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Kuroda’s Translation for Higher-Order Logic

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Abstract

In 1951, Kuroda defined an embedding of classical first-order logic into intuitionistic logic, such that a formula and its translation are equivalent in classical logic. Recently, Brown and Rizkallah extended this translation to higher-order logic, but did not prove the classical equivalence, and showed that the embedding fails in the presence of functional extensionality. We prove that functional extensionality and propositional extensionality are sufficient to derive the classical equivalence between a higher-order formula and its translation. We emphasize a condition under which Kuroda’s translation works with functional extensionality.

1 Introduction

The principle of excluded middle $A \vee \neg A$, or equivalently the double-negation elimination $\neg\neg A \Rightarrow A$, are classical logic postulates which are not derivable in intuitionistic logic. Glivenko [Gli28] proved that, for any provable formula A in classical propositional logic, there exists an intuitionistic proof of its double negation $\neg\neg A$. Kolmogorov [Kol25], Gödel [Gö33], Gentzen [Gen36], and Kuroda [Kur51] developed translations $A \mapsto A^*$ that insert double negations inside formulas. For any formula A in first-order logic, such translations satisfy two properties:

- (i) if A is provable in classical logic then A^* is provable in intuitionistic logic,
- (ii) A^* and A are equivalent in classical logic.

Brown and Rizkallah [BR14] recently showed that, unlike Kolmogorov’s and Gödel-Gentzen’s translations, Kuroda’s one can be extended to higher-order logic so that it satisfies Property (i), but they did not prove Property (ii). Moreover, they showed that such translation fails in the presence of functional extensionality.

In this paper, we propose to fill these two gaps. First, we investigate cases in which it is possible to prove the classical equivalence between a higher-order formula and its Kuroda’s translation. Second, we prove that Kuroda’s translation works in the presence of functional extensionality, assuming the double-negation elimination on equality predicates.

2 Higher-Order Logic

In higher-order logic, functions, predicates and propositions are all terms of Church's simple type theory [Chu40]. Types are defined inductively: ι is the type of individuals, o is the type of propositions, and if τ and σ are types then $\tau \rightarrow \sigma$ is a type. For every type τ , let \mathcal{V}_τ be the set of variables of type τ and \mathcal{C}_τ be a set of constants of type τ . The set of variables $\mathcal{V} := \bigcup_\tau \mathcal{V}_\tau$ and the set of constants $\mathcal{C} := \bigcup_\tau \mathcal{C}_\tau$ are assumed to be disjoint. For any set of constants \mathcal{C} , the sets $\Lambda_\tau^{\mathcal{C}}$ of terms of type τ are defined by induction:

- For every $x \in \mathcal{V}_\tau$, $x \in \Lambda_\tau^{\mathcal{C}}$.
- For every $c \in \mathcal{C}_\tau$, $c \in \Lambda_\tau^{\mathcal{C}}$.
- For every $x \in \mathcal{V}_\tau$ and $t \in \Lambda_\sigma^{\mathcal{C}}$, then $(\lambda x.t) \in \Lambda_{\tau \rightarrow \sigma}^{\mathcal{C}}$.
- For every $t \in \Lambda_{\tau \rightarrow \sigma}^{\mathcal{C}}$ and $u \in \Lambda_\tau^{\mathcal{C}}$, then $(tu) \in \Lambda_\sigma^{\mathcal{C}}$.

$\lambda x.t$ is a λ -abstraction and tu is an application. Formulas are terms of type o . There are particular constants defining the logical connectives and quantifiers: tautology \top and contradiction \perp of type o , negation \neg of type $o \rightarrow o$, implication \Rightarrow , conjunction \wedge and disjunction \vee of type $o \rightarrow o \rightarrow o$, and quantifiers \forall_τ and \exists_τ of type $(\tau \rightarrow o) \rightarrow o$. For convenience, terms of the form $\forall_\tau(\lambda x.A)$ and $\exists_\tau(\lambda x.A)$ are simply abbreviated as $\forall x.A$ and $\exists x.A$. The logical biconditional \Leftrightarrow is defined by $A \Leftrightarrow B := (A \Rightarrow B) \wedge (B \Rightarrow A)$. For every type τ , we define an equality symbol $=_\tau$ of type $\tau \rightarrow \tau \rightarrow o$. The symbols are infix, and we write $t = u$ when there is no ambiguity on the type τ .

Computation is introduced in this λ -calculus thanks to the β -reduction rule $(\lambda x.t)u \hookrightarrow t[x \leftarrow u]$, where $t[x \leftarrow u]$ corresponds to the term t in which x has been substituted by u . We denote \equiv_β the congruence generated by β -reduction. A context Γ is a finite sequence of formulas. We write $FV(t_1, \dots, t_n)$ for the set of free variables that occur in the terms t_1, \dots, t_n . The natural deduction rules for **classical logic** are given in Figure 1. The principle of excluded middle is equivalent to the *double-negation elimination* $\neg\neg A \Rightarrow A$. The *double-negation equivalence* $\neg\neg A \Leftrightarrow A$ holds in classical logic. The natural deduction rules for **intuitionistic logic** are those of classical logic, except the principle of excluded middle PEM. The standard weakening inference rule is admissible in intuitionistic logic. The natural deduction rules for equality are given in Figure 2.

We write $\Gamma \vdash_i A$ when $\Gamma \vdash A$ is derivable in intuitionistic logic, and $\Gamma \vdash_c A$ when it is derivable in classical logic. For any $k \in \{c, i\}$, we write $\Gamma \vdash_k^* A$ with $*$ $\in \{\mathbf{e}, \mathbf{ep}, \mathbf{ef}, \mathbf{efp}\}$ when $\Gamma \vdash_k A$ is derivable with possibly additional inference rules: with EQ-I and EQ-E if \mathbf{e} is in $*$, with PROPEXT if \mathbf{p} is in $*$, and with FUNEXT if \mathbf{f} is in $*$.

Let us recall some well-known results about intuitionistic logic.

Proposition 1. *Let A and B be formulas, P be a predicate, and u and v be two terms.*

1. $\vdash_i \neg\neg\perp \Rightarrow \perp$
2. $\vdash_i \neg\neg\top \Rightarrow \top$
3. $\vdash_i \neg\neg(A \vee \neg A)$
4. $\vdash_i A \Rightarrow \neg\neg A$
5. $\vdash_i \neg\neg\neg A \Leftrightarrow \neg A$
6. $\vdash_i \neg\neg(A \Rightarrow B) \Leftrightarrow (\neg\neg A \Rightarrow \neg\neg B)$
7. $\vdash_i \neg\neg(A \wedge B) \Leftrightarrow (\neg\neg A \wedge \neg\neg B)$
8. $\vdash_i \neg(A \vee B) \Leftrightarrow (\neg A \wedge \neg B)$
9. $\vdash_i \neg\neg\forall x.A \Rightarrow \forall x.\neg\neg A$
10. $\vdash_i \neg\exists x.A \Leftrightarrow \forall x.\neg A$

$$\begin{array}{c}
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \text{IMP-I} \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \text{IMP-E} \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{AND-I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \text{AND-EL} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \text{AND-ER} \\
\\
\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \text{OR-IL} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \text{OR-IR} \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \text{OR-E} \\
\\
\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \text{NOT-I} \qquad \frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash \perp} \text{NOT-E} \\
\\
\frac{\Gamma \vdash \perp}{\Gamma \vdash A} \text{BOT-E} \qquad \frac{}{\Gamma \vdash \top} \text{TOP-I} \\
\\
\frac{\Gamma \vdash A \quad x \notin FV(\Gamma)}{\Gamma \vdash \forall x.A} \text{ALL-I} \qquad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[x \leftarrow t]} \text{ALL-E} \\
\\
\frac{\Gamma \vdash A[x \leftarrow t]}{\Gamma \vdash \exists x.A} \text{EX-I} \qquad \frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash C \quad x \notin FV(\Gamma, C)}{\Gamma \vdash C} \text{EX-E} \\
\\
\frac{}{\Gamma, A, \Delta \vdash A} \text{Ax} \qquad \frac{\Gamma \vdash A \quad A \equiv_{\beta} B}{\Gamma \vdash B} \text{CONV} \qquad \frac{}{\Gamma \vdash A \vee \neg A} \text{PEM}
\end{array}$$

Figure 1: Natural deduction rules for higher-order logic.

$$\begin{array}{c}
\frac{}{\Gamma \vdash u = u} \text{EQ-I} \qquad \frac{\Gamma \vdash A[x \leftarrow u] \quad \Gamma \vdash u = v}{\Gamma \vdash A[x \leftarrow v]} \text{EQ-E} \\
\\
\frac{\Gamma \vdash fx = gx \quad x \notin FV(\Gamma, f, g)}{\Gamma \vdash f = g} \text{FUNEXT} \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash B \Rightarrow A}{\Gamma \vdash A = B} \text{PROPEXT}
\end{array}$$

Figure 2: Natural deduction rules for equality.

$$11. \vdash_i^c \neg\neg(u = v) \Rightarrow \neg\neg(Pu) \Rightarrow \neg\neg(Pv)$$

$$12. \vdash_i^{cp} \neg\neg(A \Rightarrow B) \Rightarrow \neg\neg(B \Rightarrow A) \Rightarrow \neg\neg(A = B)$$

Proof. Item 3 to Item 10 are well-known, see for example [TvD88, Chapter 2]. Item 11 follows from NOT-I, NOT-E and EQ-E. Item 12 derives from the tautology $\vdash_i^{cp} (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A = B)$, Item 4 and Item 6. \square

3 Kuroda's Translation for Higher-Order Logic

Kuroda's translation for first-order logic [Kur51] inserts a double negation in front of formulas and a double negation after every universal quantifier. More formally, we first define A_{Ku} by induction on A :

$$\begin{aligned} (A \Rightarrow B)_{Ku} &:= A_{Ku} \Rightarrow B_{Ku} & (\neg A)_{Ku} &:= \neg A_{Ku} & P_{Ku} &:= P \text{ if } P \text{ atomic} \\ (A \wedge B)_{Ku} &:= A_{Ku} \wedge B_{Ku} & \top_{Ku} &:= \top & (\forall x.A)_{Ku} &:= \forall x. \neg\neg A_{Ku} \\ (A \vee B)_{Ku} &:= A_{Ku} \vee B_{Ku} & \perp_{Ku} &:= \perp & (\exists x.A)_{Ku} &:= \exists x. A_{Ku} \end{aligned}$$

and then we set $A^{Ku} := \neg\neg A_{Ku}$. For every first-order formula A , we have $\Gamma \vdash_c A$ if and only if $\Gamma^{Ku} \vdash_i A^{Ku}$. The reverse implication derives from $\vdash_c A \Leftrightarrow A^{Ku}$. We extend Kuroda's translation to the terms of higher-order logic.

Definition 1 (Kuroda's translation for higher-order logic). *Let A be a formula in higher-order logic. Its Kuroda's translation is $A^{Ku} := \neg\neg A_{Ku}$, where A_{Ku} is inductively defined by:*

$$\begin{aligned} x_{Ku} &:= x \\ c_{Ku} &:= \begin{cases} \lambda p. \forall x. \neg\neg(px) & \text{if } c = \forall \\ c & \text{otherwise} \end{cases} \\ (\lambda x.t)_{Ku} &:= \lambda x. t_{Ku} \\ (tu)_{Ku} &:= t_{Ku} u_{Ku} \end{aligned}$$

While in first-order logic we have $(A[z \leftarrow w])^{Ku} = A^{Ku}[z \leftarrow w]$, this result cannot hold anymore in higher-order logic, since w is modified when it contains \forall . Instead, we have $(A[z \leftarrow w])_{Ku} = A_{Ku}[z \leftarrow w_{Ku}]$.

Proposition 2. *For any term t , we have $(t[z \leftarrow w])_{Ku} = t_{Ku}[z \leftarrow w_{Ku}]$.*

Proof. By induction on the term t . \square

Corollary 1. *For any higher-order formula A , we have $(A[z \leftarrow w])^{Ku} = A^{Ku}[z \leftarrow w_{Ku}]$.*

Higher-order logic is defined using simple type theory, so β -conversions may be used in the derivations. As Kuroda's proof relies on the fact that we can translate each step of the derivation, each time we have $A \equiv_\beta B$ in the classical derivation, we want to use $A^{Ku} \equiv_\beta B^{Ku}$ in the intuitionistic derivation.

Proposition 3. *For any terms t and u , if $t \equiv_\beta u$ then $t_{Ku} \equiv_\beta u_{Ku}$.*

Proof. We have $((\lambda x.t)u)_{Ku} = (\lambda x.t_{Ku})u_{Ku} \hookrightarrow t_{Ku}[x \leftarrow u_{Ku}]$, so that $((\lambda x.t)u)_{Ku} \equiv_\beta (t[x \leftarrow u])_{Ku}$ using Proposition 2. Closure by context, reflexivity, symmetry, and transitivity are immediate. \square

Corollary 2. *For any higher-order formulas A and B , if $A \equiv_\beta B$ then $A^{Ku} \equiv_\beta B^{Ku}$.*

4 From Classical Logic to Intuitionistic Logic

To prove that $\Gamma \vdash_c A$ implies $\Gamma^{Ku} \vdash_i A^{Ku}$, we do not follow Brown and Rizkallah proof [BR14], which proceeds in two steps—transforming A into a formula A' that does not contain any universal quantifier and applying to A' an extension of Glivenko’s theorem to higher-order logic without universal quantifiers. We proceed to a more direct proof that follows the intuition of the first-order case.

Moreover, we want to prove that $\Gamma \vdash_c^* A$ implies $\Gamma^{Ku} \vdash_i^* A^{Ku}$ whatever $*$ $\in \{\mathbf{c}, \mathbf{cp}, \mathbf{cf}, \mathbf{cfp}\}$. Brown and Rizkallah showed that Kuroda’s translation fails for higher-order logic in the presence of functional extensionality. To get around this problem, it is sufficient to assume the axiom $\forall x \forall y. \neg \neg(x = y) \Rightarrow x = y$, written Δ_{eq} , that corresponds to the double-negation elimination on equality.

Theorem 1. *Let A be a formula and Γ be a context in higher-order logic.*

1. *If $\Gamma \vdash_c A$ then $\Gamma^{Ku} \vdash_i A^{Ku}$.*
2. *For $*$ $\in \{\mathbf{c}, \mathbf{cp}\}$, if $\Gamma \vdash_c^* A$ then $\Gamma^{Ku} \vdash_i^* A^{Ku}$.*
3. *For $*$ $\in \{\mathbf{cf}, \mathbf{cfp}\}$, if $\Gamma \vdash_c^* A$ then $\Delta_{eq}, \Gamma^{Ku} \vdash_i^* A^{Ku}$.*

Proof. We proceed by induction on the derivation. For the first item, most of the cases are direct applications of Proposition 1. CONV derives from Proposition 2. We only show the most interesting cases:

- ALL-I: By induction, we have $\Gamma^{Ku} \vdash \neg \neg A_{Ku}$. Using ALL-I, we derive $\Gamma^{Ku} \vdash \forall x. \neg \neg A_{Ku}$. By Proposition 1(4), we get $\Gamma^{Ku} \vdash \neg \neg \forall x. \neg \neg A_{Ku}$.
- ALL-E: By induction, we have $\Gamma^{Ku} \vdash \neg \neg \forall x. \neg \neg A_{Ku}$. Using Proposition 1(9), we derive $\Gamma^{Ku} \vdash \forall x. \neg \neg \neg \neg A_{Ku}$. Using ALL-E, we obtain $\Gamma^{Ku} \vdash \neg \neg \neg \neg A_{Ku}[x \leftarrow t_{Ku}]$. We get $\Gamma^{Ku} \vdash \neg \neg A_{Ku}[x \leftarrow t_{Ku}]$ using Proposition 1(5). By Proposition 1, we conclude that $\Gamma^{Ku} \vdash (A[x \leftarrow t])^{Ku}$.
- EX-I: By induction and Proposition 1, we have $\Gamma^{Ku} \vdash \neg \neg A_{Ku}[x \leftarrow t_{Ku}]$. The proof of $\Gamma^{Ku} \vdash \neg \neg \exists x. A_{Ku}$, where Δ denotes $\Gamma^{Ku}, \forall x. \neg \neg A_{Ku}$, is

$$\frac{\frac{\text{Hypothesis}}{\Delta \vdash \neg \neg A_{Ku}[x \leftarrow t_{Ku}]} \text{Weakening} \quad \frac{\frac{\overline{\Delta \vdash \forall x. \neg \neg A_{Ku}} \text{AX}}{\Delta \vdash \neg \neg A_{Ku}[x \leftarrow t_{Ku}]} \text{ALL-E}}{\Delta \vdash \perp} \text{NOT-I}}{\Gamma^{Ku} \vdash \neg \neg \exists x. A_{Ku}} \text{Proposition 1(10)} \text{NOT-E}$$

- EX-E: By induction, we have $\Gamma^{Ku} \vdash \neg \neg \exists x. A_{Ku}$ and $\Gamma^{Ku}, A^{Ku} \vdash C^{Ku}$. We want to prove $\Gamma^{Ku} \vdash \neg \neg C_{Ku}$. We use NOT-I on $\neg C_{Ku}$ and NOT-E with $\neg \forall x. \neg \neg A_{Ku}$. The proof of the first subgoal, where Δ denotes $\Gamma^{Ku}, \neg C_{Ku}$, is

$$\begin{array}{c}
\frac{\text{Hypothesis}}{\Delta, \neg\neg A_{Ku} \vdash \neg\neg C_{Ku}} \text{ Weakening} \quad \frac{}{\Delta, \neg\neg A_{Ku} \vdash \neg C_{Ku}} \text{ Ax} \\
\hline
\frac{\Delta, \neg\neg A_{Ku} \vdash \perp}{\Delta \vdash \neg\neg\neg A_{Ku}} \text{ NOT-I} \\
\frac{\Delta \vdash \neg\neg\neg A_{Ku}}{\Delta \vdash \neg A_{Ku}} \text{ Proposition 1(5)} \\
\frac{\Delta \vdash \neg A_{Ku}}{\Delta \vdash \forall x. \neg A_{Ku}} \text{ ALL-I} \\
\frac{\Delta \vdash \forall x. \neg A_{Ku}}{\Delta \vdash \neg\neg\forall x. \neg A_{Ku}} \text{ Proposition 1(4)}
\end{array}$$

and the second subgoal derives from Proposition 1(10), weakening and the hypothesis $\Gamma^{Ku} \vdash \neg\neg\exists x. A_{Ku}$.

For the second item, we complete the first item: EQ-I derives from Proposition 1(4), EQ-E from Proposition 1(11), and PROPEXT from Proposition 1(12). For the third item, we use the cases of the second item and weakening. The case FUNEXT derives from Δ_{eq} , FUNEXT and Proposition 1(4). \square

5 Back to Classical Logic

We have shown that, for any provable formula A , there exists an intuitionistic proof of A^{Ku} . We now want to prove that A and A^{Ku} are classically equivalent. Such a result is straightforward in first-order logic, but it is not generally true in higher-order logic. For instance, when C is a constant of type $o \rightarrow o$ and P is a proposition, we have $\vdash_c (CP)^{Ku} \Leftrightarrow CP_{Ku}$, but we cannot derive $\vdash_c (CP)^{Ku} \Leftrightarrow CP$ without further assumptions. Under functional extensionality and propositional extensionality, A and A^{Ku} are classically equivalent.

Lemma 1. *For any term t , we have $\vdash_c^{\text{cfp}} t_{Ku} = t$.*

Proof. We proceed by induction on the term t . We show $\vdash_c^{\text{cfp}} x^{Ku} = x$ and $\vdash_c^{\text{cfp}} c^{Ku} = c$ with $c \neq \forall$ using EQ-I. We derive $\vdash_c^{\text{cfp}} \forall^{Ku} = \forall$ from PROPEXT, FUNEXT, EQ-E and the double-negation equivalence. We have $\vdash_c^{\text{cfp}} (tu)^{Ku} = tu$ using the induction hypotheses and EQ-E. We derive $\vdash_c^{\text{cfp}} (\lambda x.t)^{Ku} = \lambda x.t$ using the induction hypothesis and FUNEXT. \square

Lemma 2. *For any higher-order formula A , we have $\vdash_c^{\text{cfp}} A^{Ku} \Leftrightarrow A$.*

Proof. By Lemma 1, $\vdash_c^{\text{cfp}} A_{Ku} = A$. Using $A \Leftrightarrow A$ and EQ-E, we derive $\vdash_c^{\text{cfp}} A_{Ku} \Leftrightarrow A$. We conclude using the double-negation equivalence. \square

Theorem 2. *Let A be a formula and Γ be a context in higher-order logic.*

1. *If $\Gamma^{Ku} \vdash_i A^{Ku}$ then $\Gamma \vdash_c^{\text{cfp}} A$.*
2. *For $* \in \{\text{c}, \text{cp}\}$, if $\Gamma^{Ku} \vdash_i^* A^{Ku}$ then $\Gamma \vdash_c^{\text{cfp}} A$.*
3. *For $* \in \{\text{cf}, \text{cjp}\}$, if $\Delta_{eq}, \Gamma^{Ku} \vdash_i^* A^{Ku}$ then $\Gamma \vdash_c^{\text{cfp}} A$.*

Proof. For the first and second item, we naturally have $\Gamma^{Ku} \vdash_c^{\text{cfp}} A^{Ku}$, and we derive $\Gamma \vdash_c^{\text{cfp}} A$ using Lemma 2. Similarly, for the third item we get $\Delta_{eq}, \Gamma \vdash_c^{\text{cfp}} A$, and we use the fact that Δ_{eq} is derivable in classical logic. \square

6 Conclusion

Brown and Rizkallah [BR14] extended Kuroda’s translation so that it embeds classical higher-order logic into intuitionistic logic. In this paper, we have shown that the classical equivalence between a higher-order formula and its translation holds under functional extensionality and propositional extensionality. While Brown and Rizkallah showed that Kuroda’s translation fails in the presence of functional extensionality, we have proved that it works when we assume the double-negation elimination on equality predicates.

The role of functional extensionality and propositional extensionality is predominant when extending Kuroda’s translation to higher-order logic. One has to be careful when using both principles in an intuitionistic context: the Diaconescu-Goodman-Myhill theorem [Dia75, GM78] states that, together with the axiom of choice, they entail the principle of excluded middle.

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