\rho\sqrt{x^2-\rho^2}} & [\rho > x] \\ 0 & [\rho \leq x] \end{cases} \quad (32)$$

and

$$\int_{\beta=0}^{\infty} J_1(\beta a)\sin(\beta x)d\beta = \frac{x}{a\sqrt{a^2-x^2}} \quad (33)$$

By replacing in (31), we get

$$T_c(a) \frac{x}{\sqrt{a^2-x^2}} - x \int_{\rho=x}^a \frac{T_c'(\rho)d\rho}{\sqrt{\rho^2-x^2}} = \frac{1}{\lambda} \int_{r=0}^x \frac{r\varphi_c(r)dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (34)$$

We thus have the following integral equation to solve

$$\int_{\rho=x}^a \frac{T_c'(\rho)d\rho}{\sqrt{\rho^2-x^2}} = T_c(a) \frac{1}{\sqrt{a^2-x^2}} - \frac{1}{\lambda x} \int_{r=0}^x \frac{r\varphi_c(r)dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (35)$$

It is the Abel's integral equation whose solution is written [15, p.35]

$$T_c'(\rho) = -\frac{2}{\pi} \frac{d}{d\rho} \int_{x=\rho}^a \frac{x dx}{\sqrt{a^2-x^2}} \left[ T_c(a) \frac{1}{\sqrt{a^2-x^2}} - \frac{1}{\lambda x} \int_{r=0}^x \frac{r\varphi_c(r)dr}{\sqrt{x^2-r^2}} \right] \quad [0 < x < a] \quad (36)$$

Eq. (36) is reduced to

$$T_c'(\rho) = \frac{2}{\pi\lambda} \frac{d}{d\rho} \int_{x=\rho}^a \frac{dx}{\sqrt{a^2-x^2}} \int_{r=0}^x \frac{r\varphi_c(r)dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (37)$$

We deduce from (37) the expression of the temperature of the contact area under the form

$$T_c(\rho) = \frac{2}{\pi\lambda} \int_{x=\rho}^a \frac{dx}{\sqrt{a^2-x^2}} \int_{r=0}^x \frac{r\varphi_c(r)dr}{\sqrt{x^2-r^2}} \quad [0 < x < a] \quad (38)$$

The average temperature of the contact area is deduced from (38) under the form

$$T_{c,av} = \frac{2}{\pi\lambda} \left\{ \frac{2\pi}{\pi a^2} \int_{\rho=0}^a \rho d\rho \int_{x=\rho}^a \frac{dx}{\sqrt{a^2-x^2}} \int_{r=0}^x \frac{r\varphi_c(r)dr}{\sqrt{x^2-r^2}} \right\} \quad [0 < x < a] \quad (39)$$

By successive permutations of the integrals of (39) we obtain a simple expression for the average temperature of the contact area

$$T_{c,av} = \frac{4}{\pi a^2 \lambda} \int_{r=0}^a r \phi_c(r) \sqrt{a^2 - r^2} dr \quad (40)$$

Considering the general case of heat flux density on the contact area expressed by

$$\phi_c(r) = \phi_0 \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^\mu \quad (41)$$

the average temperature of the contact is obtained from the relation (40) under the form

$$T_{c,av} = \frac{\phi_0 a}{\lambda} \frac{4}{\pi(2\mu+3)} \quad [\mu > -3/2] \quad (42)$$

The total heat flux is deduced from (41) as follows

$$\phi_c = \phi_0 a^2 \frac{\pi}{(\mu+1)} \quad [\mu > -1] \quad (43)$$

With eqs. (42) and (43) we determine a simple expression of the dimensionless thermal constriction resistance

$$R_c^{*(F)} = \lambda a R_c^{(F)} = \frac{4(\mu+1)}{\pi^2(2\mu+3)} \quad [\mu > -1] \quad (44)$$

### III.3- Solution for concave heat flux density or temperature on the contact ( $\mu > 0$ )

One can deduce from the particular cases developed previously the thermal constriction resistance when the boundary condition imposed on the contact is of concave shape for ( $\mu > 0$ ).

#### III.3.1- Solution for the case the contact temperature (T) with concave shape ( $\mu > 0$ )

We consider in this case

$$T_{c-c}(r) = T_0 \left\{ 1 - \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^\mu \right\} \quad [\mu > 0] \quad (45)$$

From the developments in (III.1) one deduces the thermal constriction resistance under the following form:

$$R_{c-c}^{*(T)} = \lambda a R_c^{(T)} = \frac{2\mu+1}{8(\mu+1)} \quad [\mu > 0] \quad (46)$$

### III.3.2- Solution for the case of the contact heat flux density (F) with concave shape ( $\mu > 0$ )

We consider in this case

$$\varphi_{c-c}(r) = \varphi_0 \left\{ 1 - \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^\mu \right\} \quad [\mu > 0] \quad (47)$$

From the developments in (III.2) one deduces the thermal constriction resistance under the following form

$$R_{c-c}^{*(F)} = \lambda a R_{c-c}^{(F)} = \frac{8(\mu+1)}{3\pi^2(2\mu+3)} \quad [\mu > 0] \quad (48)$$

### III.4- Solution for axisymmetric boundary conditions of the second kind

This is the case of boundary conditions with an axisymmetric heat flux density on the contact area while the remainder of the surface is adiabatic. This case does not pose any difficulty to be solved analytically. It is presented here just to be able to compare it with the cases of mixed boundary conditions, developed previously. The solution for the particular case of a uniform heat flux density over the disk is given in [e.g. 4, p. 216, eqs. (14-16)]

We consider the following surface conditions associated with eqs. (1-4).

$$-\lambda \left( \frac{\partial T}{\partial z} \right)_{z=0} = \begin{cases} \varphi_0 \left[ 1 - (r/a)^2 \right]^\mu & [0 < r < a] \\ 0 & [r > a] \end{cases} \quad (49)$$

The Hankel transform (7) of eq. (49) is [14, 6.567-1]

$$-\lambda \left( \frac{\partial \tilde{T}}{\partial z} \right)_{z=0} = \varphi_0 \int_{r=0}^a r \left[ 1 - (r/a)^2 \right]^\mu J_0(\beta r) dr = \frac{\varphi_0 a^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(\beta a)}{(\beta a)^{\mu+1}}, \quad [\mu > -1] \quad (49a)$$

Where  $\Gamma$  denotes the Gamma function.

By using the solution (8) and the transformed boundary condition (49a) we deduce the transformed surface temperature under the form

$$\tilde{T}(z=0) = \frac{\varphi_0 a^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(\beta a)}{\lambda \beta (\beta a)^{\mu+1}}, \quad [\mu > -1] \quad (49b)$$

By applying to (49b) the inverse Hankel transform (11), we deduce the surface temperature such that

$$T(r,0) = \frac{\varphi_0 a^2 2^\mu \Gamma(\mu+1)}{\lambda} \int_{\beta=0}^{\infty} \frac{J_0(\beta r) J_{\mu+1}(\beta a) d\beta}{(\beta a)^{\mu+1}}, \quad [\mu > -1] \quad (49c)$$

From (49c) we determine the average surface temperature over the disk under the form

$$T_{c,av} = \frac{\varphi_0 a 2^{\mu+1} \Gamma(\mu+1)}{\lambda} \int_{\beta=0}^{\infty} \frac{J_1(\beta a) J_{\mu+1}(\beta a) d\beta}{\beta(\beta a)^{\mu+1}} = \frac{\varphi_0 a}{\lambda} \frac{2\Gamma(\mu+1)\Gamma(\mu+2)}{(2\mu+3)\Gamma^2\left(\mu+\frac{3}{2}\right)}, \quad [\mu > -1] \quad (49d)$$

The integral (49d) is given in [14, 6.575-2]

The total heat flux through the disk is deduced from the eq. (49) such as

$$\Phi_c = 2\pi\varphi_0 \int_{r=0}^a r \left[ 1 - \left(\frac{r}{a}\right)^2 \right]^{\mu} dr = \frac{\pi\varphi_0 a^2}{(\mu+1)} \quad (49e)$$

From eqs. (49d) and (49e) we deduce the thermal constriction resistance under the following form

$$R_{c-h}^{*(F)} = \lambda a R_{c-h}^{(F)} = \frac{2\Gamma^2(\mu+2)}{\pi(2\mu+3)\Gamma^2\left(\mu+\frac{3}{2}\right)} \quad [\mu > -1] \quad (50)$$

## IV- Results and discussions

### IV.1- Verifications for some special cases

Using the relation (24) established for a contact at an axisymmetric temperature while the remainder of the surface is adiabatic, we find the known result for a uniform temperature, i.e.,  $R_c^{*(T)}(\mu=0) = 1/4$  [4,p.216,eq.(11)].

Using the relation (44) established for a contact at an axisymmetric heat flux density while the remainder of the surface is at zero temperature, we find the result for a uniform flux density, i.e.,  $R_c^{*(F)}(\mu=0) = 4/(3\pi^2)$  that can be derived from [4,p.217,eq.(17)]. Indeed, the solution for this particular case is given by the authors under the form

$$T(r,0) = \frac{2\varphi_c}{\pi\lambda} \int_{\beta=0}^{\infty} J_0(\beta r) [\sin(\beta a) - \beta a \cos(\beta a)] \frac{d\beta}{\beta^2} \quad (51)$$

We deduce from eq. (51) the average temperature over the disk such as

$$T_{c,av} = \frac{4\varphi_c}{\pi\lambda a} \int_{\beta=0}^{\infty} J_1(\beta a) [\sin(\beta a) - \beta a \cos(\beta a)] \frac{d\beta}{\beta^3} \quad (52)$$

Knowing that

$$\sin(\beta a) - \beta a \cos(\beta a) = \frac{\sqrt{\pi}}{\sqrt{2}} (\beta a)^{3/2} J_{3/2}(\beta a) \quad (53)$$

by replacing (53) in (52) we get

$$T_{c,av} = \frac{2\sqrt{2}\varphi_c\sqrt{a}}{\sqrt{\pi\lambda}} \int_{\beta=0}^{\infty} \beta^{-3/2} J_1(\beta a) J_{3/2}(\beta a) d\beta = \frac{4}{3\pi} \frac{\varphi_c a}{\lambda} \quad (54)$$

To solve the integral (54) we used [14, 6.574-2].

The thermal constriction resistance for this particular case is then

$$R_c = \frac{4}{\varphi_c \pi a^2} \frac{\varphi_c a}{\lambda} = \frac{4}{3\pi^2 \lambda a} \quad (55)$$

Using the relation (50) established for a contact at an axisymmetric heat flux density while the remainder is adiabatic, we find the known result for a uniform flux density, i.e.,  $R_{c-h}^{*(F)}(\mu=0) = 8/(3\pi^2)$  [4,p.216,eq.(16)]. It is also known that for a contact at uniform temperature  $T_0$ , the heat flux density is  $\varphi_c(r) = (2\lambda T_0 / \pi a) [1 - (r/a)^2]^{-1/2}$ , which corresponds to the case  $(\mu = -1/2)$  of relation (50). We deduce from eq. (50) that  $R_{c-h}^{*(F)}(\mu = -1/2) = 2\Gamma^2(3/2) / (2\pi\Gamma^2(1)) = 1/4$ , which is the known result in the literature for a contact at uniform temperature.

Eq. (18) satisfies the particular case of uniform temperature  $T_c(r) = T_0$ , since we find the known result in this case

$$\varphi_c(\rho) = -\frac{2\lambda}{\pi} \frac{1}{\rho} \frac{d}{d\rho} \int_{\rho}^a \frac{x dx}{\sqrt{x^2 - \rho^2}} \frac{d}{dx} \int_0^x \frac{r T_0 dr}{\sqrt{x^2 - r^2}} = \frac{2\lambda T_0}{\pi \sqrt{a^2 - \rho^2}} \quad (18a)$$

Equation (18a) is similar to that which can be derived from the solution given in [4,p.215, eq.(8)]. The authors give the expression of the temperature under the form

$$T(r, z) = \frac{2T_0}{\pi} \int_{\beta=0}^{\infty} \frac{e^{-\beta z} J_0(\beta r) \sin(\beta a) d\beta}{\beta} \quad (56)$$

From eq. (56) we deduce the heat flux density over the disk such as

$$\varphi_c = \frac{2\lambda T_0}{\pi} \int_{\beta=0}^{\infty} J_0(\beta r) \sin(\beta a) d\beta = \frac{2\lambda T_0}{\pi} \frac{1}{\sqrt{a^2 - r^2}} \quad [0 < r < a] \quad (57)$$

The integral (57) is given in [14, 6.671-1].

By considering the particular case of uniform heat flux density  $\varphi_c(r) = \varphi_0$  in eq. (38), we deduce the expression of the surface temperature over the disk such as

$$T_c(\rho) = \frac{2}{\pi\lambda} \int_{x=\rho}^a \frac{dx}{\sqrt{a^2 - x^2}} \int_{r=0}^x \frac{r\varphi_0 dr}{\sqrt{x^2 - r^2}} = \frac{2\varphi_0}{\pi\lambda} \sqrt{a^2 - \rho^2} \quad (38a)$$

Eq. (38a) is similar to that which can be derived from the solution given in [4,p.217, eq.(17)]. For this particular case, the authors give the solution (51). By using the relation (53), eq. (51) can be written as follows

$$T(r,0) = \frac{\sqrt{2}\varphi_c a^{3/2}}{\sqrt{\pi\lambda}} \int_{\beta=0}^{\infty} \beta^{-1/2} J_0(\beta r) J_{3/2}(\beta a) d\beta = \frac{\sqrt{2}\varphi_c a^{3/2}}{\sqrt{\pi\lambda}} \sqrt{\frac{2}{\pi a}} F\left(1, -\frac{1}{2}; 1; \frac{r^2}{a^2}\right) \quad [0 < r < a] \quad (51a)$$

In eq.(51a) the integral is given in [14, 6.574-1], where F is the hypergeometric function. We also have from [16, p. 467,eq.(200)]:

$$F\left(1, -\frac{1}{2}; 1; \frac{r^2}{a^2}\right) = P_1\left(\sqrt{1 - \frac{r^2}{a^2}}\right) = \sqrt{1 - \frac{r^2}{a^2}} \quad (58)$$

Where  $P_1$  is the Legendre polynomial of order 1;  $P_1(x) = x$ . By replacing (58) into (51a), we find the surface temperature over the disk such as

$$T(r,0) = \frac{2\varphi_c}{\pi\lambda} \sqrt{a^2 - r^2} \quad [0 < r < a] \quad (51a)$$

## IV.2- Analysis of the behavior of the thermal constriction resistance

Figure 2 compares the values of the thermal constriction resistance as a function of the curvature of the source on the contact area (characterized by the parameter  $\mu$ ) and of the kind of boundary conditions imposed on the surface.

(i)- Let us first consider the case of the contact with imposed temperature  $R_c^{*(T)}$  and that of the contact with imposed flux density  $R_{c-h}^{*(F)}$ , while for both the remainder of the surface is adiabatic. We can note that  $R_{c-h}^{*(F)} > R_c^{*(T)}$  for  $0 \leq \mu \ll 0.1$  and this tendency is reversed for  $\mu \gg 0.1$ .

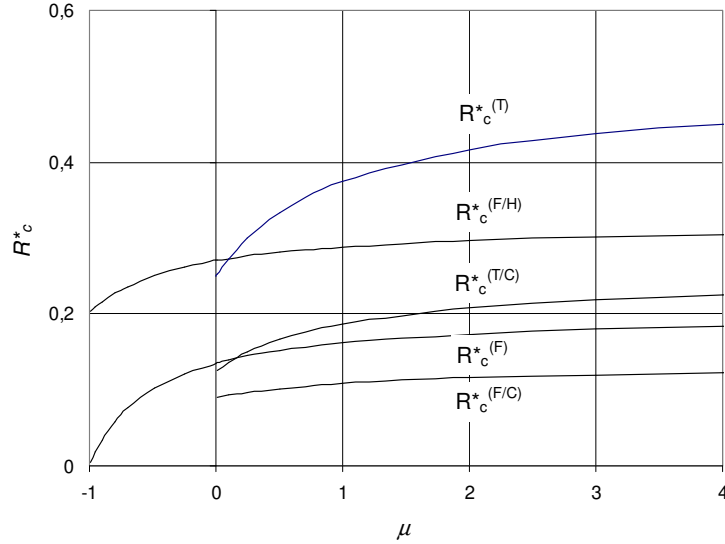


Figure 2: Dimensionless thermal constriction resistance for all studied cases as a function of the curvature parameter

(ii)- Let us compare the first two cases of this study, i.e. the contact at imposed temperature while the remainder of the surface is adiabatic  $R_c^{*(T)}$  to that of the contact at imposed heat flux density while the remainder of the surface is at zero temperature  $R_c^{*(F)}$ . We have for any value of  $\mu > 0$ ,  $R_c^{*(T)} > R_c^{*(F)}$ .

(iii)- We now compare the effect of the curvature of the source (i.e., convex or concave) for the same two cases of boundary conditions as in (ii). It can be seen that the thermal constriction resistance is in both cases greater for convex sources, i.e.  $R_c^{*(T)}(\mu > 0) > R_{c-c}^{*(T)}(\mu > 0)$  and  $R_c^{*(F)}(\mu > 0) > R_{c-c}^{*(F)}(\mu > 0)$ . This behavior is explained by the fact that when the source is concave, its intensity is mainly concentrated on its edges, thus facilitating its dissipation in the solid.

(iv)- For all the studied configurations, the thermal constriction resistance increases with the increase in the value of  $\mu$ . In fact, the greater the value of  $\mu$  the more the source is pointed and concentrated around its center. The heat passage therefore becomes narrower, involving in greater resistance.

## V- Conclusions

In this article, accurate analytical solutions are developed to calculate the thermal constriction resistance for cases of mixed boundary conditions, generating singularities. It is shown that the expression of thermal constriction resistance can be determined under a simple form. The proposed solutions can be implemented easily. The analysis of the results shows that it is possible to highlight the effect of the kind of mixed boundary conditions, and also of the curvatures of the heat sources, on the behavior of the thermal constriction resistance.

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