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► **To cite this version:**

Pierre-A Vuillermot. Compact embeddings and Pitt's property for weighted sequence spaces of Sobolev type. 2024. ⟨hal-04561466⟩

HAL Id: hal-04561466

<https://hal.science/hal-04561466v1>

Preprint submitted on 27 Apr 2024

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Compact embeddings and Pitt's property for weighted sequence spaces of Sobolev type

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Abstract

In this article we introduce a new class of weighted sequence spaces of Sobolev type and prove several compact embedding theorems for them. In all cases our proofs are based on the existence of a Schauder basis that spans all of the spaces under consideration. Our choice of spaces is primarily motivated by earlier work on infinite-dimensional dynamical systems of hyperbolic type arising in non-equilibrium statistical mechanics. We also prove a theorem of Pitt's type asserting that under some conditions every linear bounded transformation from one weighted sequence space of the class into another is compact.

Keywords: Sobolev Sequence Spaces, Pitt's Property
MSC 2020: primary 46B50; secondary 46E35, 47B37
Abbreviated Title: Compact Embeddings

1 Outline and main results

The essential role played by compact embeddings of Sobolev spaces of various kinds in the analysis of initial- and boundary-value problems for ordinary and partial differential equations is well known (see, e.g., [1] or [7] and the numerous references therein). Of equal importance are certain Hilbert spaces of sequences and their relation to Sobolev spaces of periodic functions as in [3]. In this article we introduce a new scale of weighted sequence spaces of Sobolev type and prove compact embedding results for them. In all cases our arguments are based on the existence of a Schauder basis in the spaces under consideration, while the choice of the scale is primarily motivated by earlier investigations of some initial-value problems for master equations arising in non-equilibrium statistical mechanics (see [4], [5] and [12]). We also prove a Pitt's type theorem asserting that under some restrictions, every linear bounded operator between two spaces of the scale is necessarily compact. We refer the reader to [11] for

the original statement involving a linear bounded operator acting between two spaces of summable sequences, to [6] and [8] for much shorter proofs thereof and to Theorem 2.1.4 in [2] or Proposition 2.c.3 in [10] for yet more condensed arguments.

Thus, with $s \in [1, +\infty)$, $k \in \mathbb{R}$ and $w = (w_m)_{m \in \mathbb{Z}}$ a sequence of weights satisfying $w_m > 0$ for every m , let us consider the separable Banach space $h_{\mathbb{C},w}^{k,s}$ of Sobolev type consisting of all complex sequences $\mathbf{p} = (p_m)$ endowed with the usual algebraic operations and the norm

$$\|\mathbf{p}\|_{k,s,w} := \left(\sum_{m \in \mathbb{Z}} w_m (1 + |m|^s)^k |p_m|^s \right)^{\frac{1}{s}} < +\infty. \quad (1)$$

If $k = 0$ we simply write $l_{\mathbb{C},w}^s := h_{\mathbb{C},w}^{0,s}$ and

$$\|\mathbf{p}\|_{s,w} := \left(\sum_{m \in \mathbb{Z}} w_m |p_m|^s \right)^{\frac{1}{s}} \quad (2)$$

for the corresponding norm. We may refer to s as the *degree of summability* of \mathbf{p} and to k as its *generalized order of differentiability*, a terminology justified by the analogy with the usual Sobolev space theory and its relation to Fourier analysis on \mathbb{R}^d (see, e.g., Chapter VI in [13]). Unless $s = 1$, the spaces $h_{\mathbb{C},w}^{k,s}$ are reflexive and it is also easily determined by means of the parallelogram law that $h_{\mathbb{C},w}^{k,s}$ is a Hilbert space if and only if $s = 2$, in which case (1) is related to the sesquilinear form

$$(\mathbf{p}, \mathbf{q})_{k,2,w} := \sum_{m \in \mathbb{Z}} w_m (1 + m^2)^k p_m \bar{q}_m$$

in the usual way. We then write $h_{\mathbb{C},w}^k := h_{\mathbb{C},w}^{k,2}$, noting also that the role of the sequence of weights (w_m) generally speaking amounts to making certain initial value problems self-adjoint, as was the case in the last two aforementioned references on statistical mechanics.

As already mentioned, our proofs of various compactness criteria for the space $h_{\mathbb{C},w}^{k,s}$ rest in an essential way on the existence of a very concrete and simple Schauder basis therein (for the definition and various properties of such bases see, e.g., [2] and [10]). Thus, for any $m \in \mathbb{Z}$ we define \mathbf{e}_m by $(\mathbf{e}_m)_n = \delta_{m,n}$ for every $n \in \mathbb{Z}$ so that $\mathbf{e}_m \in h_{\mathbb{C},w}^{k,s}$ for each m with

$$\|\mathbf{e}_m\|_{k,s,w} = w_m^{\frac{1}{s}} (1 + |m|^s)^{\frac{k}{s}}.$$

Cauchy's criterion then shows that every $\mathbf{p} = (p_m) \in h_{\mathbb{C},w}^{k,s}$ may be expanded as the norm-convergent series

$$\mathbf{p} = \sum_{m \in \mathbb{Z}} p_m \mathbf{e}_m, \quad (3)$$

a fact that will be crucial in what follows. As a preliminary remark we note that if $k \leq k'$, there exists the continuous embedding

$$h_{\mathbb{C},w}^{k',s} \rightarrow h_{\mathbb{C},w}^{k,s} \quad (4)$$

by virtue of the obvious inequality

$$\|p\|_{k,s,w}^s = \sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \leq \sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^{k'} |p_{\mathbf{m}}|^s = \|p\|_{k',s,w}^s.$$

Our first result is then:

Theorem 1. *The following statements hold:*

(a) *If the strict inequality $k < k'$ holds, then embedding (4) is compact, in which case we write*

$$h_{\mathbb{C},w}^{k',s} \hookrightarrow h_{\mathbb{C},w}^{k,s}. \quad (5)$$

(b) *Let us assume that $0 \leq k < k'$, and that the sequence of weights $w = (w_{\mathbf{m}})$ satisfies the additional constraint $w_{\mathbf{m}} \geq 1$ for every \mathbf{m} . Then if $s \geq t \geq 1$, there exists the compact embedding*

$$h_{\mathbb{C},w}^{k',t} \hookrightarrow h_{\mathbb{C},w}^{k,s}. \quad (6)$$

In particular, the embedding

$$h_{\mathbb{C},w}^{k',t} \hookrightarrow l_{\mathbb{C},w}^s$$

is compact.

Proof. Let \mathcal{K} be a bounded set in $h_{\mathbb{C},w}^{k',s}$ and let $\kappa > 0$ be the radius of a ball centered at the origin of $h_{\mathbb{C},w}^{k',s}$ and containing \mathcal{K} . Then for each $\varepsilon > 0$ there exists $\mathbf{m}^* \in \mathbb{Z}^+$ such that for every $|\mathbf{m}| \geq \mathbf{m}^*$ we have

$$\frac{1}{(1 + |\mathbf{m}|^s)^{k'-k}} \leq \left(\frac{\varepsilon}{2\kappa}\right)^s$$

since $k' - k > 0$. Therefore, for every $p \in \mathcal{K}$ we obtain

$$\begin{aligned} & \sum_{|\mathbf{m}| \geq \mathbf{m}^*} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \\ = & \sum_{|\mathbf{m}| \geq \mathbf{m}^*} \frac{1}{(1 + |\mathbf{m}|^s)^{k'-k}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^{k'} |p_{\mathbf{m}}|^s \\ \leq & \left(\frac{\varepsilon}{2\kappa}\right)^s \sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^{k'} |p_{\mathbf{m}}|^s \leq \left(\frac{\varepsilon}{2\kappa}\right)^s \kappa^s = \left(\frac{\varepsilon}{2}\right)^s, \end{aligned}$$

that is,

$$\left(\sum_{|\mathbf{m}| \geq \mathbf{m}^*} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \right)^{\frac{1}{s}} \leq \frac{\varepsilon}{2}. \quad (7)$$

Now let $\bar{\mathcal{K}}$ be the closure of \mathcal{K} in $h_{\mathbb{C},w}^{k,s}$. In order to show that $\bar{\mathcal{K}}$ is compact it is necessary and sufficient to show that $\bar{\mathcal{K}}$ is totally bounded or, equivalently, that there exists a finite-dimensional subspace $\mathcal{S}_{\varepsilon,\kappa} \subset h_{\mathbb{C},w}^{k,s}$ such that the distance of every $\mathbf{p} \in \bar{\mathcal{K}}$ to $\mathcal{S}_{\varepsilon,\kappa}$ satisfies

$$\text{dist}(\mathbf{p}, \mathcal{S}_{\varepsilon,\kappa}) := \inf_{\mathbf{q} \in \mathcal{S}_{\varepsilon,\kappa}} \|\mathbf{p} - \mathbf{q}\|_{k,s,w} \leq \varepsilon$$

(see, e.g., Proposition 2.1 in [3], or [9]). To this end we choose

$$\mathcal{S}_{\varepsilon,\kappa} = \text{span} \{\mathbf{e}_{-m^*}, \dots, \mathbf{e}_{m^*}\}$$

where \mathbf{e}_m is as in (3). Furthermore, having (3) in mind we define

$$\mathbf{p}_{m^*} := \sum_{|m| < m^*} p_m \mathbf{e}_m.$$

Then $\mathbf{p}_{m^*} \in \mathcal{S}_{\varepsilon,\kappa}$ and for every $\mathbf{p} \in \mathcal{K}$ we obtain

$$\text{dist}(\mathbf{p}, \mathcal{S}_{\varepsilon,\kappa}) \leq \|\mathbf{p} - \mathbf{p}_{m^*}\|_{k,s,w} = \left(\sum_{|m| \geq m^*} w_m (1 + |m|^s)^k |p_m|^s \right)^{\frac{1}{s}} \leq \frac{\varepsilon}{2} \quad (8)$$

according to (7). Finally, for every $\mathbf{p} \in \bar{\mathcal{K}} \setminus \mathcal{K}$ there exists $\mathbf{p}_\varepsilon \in \mathcal{K}$ such that $\|\mathbf{p} - \mathbf{p}_\varepsilon\|_{s,w} \leq \frac{\varepsilon}{2}$ so that we obtain

$$\text{dist}(\mathbf{p}, \mathcal{S}_{\varepsilon,\kappa}) \leq \frac{\varepsilon}{2} + \inf_{\mathbf{q} \in \mathcal{S}_{\varepsilon,\kappa}} \|\mathbf{p}_\varepsilon - \mathbf{q}\|_{s,w} \leq \varepsilon$$

as a consequence of (8), which proves Statement (a).

In order to prove Statement (b), we only consider $s > t$ since the case $s = t$ has already been dealt with. Since $\frac{s}{t} > 1$ we have successively

$$\begin{aligned} & \sum_{m \in \mathbb{Z}} w_m (1 + |m|^s)^k |p_m|^s \\ &= \sum_{m \in \mathbb{Z}} \left(w_m^{\frac{t}{s}} \left(1 + (|m|^t)^{\frac{s}{t}} \right)^{\frac{kt}{s}} |p_m|^t \right)^{\frac{s}{t}} \\ &\leq \left(\sum_{m \in \mathbb{Z}} w_m^{\frac{t}{s}} \left(1 + (|m|^t)^{\frac{s}{t}} \right)^{\frac{kt}{s}} |p_m|^t \right)^{\frac{s}{t}} \leq \left(\sum_{m \in \mathbb{Z}} w_m (1 + |m|^t)^k |p_m|^t \right)^{\frac{s}{t}} \end{aligned}$$

where the last inequality indeed requires $w_m \geq 1$ and $k \geq 0$. This gives

$$\|\mathbf{p}\|_{k,s,w} \leq \|\mathbf{p}\|_{k,t,w}$$

so that the embedding

$$h_{\mathbb{C},w}^{k,t} \rightarrow h_{\mathbb{C},w}^{k,s} \quad (9)$$

is continuous. Therefore, (6) may be viewed as the composition

$$h_{\mathbb{C},w}^{k',t} \hookrightarrow h_{\mathbb{C},w}^{k,t} \rightarrow h_{\mathbb{C},w}^{k,s}$$

where the first embedding is compact according to Statement (a). ■

It is interesting to note the interlacing properties of $l_{\mathbb{C},w}^s$ with respect to $h_{\mathbb{C},w}^{k',s}$ and $h_{\mathbb{C},w}^{k,s}$ as functions of the signs of k and k' :

Corollary 1. *The following statements hold:*

(a) *If $k < k' \leq 0$, we have the embeddings*

$$l_{\mathbb{C},w}^s \rightarrow h_{\mathbb{C},w}^{k',s} \hookrightarrow h_{\mathbb{C},w}^{k,s} \quad (10)$$

where the first one is continuous and the second one compact. In particular, the embedding

$$l_{\mathbb{C},w}^s \hookrightarrow h_{\mathbb{C},w}^{k,s}$$

is compact.

(b) *If $k \leq 0 < k'$, the above chain of embeddings is changed into*

$$h_{\mathbb{C},w}^{k',s} \hookrightarrow l_{\mathbb{C},w}^s \rightarrow h_{\mathbb{C},w}^{k,s} \quad (11)$$

where the first embedding is compact and the second one continuous.

(c) *If $0 \leq k < k'$, the chain of embeddings becomes*

$$h_{\mathbb{C},w}^{k',s} \hookrightarrow h_{\mathbb{C},w}^{k,s} \rightarrow l_{\mathbb{C},w}^s$$

where the first embedding is compact and the second one continuous.

Proof. The continuity of the first embedding in (10) is a consequence of the inequality

$$\|\mathbf{p}\|_{k',s,w}^s = \sum_{\mathbf{m} \in \mathbb{Z}} \frac{w_{\mathbf{m}}}{(1 + |\mathbf{m}|^s)^{|k'|}} |p_{\mathbf{m}}|^s \leq \sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} |p_{\mathbf{m}}|^s = \|\mathbf{p}\|_{s,w}^s,$$

so that the remaining part of Statement (a) then follows from Theorem 1.

As for Statement (b), the compactness of the first embedding in (11) follows from (5) with $k = 0$ while the remaining part of the statement is a consequence of what has already been proved, as is the case for Statement (c). ■

At this point it is worth illustrating some of the above results by mentioning an example that recently played an important role in [12]. Let us first consider

$$\hat{h}_{\mathbb{C},w}^{k,s} := \left\{ \mathbf{p} \in h_{\mathbb{C},w}^{k,s} : p_{\mathbf{m}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^- \right\},$$

which is a closed subspace of $h_{\mathbb{C},w}^{k,s}$ and thereby a Banach space for which all of the above embedding statements hold. With $\beta > 0$, let $w_{\beta} := (w_{\beta,\mathbf{m}})$ be

the sequence of weights given by $w_{\beta, \mathbf{m}} = \exp[\beta \mathbf{m}]$ for every $\mathbf{m} \in \mathbb{N}$, and let us consider the separable Hilbert spaces $\hat{l}_{\mathbb{C}, w_\beta}^2$ and $\hat{h}_{\mathbb{C}, w_\beta}^1$ which, according to the above definitions and notations, consist of all complex sequences $\mathbf{p} = (p_{\mathbf{m}})$ satisfying

$$\|\mathbf{p}\|_{2, w_\beta}^2 = \sum_{\mathbf{m}=0}^{+\infty} w_{\mathbf{m}, \beta} |p_{\mathbf{m}}|^2 < +\infty$$

and

$$\|\mathbf{p}\|_{1, 2, w_\beta}^2 = \sum_{\mathbf{m}=0}^{+\infty} w_{\mathbf{m}, \beta} (1 + \mathbf{m}^2) |p_{\mathbf{m}}|^2 < +\infty,$$

respectively. An application of (5) with $k = 0$, $k' = 1$ and $s = 2$ then shows that the embedding

$$\hat{h}_{\mathbb{C}, w_\beta}^1 \hookrightarrow \hat{l}_{\mathbb{C}, w_\beta}^2 \quad (12)$$

is compact, as is the embedding

$$\hat{h}_{\mathbb{C}, w_\beta}^1 \hookrightarrow \hat{l}_{\mathbb{C}, w_\beta}^s$$

for every $s \in (2, +\infty)$ as an application of (6) with $k = 0$, $k' = 1$ and $t = 2$. In fact, a stronger result than (12) was proved in the appendix of [12], namely, that

$$\hat{h}_{\mathbb{C}, w_\beta}^1 \hookrightarrow \hat{l}_{\mathbb{C}, w_{\frac{\beta}{2}}}^1 \rightarrow \hat{l}_{\mathbb{C}, w_\beta}^2 \quad (13)$$

where the first embedding is compact and the second one continuous.

Considering again $h_{\mathbb{C}, w}^{k, s}$ it is then natural to generalize (13), which obviously calls for embedding properties between spaces whose norms involve different weights. Thus, we consider another sequence of positive weights $\hat{w} = (\hat{w}_{\mathbf{m}})$ related to w by the inequalities

$$c_1 \leq \frac{w_{\mathbf{m}}^{\frac{t}{s}}}{\hat{w}_{\mathbf{m}}} \leq c_2 \quad (14)$$

for every $\mathbf{m} \in \mathbb{Z}$, where the constants $c_{1,2} > 0$ are finite, independent of \mathbf{m} and $s, t \in [1, +\infty)$ as before. Then we have:

Theorem 2. *Let us assume that (14) holds with $s > t$, and that $k \in (\frac{s-t}{st}, +\infty)$. Then there exist the embeddings*

$$h_{\mathbb{C}, w}^{k, s} \hookrightarrow l_{\mathbb{C}, \hat{w}}^t \rightarrow l_{\mathbb{C}, w}^s \quad (15)$$

where the first one is compact and the second one continuous.

Proof. We begin by proving the continuity of the first embedding in (15). Let us set

$$r := \frac{st}{s-t} \quad (16)$$

so that

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$$

and $r \in (1, +\infty)$. Owing to Hölder's inequality and the first inequality in (14) we then get

$$\begin{aligned} & \sum_{\mathbf{m} \in \mathbb{Z}} \hat{w}_{\mathbf{m}} |p_{\mathbf{m}}|^t \\ &= \sum_{\mathbf{m} \in \mathbb{Z}} \left(\frac{1}{(1 + |\mathbf{m}|^s)^{\frac{k}{s}}} \right)^t \left(\hat{w}_{\mathbf{m}}^{\frac{1}{t}} (1 + |\mathbf{m}|^s)^{\frac{k}{s}} |p_{\mathbf{m}}| \right)^t \\ &\leq \left(\sum_{\mathbf{m} \in \mathbb{Z}} \frac{1}{(1 + |\mathbf{m}|^s)^{\frac{kr}{s}}} \right)^{\frac{t}{r}} \left(\sum_{\mathbf{m} \in \mathbb{Z}} \hat{w}_{\mathbf{m}}^{\frac{s}{t}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \right)^{\frac{t}{s}} \\ &\leq c_1^{-1} \left(\sum_{\mathbf{m} \in \mathbb{Z}} \frac{1}{(1 + |\mathbf{m}|^s)^{\frac{kr}{s}}} \right)^{\frac{t}{r}} \left(\sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \right)^{\frac{t}{s}}. \end{aligned} \quad (17)$$

Furthermore we have

$$\sum_{\mathbf{m} \in \mathbb{Z}} \frac{1}{(1 + |\mathbf{m}|^s)^{\frac{kr}{s}}} < +\infty \quad (18)$$

since $kr > r \frac{s-t}{st} = 1$ by virtue of the hypothesis and (16), and therefore

$$\sum_{\mathbf{m} \in \mathbb{Z}} \hat{w}_{\mathbf{m}} |p_{\mathbf{m}}|^t \leq c_{k,s,t} \left(\sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \right)^{\frac{t}{s}}$$

for some constant $c_{k,s,t} > 0$ depending solely on k , s and t . Consequently, changing the value of $c_{k,s,t}$ as necessary we have

$$\|\mathbf{p}\|_{t,\hat{w}} \leq c_{k,s,t} \|\mathbf{p}\|_{k,s,w} \quad (19)$$

for every $\mathbf{p} \in h_{\mathbb{C},w}^{k,s}$, which is the desired continuity.

As for compactness, if \mathcal{K} is a bounded set in $h_{\mathbb{C},w}^{k,s}$ and if $\kappa > 0$ is the radius of a ball centered at the origin of $h_{\mathbb{C},w}^{k,s}$ and containing \mathcal{K} , then for each $\varepsilon > 0$ there exists $\mathbf{m}^* \in \mathbb{Z}^+$ such that

$$\sum_{|\mathbf{m}| \geq \mathbf{m}^*} \frac{1}{(1 + |\mathbf{m}|^s)^{\frac{kr}{s}}} \leq \left(\frac{\varepsilon c_1^{\frac{1}{t}}}{2\kappa} \right)^r$$

because of (18), where $c_1 > 0$ is chosen to be the constant that appears in (14) or on the right-hand side of the last inequality in (17). Consequently, from an

estimate similar to (17) we obtain for every $\mathbf{p} \in \mathcal{K}$ the inequalities

$$\begin{aligned}
& \sum_{|\mathbf{m}| \geq \mathbf{m}^*} \hat{w}_{\mathbf{m}} |p_{\mathbf{m}}|^t \\
& \leq c_1^{-1} \left(\sum_{|\mathbf{m}| \geq \mathbf{m}^*} \frac{1}{(1 + |\mathbf{m}|^s)^{\frac{kr}{s}}} \right)^{\frac{t}{r}} \left(\sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} (1 + |\mathbf{m}|^s)^k |p_{\mathbf{m}}|^s \right)^{\frac{t}{s}} \\
& \leq c_1^{-1} \left(\frac{\varepsilon c_1^{\frac{1}{t}}}{2\kappa} \right)^t \kappa^t = \left(\frac{\varepsilon}{2} \right)^t,
\end{aligned}$$

that is,

$$\left(\sum_{|\mathbf{m}| \geq \mathbf{m}^*} \hat{w}_{\mathbf{m}} |p_{\mathbf{m}}|^t \right)^{\frac{1}{t}} \leq \frac{\varepsilon}{2}.$$

Compactness of the first embedding in (15) then follows from an argument entirely similar to that set forth in the proof of Theorem 1, based on the existence of the Schauder basis $(\mathbf{e}_{\mathbf{m}})_{\mathbf{m} \in \mathbb{Z}}$. The second embedding is an easy consequence of the second inequality in (14), for we have successively

$$\begin{aligned}
& \sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}} |p_{\mathbf{m}}|^s \\
& = \sum_{\mathbf{m} \in \mathbb{Z}} \left(w_{\mathbf{m}}^{\frac{t}{s}} |p_{\mathbf{m}}|^t \right)^{\frac{s}{t}} \leq \left(\sum_{\mathbf{m} \in \mathbb{Z}} w_{\mathbf{m}}^{\frac{t}{s}} |p_{\mathbf{m}}|^t \right)^{\frac{s}{t}} \\
& \leq \left(c_2 \sum_{\mathbf{m} \in \mathbb{Z}} \hat{w}_{\mathbf{m}} |p_{\mathbf{m}}|^t \right)^{\frac{s}{t}} = c_{s,t} \left(\sum_{\mathbf{m} \in \mathbb{Z}} \hat{w}_{\mathbf{m}} |p_{\mathbf{m}}|^t \right)^{\frac{s}{t}}
\end{aligned}$$

with an obvious choice for $c_{s,t} > 0$. ■

As the preceding result remains true for $\hat{h}_{\mathbb{C},w}^{k,s}$, $\hat{l}_{\mathbb{C},\hat{w}}^t$ and $\hat{l}_{\mathbb{C},w}^s$, that is,

$$\hat{h}_{\mathbb{C},w}^{k,s} \hookrightarrow \hat{l}_{\mathbb{C},\hat{w}}^t \rightarrow \hat{l}_{\mathbb{C},w}^s, \tag{20}$$

it is clear that it is sufficient to choose $s = 2$, $t = 1$ and $k = 1$ in order to get (13) since the exponential weights $w_{\beta,\mathbf{m}} = \exp[\beta\mathbf{m}]$ and $\hat{w}_{\beta,\mathbf{m}} := w_{\frac{\beta}{2},\mathbf{m}}$ obviously satisfy (14). Similarly, Theorem 3.1 in [3] is also a particular case of (15) corresponding to the same values of s , t and k , with $w_{\mathbf{m}} = \hat{w}_{\mathbf{m}} = 1$ for every $\mathbf{m} \in \mathbb{Z}$.

We complete this article by proving a result of Pitt's type for the spaces $h_{\mathbb{C},w}^{k,s}$. Recall that if C denotes any linear bounded operator from $l_{\mathbb{C}}^s$ into $l_{\mathbb{C}}^t$ with $s > t \geq 1$, where $l_{\mathbb{C}}^s$ carries the norm (2) with $w_{\mathbf{m}} = 1$ for every \mathbf{m} , then

C is necessarily compact (see the aforementioned references on this theme). Remembering that embedding (9) holds we then have:

Theorem 3. *Let us assume that $s > t \geq 1$. Then every linear bounded operator $T : h_{\mathbb{C},w}^{k,s} \rightarrow h_{\mathbb{C},w}^{k,t}$ is compact.*

Proof. Let us define $J_{k,s,w} : h_{\mathbb{C},w}^{k,s} \rightarrow l_{\mathbb{C}}^s$ by

$$(J_{k,s,w}\mathbf{p})_m := w_m^{\frac{1}{s}} (1 + |\mathbf{m}|^s)^{\frac{k}{s}} p_m$$

for every $\mathbf{m} \in \mathbb{Z}$. It is then plain that $J_{k,s,w}$ is an isometric isomorphism with $J_{k,s,w} \left(h_{\mathbb{C},w}^{k,s} \right) = l_{\mathbb{C}}^s$ and $J_{k,s,w}^{-1} \left(l_{\mathbb{C}}^s \right) = h_{\mathbb{C},w}^{k,s}$, so that

$$C := J_{k,t,w} T J_{k,s,w}^{-1}$$

is a linear bounded operator from $l_{\mathbb{C}}^s$ into $l_{\mathbb{C}}^t$, hence compact as a consequence of Pitt's theorem. Therefore

$$T = J_{k,t,w}^{-1} C J_{k,s,w}$$

is also compact as the composition of C with linear bounded operators. ■

REMARK. An interesting open problem is that of extending the results of this article to appropriate scales of weighted sequence spaces of Orlicz-Sobolev type by invoking the generalization of Pitt's result to certain classes of Orlicz sequence spaces set forth in Section 4.b of Chapter 4 in [10].

Statements and declarations

DECLARATION OF COMPETING INTEREST: The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CONFLICT OF INTEREST: The author states that there is no conflict of interest regarding the content of this paper.

Acknowledgements. The author would like to thank the Fundação para a Ciência e a Tecnologia (FCT) of the Portuguese Government for its financial support under grant UIDB/04561/2020.

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