



HAL
open science

Introduction to Lorentzian and Flat Affine Geometry of $GL(2, \mathbb{R})$

Alberto Medina, Andrés Villabon

► **To cite this version:**

Alberto Medina, Andrés Villabon. Introduction to Lorentzian and Flat Affine Geometry of $GL(2, \mathbb{R})$. 2024. hal-04561292v3

HAL Id: hal-04561292

<https://hal.science/hal-04561292v3>

Preprint submitted on 14 Sep 2024 (v3), last revised 28 Sep 2024 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Lorentzian and flat affine geometry on $\mathbf{GL}(2, \mathbb{R})$

Alberto Medina.

Université de Montpellier, Institute A. Grothendieck, France

Email: alberto.medina@umontpellier.fr

Email: albertomdn47@gmail.com

Andrés Villabón.

Universidad Nacional Abierta y a Distancia, Colombia

Email: edgar.villabon@unad.edu.co

Abstract

The goal of this paper is to study the geometry of the connected unit component of the real general linear Lie group four dimensional G_0 as a Lorentzian and flat affine manifold.

As the group G_0 is naturally equipped with a bi-invariant Hessian metric k^+ , relative to the natural bi-invariant flat affine structure ∇ (see [1]), we examine these structures and the relationships between them.

The curvatures, tidal force, and Jacobi vector fields of (G_0, k^+) are determined in Section 1. Section 2 discusses the causal structure of (G_0, k^+) , while Section 3 focuses on the developed map relative to ∇ in the sense of C. Ehresmann.

keywords Lorentzian metric, flat affine structure, hessian structure, quadratic Lie groups, curvatures, Weyl tensor.

1 Introduction

Let G_0 be the connected unit component of the Lie group of isomorphisms of the real vector space \mathbb{R}^2 and \mathfrak{g} its Lie algebra. We will identify \mathfrak{g} with $T_\varepsilon(G_0)$, the tangent space of G_0 in the identity element ε , or with the space of left-invariant vector fields on G_0 .

Consider the left-invariant metric k^+ on G_0 given by the inner product k on \mathfrak{g} defined by the trace as

$$k(u, v) := \text{tr}(u \circ v)$$

and the natural bi-invariant flat affine structure ∇ given by

$$\nabla_{u+v^+} := (u \circ v)^+$$

where $u, v \in \mathfrak{g}$ and u^+, v^+ are the left-invariant vector fields on G_0 determined by $u, v \in \mathfrak{g}$. Since the operators ad_{u^+} are k^+ -antisymmetric, the right translations on G_0 are also isometries. Then k^+ is a bi-invariant metric, i.e. (G_0, k^+) is a (orthogonal or) quadratic Lie group in the Medina-Revoy's sense (see [12]).

2 The Lorentzian Structure

Consider the following k -orthonormal basis \mathcal{B} of \mathfrak{g} given by

$$\begin{aligned} e_1 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & e_2 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ e_3 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & e_4 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (1)$$

that it will be used throughout the paper.

The Lie brackets, except antisymmetries, relatives to \mathcal{B} become

$$[e_1, e_2] = \sqrt{2}e_3, \quad [e_1, e_3] = -\sqrt{2}e_2, \quad [e_2, e_3] = -\sqrt{2}e_1.$$

As the product k is given by the table

k	e_1	e_2	e_3	e_4
e_1	-1	0	0	0
e_2	0	1	0	0
e_3	0	0	1	0
e_4	0	0	0	1

k^+ is Lorentzian.

Recall that, a tangent vector u to G_0 is called timelike (respectively lightlike, spacelike) if $k(u, u) < 0$ (respectively $k(u, u) = 0$, $k(u, u) > 0$). In particular, the vector e_1 is timelike, and the others e_i are spacelike.

Putting

$$(x_1, x_2, x_3, x_4) \equiv \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad (2)$$

the metric becomes

$$\begin{aligned} k^+ &= -(e_1^+)^*(e_1^+)^* + (e_2^+)^*(e_2^+)^* + (e_3^+)^*(e_3^+)^* + (e_4^+)^*(e_4^+)^* \\ &= \frac{1}{(x_1x_4 - x_2x_3)^2} (x_4^2dx_1^2 - 2x_3x_4dx_1dx_2 - 2x_2x_4dx_1dx_3 + 2x_2x_3dx_1dx_4 + x_3^2dx_2^2 \\ &\quad + 2x_1x_4dx_2dx_3 - 2x_1x_3dx_2dx_4 + x_2^2dx_3^2 - 2x_1x_2dx_3dx_4 + x_1^2dx_4^2) \end{aligned}$$

where

$$\begin{aligned}
(e_1^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_3dx_1 + x_4dx_2 - x_1dx_3 - x_2dx_4) \\
(e_2^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (-x_3dx_1 + x_4dx_2 + x_1dx_3 - x_2dx_4) \\
(e_3^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_4dx_1 + x_3dx_2 - x_2dx_3 - x_1dx_4) \\
(e_4^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4)
\end{aligned}$$

is the dual coframe of the smooth orthogonal frame

$$\begin{aligned}
e_1^+ &= \frac{\sqrt{2}}{2} \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) \\
e_2^+ &= \frac{\sqrt{2}}{2} \left(x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) \\
e_3^+ &= \frac{\sqrt{2}}{2} \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right) \\
e_4^+ &= \frac{\sqrt{2}}{2} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \right)
\end{aligned}$$

A direct calculation allows us to obtain the following result.

Lemma 1. *Let φ, ψ be a k -symmetric linear isomorphisms. Then φ^{-1}, ψ^{-1} are k -symmetrics and $\varphi \circ \psi, \psi \circ \varphi$ are k -symmetrics, if $\varphi \circ \psi = \psi \circ \varphi$. Moreover, every polynomial $p(\varphi)$ with real coefficients is k -symmetric.*

Proof. For any $u, v \in \mathfrak{g}$ we have

$$\begin{aligned}
k(\varphi(\varphi^{-1}(u)), v) &= k(\varphi^{-1}(u), \varphi(v)) \\
&= k(u, v) \\
&= k(u, \varphi^{-1}(\varphi(v))) =
\end{aligned}$$

Since v is arbitrary and φ is isomorphism, then

$$k(u, \varphi^{-1}(w)) = k(\varphi^{-1}(u), w) \text{ for all } u, w \in \mathfrak{g}$$

On the other hand

$$\begin{aligned}
k((\varphi \circ \psi)(u), v) &= k(\varphi(\psi(u)), v) = k(\psi(u), \varphi(v)) \\
&= k(u, \psi(\varphi(v))) \\
&= k(u, \varphi(\psi(v))) = k(u, (\varphi \circ \psi)(v))
\end{aligned}$$

□

The next assertion is clear.

Lemma 2. *Any left invariant semi-Riemannian metric on a quadratic Lie group (H, g) is given by a g -symmetric linear isomorphism φ on its Lie algebra \mathfrak{h} by the formula*

$$\langle u, v \rangle_\varphi := g(\varphi(u), v), \quad u, v \in \mathfrak{h}$$

From the above lemmas, we have

Corollary 1. *Let φ, ψ be k -symmetric linear isomorphisms of \mathfrak{g} . Then the following operators determine a left-invariant (respectively right-invariant) semi-Riemannian metric on G_0*

- a) $\varphi \circ \psi$, if $\varphi \circ \psi = \psi \circ \varphi$.
- b) φ^{-1} and $p(\varphi)$, where p is a polynomial with real coefficients.

In Section 4, we use Python to find all k -symmetric operators on the Lie algebra \mathfrak{g} . From these computations, we derive the following result.

Theorem 1. *In the basis \mathcal{B} , the k -symmetric linear isomorphisms φ of \mathfrak{g} are given by the matrix representation*

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{pmatrix}$$

Moreover, the metrics given by $\langle u, v \rangle := k(\varphi(u), v)$ for any k -symmetric φ are not flat (see [1], Theorem 4.1).

Example 1. *The left-invariant semi-Riemannian metrics on G_0 given by the following k -symmetric linear isomorphisms of \mathfrak{g}*

$$\begin{aligned} \varphi_0(e_1) &= -e_1, \quad \varphi_0(e_2) = e_2, \quad \varphi_0(e_3) = e_3, \quad \varphi_0(e_4) = e_4 \\ \varphi_1(e_1) &= e_1, \quad \varphi_1(e_2) = e_2, \quad \varphi_1(e_3) = e_3, \quad \varphi_1(e_4) = e_4 \\ \varphi_2(e_1) &= e_1, \quad \varphi_2(e_2) = -e_2, \quad \varphi_2(e_3) = e_3, \quad \varphi_2(e_4) = e_4 \end{aligned}$$

have index 0, 1, 2 respectively and $\langle u, v \rangle_i := k(\varphi_i(u), v)$, for $i = 0, 1, 2$, are not flat.

For the study of the movement space in (G_0, k^+) , in addition to the tangent bundle, it is convenient to introduce other principal fiber bundles. For instance, the orthonormal fiber bundle over (G_0, k^+) facilitates the understanding of parallel transport along a curve, the holonomy groups, and the corresponding holonomy bundles.

Let $P = L(G_0)$ be the principal fiber bundle of the linear frames over G_0 . Since $\text{GL}(4, \mathbb{R})$ is 16-dimensional, P has a dimension of 20. On the other hand, as the orthogonal group $O(1, 3)$ is the dimension 6, the principal fiber bundle of the orthogonal frames over G_0 is 10-dimensional.

Now, as the Levi-Civita connection D of (G_0, k^+) can be expressed using left-invariant fields as

$$D_{u+v^+} := \frac{1}{2}[u, v]^+$$

the curvature, sectional curvature, scalar curvature, and Ricci tensors become

$$\begin{aligned} R_{u^+v^+w^+} &:= \frac{1}{4} [[u^+, v^+], w^+] \\ K(u^+, v^+) &:= \frac{1}{4} \frac{k^+([u^+, v^+], [u^+, v^+])}{k^+(u^+, u^+)k^+(v^+, v^+) - k^+(u^+, v^+)^2} \end{aligned}$$

with $\text{span}\{u^+, v^+\}$ a nondegenerate plane,

$$\begin{aligned} S &:= 2 \sum_{i < j} K(e_i^+, e_j^+) \\ R_{\text{icci}}(u, v) &:= -\frac{B(u, v)}{4} = -\frac{1}{4} \text{tr}(\text{ad}_u \circ \text{ad}_v) \end{aligned}$$

respectively. Here B is known as the Cartan-Killing form.

Consequently, we have

Lemma 3. *In terms of the above orthonormal basis (1) of (\mathfrak{g}, k) , the curvature tensor, the sectional curvature, the scalar curvature, the Cartan-Killing form and the Ricci tensor of (G_0, k^+, D) are given respectively by*

$$\begin{aligned} R(e_1^+, e_2^+, e_1^+) &= \frac{1}{2}e_2^+, R(e_1^+, e_2^+, e_2^+) = \frac{1}{2}e_1^+, R(e_1^+, e_3^+, e_1^+) = \frac{1}{2}e_3^+, \\ R(e_1^+, e_3^+, e_3^+) &= \frac{1}{2}e_1^+, R(e_2^+, e_1^+, e_1^+) = -\frac{1}{2}e_2^+, R(e_2^+, e_1^+, e_2^+) = -\frac{1}{2}e_1^+, \\ R(e_2^+, e_3^+, e_2^+) &= -\frac{1}{2}e_3^+, R(e_2^+, e_3^+, e_3^+) = \frac{1}{2}e_2^+, R(e_3^+, e_1^+, e_1^+) = -\frac{1}{2}e_3^+, \\ R(e_3^+, e_1^+, e_3^+) &= -\frac{1}{2}e_1^+, R(e_3^+, e_2^+, e_2^+) = e_3^+, R(e_3^+, e_2^+, e_3^+) = -\frac{1}{2}e_2^+ \\ R(e_i^+, e_j^+, e_k^+) &= 0 \text{ in the other cases.} \end{aligned}$$

$$K(e_1^+, e_2^+) = K(e_1^+, e_3^+) = K(e_2^+, e_3^+) = -\frac{1}{2},$$

$$K(e_i^+, e_4^+) = 0, \text{ for } i = 1, 2, 3,$$

$$S = -3$$

$$B(e_1, e_1) = -4, B(e_2, e_2) = B(e_3, e_3) = 4, B(e_4, e_4) = 0$$

$$B(e_i, e_j) = 0 \text{ for } i \neq j$$

$$R_{\text{icci}}(e_1^+, e_1^+) = 1, R_{\text{icci}}(e_2^+, e_2^+) = R_{\text{icci}}(e_3^+, e_3^+) = -1,$$

$$R_{\text{icci}}(e_4^+, e_4^+) = R_{\text{icci}}(e_i^+, e_j^+) = 0, \text{ for } i \neq j$$

Using (2), we have

$$\begin{aligned}
R_{icci} = & \left(-\frac{x_1x_2}{2} + \frac{x_1x_4}{2} + \frac{x_2^2}{2} \right) dx_1^2 + \left(\frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_2}{2} \right) dx_1dx_2 \\
& + \left(\frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_4}{2} \right) dx_1dx_3 + \left(\frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_3}{2} \right) dx_1dx_4 \\
& + \left(\sqrt{2}x_1 \right) dx_2^2 + \left(-\frac{\sqrt{2}x_2}{2} + \frac{\sqrt{2}x_4}{2} \right) dx_2dx_3 \\
& + \left(\frac{\sqrt{2}x_2}{2} + \frac{\sqrt{2}x_3}{2} \right) dx_2dx_4 + \left(\sqrt{2}x_3 \right) dx_3^2 \\
& + \left(\frac{\sqrt{2}x_3}{2} - \frac{\sqrt{2}x_4}{2} \right) dx_3dx_4 + \left(\sqrt{2}x_4 \right) dx_4^2
\end{aligned}$$

It is clear that the curvature tensor (respectively the tensor Ricci), viewed in the principal fiber bundle of the orthogonal frames, has 12 components (respectively 3).

Recall that the structure of the gravitational field, without matter (or antimatter), is determined by the Weyl tensor. The next result exhibited the components of the Weyl tensor.

Theorem 2. *Using the orthonormal basis (1), the components of the Weyl tensor W are given by*

$$W(e_i^+, e_j^+, e_k^+, e_l^+) = \begin{cases} 0 & \text{if } i = j = k = l, \text{ or } i = j = k \neq l, \text{ or } i \neq j \neq k \neq l \\ \frac{3}{2} & \text{if } i = k = 1, j = l = 2, \text{ or } i = k = 1, j = l = 3, \\ & \text{or } i = l = 2, j = k = 3 \\ -\frac{3}{2} & \text{if } i = l = 1, j = k = 2, \text{ or } i = l = 1, j = k = 3, \\ & \text{or } i = k = 2, j = l = 3 \\ \frac{1}{2} & \text{if } i = l = 2, 3, j = k = 4, \text{ or } i = l = 4, j = k = 2, 3, \\ & \text{or } i = k = 1, j = l = 4, \text{ or } i = k = 4, j = l = 1 \\ -\frac{1}{2} & \text{if } i = l = 1, j = k = 4, \text{ or } i = l = 4, j = k = 1, \\ & \text{or } i = k = 2, 3, j = l = 4, \text{ or } i = k = 4, j = l = 2, 3 \end{cases}$$

Proof. Recall, the Kulkarni–Nomizu product of two symmetric covariant 2-tensor h and l is defined by

$$\begin{aligned}
(h \otimes l)(w, x, y, z) := & h(w, z)l(x, y) + h(x, y)l(w, z) \\
& - h(w, y)l(x, z) - h(x, z)l(w, y)
\end{aligned} \tag{3}$$

Using (3) we can write de Wey tensor as

$$\begin{aligned} W &= R_m - \frac{1}{2}R_{icci} \otimes k^+ + \frac{S}{12}k^+ \otimes k^+ \\ &= R_m - \frac{1}{2}R_{icci} \otimes k^+ - \frac{1}{4}k^+ \otimes k^+ \end{aligned}$$

where $R_m(X_1, X_2, X_3, X_4) = \langle R(X_1, X_2, X_3), X_4 \rangle_{k^+}$ for all $X_1, X_2, X_3, X_4 \in \mathfrak{X}(G_0)$ (see [9]). From a direct calculus, the result follows. \square

2.1 Tidal Force and Jacobi Vector Fields

Definition 1. For a vector $0 \neq v \in T_\sigma(G_0)$ the tidal force operator $F_v : v^\perp \rightarrow v^\perp$ according to v is given by $F_v(y) = R_{yv}v$.

The operator F_v is k^+ -self-adjoint on v^\perp and it verifies the condition $\text{tr}(F_v) = -R_{icci}(v, v)$. A direct calculation gives

Lemma 4. For $v_\sigma = \sum_{i=1}^4 f_i e_{i,\sigma}^+$, with $\sigma \in G_0$ we have

$$\begin{aligned} \text{tr}(F_{v_\sigma}) &= -R_{icci}(v_\sigma, v_\sigma) \\ &= -R_{icci} \left(\sum_{i=1}^4 f_i e_{i,\sigma}^+, \sum_{j=1}^4 f_j e_{j,\sigma}^+ \right) \\ &= - \sum_{i,j=1}^4 f_i f_j R_{icci}(e_{i,\sigma}^+, e_{j,\sigma}^+) \\ &= -f_1^2 + f_2^2 + f_3^2. \end{aligned}$$

Jacobi Fields. If $\tau(t)$ is a geodesic, the curvature variations along of τ are given by vector fields Y , called Jacobi fields, solutions of the differential equation

$$\frac{D^2 Y}{dt^2} = R(Y, \tau', \tau') \quad (4)$$

where $\frac{DY}{dt}$ denotes the affine covariant derivative relative to D along τ .

Now, we will focus on the Jacobi equation in a Lie Group (G_0, k^+) using the notion of **reflection** introduced by Bromberg-Median in [4].

Every vector field X on G_0 defines a map

$$\begin{aligned} \tilde{X} : G_0 &\rightarrow \mathfrak{g} \\ \sigma &\mapsto (L_{\sigma^{-1}})_{*,\sigma} X_\sigma. \end{aligned}$$

Obviously, a vector field is left invariant if and only if the associated map is constant.

Given a curve $\sigma : [t_0, t_1] \rightarrow G_0$, every vector field Y along σ defines a curve in \mathfrak{g} :

$$\tilde{Y}(t) = (\mathbb{L}_{\sigma(t)^{-1}})_{*,\sigma(t)} Y(t)$$

and conversely, every curve in \mathfrak{g} defined on $[0, 1]$ determines a vector field along σ . We say that one is the **reflection** of the other and we write either $y^\sim = Y$ or $Y^\sim = y$. Notice that $y(t) = (Y(t))^\sim$ is equivalent to $y(t)_{\sigma(t)}^+ = Y(t)$.

Theorem 3. *The functions components of the reflection field $y(t) = y_1 e_1^+ + y_2 e_2^+ + y_3 e_3^+ + y_4 e_4^+$ in \mathfrak{g} of the Jacobi field along a geodesic $\sigma : [0, 1] \rightarrow G$ with $\sigma(0) = \varepsilon$ and initial velocity $\sigma'(0) = ae_1 + be_2 + ce_3 + de_4$ are given by*

$$\begin{aligned} y_1(t) &= A_0 t + A_1 e^{\alpha t} + A_2 e^{-\alpha t} \\ y_2(t) &= B_0 + B_1 e^{\alpha t} + A_2 e^{-\alpha t} \\ y_3(t) &= C_0 + C_1 e^{\alpha t} + C_2 e^{-\alpha t} \\ y_4(t) &= D_0 + D_1 t \end{aligned}$$

where A_i, B_i, C_i, D_i and α are constants that depend on a, b, c, d .

Proof. As Lemma 3 in [4] establish that

$$\frac{DY}{dt} = (y' + xy)^\sim \tag{5}$$

$$\frac{D^2 Y}{dt^2} = (y'' + 2xy' + x'y + x(xy))^\sim \tag{6}$$

from (4), we obtain the differential system on \mathfrak{g}

$$\begin{aligned} y''_1 &= -\sqrt{2} cy'_2 + \sqrt{2} by'_3 \\ y''_2 &= -\sqrt{2} cy'_1 + \sqrt{2} ay'_3 \\ y''_3 &= \sqrt{2} by'_1 - \sqrt{2} ay'_2 \\ y''_4 &= 0 \end{aligned}$$

A direct calculation gives

$$\begin{aligned}
y_1(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left(2at(aC_1 - bC_2 - cC_3) + \right. \\
&\quad + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left(b^2C_1 + c^2C_1 - abC_2 + c\sqrt{-a^2+b^2+c^2}C_2 - acC_3 - b\sqrt{-a^2+b^2+c^2}C_3 \right) + \\
&\quad \left. + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left(-b^2C_1 - c^2C_1 + abC_2 + c\sqrt{-a^2+b^2+c^2}C_2 + acC_3 - b\sqrt{-a^2+b^2+c^2}C_3 \right) \right) \\
y_2(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left(-2bt(-aC_1 + bC_2 + cC_3) + \right. \\
&\quad + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left(abC_1 + c\sqrt{-a^2+b^2+c^2}C_1 - a^2C_2 + c^2C_2 - bcC_3 - a\sqrt{-a^2+b^2+c^2}C_3 \right) + \\
&\quad \left. + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left(-abC_1 + c\sqrt{-a^2+b^2+c^2}C_1 + a^2C_2 - c^2C_2 + bcC_3 - a\sqrt{-a^2+b^2+c^2}C_3 \right) \right) \\
y_3(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left(-2ct(-aC_1 + bC_2 + cC_3) + \right. \\
&\quad + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left(-acC_1 - b\sqrt{-a^2+b^2+c^2}C_1 + bcC_2 + a\sqrt{-a^2+b^2+c^2}C_2 + a^2C_3 - b^2C_3 \right) + \\
&\quad \left. + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} \left(acC_1 - b\sqrt{-a^2+b^2+c^2}C_1 - bcC_2 + a\sqrt{-a^2+b^2+c^2}C_2 - a^2C_3 + b^2C_3 \right) \right) \\
y_4(t) &= C_4t + C_5
\end{aligned}$$

In the case of $a^2 = b^2 + c^2$, the components of the reflection of Jacobi fields become

$$\begin{aligned}
y_1(t) &= C_1 + \frac{1}{3}(3t + b^2t^3 + c^2t^3)C_2 + \frac{1}{6}(-3\sqrt{2}ct^2 \pm 2b\sqrt{b^2 + c^2}t^3)C_4 \\
&\quad + \frac{1}{6}(3\sqrt{2}bt^2 \pm 2c\sqrt{b^2 + c^2}t^3)C_6 \\
y_2(t) &= \frac{1}{6}(-3\sqrt{2}ct^2 \mp 2b\sqrt{b^2 + c^2}t^3)C_2 + C_3 + \frac{1}{3}(3t - b^2t^3)C_4 \\
&\quad + \frac{1}{6}(\mp 3\sqrt{2}\sqrt{b^2 + c^2}t^2 - 2bct^3)C_6 \\
y_3(t) &= \frac{1}{6}(3\sqrt{2}bt^2 \mp 2c\sqrt{b^2 + c^2}t^3)C_2 + \frac{1}{6}(\mp 3\sqrt{2}\sqrt{b^2 + c^2}t^2 - 2bct^3)C_4 \\
&\quad + C_5 + \frac{1}{3}(3t - c^2t^3)C_6 \\
y_4(t) &= C_7t + C_8
\end{aligned}$$

□

2.2 Parallel transport in terms of reflections

A smooth vector field Y along a smooth curve $\alpha : [a, b] \rightarrow G_0$ is said to be parallel relatively to connection D if its variation is null, i.e. $\frac{DY}{dt} = 0$.

The map $P_a^b : T_{\alpha(a)}(G_0) \rightarrow T_{\alpha(b)}(G_0)$ sending each $v_{\alpha(a)}$ to a parallel vector Y on α such that $Y(a) = v_{\alpha(a)}$ is called **parallel transport along** α . In what follows we give an outline of the parallel transport starting at ε , using the notion of reflection

Proposition 1. *Let γ be a smooth curve with initial direction $x_0 \neq 0$ starting in the unit of G_0 with tangent vector field γ' and $x = \sum_{i=1}^4 x_i e_i$ the reflection of γ' . Then the reflection $y := Y^\sim$ of the parallel transport of Y along γ is given by the solution of the following system of the differential equations on \mathfrak{g}*

$$y'_1 = \frac{\sqrt{2}}{2} (x_3 y_2 - x_2 y_3)$$

$$y'_2 = \frac{\sqrt{2}}{2} (x_3 y_1 - x_1 y_3)$$

$$y'_3 = \frac{\sqrt{2}}{2} (x_1 y_2 - x_2 y_1)$$

$$y'_4 = 0$$

To illustrate this notion, we give some examples

Example 2. *If γ is an integral curve of e_1^+ (timelike vector) through ε and $y(0) = ae_1 + be_2 + ce_3 + de_4 \in T_{\gamma(0)}G_0$, then*

$$\gamma(s) = \begin{pmatrix} \cos\left(\frac{\sqrt{2}}{2}s\right) & \sin\left(\frac{\sqrt{2}}{2}s\right) \\ -\sin\left(\frac{\sqrt{2}}{2}s\right) & \cos\left(\frac{\sqrt{2}}{2}s\right) \end{pmatrix}$$

and the reflection of $\gamma'(s)$ on \mathfrak{g} is $x(s) = e_1$. So, we have the following system of differential equations

$$y'_1 = \frac{1}{2} (y_3 + y_2)$$

$$y'_2 = -\frac{1}{2} y_1$$

$$y'_3 = -\frac{1}{2} y_1$$

$$y'_4 = 0$$

Taking as initial condition $y(0)$, the components of the reflection $y(s)$ in \mathfrak{g} are

$$y_1(s) = a \cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}}{2} (b + c) \sin\left(\frac{\sqrt{2}}{2}s\right)$$

$$y_2(s) = \frac{b - c}{2} + \frac{b + c}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2} a \sin\left(\frac{\sqrt{2}}{2}s\right)$$

$$y_3(s) = \frac{-b + c}{2} + \frac{b + c}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2} a \sin\left(\frac{\sqrt{2}}{2}s\right)$$

$$y_4(s) = d$$

Hence, the components of the parallel vector field Y along γ such that $Y(0) = y^+(0)_{\gamma(0)}$ are given by

$$\begin{aligned} Y_1(s) &= a \cos^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}a}{2} \sin^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{(b+c)(\sqrt{2}+2)}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) \cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{c-b}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) \\ Y_2(s) &= \frac{b+c}{2} \cos^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{b-c}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}a}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) \sin\left(\frac{\sqrt{2}}{2}s\right) + d \sin\left(\frac{\sqrt{2}}{2}s\right) \\ Y_3(s) &= \frac{b+c}{2} \cos^2\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}(b+c) \sin^2\left(\frac{\sqrt{2}}{2}s\right) - \left(\frac{\sqrt{2}}{2}+1\right) a \sin\left(\frac{\sqrt{2}}{2}s\right) \cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{c-b}{2} \cos\left(\frac{\sqrt{2}}{2}s\right) \\ Y_4(s) &= \frac{\sqrt{2}}{2} a \sin^2\left(\frac{\sqrt{2}}{2}s\right) + \frac{c-b}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) - \frac{b+c}{2} \sin\left(\frac{\sqrt{2}}{2}s\right) \cos\left(\frac{\sqrt{2}}{2}s\right) + d \cos\left(\frac{\sqrt{2}}{2}s\right) \end{aligned}$$

Example 3. If $\gamma(s)$ is an integral curve of Y through ε , such that $Y_\gamma(0)$ is a lightlike vector $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ i.e. $a^2 + 2bc + d^2 = 0$, then

$$\gamma(s) = e^{\frac{a+d}{2}s} \begin{pmatrix} \cos\left(\frac{\theta}{2}s\right) + \frac{(a-d)}{\theta} \sin\left(\frac{\theta}{2}s\right) & \frac{2b}{\theta} \sin\left(\frac{\theta}{2}s\right) \\ \frac{2c}{\theta} \sin\left(\frac{\theta}{2}s\right) & \cos\left(\frac{\theta}{2}s\right) - \frac{(a-d)}{\theta} \sin\left(\frac{\theta}{2}s\right) \end{pmatrix}$$

and the reflection of $\gamma'(t)$ on \mathfrak{g} will be given by

$$\begin{aligned} x(s) &= \frac{a+d}{2} e^{\frac{a+d}{2}s} I_{2 \times 2} + \\ &+ \frac{e^{(a+d)s}}{\det \gamma} \begin{pmatrix} \frac{a-d}{2} - \left(\frac{\theta}{2} + \frac{(a-d)^2}{2\theta} + \frac{2bc}{\theta}\right) \sin\left(\frac{\theta}{2}s\right) \cos\left(\frac{\theta}{2}s\right) & b \\ c & -\frac{a-d}{2} - \left(\frac{\theta}{2} + \frac{(a-d)^2}{2\theta} + \frac{2bc}{\theta}\right) \sin\left(\frac{\theta}{2}s\right) \cos\left(\frac{\theta}{2}s\right) \end{pmatrix} \end{aligned}$$

where $\theta = \operatorname{Re}(\sqrt{2bc - 2ad})$ and $\det(\gamma(s)) = e^{(a+d)s} \left(\cos\left(\frac{\theta}{2}s\right) + 2\left(\frac{ad-bc}{\theta^2}\right) \sin\left(\frac{\theta}{2}s\right) \right)$

The writing and solution of the system of differential equations are left to the reader.

2.3 Isometries of (G_0, k^+)

Recall that we have

Lemma 5. The left and right multiplication are isometries of (G_0, k^+) , and consequently the inversion map and the map I_σ , defined by $I_\sigma(\tau) := \sigma\tau^{-1}\sigma$, for any $\sigma, \tau \in G_0$, is an isometry. Consequently, (G_0, k^+) is a **symmetric space**.

In fact, I_σ inverse the geodesic, this means that

$$I_\sigma(\gamma(s)) = \sigma\gamma(-s)\sigma$$

where $\gamma(s) = e^{su}$ is a geodesic with $\gamma(0) = I_\varepsilon \in G_0$ and $\gamma'(0) = u \in \mathfrak{g}$.

In particular for $\sigma = \varepsilon$ and $u = e_1$

$$I_\varepsilon(\gamma(s)) = \begin{pmatrix} \cos\left(\frac{\sqrt{2}}{2}s\right) & -\sin\left(\frac{\sqrt{2}}{2}s\right) \\ \sin\left(\frac{\sqrt{2}}{2}s\right) & \cos\left(\frac{\sqrt{2}}{2}s\right) \end{pmatrix}$$

On the other hand, the metric k^+ can be extended to the Lorentzian metric \tilde{k}^+ over the universal covering \tilde{G} in such a way that the covering map p acts as a local isometry. Examining the isometries of the universal covering space can help us understand the isometries of (G_0, k^+) (for instance see Proposition 2.1 and Theorem 2.2 of [3]).

2.4 Causal Structure

Let W be a subspace of the Lorentz vector space $(T_\varepsilon(G_0), k)$. The causal character of W can be classified into three types, Spacelike, Timelike, or Lightlike. The set of all lightlike vectors in $T_\sigma(G_0)$ is called the lightcone at $\sigma \in G_0$, this one determined the causal character of the event σ .

In particular, a real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $T_\varepsilon(G_0)$ is lightlike (respectively timelike, spacelike) if it verifies the condition $a^2 + 2bc + d^2 = 0$ (respectively $a^2 + 2bc + d^2 < 0$, $a^2 + 2bc + d^2 > 0$).

Now, the timecone containing $u \in T_\varepsilon(G_0)$ is the set $C(u)$ of timelike vectors $v \in T_\varepsilon(G_0)$ such that $k(u, v) < 0$. Hence, the timecone containing $e_1 \in T_\varepsilon(G_0)$ is

$$C(e_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c < b, a^2 + 2bc + d^2 < 0 \right\}$$

whereas

$$-C(e_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c > b, a^2 + 2bc + d^2 < 0 \right\}$$

it is the opposite timecone.

Note that the function f from matrices 2×2 to real numbers defined as $f(A) = a^2 + 2bc + d^2$ is a smooth map with constant rank 1, then the lightcone, $f^{-1}(0)$, is an embedded submanifold of dimension 3.

3 Flat Affine Structure on $\text{GL}(2, \mathbb{R})_0$

A flat affine connections on manifolds can be given by a natural affine representation of the universal covering of the Lie group of diffeomorphisms preserving the connection (see [11]).

Since the matrix multiplication is associative, the above Lie group G_0 is endowed with a bi-invariant flat affine structure ∇ i.e. a linear connection without torsion and curvature.

It is well known that ∇ is highly geodesically incomplete (see [2]).

The universal covering group of G_0 is the product manifold $\tilde{G} = \mathbb{R} \times \text{SDP}(2)$, where $\text{SDP}(2)$ are the symmetric definite positive matrices and the covering map is

$$\begin{aligned} p: \mathbb{R} \times \text{SDP}(2) &\rightarrow G_0 \\ (t, T) &\mapsto O_t T \end{aligned} \tag{7}$$

here O_t denotes an orthogonal matrix.

The product of G_0 rises to a product in \tilde{G} such that \tilde{G} is a Lie group and p is a homomorphism of Lie groups (see [5] pag. 53).

To find the multiplication in \tilde{G} , we must calculate the polar decomposition of $O_t T O_r R$, this is

$$\underbrace{O_t T O_r R (R O_{-r} T^2 O_r R)^{-\frac{1}{2}}}_{O_s} \underbrace{(R O_{-r} T^2 O_r R)^{\frac{1}{2}}}_S$$

Then the multiplication of two elements in \tilde{G} is given as follows:

$$(t, T) \cdot (r, R) = \left(s(t, r), (R O_{-r} T^2 O_r R)^{\frac{1}{2}} \right) \quad (8)$$

where $s(t, r) \in \mathbb{R}$ can be calculated using the orthogonal matrix of order 2×2

$$O_{s(t,r)} = O_t T O_r R (R O_{-r} T^2 O_r R)^{-\frac{1}{2}}$$

The multiplication (8) can be written as

$$(t, T) \cdot (r, R) = (\arctan(\theta) + t + r, O_{-\arctan(\theta)-t-r} T O_r R) \quad (9)$$

where

$$\theta = \frac{\text{tr}(O_{-r} T O_r R \omega)}{\text{tr}(O_{-r} T O_r R)} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A direct calculation shows that (\tilde{G}, \cdot) is a Lie group.

Theorem 4. *The developed map of the flat affine manifold (G_0, ∇) in the Ehresmann's sense [6], following Koszul method [8] (see also [7]), is given by*

$$\begin{aligned} Dev : \quad \tilde{G}_0 &\rightarrow \mathbb{R}^4 \\ (y_1, y_2, y_3, y_4) &\mapsto (y_1, y_2 - 1, y_3, y_4 - 1) \end{aligned}$$

Proof. Let (x_1, x_2, x_3, x_4) be the natural coordinates (2) of G_0 . We know that ∇ is locally isomorphic to usual connection ∇^0 on \mathbb{R}^4 , then its 1-form connection ω can be written in local coordinates $(x_1, x_2, x_3, x_4, (X)_{ij})$, like

$$\omega = (\omega_{ij}) = \left(\sum_{k=1}^4 Y_{ik} dX_{kj} \right), \quad \text{where } (Y_{ij}) = (X_{ij})^{-1}$$

Let (\tilde{G}, p) be the universal covering group of G_0 , where p is the covering map (7), and $\tilde{\mathfrak{g}}$ the Lie algebra of \tilde{G} .

We can lift the affine structure to \tilde{G} through $\tilde{\omega} := p^* \omega$. Now, we will compute the developed map of $(\tilde{G}, \tilde{\nabla})$. First, we find the vector fields Y on \tilde{G} such that $\tilde{\nabla} Y = 0$. In the local coordinates

$$\left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_3 & y_4 \end{pmatrix} \right) \cong (y_1, y_2, y_3, y_4)$$

they are

$$Y = \sum_{i=1}^4 a_i \frac{\partial}{\partial y_i}, \quad \text{with } a_i \in \mathbb{R}$$

Second, we shall calculate the close $\tilde{\mathfrak{g}}$ -valued 1-form η on \tilde{G} such that $\tilde{\nabla}\eta = 0$ and $\eta(Y) = Y_{\tilde{\varepsilon}}$. A direct calculation shows that $\eta = (dy_1, dy_2, dy_3, dy_4)$

Finally, the developed map is given by $Dev(\tilde{\sigma}) = \int_C \eta$, where C is a curve joining $\tilde{\varepsilon} = (0, 1, 0, 1)$ and $\tilde{\sigma}$. Then

$$\begin{aligned} Dev : \quad \tilde{G}_0 &\rightarrow \mathbb{R}^4 \\ (y_1, y_2, y_3, y_4) &\mapsto (y_1, y_2 - 1, y_3, y_4 - 1) \end{aligned}$$

□

We have the following result (See [1])

Theorem 5. (G_0, ∇, k^+) is a local (in fact global) Hessian manifold.

Proof. A potential function for k^+ is

$$\begin{aligned} f : G_0 &\rightarrow \mathbb{R} \\ M &\mapsto \frac{1}{2} \text{tr}(M^2) \end{aligned}$$

In natural coordinates, the gradient has the expression

$$\begin{aligned} \text{grad } f &= \sum_{i=1}^4 \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \\ &= x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \end{aligned}$$

Hence

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

4 Python Computes

```
[1]: import numpy as np
from sympy import symbols, Matrix, simplify, sqrt, solve, Transpose, \
↳Trace
```

4.1 k-symmetric Operators

```
[2]: # Defining metric k = Traza(AB) respect to Orthonormal basis
k = Matrix([[[-1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])

# Defining variables
u11, u12, u13, u14, u21, u22, u23, u24, u31, u32, u33, u34, u41, u42, u43, u44 = symbols('u11 u12 u13 u14 u21 u22 u23 u24 u31 u32 u33 u34 u41 u42 u43 u44')

# Matrix representation of an operator in the natural basis of GL_2
u=Matrix([[u11, u12, u13, u14], [u21, u22, u23, u24], [u31, u32, u33, u34], [u41, u42, u43, u44]])
```

```
[3]: # Orthonormal basis respect to metric k=Traza(AB)
E1=Matrix([[0,sqrt(2)/2],[-sqrt(2)/2,0]]) #Timelike
E2=Matrix([[0,sqrt(2)/2],[sqrt(2)/2,0]]) #Spacelike
E3=Matrix([[sqrt(2)/2,0],[0,-sqrt(2)/2]]) #Spacelike
E4=Matrix([[sqrt(2)/2,0],[0,sqrt(2)/2]]) #Spacelike

#Compute k(u(a),b)=k(a,u(b)) on orthonormal basis
K0=Transpose(u)*k-k*u

solve([K0], u11, u12, u13, u14, u21, u22, u23, u24, u31, u32, u33, u34, u41, u42, u43, u44, dict=True)
```

```
[3]: [{u12: -u21, u13: -u31, u14: -u41, u23: u32, u24: u42, u34: u43}]
```

```
[4]: # Matrix representation of the k-symmetric operator in the orthonormal basis
ku=Matrix([[u11,u12,u13,u14],[-u12,u22,u23,u24],[-u13,u23,u33,u34],[-u14,u24,u34,u44]])
print('k-symmetric operator in the orthonormal basis:')
ku
```

The k-symmetric operator on the orthonormal basis:

```
[4]: 
$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{bmatrix}$$

```

```
[5]: # Metrics given by the k-symmetrics operator in the orthonormal basis:
Transpose(ku)*k
```


[5]:
$$\begin{bmatrix} -u_{11} & -u_{12} & -u_{13} & -u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{bmatrix}$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

All sources of funding of the study must be disclosed.

Conflict of interest

The authors declare there is no conflicts of interest.

References

- [1] A. Aubert, A. Medina, A, Groupes de Lie semi-Riemanniens Plats, *Tohoku Math. J.*, **55** (2003), 487–506.
- [2] M. Bordemann, A. Medina, Le groupe des transformations affines d'un groupe de Lie à structure affine bi-invariante, *Research and Exposition in Mathematics*, **25** (2002), 149–179.
- [3] S. Bromberg, A. Medina, Geometry of Oscillator Groups and Locally Symmetric Manifolds, *Geometriae Dedicata*, **106** (2011), 97–111.
- [4] S. Bromberg, A. Medina, Left invariant semi Riemannian metrics on quadratic Lie groups, preprint, arXiv:1103.1356.
- [5] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, 1946.
- [6] C. Ehresmann, Sur les espaces localement homogènes, *L'Ens Math*, **35** (1936), 317–333.
- [7] W. Goldman, *Geometric structures on manifolds*, American Mathematical Society, 2023.
- [8] J. Koszul, Variétés localement plates et convexité, *Osaka J. Math*, **2** (1965), 285-290.
- [9] J. Lee, *Introduction to Riemannian Manifolds*, Springer, (2028).

- [10] A. Medina, Symmetric Space Time Model, *Institute A. Grothendieck. Université de Montpellier*. [To appear].
- [11] A. Medina, O. Saldarriaga, and A. Villabon, Flat Affine Manifolds And Their Transformations, *Manuscripta Mathematica*, **166** (2021), 1-19.
- [12] A. Medina, Ph. Revoy, Algèbres de Lie et produit scalaire invariant, *Annales scientifiques de l'École Normale Supérieure, Série 4, Tome 18*, **3** (1085), 553-561.
- [13] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic press, (1983).