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# PROLONGATION OF REGULAR-SINGULAR CONNECTIONS ON PUNCTURED AFFINE LINE OVER A HENSELIAN RING

PHÙNG HỒ HAI, JOÃO PEDRO DOS SANTOS, PHẠM THANH TÂM, AND ĐÀO VĂN THỊNH

ABSTRACT. We generalize Deligne’s equivalence between the categories of regular-singular connections on the formal punctured disk and on the punctured affine line to the case where the base is a strictly Henselian discrete valuation ring of equal characteristic 0. We also provide a weaker result when the base is higher dimensional.

## 1. INTRODUCTION

Let  $C$  be an algebraically closed field of characteristic 0 and  $x$  be a variable. The formal punctured disk,  $\text{Spec } C((x))$ , is equipped with the “vector field”  $\vartheta := x \frac{d}{dx}$ . In [Del87, Proposition 13.35], Deligne established an equivalence between regular-singular connections on the formal punctured disk and on the punctured affine line  $\mathbb{P}_C^1 \setminus \{0, \infty\}$ :

$$\left\{ \begin{array}{l} \text{Regular-singular connections} \\ \text{on the punctured formal disk} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{Regular-singular connections} \\ \text{on the punctured affine line} \end{array} \right\}.$$

Deligne uses this equivalence to produce “fiber functors” from the category of regular-singular connections on the formal punctured disk, the tangential fiber functors.

Deligne’s equivalence was also considered by Katz in a more general setting [Kat87]. The analogs in characteristic  $p$  were essentially established by Gieseker in [Gie75] and further developed by Kindler in [Kin15]. There is also a generalization to the  $p$ -adic setting by Matsuda [Mat02], see also [And02].

If we now replace  $C$  by a complete local noetherian  $C$ -algebra  $R$ , Deligne’s equivalence possesses a clear analogue, which was proved in [HdST23, Theorem 10.1]. The main idea behind the proof of this result is to make use of the fact that  $R$  is a limit of finite dimensional local  $C$ -algebras and then, turning attention to objects with “truncated actions” of  $R$ , to “pass to the limit.” To be more specific, let  $\mathfrak{r}$  stand for the maximal ideal of  $R$ , so that  $R_k := R/\mathfrak{r}^{k+1}$  is a finite  $C$ -algebra. Now, given a  $C$ -linear category  $\mathfrak{C}$  (such as the category of connections on the punctured affine line), we restrict our attention to the category  $\mathfrak{C}_{(R_k)}$  of objects in  $\mathfrak{C}$  which have an action of  $R_k$ . If  $\mathfrak{C}'$  is another  $C$ -linear category (e.g. the category of connections on the formal punctured disk) and if we are able to obtain compatible equivalences  $\mathfrak{C}_{(R_k)} \simeq \mathfrak{C}'_{(R_k)}$ , it is to be hoped that “passing to the limit” will

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give us a means to produce equivalences of  $R$ -linear categories. Clearly, this idea relies heavily on the completeness of  $R$ .

In this manuscript, we deal with the case where  $R$  is a noetherian Henselian local  $C$ -algebra. Our main results are Theorem 4.1 and Theorem 4.2. In a nutshell:

**Theorem 1.1.** *Let  $\mathbf{MC}_{\text{rs}}(* / R)$  denote the category of regular-singular connections on  $*$  over  $R$ , and  $\mathbf{MC}_{\text{rs}}^{\circ}(* / R)$  denote the full subcategory of objects whose underlying modules are  $R$ -flat. Then, the restriction functor*

$$\mathbf{r} : \mathbf{MC}_{\text{rs}}^{\circ}(R[x^{\pm}] / R) \longrightarrow \mathbf{MC}_{\text{rs}}^{\circ}(R((x)) / R)$$

*is an equivalence provided that  $R$  is a  $G$ -ring. If  $R$  is, moreover, a discrete valuation ring, then*

$$\mathbf{r} : \mathbf{MC}_{\text{rs}}(R[x^{\pm}] / R) \longrightarrow \mathbf{MC}_{\text{rs}}(R((x)) / R)$$

*is an equivalence.*

The relevance of this result is twofold. On the one hand, according to Deligne's point of view [Del87], it produces fiber functors for the category  $\mathbf{MC}_{\text{rs}}(R((x)) / R)$ : we compose the aforementioned equivalence with a fiber functor at an  $R$ -point of  $\text{Spec } R[x^{\pm}]$ . (Deligne calls these "tangential" fiber functors.) On the other hand, this equivalence describes the structure of  $\mathbf{MC}_{\text{rs}}(R[x^{\pm}] / R)$  in terms of  $\mathbf{MC}_{\text{rs}}(R((x)) / R)$ , which is easier to grasp. Finally, the reduction from a complete noetherian local ring to a noetherian Henselian local ring should be an important step toward the case of an arbitrary noetherian local ring. (It is perhaps useful to observe that the class of  $G$ -rings is a broad and reasonable one. More on this will be found in the body of the text.)

Our approach is based on Deligne's equivalence as presented in [HdST23, Theorem 10.1] and Popescu approximation. While the proof of [HdST23, Theorem 10.1] relied on the accessory category of representations of the group  $\mathbb{Z}$ , we have found no reasonable way to include this actor in the present picture; instead, we have made use of its "Lie version", which is the category of endomorphisms of  $R$ -modules. The part concerning Popescu approximation is of course important, but its employment is more straightforward.

The paper is organized as follows. Section 2 is devoted to the category of regular-singular connections on the formal relative punctured disk. We show that each connection on a flat  $R$ -module admits an Euler form. Section 3 is devoted to the category of regular-singular connections on the punctured relative affine line. Similarly, we show that a connection on an  $R$ -flat module admits an Euler form. The results obtained in these two sections are then used to prove Theorem 1.1 in Section 4.

### 1.1. Notation and conventions.

$C$  is a fixed algebraically closed field of characteristic 0.

$R$  is an integral noetherian local  $C$ -algebra with maximal ideal  $\mathfrak{t}$  and residue field isomorphic to  $C$ .

$R_k$  is the truncation  $R / \mathfrak{t}^{k+1}$ .

$\widehat{R}$  is the  $\mathfrak{t}$ -adic completion of  $R$ .

$R((x))$  denotes the ring of formal Laurent series with coefficients in  $R$ : we have  $R((x)) = R[[x]][x^{-1}]$ .

$(0 : \mathfrak{a})_M$  is the submodule of all  $m \in M$  annihilated by the ideal  $\mathfrak{a}$ . If  $\mathfrak{a} = (a)$ , we shall abbreviate  $(0 : \mathfrak{a})_M$  to  $(0 : a)_M$ .

$\vartheta$  denotes  $R$ -linear derivation on  $R((x))$  given by

$$\vartheta \sum a_n x^n = x \frac{d}{dx} \sum a_n x^n = \sum n a_n x^n.$$

$\text{Sp}_{\varphi}$  denotes the spectrum of the endomorphism  $\varphi : V \rightarrow V$  of vector space over  $C$ .

$\tau$  denotes a subset of  $C$  such that the natural map  $\tau \rightarrow C/\mathbb{Z}$  is bijective. In some cases, we shall assume that  $0 \in \tau$ .

$\mathbf{End}_R$  denotes the category of couples  $(V, A)$ , consisting of a finite  $R$ -module  $V$  and an  $R$ -linear endomorphism  $A : V \rightarrow V$ ; *arrows* from  $(V, A)$  to  $(V', A')$  are  $R$ -linear morphisms  $\varphi : V \rightarrow V'$  such that  $A'\varphi = \varphi A$ . See also [HdST23].

$\mathbf{End}_R^\circ$  denotes the full subcategory of  $\mathbf{End}_R$  whose objects are flat (and hence free)  $R$ -modules.

## 2. REGULAR-SINGULAR CONNECTIONS ON A PUNCTURED FORMAL DISK

**2.1. Definitions and properties.** We review in this subsection the definitions and main properties of regular-singular connections on the relative formal punctured disk  $\mathrm{Spec} R((x))$ . Our reference is [HdST23]. We notice that although in op. cit. the ring  $R$  is assumed to be complete, many results hold in more generality.

**Definition 2.1** (Connections on the punctured formal disk). The *category of connections* on the punctured formal disk over  $R$ , or on  $R((x))$  over  $R$ , or on  $R((x))/R$ , denoted  $\mathbf{MC}(R((x))/R)$ , has for

*objects* those couples  $(M, \nabla)$  consisting of a finite  $R((x))$ -module  $M$  and an  $R$ -linear endomorphism  $\nabla : M \rightarrow M$ , called *the derivation*, satisfying Leibniz's rule  $\nabla(fm) = \vartheta(f)m + f\nabla(m)$ , and the  
*arrows* from  $(M, \nabla)$  to  $(M', \nabla')$  are  $R((x))$ -linear morphisms  $\varphi : M \rightarrow M'$  such that  $\nabla'\varphi = \varphi\nabla$ .

The  $R$ -flat connections on  $R((x))/R$  enjoy the following remarkable property which is employed further ahead.

**Proposition 2.2** ([HdST23, Theorem 8.18]). *Let  $(M, \nabla)$  be a connection on  $R((x))$  over  $R$  such that  $M$  is  $R$ -flat. Then,  $M$  is a flat  $R((x))$ -module.*

**Definition 2.3** (Logarithmic connections). The category of *logarithmic connections*, denoted  $\mathbf{MC}_{\log}(R[[x]]/R)$ , has for

*objects* those couples  $(\mathcal{M}, \nabla)$  consisting of a finite  $R[[x]]$ -module and an  $R$ -linear endomorphism  $\nabla : \mathcal{M} \rightarrow \mathcal{M}$ , called *the derivation*, satisfying Leibniz's rule  $\nabla(fm) = \vartheta(f)m + f\nabla(m)$ , and  
*arrows* from  $(\mathcal{M}, \nabla)$  to  $(\mathcal{M}', \nabla')$  are  $R[[x]]$ -linear morphisms  $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$  such that  $\nabla'\varphi = \varphi\nabla$ .

The two categories  $\mathbf{MC}(R((x))/R)$  and  $\mathbf{MC}_{\log}(R[[x]]/R)$  are abelian categories and there is an evident  $R$ -linear functor

$$\gamma : \mathbf{MC}_{\log}(R[[x]]/R) \longrightarrow \mathbf{MC}(R((x))/R).$$

**Definition 2.4** (Regular-singular connections and models).

- (1) An object  $M \in \mathbf{MC}(R((x))/R)$  is said to be *regular-singular* if it is isomorphic to a certain  $\gamma(\mathcal{M})$  for some  $\mathcal{M} \in \mathbf{MC}_{\log}(R[[x]]/R)$ . The full subcategory of regular-singular connections will be denoted by  $\mathbf{MC}_{\mathrm{rs}}(R((x))/R)$ .
- (2) Given  $M$  in  $\mathbf{MC}_{\mathrm{rs}}(R((x))/R)$ , any object  $\mathcal{M} \in \mathbf{MC}_{\log}(R[[x]]/R)$  such that  $\gamma(\mathcal{M}) \simeq M$  is called a *logarithmic model* of  $M$ .
- (3) Let  $\mathbf{MC}_{\mathrm{rs}}^\circ(R((x))/R)$ , respectively  $\mathbf{MC}_{\log}^\circ(R[[x]]/R)$ , stand for the full subcategory of  $\mathbf{MC}_{\mathrm{rs}}(R((x))/R)$ , respectively  $\mathbf{MC}_{\log}(R[[x]]/R)$ , consisting of those objects  $(M, \nabla)$  for which  $M$  is a flat  $R((x))$ -module, respectively flat  $R[[x]]$ -module.

**Remark 2.5.** Let  $\mathcal{M} \in \mathbf{MC}_{\log}(R[[x]]/R)$  be a model of  $M$ . Since  $(0 : x)_{\mathcal{M}} \subset \mathcal{M}$  is preserved by the derivation, it is clear that  $M$  possesses a model  $\mathcal{M}'$  such that  $(0 : x)_{\mathcal{M}'} = 0$ .

**Example 2.6** (Euler connections). Let  $(V, A) \in \mathbf{End}_R$  be given. The logarithmic connection associated with the couple  $(V, A)$  is defined by the couple  $(R[[x]] \otimes_R V, D_A)$ , where

$$D_A(f \otimes v) = \vartheta(f) \otimes v + f \otimes Av.$$

This logarithmic connection is called the *Euler connection associated with  $(V, A)$* . Notation:  $\text{eul}_{R[[x]]}(V, A)$ .

The Euler connections yield a functor, denoted  $\text{eul}_{R[[x]]}$  or simply  $\text{eul}$  when no confusion may appear:

$$\text{eul} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\log}(R[[x]]/R).$$

It is straightforward to check that this is an  $R$ -linear, exact, and faithful tensor functor. Combining  $\text{eul}$  with  $\gamma$  we have a functor

$$\gamma\text{eul} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\text{rs}}(R((x))/R).$$

The main aim of this section is to show that this functor produces an equivalence when restricted to objects with *exponents* lying in  $\tau \subset C$  (Theorem 2.15). We first introduce the exponents.

Let  $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[[x]]/R)$ . The Leibniz rule implies that  $\nabla(x\mathcal{M}) \subset x\mathcal{M}$ . Hence, we obtain an  $R$ -linear endomorphism

$$(1) \quad \text{res}_{\nabla} : \mathcal{M}/(x) \longrightarrow \mathcal{M}/(x),$$

given by

$$(2) \quad \text{res}_{\nabla}(m + (x)) = \nabla(m) + (x).$$

Further, taking residue modulo  $\mathfrak{r}$  we have the map

$$(3) \quad \overline{\text{res}}_{\nabla} : \mathcal{M}/(\mathfrak{r}, x) \longrightarrow \mathcal{M}/(\mathfrak{r}, x).$$

**Definition 2.7** (Residue and exponents). Let  $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[[x]]/R)$ .

- (1) The  $R$ -linear map (1) is called the *residue* of  $\nabla$ .
- (2) The eigenvalues of  $\overline{\text{res}}_{\nabla}$  are called the (reduced) *exponents* of  $\nabla$ . The set of exponents will be denoted by  $\text{Exp}(\mathcal{M}, \nabla)$ ,  $\text{Exp}(\nabla)$  or  $\text{Exp}(\mathcal{M})$  if no confusion may appear.

The following result was obtained in [HdST23] for  $R$  being a complete local  $C$ -algebra, but the proof works indeed for any local  $C$ -algebra.

**Theorem 2.8** ([HdST23, Theorem 8.10]). *Let  $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[[x]]/R)$  be such that  $\mathcal{M}$  is a free  $R[[x]]$ -module. If  $\text{Exp}(\mathcal{M}) \subset \tau$ , then  $(\mathcal{M}, \nabla)$  is isomorphic to  $\text{eul}_{R[[x]]}(\mathcal{M}/(x), \text{res}_{\nabla})$ .*  $\square$

**2.2. Euler form for connections of  $\mathbf{MC}_{\text{rs}}^{\circ}(R((x))/R)$ .** *We now suppose, until the end of Section 2.2, that  $R$  is in addition Henselian.* With the preparation in the previous subsection, we show now that any regular-singular connection  $(M, \nabla)$ , where  $M$  is a flat  $R$ -module, is isomorphic to an Euler connection. This is an extension of [HdST23, Corollary 9.4] to the case where  $R$  is solely Henselian. The idea behind the proof is to show that logarithmic models with exponents in  $\tau$  exist in all generality (cf. Proposition 2.12) and that these models, when  $R$  is complete, are sufficient to characterize the Deligne-Manin models appearing in the central result [HdST23, Theorem 9.1] (this is the content of Theorem 2.9). Then, basic Commutative Algebra (cf. Lemma 2.13) allows us to find *free* logarithmic models for  $(M, \nabla)$  by using free logarithmic models of  $\widehat{R}((x)) \otimes M$ .

In order to present a clear argument, we require the following notations and terminology from [HdST23]. Given  $k \in \mathbb{N}$  and an object  $(\mathcal{M}, \nabla)$  of  $\mathbf{MC}_{\log}(R[[x]]/R)$ , or

of  $\mathbf{MC}_{\text{rs}}(R[[x]]/R)$ , we let  $(\mathcal{M}, \nabla)|_k$ , or  $\mathcal{M}|_k$  if no confusion is likely, stand for the object of  $\mathbf{MC}_{\text{log}}(C[[x]]/C)$ , respectively  $\mathbf{MC}_{\text{rs}}(C((x))/C)$ , obtained from the induced map  $\nabla : \mathcal{M}/\mathfrak{r}^{k+1} \rightarrow \mathcal{M}/\mathfrak{r}^{k+1}$ .

We begin by showing that the Deligne-Manin model constructed in [HdST23, Theorem 9.1] can be singled-out by a much simpler condition.

**Theorem 2.9.** *We assume that  $R$  is complete for the moment. Let  $(M, \nabla) \in \mathbf{MC}_{\text{rs}}(R((x))/R)$  posses a logarithmic model  $\mathcal{E} \in \mathbf{MC}_{\text{log}}(R[[x]]/R)$  enjoying the following properties:*

- (1) *All its exponents are on  $\tau$ .*
- (2) *We have  $(0 : x)_{\mathcal{E}} = 0$ .*

*Then  $\mathcal{E}$  is isomorphic to the Deligne-Manin logarithmic model described in [HdST23, Theorem 9.1]. In particular,*

- (a) *the  $C[[x]]$ -module  $\mathcal{E}|_k$  is free for any given  $k \in \mathbb{N}$ .*
- (b) *If, in addition  $M$  is  $R$ -flat, then  $\mathcal{E}$  is a **free**  $R[[x]]$ -module.*

*Proof.* Let  $\mathcal{M} \in \mathbf{MC}_{\text{log}}(R[[x]]/R)$  be a model of  $M$  as in [HdST23, Theorem 9.1]. Fix  $k \in \mathbb{N}$ ; we know that  $\mathcal{M}|_k$  is a free  $C[[x]]$ -module. Hence, we obtain an arrow  $\varphi_k : \mathcal{E}|_k \rightarrow \mathcal{M}|_k$  of  $\mathbf{MC}_{\text{log}}(C[[x]]/C)$  which fits into

$$(4) \quad \begin{array}{ccc} \mathcal{E}|_k & \xrightarrow{\varphi_k} & \mathcal{M}|_k \\ \text{natural} \downarrow & & \downarrow \text{natural} \\ M|_k & \xlongequal{\quad} & M|_k \end{array}$$

because of [HdST23, Proposition 4.4(3)]. Note that,  $\varphi_k$  is the unique arrow rendering diagram (4) commutative. By this reason and the fact that  $\mathcal{E}$  and  $\mathcal{M}$  are  $\mathfrak{r}$ -adically complete (see Exercise 8.2 and Theorem 8.7 in [Mat86]), we obtain an arrow of  $\mathbf{MC}_{\text{log}}(R[[x]]/R)$

$$\varphi : \mathcal{E} \longrightarrow \mathcal{M}$$

enjoying the following properties.

- (i) For each  $k \in \mathbb{N}$ , we have  $\varphi|_k = \varphi_k$ .
- (ii) The following diagram commutes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\varphi} & \mathcal{M} \\ \text{natural} \downarrow & & \downarrow \text{natural} \\ \varprojlim_k M|_k & \xlongequal{\quad} & \varprojlim_k M|_k. \end{array}$$

Let us now observe that  $\varphi$  is *surjective*. Indeed,  $\mathcal{E}|_0 \in \mathbf{MC}_{\text{log}}(C[[x]]/C)$  is a model of  $M|_0 \in \mathbf{MC}_{\text{rs}}(C((x))/C)$  having exponents in  $\tau$  so that,

$$\frac{\mathcal{E}|_0}{(0 : x)_{\mathcal{E}|_0}}$$

is, being a *quotient* of  $\mathcal{E}|_0$ , a model of  $M|_0$  with exponents in  $\tau$ . Therefore, the  $C[[x]]$ -linear mapping

$$\frac{\mathcal{E}|_0}{(0 : x)_{\mathcal{E}|_0}} \longrightarrow \mathcal{M}|_0$$

which is induced by  $\varphi_0$  is an isomorphism [HdST23, Proposition 4.4(3)]; consequently,  $\varphi|_0$  is surjective. Because  $R[[x]]$ ,  $\mathcal{E}$  and  $\mathcal{M}$  are  $\mathfrak{r}$ -adically complete, and because of [EGA, 0<sub>I</sub>, 7.1.14], we conclude that  $\varphi$  is also surjective.

We need to show that  $\varphi$  is also *injective*. It is tempting to argue with the completion  $\varprojlim_k M|_k$ , but this is a complicated object and we proceed as at the end of the proof of [HdST23, Theorem 9.1]: *We show that*

$$\varphi[x^{-1}] : \mathcal{E}[x^{-1}] \longrightarrow \mathcal{M}[x^{-1}]$$

is an isomorphism. Once this is guaranteed, injectivity of  $\varphi$  is a consequence of the fact that  $\mathcal{E} \rightarrow \mathcal{E}[x^{-1}]$  is injective (by construction) and that

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\varphi} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{E}[x^{-1}] & \xrightarrow[\varphi[x^{-1}]]{\sim} & \mathcal{M}[x^{-1}] \end{array}$$

commutes.

That  $\varphi[x^{-1}]$  is an isomorphism is verified by the ensuing arguments. We start by observing that  $\varphi[x^{-1}]|_k$  is an isomorphism for all  $k$ , so that  $\varphi[x^{-1}]$  is an isomorphism in a *neighborhood of the closed fiber* of  $\mathrm{Spec} R((x)) \rightarrow \mathrm{Spec} R$ . This implies that  $N := \mathrm{Ker} \varphi[x^{-1}]$  and  $Q := \mathrm{Coker} \varphi[x^{-1}]$  vanish on an open neighborhood of the aforementioned closed fiber. Now,  $N$  and  $Q$  are objects of  $\mathbf{MC}(R((x))/R)$ , so that for each  $\mathfrak{p} \in \mathrm{Spec} R$ , the fibers  $N \otimes_R \mathbf{k}(\mathfrak{p})$  and  $Q \otimes_R \mathbf{k}(\mathfrak{p})$  are flat as  $R((x)) \otimes_R \mathbf{k}(\mathfrak{p})$ -modules [HdST23, Theorem 8.19]. A simple argument in Commutative Algebra [HdST23, Lemma 9.2] now shows that  $N = Q = 0$ , thus assuring that  $\varphi[x^{-1}]$  is an isomorphism.

To end, if  $M$  is  $R$ -flat, Corollary 9.4 of [HdST23] is enough to conclude the prof of item (b).  $\square$

In view of Theorem 2.9, it becomes important, even when  $R$  is solely Henselian, to construct logarithmic models with exponents in  $\tau$ . The construction follows the classical method of using ‘‘Jordan subspaces’’ (generalized eigenspaces) to adjust the exponents [Was76, Section 17.4] but, in the present case it is necessary to have such a decomposition for  $R$ -linear endomorphisms. This is a consequence of the following lemma which was mentioned in Remarks 8.15(a) of [HdST23].

**Lemma 2.10** (‘‘Jordan decomposition over  $R$ ’’). *Let  $(V, \varphi) \in \mathbf{End}_R$  and denote by  $\overline{\varphi} : \overline{V} \rightarrow \overline{V}$  the reduction of  $\varphi : V \rightarrow V$  modulo  $\mathfrak{r}$ . Let  $\{\varrho_1, \dots, \varrho_r\}$  be the spectrum of  $\overline{\varphi}$  and write*

$$\overline{V} = \bigoplus_{i=1}^r \mathrm{Ker}(\overline{\varphi} - \varrho_i)^{\mu_i}$$

Then, there exists a direct sum

$$V = V_1 \oplus \dots \oplus V_r,$$

where  $V_i$  is  $\varphi$ -invariant  $R$ -submodule of  $V$ , such that its reduction modulo  $\mathfrak{r}$  is  $\mathrm{Ker}(\overline{\varphi} - \varrho_i)^{\mu_i}$ , for each  $1 \leq i \leq r$ .

*Proof.* Let  $R^n \rightarrow V$  be a surjection inducing an isomorphism  $C^n \rightarrow V/\mathfrak{r}$ . Then  $\varphi$  lifts to  $\tilde{\varphi} : R^n \rightarrow R^n$  and the residue of the characteristic polynomial of  $\tilde{\varphi}$  equals the characteristic polynomial of  $\overline{\varphi}$ :

$$\overline{P_{\tilde{\varphi}}(T)} = P_{\overline{\varphi}}(T).$$

As  $R$  is Henselian, the factorization

$$P_{\overline{\varphi}}(T) = \prod_{i=1}^r (T - \varrho_i)^{\mu_i}$$

lifts to a factorization

$$P_{\bar{\varphi}}(T) = \prod_{i=1}^r g_i(T),$$

where, for any  $1 \leq i \leq r$ , the polynomials  $g_i$  and  $\hat{g}_i := \prod_{j \neq i} g_j$  are *strictly coprime*, i.e.

$$(\star) \quad R[T] \cdot g_i + R[T] \cdot \hat{g}_i = (1),$$

cf. [Mi80, Ch. I, Section 4, p.32]. Let  $V_i = \text{Ker } g_i(\varphi)$ ; then  $V_i$  are  $\varphi$ -invariant  $R$ -submodules of  $V$ . From  $(\star)$  and the fact that  $P_{\bar{\varphi}}(\varphi)$  vanishes identically on  $V$ , it is easy to see that

$$\begin{aligned} V &= \text{Ker } P_{\bar{\varphi}}(\varphi) \\ &= V_1 \oplus \cdots \oplus V_r. \end{aligned}$$

Now, the composition  $V_i \rightarrow V \rightarrow \bar{V}$  sends  $V_i$  to  $\text{Ker}(\bar{\varphi} - \varrho_i)^{\mu_i}$  and has kernel  $\mathfrak{r}V \cap V_i = \mathfrak{r}V_i$ . It is easily verified that  $V_i \rightarrow \text{Ker}(\bar{\varphi} - \varrho_i)^{\mu_i}$  must be surjective as well, so that the last claim is verified.  $\square$

**Example 2.11.** If we drop the assumption that  $R$  be Henselian, the above result certainly fails. Suppose for example that  $R = \{a/b : a, b \in C[t], b(0) \neq 0\}$ . Define  $\varphi = \begin{pmatrix} 0 & 1+t \\ 1 & 0 \end{pmatrix} : R^2 \rightarrow R^2$ . Then  $\bar{\varphi} : C^2 \rightarrow C^2$  acts by multiplication by 1 on  $C(\bar{e}_1 + \bar{e}_2)$  and by multiplication by  $-1$  on  $C(\bar{e}_1 - \bar{e}_1)$ . On the other hand, it is not possible to decompose  $R^2$  into a direct sum of submodules of rank one.

We are now ready to show the existence of logarithmic models having exponents on  $\tau$ .

**Proposition 2.12** (Shearing). *Let  $(M, \nabla)$  be the regular-singular connection on  $R((x))/R$ . Then there exists a logarithmic model  $\mathcal{M} \in \mathbf{MC}_{\log}(R[[x]]/R)$  of  $(M, \nabla)$  such that  $\text{Exp}(\mathcal{M}) \subset \tau$  and  $(0 : x)_{\mathcal{M}} = 0$ .*

*Proof.* Let  $\mathcal{E}$  be an arbitrary logarithmic model of  $(M, \nabla)$  such that  $(0 : x)_{\mathcal{E}} = 0$  — and hence  $\mathcal{E} \subset M$ . We shall proceed by reverse induction on the non-negative integer

$$b(\mathcal{E}) := \# \text{Exp}(\mathcal{E}) \setminus \tau.$$

Let us assume that  $b(\mathcal{E}) > 0$  and let  $\varrho \in \text{Exp}(\mathcal{E}) \setminus \tau$ . We shall construct a logarithmic model  $\mathcal{E}'$  such that  $b(\mathcal{E}') < b(\mathcal{E})$  and  $(0 : x)_{\mathcal{E}'} = 0$ . Let  $V := \mathcal{E}/x\mathcal{E}$ , this is an  $R$ -module, and consider its decomposition into “generalized eigenspaces”, with respect to the residue morphism  $\text{res}_{\mathcal{E}}$ ,

$$V = \bigoplus_{\sigma \in \text{Exp}(\mathcal{E})} V_{\sigma}$$

as in Lemma 2.10. (Each  $V_{\sigma}$  is an  $R$ -module.) Note that there exists  $\mu \in \mathbb{N}$  such that, for each  $\sigma$  and  $v \in V_{\sigma}$ , we have

$$(\text{res}_{\mathcal{E}} - \sigma)^{\mu} v \in \mathfrak{r}V_{\sigma}.$$

In particular

$$\prod_{\sigma \in \text{Exp}(\mathcal{E})} (\text{res}_{\mathcal{E}} - \sigma)^{\mu}(V) \subset \mathfrak{r}V.$$

The reduction map  $\mathcal{E} \rightarrow V$  shall be denoted by  $e \mapsto \tilde{e}$ . Then, if  $\mathcal{E}_{\sigma} := \{e \in \mathcal{E} : \tilde{e} \in V_{\sigma}\}$ , we have

$$\mathcal{E} = \sum_{\sigma} \mathcal{E}_{\sigma}.$$



Each  $\mathcal{E}_\sigma$  is stable under  $\nabla$  because  $\widetilde{\nabla}e = \text{res}(\tilde{e})$ . In addition, each  $\mathcal{E}_\sigma$  is an  $R[[x]]$ -submodule of  $\mathcal{E}$  and

$$(\star) \quad (\nabla - \sigma)^\mu(\mathcal{E}_\sigma) \subset (x, \mathfrak{r})\mathcal{E}_\sigma.$$

Let  $k \in \mathbb{Z}$  be such that  $\varrho + k \in \tau$ . Define  $\mathcal{E}'_\varrho = x^k \mathcal{E}_\varrho$  and

$$\mathcal{E}' := \mathcal{E}'_\varrho + \sum_{\sigma \neq \varrho} \mathcal{E}_\sigma,$$

which is an  $R[[x]]$ -submodule of  $M$ , stable under the action of  $\nabla$ . We now choose  $e \in \mathcal{E}_\varrho$  and let  $e' := x^k e \in \mathcal{E}'_\varrho$ . By a direct verification, we know that

$$[\nabla - (\varrho + k)]^\mu(e') = x^k [\nabla - \varrho]^\mu(e).$$

Because  $[\nabla - \varrho]^\mu(e) \in (x, \mathfrak{r})\mathcal{E}_\varrho$ , we then have

$$(\dagger) \quad [\nabla - (\varrho + k)]^\mu(e') \in x^k \cdot (x, \mathfrak{r})\mathcal{E}_\varrho \subset (x, \mathfrak{r})\mathcal{E}'.$$

Consequently, from  $(\star)$  and  $(\dagger)$ , the  $R$ -linear map

$$[\nabla - (\varrho + k)]^\mu \prod_{\sigma \neq \varrho} (\nabla - \sigma)^\mu$$

sends  $\mathcal{E}'$  into  $(x, \mathfrak{r})\mathcal{E}'$ . Letting  $V' = \mathcal{E}'/x\mathcal{E}'$ , we conclude that

$$[\text{res}_{\mathcal{E}'} - (\varrho + k)]^\mu \prod_{\sigma \neq \varrho} (\text{res}_{\mathcal{E}'} - \sigma)^\mu(V') \subset \mathfrak{r}V',$$

showing that  $\text{Exp}(\mathcal{E}') \subset (\text{Exp}(\mathcal{E}) \setminus \{\varrho\}) \cup \{\varrho + k\}$ , which in particular proves that  $b(\mathcal{E}') < b(\mathcal{E})$ . Obviously,  $\mathcal{E}'$ , being contained in  $M$  is such that  $(0 : x)_{\mathcal{E}'} = 0$ . □

We now require a result in Commutative Algebra.

**Lemma 2.13.** *The following claims are true.*

- (i) *The homomorphisms  $R[[x]] \rightarrow \widehat{R}[[x]]$  and  $R((x)) \rightarrow \widehat{R}((x))$  are faithfully flat.*
- (ii) *A finite  $R((x))$ -module  $E$  is flat if and only if  $\widehat{R}((x)) \otimes_{R((x))} E$  is  $\widehat{R}((x))$ -flat. A finite  $R[[x]]$ -module  $\mathcal{E}$  is free if and only if  $\widehat{R}[[x]] \otimes_{R[[x]]} \mathcal{E}$  is  $\widehat{R}[[x]]$ -flat.*

*Proof.* (i) Firstly,  $\widehat{R}[[x]]$  is  $(\mathfrak{r}, x)$ -adically complete [Mat86, Exercise 8.6]. Thus, we can view  $\widehat{R}[[x]]$  as the  $(\mathfrak{r}, x)$ -adic completion of  $R[[x]]$ . As  $R[[x]]$  is a noetherian local ring, we conclude that  $\widehat{R}[[x]]$  is faithfully flat over  $R[[x]]$  [Mat86, Theorem 8.14]. The fact that  $R((x)) \rightarrow \widehat{R}((x))$  is faithfully flat is a consequence of the fact that this mapping is obtained from  $R[[x]] \rightarrow \widehat{R}[[x]]$  by inverting  $x$ .

(ii) This is [Mat86, Exercise 7.1] together with the fact that a finite module over a local noetherian ring is flat if and only if it is free [Mat86, Theorem 7.10]. □

**Theorem 2.14.** *Let  $(M, \nabla)$  be a regular-singular connection of  $R((x))/R$ , with  $M$  being flat as a  $R((x))$ -module. (That is,  $(M, \nabla)$  is an object of  $\mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$ .) Then,  $M$  possesses a logarithmic model  $\mathcal{M}$  which, as an  $R[[x]]$ -module, is **free**, and in particular  $M$  is a free  $R((x))$ -module. Moreover, the model  $\mathcal{M}$  can be chosen of the form  $\text{eul}(V, A)$ , with  $\text{Exp}(\mathcal{M}) \subset \tau$ .*

*Proof.* Let  $\mathcal{M} \in \mathbf{MC}_{\log}(R[[x]]/R)$  be a model of  $M$  as in Proposition 2.12. Then,  $\widehat{R}[[x]] \otimes_{R[[x]]} \mathcal{M} \in \mathbf{MC}_{\log}(\widehat{R}[[x]]/R)$  is a model of  $\widehat{R}((x)) \otimes_{R((x))} M$  as in Theorem 2.9 and hence,  $\widehat{R}((x)) \otimes_{R((x))} M$  being flat over  $\widehat{R}$ , it must be that  $\widehat{R}[[x]] \otimes_{R[[x]]} \mathcal{M}$  is free over  $\widehat{R}[[x]]$ . It then follows that

$\mathcal{M}$  is  $R[[x]]$ -free, by Lemma 2.13-(ii). Consequently,  $M$  is free over  $R((x))$ . To verify the last claim, it suffices to employ Theorem 2.8.  $\square$

**Theorem 2.15.** *The functor*

$$\gamma\text{eul}_{R[[x]]} : \mathbf{End}_R^\circ \longrightarrow \mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$$

*is faithful and essentially surjective. This functor is not full. Assume that  $0 \in \tau$ ; then restriction of  $\gamma\text{eul}_{R[[x]]}$  to the full subcategory of all objects  $(V, A)$  such that the spectrum of  $A : V/\mathfrak{t} \rightarrow V/\mathfrak{t}$  is contained in  $\tau$ , is indeed full.*

*Proof.* Essential surjectivity is already verified by Theorem 2.14, while faithfulness is obvious. We then concentrate on the verification of the last claim. Let  $(V, A)$  and  $(W, B)$  be objects of  $\mathbf{End}_R^\circ$  and suppose that the eigenvalues of the  $C$ -linear endomorphisms  $A_0 : V/\mathfrak{t} \rightarrow V/\mathfrak{t}$  and  $B_0 : W/\mathfrak{t} \rightarrow W/\mathfrak{t}$  associated respectively to  $A$  and  $B$  lie in  $\tau$ . On  $H = \text{Hom}_R(V, W)$ , consider the endomorphism  $T : h \mapsto hA - Bh$ ; we then obtain an object  $(H, T)$  of  $\mathbf{End}_R^\circ$ . The spectrum of the  $C$ -linear endomorphism  $T_0 : H/\mathfrak{t} \rightarrow H/\mathfrak{t}$  is built up from the differences of eigenvalues of  $A_0$  and  $B_0$  [Was76, II, Problem 4.1], so that  $\text{Sp}_{T_0} \cap \mathbb{Z} \subset \{0\}$ . Consequently, for each  $k \in \mathbb{N}$ , the spectrum of the  $C$ -linear endomorphism  $T_k : H/\mathfrak{t}^{k+1} \rightarrow H/\mathfrak{t}^{k+1}$  contains no integers except perhaps 0. This is because  $\text{Sp}_{T_k} = \text{Sp}_{T_0}$  [HdST23, Prp. 8.11]. It is a simple matter to see that  $\text{Hom}_{\mathbf{MC}}(\gamma\text{eul}(V, A), \gamma\text{eul}(W, B))$  corresponds to the horizontal elements of  $\gamma\text{eul}(H, T)$ . After picking a basis of  $H$ , a horizontal section of  $\gamma\text{eul}(H, T)$  amounts to a vector  $\mathbf{h} \in R((x))^r$  such that

$$\vartheta\mathbf{h} = -T\mathbf{h}.$$

Writing  $\mathbf{h} = \sum_{i \geq i_0} \mathbf{h}_i x^i$ , we see that

$$T\mathbf{h}_i = -i\mathbf{h}_i.$$

Now, let  $i \neq 0$ . Then the image of  $\mathbf{h}_i$  in  $R_k^{\oplus r}$  must be zero, since  $i \notin \text{Sp}_{T_k}$ . Hence,  $\mathbf{h}_i = 0$  [Mat86, Theorem 8.10(i)] except perhaps for  $i = 0$ . This proves that any arrow

$$h : \gamma\text{eul}(V, A) \longrightarrow \gamma\text{eul}(W, B)$$

comes from an arrow  $V \rightarrow W$ .  $\square$

**2.3. The case where  $R$  is a DVR.** Previously, we described the objects of  $\mathbf{MC}_{\text{rs}}^\circ$ , but we still have no conclusions in general. So let us, in this section, add to the assumption that  $R$  is Henselian the hypothesis

$$R \text{ is a DVR and } \mathfrak{t} = Rt.$$

In this setting, we shall show that the functor  $\gamma\text{eul}_{R[[x]]} : \mathbf{End}_R \rightarrow \mathbf{MC}_{\text{rs}}(R((x))/R)$  is *essentially surjective*. See Corollary 2.17. Part of this result was already achieved by Theorem 2.15. It is a technique from [DH18], expressed in Proposition 2.16 below, which allows the deduction of the general case.

**Proposition 2.16.** *Each object of  $\mathbf{MC}_{\text{rs}}(R((x))/R)$  is a quotient of a certain  $(E, \nabla) \in \mathbf{MC}_{\text{rs}}(R((x))/R)$  such that  $E$  is a free  $R((x))$ -module.*

*Proof.* The proof is almost identical to that of [DH18, Proposition 5.2.2], but some care has to be taken to assure that the connections constructed are regular-singular.

Let  $(M, \nabla) \in \mathbf{MC}_{\text{rs}}(R((x))/R)$  be given. The reader should recall that in view of Proposition 2.2, a necessary and sufficient condition for  $N \in \mathbf{MC}_{\text{rs}}(R((x))/R)$  to belong to  $\mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$  is that  $(0 : t)_N = 0$ .

Let us introduce, for a given finite  $R((x))$ -module  $W$ , the submodule

$$\begin{aligned} W_{\text{tors}} &= \bigcup_{k=1}^{\infty} (0 : t^k)_W \\ &= \{w \in W : \text{some power of } t \text{ annihilates } w\}. \end{aligned}$$

Noetherianity assures that  $W_{\text{tors}} = (0 : t^\ell)_W$  for some  $\ell$  and we define

$$\begin{aligned} r(W) &= \min\{k \in \mathbb{N} : t^k W_{\text{tors}} = 0\} \\ &= \min\{k \in \mathbb{N} : W_{\text{tors}} = (0 : t^k)_W\}. \end{aligned}$$

The proposition is to be proved by induction on  $r(M)$ . (Note that for each  $k$ , the submodule  $(0 : t^k)_M$  is stable under  $\nabla$ .) If  $r(M) = 0$ , then  $M_{\text{tors}} = 0$  and there is nothing to be done. Assume  $r(M) = 1$ , so that  $(tM)_{\text{tors}} = 0$ . Let  $q : M \rightarrow Q$  be the quotient by  $tM$ ; since  $Q$  is annihilated by  $t$ , this is an object of  $\mathbf{MC}_{\text{rs}}(C((x))/C)$  and as such has the form  $\gamma\text{eul}(V, A)$ , where  $V$  is a  $C$ -vector space [HdST23, Cor. 4.3]. This connection is certainly a quotient of the Euler connection

$$\tilde{Q} := \gamma\text{eul}(R \otimes_C V, \text{id}_R \otimes A),$$

which is an object of  $\mathbf{MC}_{\text{rs}}(R((x))/R)$ . We then have a diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & tM & \longrightarrow & M & \xrightarrow{q} & Q \longrightarrow 0 \\ & & \sim \uparrow & & \uparrow & \square & \uparrow \\ 0 & \longrightarrow & tM & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{Q} \longrightarrow 0, \end{array}$$

where the rightmost square is cartesian and  $\tilde{M} \rightarrow M$  is in fact surjective. Since  $(tM)_{\text{tors}} = \tilde{Q}_{\text{tors}} = 0$ , we have  $\tilde{M}_{\text{tors}} = 0$ . Since  $\tilde{M}$  is a subobject of  $\tilde{Q} \oplus M$ , we can appeal to [HdST23, Proposition 8.3] to assure that it is regular-singular. In conclusion,  $\tilde{M} \in \mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$ .

Let us now assume that  $r(M) > 1$ . Let  $N = (0 : t)_M$  and observe that  $r(N) = 1$ . Denote by  $q : M \rightarrow Q$  the quotient by  $N$ . It then follows that  $t^{r(M)-1}Q_{\text{tors}} = 0$ , so that  $r(Q) \leq r(M) - 1$ . By induction, there exists  $\tilde{Q} \in \mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$  and a surjection  $\tilde{Q} \rightarrow Q$ . We arrive at a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{q} & Q \longrightarrow 0 \\ & & \sim \uparrow & & \uparrow & \square & \uparrow \\ 0 & \longrightarrow & N & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{Q} \longrightarrow 0, \end{array}$$

where the rightmost square is cartesian so that  $\tilde{M} \rightarrow M$  is surjective. Since  $\tilde{Q}_{\text{tors}} = 0$ , we conclude that  $\tilde{M}_{\text{tors}} = N$ , so that  $r(\tilde{M}) = 1$ . We can therefore find  $\tilde{M}_1 \in \mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$  and a surjection  $\tilde{M}_1 \rightarrow \tilde{M}$  and consequently a surjection  $\tilde{M}_1 \rightarrow M$ .  $\square$

**Corollary 2.17.** *The functor  $\gamma\text{eul}_{R[[x]]} : \mathbf{End}_R \rightarrow \mathbf{MC}_{\text{rs}}(R((x))/R)$  is essentially surjective.*

*Proof.* We assume that  $0 \in \tau$ . Let  $M \in \mathbf{MC}_{\text{rs}}(R((x))/R)$  be given; because of Proposition 2.16, we can find an exact sequence in  $\mathbf{MC}_{\text{rs}}(R((x))/R)$ :

$$E \xrightarrow{\Phi} F \longrightarrow M \longrightarrow 0$$

where  $E$  and  $F$  belong to  $\mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$ . According to Theorem 2.15, we can assume that

$$E = \gamma\text{eul}_{R[[x]]}(V, A) \quad \text{and} \quad F = \gamma\text{eul}_{R[[x]]}(W, B),$$

where  $(V, A)$  and  $(W, B)$  belong to  $\mathbf{End}_R^\circ$  and the spectra of

$$V/(t) \xrightarrow{A} V/(t) \quad \text{and} \quad W/(t) \xrightarrow{B} W/(t)$$

are all contained in  $\tau$ . In this case,  $\Phi = \gamma_{\text{eul}_{R[[x]]}}(\varphi : V \rightarrow W)$ , again by Theorem 2.15, and hence  $M$  is isomorphic to  $\gamma_{\text{eul}_{R[[x]]}}(\text{Coker } \varphi)$ .  $\square$

### 3. STRUCTURE OF $\mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$

Our aim in this section is to relate  $\mathbf{End}_R$  and  $\mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$  and obtain the equivalent of Corollary 2.17 in this setting. Our strategy is different from the one in the previous section. Instead of using the shearing technique, we rely on Popescu's approximation theorem to descend from  $\widehat{R}$  to  $R$ .

We fix a choice of local coordinates of  $\mathbb{P}_R^1$  as follows: write  $\mathbb{P}_R^1$  as the union of two affine lines  $\mathbb{A}_0$  and  $\mathbb{A}_\infty$ , where  $\mathbb{A}_0 = \text{Spec}(R[x])$  and  $\mathbb{A}_\infty = \text{Spec}(R[y])$ , with the transition function on their intersection  $R[x^\pm] = R[y^\pm]$  being  $y = x^{-1}$ .

By the equality  $y = x^{-1}$  we have

$$x \frac{d}{dx} = -y \frac{d}{dy},$$

and therefore  $\vartheta : R[x^\pm] \rightarrow R[x^\pm]$  can be extended canonically to a global section, still denoted by  $\vartheta$ , of the tangent sheaf of  $\mathbb{P}_R^1$ .

**Definition 3.1** (Connection on the punctured affine line). The category of  $R$ -connections on  $R[x^\pm]$ , or of connections on  $R[x^\pm]/R$ , or on the punctured affine line  $\mathbb{P}_R^1 \setminus \{0, \infty\}$ , etc, denoted  $\mathbf{MC}(R[x^\pm]/R)$ , has for

*objects* those couples  $(M, \nabla)$  consisting of a  $R[x^\pm]$ -module of finite presentation and an  $R$ -linear endomorphism  $\nabla : M \rightarrow M$  satisfying Leibniz's rule

$$\nabla(fm) = \vartheta(f)m + f\nabla(m);$$

*arrows* between  $(M, \nabla)$  and  $(M', \nabla')$  are just  $R[x^\pm]$ -linear maps  $\varphi : M \rightarrow M'$  satisfying  $\nabla'\varphi = \varphi\nabla$ .

It is well-known that for a connection  $(M, \nabla)$  on  $R[x^\pm]/R$ , a necessary and sufficient condition for  $M$  to be  $R[x^\pm]$ -flat is that it be  $R$ -flat, cf., e.g. [dS09, p.82] or [DH18, Proposition 5.1.1]. (We profit to note that in the proof of Proposition 5.1.1 in [DH18], we need to employ the ‘‘fiber-by-fiber flatness criterion’’ [EGA, IV<sub>3</sub>, 11.3.10] and not the ‘‘local flatness criterion.’’) Moreover, although these references are written in the context where  $R$  is a DVR, the idea of proof applies in more generality since it is a consequence of the ‘‘fiber-by-fiber flatness criterion’’ [EGA, IV<sub>3</sub>, 11.3.10] and the well-documented case of a base field of characteristic zero. See [HdST23, Remark 8.20] for more details and references.

**Definition 3.2** (Logarithmic connections on the punctured affine line). The category of *logarithmic connections* on the punctured affine line, denoted  $\mathbf{MC}_{\log}(\mathbb{P}_R^1/R)$ , has for

*objects* those couples  $(\mathcal{M}, \nabla)$  consisting of a coherent  $\mathcal{O}_{\mathbb{P}_R^1}$ -module and an  $R$ -linear endomorphism  $\nabla : \mathcal{M} \rightarrow \mathcal{M}$  satisfying Leibniz's rule  $\nabla(fm) = \vartheta(f)m + f\nabla(m)$  on all open subsets; and

*arrows* between  $(\mathcal{M}, \nabla)$  and  $(\mathcal{M}', \nabla')$  are  $\mathcal{O}_{\mathbb{P}_R^1}$ -linear maps  $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$  satisfying  $\nabla'\varphi = \varphi\nabla$ .

We let

$$\gamma : \mathbf{MC}_{\log}(\mathbb{P}_R^1/R) \longrightarrow \mathbf{MC}(R[x^\pm]/R)$$

be the natural restriction functor.

**Definition 3.3** (Regular-singular connections on the punctured affine line).

- (1) A connection  $(M, \nabla)$  in  $\mathbf{MC}(R[x^\pm]/R)$  is *regular-singular* if  $\gamma(\mathcal{M}) \simeq M$  for a certain  $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}_R^1/R)$ ; in this case, any such  $\mathcal{M}$  is a *logarithmic model* of  $M$ .
- (2) The full subcategory of  $\mathbf{MC}(R[x^\pm]/R)$  having regular-singular connections as objects is denoted by  $\mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$ .
- (3) The full subcategory of  $\mathbf{MC}(R[x^\pm]/R)$  having as objects those connections  $(M, \nabla)$  with  $M$  being a flat  $R[x^\pm]$ -module is denoted by  $\mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$ .

The prime example of regular-singular connections is described now:

**Example 3.4** (Euler connections). For an object  $(V, A) \in \mathbf{End}_R$ , we set

$$\text{eul}_{\mathbb{P}_R^1}(V, A) := (\mathcal{O}_{\mathbb{P}_R^1} \otimes_R V, D_A),$$

where  $D_A : \mathcal{O}_{\mathbb{P}_R^1} \otimes_R V \rightarrow \mathcal{O}_{\mathbb{P}_R^1} \otimes_R V$  is  $R$ -linear and defined by

$$D_A(f \otimes m) = \vartheta(f) \otimes v + f \otimes Av$$

on any open subsets of  $\mathbb{P}_R^1$ .

Thus we have a functor  $\text{eul}_{\mathbb{P}_R^1} : \mathbf{End}_R \rightarrow \mathbf{MC}_{\log}(\mathbb{P}_R^1/R)$  and, composing it with  $\gamma$ , another the functor

$$\gamma \text{eul}_{\mathbb{P}_R^1} : \mathbf{End}_R \rightarrow \mathbf{MC}_{\text{rs}}(R[x^\pm]/R).$$

For the next theorem, we shall require the notion of  $G$ -rings [Mat86, Section 32]. A field is a  $G$ -ring as is a discrete valuation ring of characteristic zero. Other relevant  $G$ -rings are noetherian complete local rings [Mat86, Theorem 32.3], rings of finite type over  $G$ -rings [Mat70, 33.G] and Henselizations of local  $G$ -rings (use [Mat86, Theorem 32.1] and [Mat86, Theorem 32.2]). That this concept is necessary here comes from its role in the Popescu approximation theorem.

**Theorem 3.5.** *We assume that  $R$  is Henselian in all that follows.*

(i) *Suppose that  $R$  is a  $G$ -ring. Then the functor*

$$\gamma \text{eul}_{\mathbb{P}_R^1} : \mathbf{End}_R^\circ \rightarrow \mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$$

*is faithful and essentially surjective.*

(ii) *Suppose that  $R$  is a discrete valuation ring. (In which case  $R$  is also a  $G$ -ring.) Then the functor*

$$\gamma \text{eul}_{\mathbb{P}_R^1} : \mathbf{End}_R \rightarrow \mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$$

*is faithful and essentially surjective.*

*Proof.* Faithfulness is obvious, in any case, and we proceed to verify essential surjectivity. We shall deal with cases (i) and (ii) at the same time. The idea is to first base change to  $\widehat{R}$ , use the known results from [HdST23], and then descend back to  $R$  by means of Popescu's theorem.

The map  $R \rightarrow \widehat{R}$  is regular, by assumption in case (i), and because  $R$  is of characteristic zero in case (ii). According to Popescu (see [Pop86, Theorem 2.5] or [Spi99, Theorem 1.1]),

$$\widehat{R} = \varinjlim_{\lambda \in L} S_\lambda$$

where each  $S_\lambda$  is a *smooth*  $R$ -algebra.

Let  $(M, \nabla) \in \mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$ . For each  $\lambda \in L$ , we let  $(M_\lambda, \nabla_\lambda)$  stand for the object of  $\mathbf{MC}(S_\lambda[x^\pm]/S_\lambda)$  defined, in an evident manner, by employing the functor  $S_\lambda \otimes_R (-)$ . We define  $(\widehat{M}, \widehat{\nabla})$  in similar fashion.

Let us for a moment assume that  $(M, \nabla) \in \mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$  to treat case (i). Let  $\mathfrak{A} : \mathfrak{Y} \rightarrow \mathfrak{Y}$  be an endomorphism of a certain finite  $\widehat{R}$ -module  $\mathfrak{Y}$  such that there exists an *isomorphism*

$$\begin{aligned} (\widehat{M}, \widehat{\nabla}) &\xrightarrow{\mathfrak{f}} (\widehat{R}[x^\pm] \otimes_{\widehat{R}} \mathfrak{Y}, D_{\mathfrak{A}}) \\ &= \gamma_{\text{eul}_{\mathbb{P}_R^1}}(\mathfrak{Y}, \mathfrak{A}) \end{aligned}$$

in  $\mathbf{MC}(\widehat{R}[x^\pm]/\widehat{R})$ . The existence of this arrow is a consequence of [HdST23, Theorem 10.1] and Theorem 2.15. Clearly,  $(\mathfrak{Y}, \mathfrak{A}) \in \mathbf{End}_R^\circ$  in this case.

If we now drop the assumption that  $(M, \nabla) \in \mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$ , but decree that  $R$  is a DVR in order to work on case (ii), then [HdST23, Theorem 10.1], in conjunction with Corollary 2.17, ensure the existence of the isomorphism  $\mathfrak{f}$  as before. Granted the existence of  $\mathfrak{f}$ , the hypothesis in (i) and (ii) now have little bearing on what follows.

There exists  $\alpha$  such that  $\mathfrak{A} : \mathfrak{Y} \rightarrow \mathfrak{Y}$  is of the form

$$\text{id}_{\widehat{R}} \otimes_{S_\alpha} A_\alpha : \widehat{R} \otimes_{S_\alpha} V_\alpha \longrightarrow \widehat{R} \otimes_{S_\alpha} V_\alpha$$

where  $A_\alpha$  is an  $S_\alpha$ -linear endomorphism of the finite  $S_\alpha$ -module  $V_\alpha$ , see [EGA, IV<sub>3</sub>, 8.5.2(i)-(ii), p.20]. Given  $\lambda \geq \alpha$ , let  $A_\lambda : V_\lambda \rightarrow V_\lambda$  be the base-changed endomorphism

$$\text{id} \otimes A_\alpha : S_\lambda \otimes_{S_\alpha} V_\alpha \longrightarrow S_\lambda \otimes_{S_\alpha} V_\alpha.$$

This allows us to define objects

$$(S_\lambda[x^\pm] \otimes_{S_\lambda} V_\lambda, D_{A_\lambda})$$

from  $\mathbf{MC}(S_\lambda[x^\pm]/S_\lambda)$ , for all  $\lambda \geq \alpha$ , along the lines of Example 3.4.

There exists  $\beta \geq \alpha$  such that  $\mathfrak{f}$  is obtained from a certain mapping of  $S_\beta[x^\pm]$ -modules

$$f_\beta : M_\beta \longrightarrow S_\beta[x^\pm] \otimes_{S_\alpha} V_\alpha$$

by base change  $S_\beta \rightarrow \widehat{R}$ , see [EGA, IV<sub>3</sub>, 8.5.2.1, p.20]. Note, in addition, that  $f_\beta$  can be taken to be an *isomorphism* of  $S_\beta[x^\pm]$ -modules. Let  $f_\lambda$  be the base change of  $f_\beta$  for  $\lambda \geq \beta$ .

Let now  $\{m_i\} \in M$  be a set of  $R[x^\pm]$ -module generators for  $M$  and write  $m_i^\lambda$  for the image of  $m_i$  in  $M_\lambda$  via the natural arrow  $M \rightarrow M_\lambda$ . Consider, for each  $\lambda \geq \beta$ , the elements

$$\delta_i^\lambda := f_\lambda(\nabla_\lambda(m_i^\lambda)) - D_{A_\lambda}(f_\lambda(m_i^\lambda))$$

of  $S_\lambda[x^\pm] \otimes_{S_\alpha} V_\alpha$ . We then conclude that for some  $\gamma \geq \beta$ , the elements  $\delta_i^\gamma$  are all zero, and hence the arrow

$$f_\gamma : M_\gamma \longrightarrow S_\gamma[x^\pm] \otimes_{S_\alpha} V_\alpha$$

is horizontal, as is verified without much effort.

Because  $C$  is algebraically closed, it is clear that  $C \rightarrow C \otimes_R S_\gamma$  has a section  $C \otimes_R S_\gamma \rightarrow C$  and hence ‘‘Hensel’s Lemma’’ [EGA, IV<sub>4</sub>, Theorem 18.5.17] shows that  $R \rightarrow S_\gamma$  also has a section  $\xi_\gamma : S_\gamma \rightarrow R$ . Base changing the morphism  $f_\gamma$  through  $\xi_\gamma$ , we obtain an isomorphism of connections  $M \rightarrow R[x^\pm] \otimes V$ . It is clear that if  $M$  is  $R$ -flat, then  $V$  is also  $R$ -flat.  $\square$

**Corollary 3.6.** *Let us instate the assumptions of Theorem 3.5. Then, if  $(M, \nabla)$  is an object of  $\mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$ , it follows that  $M$  is in fact a free  $R[x^\pm]$ -module.  $\square$*

**Corollary 3.7.** *Let us instate the assumptions of Theorem 3.5-(ii). Then, each object of  $\mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$  is a quotient of an object of  $\mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$ .*

*Proof.* Any object of  $\mathbf{End}_R$  is a quotient of an object of  $\mathbf{End}_R^\circ$  and the result follows from Theorem 3.5.  $\square$

## 4. DELIGNE'S EQUIVALENCE

In this section, we put things together to obtain an analogue of Deligne's equivalence in the case of a strict Henselian discrete valuation ring. Recall that Deligne proved in [Del87, Proposition 15.35] that for any field  $k$  of characteristic 0, the functor

$$\mathbf{r} : \mathbf{MC}_{\text{rs}}(k[x^\pm]/k) \longrightarrow \mathbf{MC}_{\text{rs}}(k((x))/k)$$

given by base change is indeed an equivalence. When  $k$  is replaced by a  $C$ -algebra of the form  $C[[t_1, \dots, t_r]]/\mathfrak{a}$ , the analogous equivalence has been established in [HdST23]. We want to establish an analogue in the case where  $k$  is replaced by our  $R$  (assumptions on it will be made as needed).

**Theorem 4.1.** *Let  $R$  be Henselian  $G$ -ring. Then the restriction functor*

$$\mathbf{r} : \mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R) \longrightarrow \mathbf{MC}_{\text{rs}}^\circ(R((x))/R)$$

*is an equivalence.*

*Proof. Essential surjectivity.* This follows from Theorem 2.15 without much difficulty since  $\mathbf{r}\gamma\text{eul}_{\mathbb{P}_R^1}(V, A) = \gamma\text{eul}_{R[[x]]}(V, A)$ .

*Faithfulness.* This is simple, as for any  $N \in \mathbf{MC}_{\text{rs}}^\circ(R[x^\pm]/R)$ , the natural map

$$R[x^\pm] \otimes_{R[[x^\pm]]} N \longrightarrow R((x)) \otimes_{R[[x^\pm]]} N$$

is injective.

*Fullness.* By Theorem 3.5-(i), we need to prove the following. Let  $(V, A)$  and  $(W, B)$  be objects of  $\mathbf{End}_R$ . Then the natural map

$$\text{Hom}(\gamma\text{eul}_{\mathbb{P}_R^1}(V, A), \gamma\text{eul}_{\mathbb{P}_R^1}(W, B)) \longrightarrow \text{Hom}(\gamma\text{eul}_{R[[x]]}(V, A), \gamma\text{eul}_{R[[x]]}(W, B))$$

is surjective. Fixing bases  $\{v_i\}_{i=1}^m$ , resp.  $\{w_i\}_{i=1}^n$ , of  $V$ , resp.  $W$ , over  $R$ , any

$$\varphi \in \text{Hom}(\gamma\text{eul}_{R[[x]]}(V, A), \gamma\text{eul}_{R[[x]]}(W, B))$$

is defined by an  $n \times m$  matrix  $\Phi$  with coefficients in  $R((x))$ . On the other hand, after base-changing to  $\widehat{R}((x))$ , [HdST23, Theorem 10.1] tells us that  $\Phi \in \text{Mat}_{n \times m}(\widehat{R}[[x^\pm]])$ . Since  $R((x)) \cap \widehat{R}[[x^\pm]] = R[[x^\pm]]$ , we are done.  $\square$

As a consequence, we obtain a ‘‘full’’ Deligne equivalence as follows.

**Corollary 4.2.** *If  $R$  is a Henselian discrete valuation ring, then the restriction functor*

$$\mathbf{r} : \mathbf{MC}_{\text{rs}}(R[x^\pm]/R) \longrightarrow \mathbf{MC}_{\text{rs}}(R((x))/R)$$

*is an equivalence.*

*Proof.* It is clear that Corollary 2.17 implies that the functor  $\mathbf{r}$  is *essentially surjective*. Further, using Theorem 3.5, it sends non-zero objects to non-zero ones. Since the mapping  $R[x^\pm] \rightarrow R((x))$  is flat, the functor is exact and the standard criterion to verify *faithfulness* is assured.

To end the proof, we establish *fullness* by following the idea behind the proof of Proposition 2.16. Let then  $M$  and  $N$  be objects of  $\mathbf{MC}_{\text{rs}}(R[x^\pm]/R)$  and let

$$\mathbf{r}_{M,N} : \text{Hom}_{\mathbf{MC}}(M, N) \longrightarrow \text{Hom}_{\mathbf{MC}}(\mathbf{r}M, \mathbf{r}N)$$

be the map which we want to show is surjective. We proceed in several steps. Let  $\varphi \in \text{Hom}_{\mathbf{MC}}(M, N)$ .

*First case:  $M$  and  $N$  are  $R$ -flat.* Surjectivity of  $\mathbf{r}_{M,N}$  was verified in Theorem 4.1.

*Second case:  $M$  is  $R$ -flat.* Let  $q : N' \rightarrow N$  be a surjection with  $N' \in \mathbf{MC}_{\text{rs}}^{\circ}(R[x^{\pm}]/R)$ , see Corollary 3.7. We then construct the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}'' & \xrightarrow{U} & \mathfrak{M}' & \xrightarrow{P} & \mathbf{r}M & \longrightarrow & 0 \\ & & \downarrow \sim & & \downarrow \psi & \square & \downarrow \varphi & & \\ 0 & \longrightarrow & \mathbf{r}N'' & \xrightarrow{\mathbf{r}v} & \mathbf{r}N' & \xrightarrow{\mathbf{r}q} & \mathbf{r}N & \longrightarrow & 0, \end{array}$$

where the rightmost square is cartesian. (We slightly abuse notation and denote by  $\mathbf{r}$  the map between sets of morphisms if no confusion is likely.) It follows that  $(0 : t)_{\mathfrak{M}'} = 0$ , so that  $\mathfrak{M}' = \mathbf{r}M'$  for some  $M' \in \mathbf{MC}_{\text{rs}}^{\circ}(R[x^{\pm}]/R)$  and  $\psi = \mathbf{r}(\Psi)$ , and  $U = \mathbf{r}u$ , by the preceding case. It is also true that  $P = \mathbf{r}p$ , so that we conclude that  $\varphi$  belongs to the image of  $\mathbf{r}_{M,N}$ .

*Most general case.* Let  $p : M' \rightarrow M$  be a surjection in  $\mathbf{MC}_{\text{rs}}(R[x^{\pm}]/R)$  with  $M' \in \mathbf{MC}_{\text{rs}}^{\circ}(R[x^{\pm}]/R)$  (cf. Corollary 3.7). From the first case, there exists an arrow  $\Psi : M' \rightarrow N$  such that

$$\begin{array}{ccc} \mathbf{r}M' & \xrightarrow{\mathbf{r}p} & \mathbf{r}M \\ & \searrow \mathbf{r}\Psi & \downarrow \varphi \\ & & \mathbf{r}N \end{array}$$

commutes. Now, let  $\iota : M'' \rightarrow M'$  be the kernel of  $p$  and note that  $\mathbf{r}(\Psi)\mathbf{r}(\iota) = 0$  and hence  $\Psi\iota = 0$ . Let  $\Phi : M \rightarrow N$  be the unique arrow rendering commutative the following diagram:

$$\begin{array}{ccc} M' & \xrightarrow{\Psi} & N \\ & \searrow p & \uparrow \Phi \\ & & M. \end{array}$$

Because  $\mathbf{r}p$  is an epimorphism, we conclude that  $\mathbf{r}\Phi = \varphi$ . □

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