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Alberto Medina, Andrés Villabon

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# Lorentzian and flat affine geometry on $\mathbf{GL}(2, \mathbb{R})$

Alberto Medina.

Université de Montpellier, Institute A. Grothendieck, France

Email: alberto.medina@umontpellier.fr

Email: albertomdn47@gmail.com

Andrés Villabón.

Universidad Nacional Abierta y a Distancia, Colombia

Email: edgar.villabon@unad.edu.co

## Abstract

The goal of this paper is to study the geometry of the connected unit component of the real general linear Lie group four dimensional  $G_0$  as a Lorentzian and flat affine manifold.

As the group  $G_0$  is naturally equipped with a bi-invariant Hessian metric  $k^+$ , relative to the natural bi-invariant flat affine structure  $\nabla$  (see [1]), we examine these structures and the relationships between them.

The curvatures, tidal force, and Jacobi vector fields of  $(G_0, k^+)$  are determined in Section 1. Section 2 discusses the causal structure of  $(G_0, k^+)$ , while Section 3 focuses on the developed map relative to  $\nabla$  in the sense of C. Ehresmann.

keywords Lorentzian metric, flat affine structure, hessian structure, quadratic Lie groups, curvatures, Weyl tensor.

## 1 Introduction

Let  $G_0$  be the connected unit component of the Lie group of isomorphisms of the real vector space  $\mathbb{R}^2$  and  $\mathfrak{g}$  its Lie algebra. We will identify  $\mathfrak{g}$  with  $T_\varepsilon(G_0)$ , the tangent space of  $G_0$  in the identity element  $\varepsilon$ , or with the space of left-invariant vector fields on  $G_0$ .

Consider the left-invariant metric  $k^+$  on  $G_0$  given by the inner product  $k$  on  $\mathfrak{g}$  defined by the trace as

$$k(u, v) := \text{tr}(u \circ v)$$

and the natural bi-invariant flat affine structure  $\nabla$  given by

$$\nabla_{u^+} v^+ := (u \circ v)^+$$

where  $u, v \in \mathfrak{g}$  and  $u^+, v^+$  are the left-invariant vector fields on  $G_0$  determined by  $u, v \in \mathfrak{g}$ . Since the operators  $ad_{u^+}$  are  $k^+$ -antisymmetric, the right translations on  $G_0$  are also isometries. Then  $k^+$  is a bi-invariant metric, i.e.  $(G_0, k^+)$  is a (orthogonal or) quadratic Lie group in the Medina-Revoy's sense (see [12]).

## 2 The Lorentzian Structure

Consider the following  $k$ -orthonormal basis  $\mathcal{B}$  of  $\mathfrak{g}$  given by

$$\begin{aligned} e_1 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & e_2 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ e_3 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & e_4 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (1)$$

that it will be used throughout the paper.

The Lie brackets, except antisymmetries, relatives to  $\mathcal{B}$  become

$$[e_1, e_2] = \sqrt{2}e_3, \quad [e_1, e_3] = -\sqrt{2}e_2, \quad [e_2, e_3] = -\sqrt{2}e_1.$$

As the product  $k$  is given by the table

$k$	$e_1$	$e_2$	$e_3$	$e_4$
$e_1$	-1	0	0	0
$e_2$	0	1	0	0
$e_3$	0	0	1	0
$e_4$	0	0	0	1

$k^+$  is Lorentzian.

Recall that, a tangent vector  $u$  to  $G_0$  is called timelike (respectively lightlike, spacelike) if  $k(u, u) < 0$  (respectively  $k(u, u) = 0$ ,  $k(u, u) > 0$ ). In particular, the vector  $e_1$  is timelike, and the others  $e_i$  are spacelike.

Putting

$$(x_1, x_2, x_3, x_4) \equiv \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad (2)$$

the metric becomes

$$\begin{aligned} k^+ &= -(e_1^+)^*(e_1^+)^* + (e_2^+)^*(e_2^+)^* + (e_3^+)^*(e_3^+)^* + (e_4^+)^*(e_4^+)^* \\ &= \frac{1}{(x_1x_4 - x_2x_3)^2} (x_4^2dx_1^2 - 2x_3x_4dx_1dx_2 - 2x_2x_4dx_1dx_3 + 2x_2x_3dx_1dx_4 + x_3^2dx_2^2 \\ &\quad + 2x_1x_4dx_2dx_3 - 2x_1x_3dx_2dx_4 + x_2^2dx_3^2 - 2x_1x_2dx_3dx_4 + x_1^2dx_4^2) \end{aligned}$$

where

$$\begin{aligned}
(e_1^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_3dx_1 + x_4dx_2 - x_1dx_3 - x_2dx_4) \\
(e_2^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (-x_3dx_1 + x_4dx_2 + x_1dx_3 - x_2dx_4) \\
(e_3^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_4dx_1 + x_3dx_2 - x_2dx_3 - x_1dx_4) \\
(e_4^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4)
\end{aligned}$$

is the dual coframe of the smooth orthogonal frame

$$\begin{aligned}
e_1^+ &= \frac{\sqrt{2}}{2} \left( -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) \\
e_2^+ &= \frac{\sqrt{2}}{2} \left( x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) \\
e_3^+ &= \frac{\sqrt{2}}{2} \left( x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right) \\
e_4^+ &= \frac{\sqrt{2}}{2} \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \right)
\end{aligned}$$

A direct calculation allows us to obtain the following result.

**Lemma 1.** *Let  $\varphi, \psi$  be a  $k$ -symmetric linear isomorphisms. Then  $\varphi^{-1}, \psi^{-1}$  are  $k$ -symmetrics and  $\varphi \circ \psi, \psi \circ \varphi$  are  $k$ -symmetrics, if  $\varphi \circ \psi = \psi \circ \varphi$ . Moreover, every polynomial  $p(\varphi)$  with real coefficients is  $k$ -symmetric.*

*Proof.* For any  $u, v \in \mathfrak{g}$  we have

$$\begin{aligned}
k(\varphi(\varphi^{-1}(u)), v) &= k(\varphi^{-1}(u), \varphi(v)) \\
&= k(u, v) \\
&= k(u, \varphi^{-1}(\varphi(v))) =
\end{aligned}$$

Since  $v$  is arbitrary and  $\varphi$  is isomorphism, then

$$k(u, \varphi^{-1}(w)) = k(\varphi^{-1}(u), w) \text{ for all } u, w \in \mathfrak{g}$$

On the other hand

$$\begin{aligned}
k((\varphi \circ \psi)(u), v) &= k(\varphi(\psi(u)), v) = k(\psi(u), \varphi(v)) \\
&= k(u, \psi(\varphi(v))) \\
&= k(u, \varphi(\psi(v))) = k(u, (\varphi \circ \psi)(v))
\end{aligned}$$

□

The next assertion is clear.

**Lemma 2.** *Any left invariant semi-Riemannian metric on a quadratic Lie group  $(H, g)$  is given by a  $g$ -symmetric linear isomorphism  $\varphi$  on its Lie algebra  $\mathfrak{h}$  by the formula*

$$\langle u, v \rangle_\varphi := g(\varphi(u), v), \quad u, v \in \mathfrak{h}$$

From the above lemmas, we have

**Corollary 1.** *Let  $\varphi, \psi$  be  $k$ -symmetric linear isomorphisms of  $\mathfrak{g}$ . Then the following operators determine a left-invariant (respectively right-invariant) semi-Riemannian metric on  $G_0$*

- a)  $\varphi \circ \psi$ , if  $\varphi \circ \psi = \psi \circ \varphi$ .
- b)  $\varphi^{-1}$  and  $p(\varphi)$ , where  $p$  is a polynomial with real coefficients.

In Section 4, we use Python to find all  $k$ -symmetric operators on the Lie algebra  $\mathfrak{g}$ . From these computations, we derive the following result.

**Theorem 1.** *In the basis  $\mathcal{B}$ , the  $k$ -symmetric linear isomorphisms  $\varphi$  of  $\mathfrak{g}$  are given by the matrix representation*

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{pmatrix}$$

Moreover, the metrics given by  $\langle u, v \rangle := k(\varphi(u), v)$  for any  $k$ -symmetric  $\varphi$  are not flat (see [1], Theorem 4.1).

**Example 1.** *The left-invariant semi-Riemannian metrics on  $G_0$  given by the following  $k$ -symmetric linear isomorphisms of  $\mathfrak{g}$*

$$\begin{aligned} \varphi_0(e_1) &= -e_1, \quad \varphi_0(e_2) = e_2, \quad \varphi_0(e_3) = e_3, \quad \varphi_0(e_4) = e_4 \\ \varphi_1(e_1) &= e_1, \quad \varphi_1(e_2) = e_2, \quad \varphi_1(e_3) = e_3, \quad \varphi_1(e_4) = e_4 \\ \varphi_2(e_1) &= e_1, \quad \varphi_2(e_2) = -e_2, \quad \varphi_2(e_3) = e_3, \quad \varphi_2(e_4) = e_4 \end{aligned}$$

have index 0, 1, 2 respectively and  $\langle u, v \rangle_i := k(\varphi_i(u), v)$ , for  $i = 0, 1, 2$ , are not flat.

For the study of the movement space in  $(G_0, k^+)$ , in addition to the tangent bundle, it is convenient to introduce other principal fiber bundles. For instance, the orthonormal fiber bundle over  $(G_0, k^+)$  facilitates the understanding of parallel transport along a curve, the holonomy groups, and the corresponding holonomy bundles.

Let  $P = L(G_0)$  be the principal fiber bundle of the linear frames over  $G_0$ . Since  $\text{GL}(4, \mathbb{R})$  is 16-dimensional,  $P$  has a dimension of 20. On the other hand, as the orthogonal group  $O(1, 3)$  is the dimension 6, the principal fiber bundle of the orthogonal frames over  $G_0$  is 10-dimensional.

Now, as the Levi-Civita connection  $D$  of  $(G_0, k^+)$  can be expressed using left-invariant fields as

$$D_{u+v^+} := \frac{1}{2}[u, v]^+$$

the curvature, sectional curvature, scalar curvature, and Ricci tensors become

$$\begin{aligned} R_{u^+v^+w^+} &:= \frac{1}{4} [[u^+, v^+], w^+] \\ K(u^+, v^+) &:= \frac{1}{4} \frac{k^+([u^+, v^+], [u^+, v^+])}{k^+(u^+, u^+)k^+(v^+, v^+) - k^+(u^+, v^+)^2} \end{aligned}$$

with  $\text{span}\{u^+, v^+\}$  a nondegenerate plane,

$$\begin{aligned} S &:= 2 \sum_{i < j} K(e_i^+, e_j^+) \\ R_{\text{icci}}(u, v) &:= -\frac{B(u, v)}{4} = -\frac{1}{4} \text{tr}(\text{ad}_u \circ \text{ad}_v) \end{aligned}$$

respectively. Here  $B$  is known as the Cartan-Killing form.

Consequently, we have

**Lemma 3.** *In terms of the above orthonormal basis (1) of  $(\mathfrak{g}, k)$ , the curvature tensor, the sectional curvature, the scalar curvature, the Cartan-Killing form and the Ricci tensor of  $(G_0, k^+, D)$  are given respectively by*

$$\begin{aligned} R(e_1^+, e_2^+, e_1^+) &= \frac{1}{2}e_2^+, R(e_1^+, e_2^+, e_2^+) = \frac{1}{2}e_1^+, R(e_1^+, e_3^+, e_1^+) = \frac{1}{2}e_3^+, \\ R(e_1^+, e_3^+, e_3^+) &= \frac{1}{2}e_1^+, R(e_2^+, e_1^+, e_1^+) = -\frac{1}{2}e_2^+, R(e_2^+, e_1^+, e_2^+) = -\frac{1}{2}e_1^+, \\ R(e_2^+, e_3^+, e_2^+) &= -\frac{1}{2}e_3^+, R(e_2^+, e_3^+, e_3^+) = \frac{1}{2}e_2^+, R(e_3^+, e_1^+, e_1^+) = -\frac{1}{2}e_3^+, \\ R(e_3^+, e_1^+, e_3^+) &= -\frac{1}{2}e_1^+, R(e_3^+, e_2^+, e_2^+) = e_3^+, R(e_3^+, e_2^+, e_3^+) = -\frac{1}{2}e_2^+ \\ R(e_i^+, e_j^+, e_k^+) &= 0 \text{ in the other cases.} \end{aligned}$$

$$K(e_1^+, e_2^+) = K(e_1^+, e_3^+) = K(e_2^+, e_3^+) = -\frac{1}{2},$$

$$K(e_i^+, e_4^+) = 0, \text{ for } i = 1, 2, 3,$$

$$S = -3$$

$$B(e_1, e_1) = -4, B(e_2, e_2) = B(e_3, e_3) = 4, B(e_4, e_4) = 0$$

$$B(e_i, e_j) = 0 \text{ for } i \neq j$$

$$R_{\text{icci}}(e_1^+, e_1^+) = 1, R_{\text{icci}}(e_2^+, e_2^+) = R_{\text{icci}}(e_3^+, e_3^+) = -1,$$

$$R_{\text{icci}}(e_4^+, e_4^+) = R_{\text{icci}}(e_i^+, e_j^+) = 0, \text{ for } i \neq j$$

Using (2), we have

$$\begin{aligned}
R_{icci} = & \left( -\frac{x_1x_2}{2} + \frac{x_1x_4}{2} + \frac{x_2^2}{2} \right) dx_1^2 + \left( \frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_2}{2} \right) dx_1dx_2 \\
& + \left( \frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_4}{2} \right) dx_1dx_3 + \left( \frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_3}{2} \right) dx_1dx_4 \\
& + \left( \sqrt{2}x_1 \right) dx_2^2 + \left( -\frac{\sqrt{2}x_2}{2} + \frac{\sqrt{2}x_4}{2} \right) dx_2dx_3 \\
& + \left( \frac{\sqrt{2}x_2}{2} + \frac{\sqrt{2}x_3}{2} \right) dx_2dx_4 + \left( \sqrt{2}x_3 \right) dx_3^2 \\
& + \left( \frac{\sqrt{2}x_3}{2} - \frac{\sqrt{2}x_4}{2} \right) dx_3dx_4 + \left( \sqrt{2}x_4 \right) dx_4^2
\end{aligned}$$

It is clear that the curvature tensor (respectively the tensor Ricci), viewed in the principal fiber bundle of the orthogonal frames, has 12 components (respectively 3).

Recall that the structure of the gravitational field, without matter (or antimatter), is determined by the Weyl tensor. The next result exhibited the components of the Weyl tensor.

**Theorem 2.** *Using the orthonormal basis (1), the components of the Weyl tensor  $W$  are given by*

$$W(e_i^+, e_j^+, e_k^+, e_l^+) = \begin{cases} 0 & \text{if } i = j = k = l, \text{ or } i = j = k \neq l, \text{ or } i \neq j \neq k \neq l \\ \frac{3}{2} & \text{if } i = k = 1, j = l = 2, \text{ or } i = k = 1, j = l = 3, \\ & \text{or } i = l = 2, j = k = 3 \\ -\frac{3}{2} & \text{if } i = l = 1, j = k = 2, \text{ or } i = l = 1, j = k = 3, \\ & \text{or } i = k = 2, j = l = 3 \\ \frac{1}{2} & \text{if } i = l = 2, 3, j = k = 4, \text{ or } i = l = 4, j = k = 2, 3, \\ & \text{or } i = k = 1, j = l = 4, \text{ or } i = k = 4, j = l = 1 \\ -\frac{1}{2} & \text{if } i = l = 1, j = k = 4, \text{ or } i = l = 4, j = k = 1, \\ & \text{or } i = k = 2, 3, j = l = 4, \text{ or } i = k = 4, j = l = 2, 3 \end{cases}$$

*Proof.* Recall, the Kulkarni–Nomizu product of two symmetric covariant 2-tensor  $h$  and  $l$  is defined by

$$\begin{aligned}
(h \otimes l)(w, x, y, z) := & h(w, z)l(x, y) + h(x, y)l(w, z) \\
& - h(w, y)l(x, z) - h(x, z)l(w, y)
\end{aligned} \tag{3}$$

Using (3) we can write de Wey tensor as

$$\begin{aligned} W &= R_m - \frac{1}{2}R_{icci} \otimes k^+ + \frac{S}{12}k^+ \otimes k^+ \\ &= R_m - \frac{1}{2}R_{icci} \otimes k^+ - \frac{1}{4}k^+ \otimes k^+ \end{aligned}$$

where  $R_m(X_1, X_2, X_3, X_4) = \langle R(X_1, X_2, X_3), X_4 \rangle_{k^+}$  for all  $X_1, X_2, X_3, X_4 \in \mathfrak{X}(G_0)$  (see [9]). From a direct calculus, the result follows.  $\square$

## 2.1 Tidal Force and Jacobi Vector Fields

**Definition 1.** For a vector  $0 \neq v \in T_\sigma(G_0)$  the tidal force operator  $F_v : v^\perp \rightarrow v^\perp$  according to  $v$  is given by  $F_v(y) = R_{yv}v$ .

The operator  $F_v$  is  $k^+$ -self-adjoint on  $v^\perp$  and it verifies the condition  $\text{tr}(F_v) = -R_{icci}(v, v)$ . A direct calculation gives

**Lemma 4.** For  $v_\sigma = \sum_{i=1}^4 f_i e_{i,\sigma}^+$ , with  $\sigma \in G_0$  we have

$$\begin{aligned} \text{tr}(F_{v_\sigma}) &= -R_{icci}(v_\sigma, v_\sigma) \\ &= -R_{icci} \left( \sum_{i=1}^4 f_i e_{i,\sigma}^+, \sum_{j=1}^4 f_j e_{j,\sigma}^+ \right) \\ &= - \sum_{i,j=1}^4 f_i f_j R_{icci}(e_{i,\sigma}^+, e_{j,\sigma}^+) \\ &= -f_1^2 + f_2^2 + f_3^2. \end{aligned}$$

**Jacobi Fields.** If  $\tau(t)$  is a geodesic, the curvature variations along of  $\tau$  are given by vector fields  $Y$ , called Jacobi fields, solutions of the differential equation

$$\frac{D^2 Y}{dt^2} = R(Y, \tau', \tau') \quad (4)$$

where  $\frac{DY}{dt}$  denotes the affine covariant derivative relative to  $D$  along  $\tau$ .

Now, we will focus on the Jacobi equation in a Lie Group  $(G_0, k^+)$  using the notion of **reflection** introduced by Bromberg-Median in [4].

Every vector field  $X$  on  $G_0$  defines a map

$$\begin{aligned} \tilde{X} : G_0 &\rightarrow \mathfrak{g} \\ \sigma &\mapsto (L_{\sigma^{-1}})_{*,\sigma} X_\sigma. \end{aligned}$$

Obviously, a vector field is left invariant if and only if the associated map is constant.



Given a curve  $\sigma : [t_0, t_1] \rightarrow G_0$ , every vector field  $Y$  along  $\sigma$  defines a curve in  $\mathfrak{g}$  :

$$\tilde{Y}(t) = (\mathbb{L}_{\sigma(t)^{-1}})_{*,\sigma(t)} Y(t)$$

and conversely, every curve in  $\mathfrak{g}$  defined on  $[0, 1]$  determines a vector field along  $\sigma$ . We say that one is the **reflection** of the other and we write either  $y^\sim = Y$  or  $Y^\sim = y$ . Notice that  $y(t) = (Y(t))^\sim$  is equivalent to  $y(t)_{\sigma(t)}^+ = Y(t)$ .

**Theorem 3.** *The functions components of the reflection field  $y(t) = y_1 e_1^+ + y_2 e_2^+ + y_3 e_3^+ + y_4 e_4^+$  in  $\mathfrak{g}$  of the Jacobi field along a geodesic  $\sigma : [0, 1] \rightarrow G$  with  $\sigma(0) = \varepsilon$  and initial velocity  $\sigma'(0) = a e_1 + b e_2 + c e_3 + d e_4$  are given by*

$$\begin{aligned} y_1(t) &= A_0 t + A_1 e^{\alpha t} + A_2 e^{-\alpha t} \\ y_2(t) &= B_0 + B_1 e^{\alpha t} + A_2 e^{-\alpha t} \\ y_3(t) &= C_0 + C_1 e^{\alpha t} + C_2 e^{-\alpha t} \\ y_4(t) &= D_0 + D_1 t \end{aligned}$$

where  $A_i, B_i, C_i, D_i$  and  $\alpha$  are constants that depend on  $a, b, c, d$ .

*Proof.* As Lemma 3 in [4] establish that

$$\frac{DY}{dt} = (y' + xy)^\sim \tag{5}$$

$$\frac{D^2 Y}{dt^2} = (y'' + 2xy' + x'y + x(xy))^\sim \tag{6}$$

from (4), we obtain the differential system on  $\mathfrak{g}$

$$\begin{aligned} y''_1 &= -\sqrt{2} c y'_2 + \sqrt{2} b y'_3 \\ y''_2 &= -\sqrt{2} c y'_1 + \sqrt{2} a y'_3 \\ y''_3 &= \sqrt{2} b y'_1 - \sqrt{2} a y'_2 \\ y''_4 &= 0 \end{aligned}$$

A direct calculation gives

$$\begin{aligned}
y_1(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left( 2at(aC_1 - bC_2 - cC_3) + \right. \\
&\quad + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left( b^2C_1 + c^2C_1 - abC_2 + c\sqrt{-a^2+b^2+c^2}C_2 - acC_3 - b\sqrt{-a^2+b^2+c^2}C_3 \right) + \\
&\quad \left. + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left( -b^2C_1 - c^2C_1 + abC_2 + c\sqrt{-a^2+b^2+c^2}C_2 + acC_3 - b\sqrt{-a^2+b^2+c^2}C_3 \right) \right) \\
y_2(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left( -2bt(-aC_1 + bC_2 + cC_3) + \right. \\
&\quad + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left( abC_1 + c\sqrt{-a^2+b^2+c^2}C_1 - a^2C_2 + c^2C_2 - bcC_3 - a\sqrt{-a^2+b^2+c^2}C_3 \right) + \\
&\quad \left. + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left( -abC_1 + c\sqrt{-a^2+b^2+c^2}C_1 + a^2C_2 - c^2C_2 + bcC_3 - a\sqrt{-a^2+b^2+c^2}C_3 \right) \right) \\
y_3(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left( -2ct(-aC_1 + bC_2 + cC_3) + \right. \\
&\quad + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left( -acC_1 - b\sqrt{-a^2+b^2+c^2}C_1 + bcC_2 + a\sqrt{-a^2+b^2+c^2}C_2 + a^2C_3 - b^2C_3 \right) + \\
&\quad \left. + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left( acC_1 - b\sqrt{-a^2+b^2+c^2}C_1 - bcC_2 + a\sqrt{-a^2+b^2+c^2}C_2 - a^2C_3 + b^2C_3 \right) \right) \\
y_4(t) &= C_4t + C_5
\end{aligned}$$

In the case of  $a^2 = b^2 + c^2$ , the components of the reflection of Jacobi fields become

$$\begin{aligned}
y_1(t) &= C_1 + \frac{1}{3}(3t + b^2t^3 + c^2t^3)C_2 + \frac{1}{6}(-3\sqrt{2}ct^2 \pm 2b\sqrt{b^2 + c^2}t^3)C_4 \\
&\quad + \frac{1}{6}(3\sqrt{2}bt^2 \pm 2c\sqrt{b^2 + c^2}t^3)C_6 \\
y_2(t) &= \frac{1}{6}(-3\sqrt{2}ct^2 \mp 2b\sqrt{b^2 + c^2}t^3)C_2 + C_3 + \frac{1}{3}(3t - b^2t^3)C_4 \\
&\quad + \frac{1}{6}(\mp 3\sqrt{2}\sqrt{b^2 + c^2}t^2 - 2bct^3)C_6 \\
y_3(t) &= \frac{1}{6}(3\sqrt{2}bt^2 \mp 2c\sqrt{b^2 + c^2}t^3)C_2 + \frac{1}{6}(\mp 3\sqrt{2}\sqrt{b^2 + c^2}t^2 - 2bct^3)C_4 \\
&\quad + C_5 + \frac{1}{3}(3t - c^2t^3)C_6 \\
y_4(t) &= C_7t + C_8
\end{aligned}$$

□

## 2.2 Parallel transport in terms of reflections

A smooth vector field  $Y$  along a smooth curve  $\alpha : [a, b] \rightarrow G_0$  is said to be parallel relatively to connection  $D$  if its variation is null, i.e.  $\frac{DY}{dt} = 0$ .

The map  $P_a^b : T_{\alpha(a)}(G_0) \rightarrow T_{\alpha(b)}(G_0)$  sending each  $v_{\alpha(a)}$  to a parallel vector  $Y$  on  $\alpha$  such that  $Y(a) = v_{\alpha(a)}$  is called **parallel transport along**  $\alpha$ . In what follows we give an outline of the parallel transport starting at  $\varepsilon$ , using the notion of reflection

**Proposition 1.** *Let  $\gamma$  be a smooth curve with initial direction  $x_0 \neq 0$  starting in the unit of  $G_0$  with tangent vector field  $\gamma'$  and  $x = \sum_{i=1}^4 x_i e_i$  the reflection of  $\gamma'$ . Then the reflection  $y := Y^\sim$  of the parallel transport of  $Y$  along  $\gamma$  is given by the solution of the following system of the differential equations on  $\mathfrak{g}$*

$$y'_1 = \frac{\sqrt{2}}{2} (x_3 y_2 - x_2 y_3)$$

$$y'_2 = \frac{\sqrt{2}}{2} (x_3 y_1 - x_1 y_3)$$

$$y'_3 = \frac{\sqrt{2}}{2} (x_1 y_2 - x_2 y_1)$$

$$y'_4 = 0$$

To illustrate this notion, we give some examples

**Example 2.** *If  $\gamma$  is an integral curve of  $e_1^+$  (timelike vector) through  $\varepsilon$  and  $y(0) = ae_1 + be_2 + ce_3 + de_4 \in T_{\gamma(0)}G_0$ , then*

$$\gamma(s) = \begin{pmatrix} \cos\left(\frac{\sqrt{2}}{2}s\right) & \sin\left(\frac{\sqrt{2}}{2}s\right) \\ -\sin\left(\frac{\sqrt{2}}{2}s\right) & \cos\left(\frac{\sqrt{2}}{2}s\right) \end{pmatrix}$$

and the reflection of  $\gamma'(s)$  on  $\mathfrak{g}$  is  $x(s) = e_1$ . So, we have the following system of differential equations

$$y'_1 = \frac{1}{2} (y_3 + y_2)$$

$$y'_2 = -\frac{1}{2} y_1$$

$$y'_3 = -\frac{1}{2} y_1$$

$$y'_4 = 0$$

Taking as initial condition  $y(0)$ , the components of the reflection  $y(s)$  in  $\mathfrak{g}$  are

$$y_1(s) = a \cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}}{2} (b + c) \sin\left(\frac{\sqrt{2}}{2}s\right)$$

$$y_2(s) = \frac{b - c}{2} + \frac{b + c}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2} a \sin\left(\frac{\sqrt{2}}{2}s\right)$$

$$y_3(s) = \frac{-b + c}{2} + \frac{b + c}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2} a \sin\left(\frac{\sqrt{2}}{2}s\right)$$

$$y_4(s) = d$$

Hence, the components of the parallel vector field  $Y$  along  $\gamma$  such that  $Y(0) = y^+(0)_{\gamma(0)}$  are given by

$$\begin{aligned} Y_1(s) &= a \cos^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}a}{2} \sin^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{(b+c)(\sqrt{2}+2)}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) \cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{c-b}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) \\ Y_2(s) &= \frac{b+c}{2} \cos^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{b-c}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}a}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) \sin\left(\frac{\sqrt{2}}{2}s\right) + d \sin\left(\frac{\sqrt{2}}{2}s\right) \\ Y_3(s) &= \frac{b+c}{2} \cos^2\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}(b+c) \sin^2\left(\frac{\sqrt{2}}{2}s\right) - \left(\frac{\sqrt{2}}{2}+1\right) a \sin\left(\frac{\sqrt{2}}{2}s\right) \cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{c-b}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) \\ Y_4(s) &= \frac{\sqrt{2}}{2} a \sin^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{c-b}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) - \frac{b+c}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) \cos\left(\frac{\sqrt{2}}{2}s\right) + d \cos\left(\frac{\sqrt{2}}{2}s\right) \end{aligned}$$

**Example 3.** If  $\gamma(s)$  is an integral curve of  $Y$  through  $\varepsilon$ , such that  $Y_\gamma(0)$  is a lightlike vector  $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  i.e.  $a^2 + 2bc + d^2 = 0$ , then

$$\gamma(s) = e^{\frac{a+d}{2}s} \begin{pmatrix} \cos\left(\frac{\theta}{2}s\right) + \frac{(a-d)}{\theta} \sin\left(\frac{\theta}{2}s\right) & \frac{2b}{\theta} \sin\left(\frac{\theta}{2}s\right) \\ \frac{2c}{\theta} \sin\left(\frac{\theta}{2}s\right) & \cos\left(\frac{\theta}{2}s\right) - \frac{(a-d)}{\theta} \sin\left(\frac{\theta}{2}s\right) \end{pmatrix}$$

and the reflection of  $\gamma'(t)$  on  $\mathfrak{g}$  will be given by

$$\begin{aligned} x(s) &= \frac{a+d}{2} e^{\frac{a+d}{2}s} I_{2 \times 2} + \\ &+ \frac{e^{(a+d)s}}{\det \gamma} \begin{pmatrix} \frac{a-d}{2} - \left(\frac{\theta}{2} + \frac{(a-d)^2}{2\theta} + \frac{2bc}{\theta}\right) \sin\left(\frac{\theta}{2}s\right) \cos\left(\frac{\theta}{2}s\right) & b \\ c & -\frac{a-d}{2} - \left(\frac{\theta}{2} + \frac{(a-d)^2}{2\theta} + \frac{2bc}{\theta}\right) \sin\left(\frac{\theta}{2}s\right) \cos\left(\frac{\theta}{2}s\right) \end{pmatrix} \end{aligned}$$

where  $\theta = \operatorname{Re}(\sqrt{2bc - 2ad})$  and  $\det(\gamma(s)) = e^{(a+d)s} \left( \cos\left(\frac{\theta}{2}s\right) + 2\left(\frac{ad-bc}{\theta^2}\right) \sin\left(\frac{\theta}{2}s\right) \right)$

The writing and solution of the system of differential equations are left to the reader.

## 2.3 Isometries of $(G_0, k^+)$

Recall that we have

**Lemma 5.** *The left and right multiplication are isometries of  $(G_0, k^+)$ , and consequently the inversion map and the map  $I_\sigma$ , defined by  $I_\sigma(\tau) := \sigma\tau^{-1}\sigma$ , for any  $\sigma, \tau \in G_0$ , is an isometry. Consequently,  $(G_0, k^+)$  is a **symmetric space**.*

In fact,  $I_\sigma$  inverse the geodesic, this means that

$$I_\sigma(\gamma(s)) = \sigma\gamma(-s)\sigma$$

where  $\gamma(s) = e^{su}$  is a geodesic with  $\gamma(0) = I_\varepsilon \in G_0$  and  $\gamma'(0) = u \in \mathfrak{g}$ .

In particular for  $\sigma = \varepsilon$  and  $u = e_1$

$$I_\varepsilon(\gamma(s)) = \begin{pmatrix} \cos\left(\frac{\sqrt{2}}{2}s\right) & -\sin\left(\frac{\sqrt{2}}{2}s\right) \\ \sin\left(\frac{\sqrt{2}}{2}s\right) & \cos\left(\frac{\sqrt{2}}{2}s\right) \end{pmatrix}$$

On the other hand, the metric  $k^+$  can be extended to the Lorentzian metric  $\tilde{k}^+$  over the universal covering  $\tilde{G}$  in such a way that the covering map  $p$  acts as a local isometry. Examining the isometries of the universal covering space can help us understand the isometries of  $(G_0, k^+)$  (for instance see Proposition 2.1 and Theorem 2.2 of [3]).

## 2.4 Causal Structure

Let  $W$  be a subspace of the Lorentz vector space  $(T_\varepsilon(G_0), k)$ . The causal character of  $W$  can be classified into three types, Spacelike, Timelike, or Lightlike. The set of all lightlike vectors in  $T_\sigma(G_0)$  is called the lightcone at  $\sigma \in G_0$ , this one determined the causal character of the event  $\sigma$ .

In particular, a real matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $T_\varepsilon(G_0)$  is lightlike (respectively timelike, spacelike) if it verifies the condition  $a^2 + 2bc + d^2 = 0$  (respectively  $a^2 + 2bc + d^2 < 0$ ,  $a^2 + 2bc + d^2 > 0$ ).

Now, the timecone containing  $u \in T_\varepsilon(G_0)$  is the set  $C(u)$  of timelike vectors  $v \in T_\varepsilon(G_0)$  such that  $k(u, v) < 0$ . Hence, the timecone containing  $e_1 \in T_\varepsilon(G_0)$  is

$$C(e_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c < b, a^2 + 2bc + d^2 < 0 \right\}$$

whereas

$$-C(e_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c > b, a^2 + 2bc + d^2 < 0 \right\}$$

it is the opposite timecone.

Note that the function  $f$  from matrices  $2 \times 2$  to real numbers defined as  $f(A) = a^2 + 2bc + d^2$  is a smooth map with constant rank 1, then the lightcone,  $f^{-1}(0)$ , is an embedded submanifold of dimension 3.

## 3 Flat Affine Structure on $\text{GL}(2, \mathbb{R})_0$

A flat affine connections on manifolds can be given by a natural affine representation of the universal covering of the Lie group of diffeomorphisms preserving the connection (see [11]).

Since the matrix multiplication is associative, the above Lie group  $G_0$  is endowed with a bi-invariant flat affine structure  $\nabla$  i.e. a linear connection without torsion and curvature.

It is well known that  $\nabla$  is highly geodesically incomplete (see [2]).

The universal covering group of  $G_0$  is the product manifold  $\tilde{G} = \mathbb{R} \times \text{SDP}(2)$ , where  $\text{SDP}(2)$  are the symmetric definite positive matrices and the covering map is

$$\begin{aligned} p: \mathbb{R} \times \text{SDP}(2) &\rightarrow G_0 \\ (t, T) &\mapsto O_t T \end{aligned} \tag{7}$$

here  $O_t$  denotes an orthogonal matrix.

The product of  $G_0$  rises to a product in  $\tilde{G}$  such that  $\tilde{G}$  is a Lie group and  $p$  is a homomorphism of Lie groups (see [5] pag. 53).

To find the multiplication in  $\tilde{G}$ , we must calculate the polar decomposition of  $O_t T O_r R$ , this is

$$\underbrace{O_t T O_r R (R O_{-r} T^2 O_r R)^{-\frac{1}{2}}}_{O_s} \underbrace{(R O_{-r} T^2 O_r R)^{\frac{1}{2}}}_S$$

Then the multiplication of two elements in  $\tilde{G}$  is given as follows:

$$(t, T) \cdot (r, R) = \left( s(t, r), (R O_{-r} T^2 O_r R)^{\frac{1}{2}} \right) \quad (8)$$

where  $s(t, r) \in \mathbb{R}$  can be calculated using the orthogonal matrix of order  $2 \times 2$

$$O_{s(t,r)} = O_t T O_r R (R O_{-r} T^2 O_r R)^{-\frac{1}{2}}$$

The multiplication (8) can be written as

$$(t, T) \cdot (r, R) = (\arctan(\theta) + t + r, O_{-\arctan(\theta)-t-r} T O_r R) \quad (9)$$

where

$$\theta = \frac{\text{tr}(O_{-r} T O_r R \omega)}{\text{tr}(O_{-r} T O_r R)} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A direct calculation shows that  $(\tilde{G}, \cdot)$  is a Lie group.

**Theorem 4.** *The developed map of the flat affine manifold  $(G_0, \nabla)$  in the Ehresmann's sense [6], following Koszul method [8] (see also [7]), is given by*

$$\begin{aligned} Dev : \quad \tilde{G}_0 &\rightarrow \mathbb{R}^4 \\ (y_1, y_2, y_3, y_4) &\mapsto (y_1, y_2 - 1, y_3, y_4 - 1) \end{aligned}$$

*Proof.* Let  $(x_1, x_2, x_3, x_4)$  be the natural coordinates (2) of  $G_0$ . We know that  $\nabla$  is locally isomorphic to usual connection  $\nabla^0$  on  $\mathbb{R}^4$ , then its 1-form connection  $\omega$  can be written in local coordinates  $(x_1, x_2, x_3, x_4, (X)_{ij})$ , like

$$\omega = (\omega_{ij}) = \left( \sum_{k=1}^4 Y_{ik} dX_{kj} \right), \quad \text{where } (Y_{ij}) = (X_{ij})^{-1}$$

Let  $(\tilde{G}, p)$  be the universal covering group of  $G_0$ , where  $p$  is the covering map (7), and  $\tilde{\mathfrak{g}}$  the Lie algebra of  $\tilde{G}$ .

We can lift the affine structure to  $\tilde{G}$  through  $\tilde{\omega} := p^* \omega$ . Now, we will compute the developed map of  $(\tilde{G}, \tilde{\nabla})$ . First, we find the vector fields  $Y$  on  $\tilde{G}$  such that  $\tilde{\nabla} Y = 0$ . In the local coordinates

$$\left( y_1, \begin{pmatrix} y_2 & y_3 \\ y_3 & y_4 \end{pmatrix} \right) \cong (y_1, y_2, y_3, y_4)$$

they are

$$Y = \sum_{i=1}^4 a_i \frac{\partial}{\partial y_i}, \quad \text{with } a_i \in \mathbb{R}$$

Second, we shall calculate the close  $\tilde{\mathfrak{g}}$ -valued 1-form  $\eta$  on  $\tilde{G}$  such that  $\tilde{\nabla}\eta = 0$  and  $\eta(Y) = Y_{\tilde{\varepsilon}}$ . A direct calculation shows that  $\eta = (dy_1, dy_2, dy_3, dy_4)$

Finally, the developed map is given by  $Dev(\tilde{\sigma}) = \int_C \eta$ , where  $C$  is a curve joining  $\tilde{\varepsilon} = (0, 1, 0, 1)$  and  $\tilde{\sigma}$ . Then

$$\begin{aligned} Dev : \quad \tilde{G}_0 &\rightarrow \mathbb{R}^4 \\ (y_1, y_2, y_3, y_4) &\mapsto (y_1, y_2 - 1, y_3, y_4 - 1) \end{aligned}$$

□

We have the following result (See [1])

**Theorem 5.**  $(G_0, \nabla, k^+)$  is a local (in fact global) Hessian manifold.

*Proof.* A potential function for  $k^+$  is

$$\begin{aligned} f : G_0 &\rightarrow \mathbb{R} \\ M &\mapsto \frac{1}{2} \text{tr}(M^2) \end{aligned}$$

In natural coordinates, the gradient has the expression

$$\begin{aligned} \text{grad } f &= \sum_{i=1}^4 \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \\ &= x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \end{aligned}$$

Hence

$$\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

## 4 Python Computes

```
[1]: import numpy as np
from sympy import symbols, Matrix, simplify, sqrt, solve, Transpose,
↳Trace
```

## 4.1 k-symmetric Operators

```
[2]: # Defining metric k = Traza(AB) respect to Orthonormal basis
k = Matrix([[[-1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])

# Defining variables
u11, u12, u13, u14, u21, u22, u23, u24, u31, u32, u33, u34, u41, u42, u43, u44 = symbols('u11 u12 u13 u14 u21 u22 u23 u24 u31 u32 u33 u34 u41 u42 u43 u44')

# Matrix representation of an operator in the natural basis of GL_2
u=Matrix([[u11, u12, u13, u14], [u21, u22, u23, u24], [u31, u32, u33, u34], [u41, u42, u43, u44]])
```

```
[3]: # Orthonormal basis respect to metric k=Traza(AB)
E1=Matrix([[0,sqrt(2)/2],[-sqrt(2)/2,0]]) #Timelike
E2=Matrix([[0,sqrt(2)/2],[sqrt(2)/2,0]]) #Spacelike
E3=Matrix([[sqrt(2)/2,0],[0,-sqrt(2)/2]]) #Spacelike
E4=Matrix([[sqrt(2)/2,0],[0,sqrt(2)/2]]) #Spacelike

#Compute k(u(a),b)=k(a,u(b)) on orthonormal basis
K0=Transpose(u)*k-k*u

solve([K0], u11, u12, u13, u14, u21, u22, u23, u24, u31, u32, u33, u34, u41, u42, u43, u44, dict=True)
```

```
[3]: [{u12: -u21, u13: -u31, u14: -u41, u23: u32, u24: u42, u34: u43}]
```

```
[4]: # Matrix representation of the k-symmetric operator in the orthonormal basis
ku=Matrix([[u11,u12,u13,u14],[-u12,u22,u23,u24],[-u13,u23,u33,u34],[-u14,u24,u34,u44]])
print('k-symmetric operator in the orthonormal basis:')
ku
```

The k-symmetric operator on the orthonormal basis:

```
[4]: 
$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{bmatrix}$$

```

```
[5]: # Metrics given by the k-symmetrics operator in the orthonormal basis:
Transpose(ku)*k
```



[5]: 
$$\begin{bmatrix} -u_{11} & -u_{12} & -u_{13} & -u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{bmatrix}$$

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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