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Introduction to Lorentzian and Flat Affine Geometry of $\mathbf{GL}(2, \mathbb{R})$

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Abstract

The goal of this paper is to study the geometry of the connected unit component of the real general linear Lie group 4 dimensional G_0 as a Lorentzian and flat affine manifold.

As the group G_0 is naturally equipped with a bi-invariant Hessian metric k^+ , relative to a bi-invariant flat affine structure ∇ , we examine these structures and the relationships between them. Both structures are defined using the Lie algebra \mathfrak{g} , the first one through the trace $k(u, v) := \text{trace}(u \circ v)$ and the second by the composition $\nabla_{u+v}^+ := (u \circ v)^+$, where $u, v \in \mathfrak{g}$.

The curvatures, tidal force, and Jacobi vector fields of (G_0, k^+) are determined in Section 1. Section 2 discusses the causal structure of (G_0, k^+) , while Section 3 focuses on the developed map relative to ∇ in the sense of C. Ehresmann.

Keywords: Lorentzian metric, flat affine structure, quadratic Lie groups, curvatures, Weyl tensor.

1 Lorentzian Structure

Let G_0 be the connected unit component of the Lie group of isomorphisms of the real vector space \mathbb{R}^2 and \mathfrak{g} its Lie algebra. We will identify \mathfrak{g} with $T_\varepsilon(G_0)$, the tangent space of G_0 in the identity element ε , or with the space of left-invariant vector fields on G_0 .

Consider the left-invariant metric k^+ on G_0 given by the inner product k on \mathfrak{g} defined by

$$k(u, v) := \text{trace}(u \circ v), \text{ where } u, v \in \mathfrak{g}.$$

If u^+ is the left-invariant vector field on G_0 determined by $u \in \mathfrak{g}$, since the operators ad_{u^+} are k^+ -antisymmetric, the right translations on G_0 are also isometries. Then k^+ is a bi-invariant metric, i.e. (G_0, k^+) is a quadratic Lie group in the Medina-Revoy's sense.

Consider the following k -orthonormal basis \mathcal{B} of \mathfrak{g} given by

$$\begin{aligned} e_1 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & e_2 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ e_3 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & e_4 &:= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (1)$$

The Lie brackets, except antisymmetries, relatives to \mathcal{B} become

$$[e_1, e_2] = \sqrt{2}e_3, \quad [e_1, e_3] = -\sqrt{2}e_2, \quad [e_2, e_3] = -\sqrt{2}e_1.$$

As k is given by the table

k	e_1	e_2	e_3	e_4
e_1	-1	0	0	0
e_2	0	1	0	0
e_3	0	0	1	0
e_4	0	0	0	1

k^+ is Lorentzian.

Recall that, a tangent vector u to G_0 is called timelike (respectively lightlike, spacelike) if $k(u, u) < 0$ (respectively $k(u, u) = 0$, $k(u, u) > 0$). In particular, the vector e_1 is timelike, and the others e_i are spacelike.

We will show that every left-invariant semi-Riemannian metric on G_0 is non-flat.

The following assertion is clear.

Lemma 1. *Any left invariant semi-Riemannian metric on a quadratic Lie group (H, g) is given by a linear isomorphism g -symmetric φ on its Lie algebra \mathfrak{h} by the formula*

$$\langle u, v \rangle_\varphi := g(\varphi(u), v), \quad u, v \in \mathfrak{h}$$

From Lemma 1, we have

Proposition 1. *Every left-invariant (respectively right-invariant) semi-Riemannian metric on G_0 is not flat.*

The proposition is a consequence of the following lemma

Lemma 2. *The left-invariant semi-Riemannian metrics on G_0 are given by the following k -symmetric linear isomorphisms of \mathfrak{g}*

$$\begin{aligned} \varphi_0(e_1) &= -e_1, \quad \varphi_0(e_2) = e_2, \quad \varphi_0(e_3) = e_3, \quad \varphi_0(e_4) = e_4 \\ \varphi_1(e_1) &= e_1, \quad \varphi_1(e_2) = e_2, \quad \varphi_1(e_3) = e_3, \quad \varphi_1(e_4) = e_4 \\ \varphi_2(e_1) &= e_1, \quad \varphi_2(e_2) = -e_2, \quad \varphi_2(e_3) = e_3, \quad \varphi_2(e_4) = e_4 \end{aligned}$$

and their index are 0, 1, 2 respectively. Moreover, the curvatures of $\langle u, v \rangle_i := k(\varphi_i(u), v)$ for $i = 0, 1, 2$ are not flat.

These computations are provided in Section 3.

For the study of the movement space in (G_0, k^+) , in addition to the tangent bundle, it is convenient to introduce some principal fiber bundles. For instance, the orthonormal fiber bundle over (G_0, k^+) facilitates the understanding of parallel transport along a curve, in particular the holonomy groups and the corresponding holonomy bundles.

Let $P = L(G_0)$ be the principal fiber bundle of the linear frames over G_0 . Since $\text{GL}(4, \mathbb{R})$ is 16-dimensional, P has a dimension of 20. On the other hand, as the orthogonal group $O(1, 3)$ is the dimension 6, the principal fiber bundle of the orthogonal frames over G_0 is 10-dimensional.

Now, the Levi-Civita connection D of (G_0, k^+) can be expressed using left-invariant fields as

$$D_{u^+}v^+ := \frac{1}{2}[u, v]^+$$

so the curvature, sectional curvature, scalar curvature, and Ricci tensors become

$$\begin{aligned} R_{u^+v^+w^+} &:= \frac{1}{4} [[u^+, v^+], w^+] \\ K(u^+, v^+) &:= \frac{1}{4} \frac{k^+([u^+, v^+], [u^+, v^+])}{k^+(u^+, u^+)k^+(v^+, v^+) - k^+(u^+, v^+)^2} \end{aligned}$$

where $\text{span}\{u^+, v^+\}$ is a nondegenerate plane,

$$\begin{aligned} S &:= 2 \sum_{i < j} K(e_i^+, e_j^+) \\ R_{\text{icci}}(u, v) &:= -\frac{B(u, v)}{4} = -\frac{1}{4} \text{trace}(\text{ad}_u \circ \text{ad}_v) \end{aligned}$$

respectively. Here B is known as the Cartan-Killing form.

Consequently, we have

Lemma 3. *In terms of the above orthonormal basis (1) of (\mathfrak{g}, k) , the curvature tensor, the sectional curvature, the scalar curvature, the Cartan-Killing form and the Ricci tensor of (G_0, k^+, D) are given respectively by*

$$\begin{aligned} R(e_1^+, e_2^+, e_1^+) &= \frac{1}{2}e_2^+, R(e_1^+, e_2^+, e_2^+) = \frac{1}{2}e_1^+, R(e_1^+, e_3^+, e_1^+) = \frac{1}{2}e_3^+, \\ R(e_1^+, e_3^+, e_3^+) &= \frac{1}{2}e_1^+, R(e_2^+, e_1^+, e_1^+) = -\frac{1}{2}e_2^+, R(e_2^+, e_1^+, e_2^+) = -\frac{1}{2}e_1^+, \\ R(e_2^+, e_3^+, e_2^+) &= -\frac{1}{2}e_3^+, R(e_2^+, e_3^+, e_3^+) = \frac{1}{2}e_2^+, R(e_3^+, e_1^+, e_1^+) = -\frac{1}{2}e_3^+, \\ R(e_3^+, e_1^+, e_3^+) &= -\frac{1}{2}e_1^+, R(e_3^+, e_2^+, e_2^+) = e_3^+, R(e_3^+, e_2^+, e_3^+) = -\frac{1}{2}e_2^+ \\ R(e_i^+, e_j^+, e_k^+) &= 0 \text{ in the other cases.} \end{aligned}$$

$$\begin{aligned}
K(e_1^+, e_2^+) &= K(e_1^+, e_3^+) = K(e_2^+, e_3^+) = -\frac{1}{2}, \\
K(e_i^+, e_4^+) &= 0, \quad \text{for } i = 1, 2, 3, \\
S &= -3 \\
B(e_1, e_1) &= -4, \quad B(e_2, e_2) = B(e_3, e_3) = 4, \quad B(e_4, e_4) = 0 \\
B(e_i, e_j) &= 0 \quad \text{for } i \neq j \\
R_{icci}(e_1^+, e_1^+) &= 1, \quad R_{icci}(e_2^+, e_2^+) = R_{icci}(e_3^+, e_3^+) = -1, \\
R_{icci}(e_4^+, e_4^+) &= R_{icci}(e_i^+, e_j^+) = 0, \quad \text{for } i \neq j
\end{aligned}$$

It is clear that the curvature tensor (respectively the tensor Ricci), viewed in the principal fiber bundle of the orthogonal frames, has 12 components (respectively 3).

Recall that the structure of the gravitational field, without matter (or antimatter), is determined by the Weyl tensor. The next result exhibited the components of the Weyl tensor.

Theorem 1. *Using the orthonormal basis (1), the Weyl tensor components are given by:*

$$W(e_i^+, e_j^+, e_k^+, e_l^+) = \begin{cases} 0 & \text{if } i = j = k = l, \text{ or } i = j = k \neq l, \text{ or } i \neq j \neq k \neq l \\ \frac{3}{2} & \text{if } i = k = 1, j = l = 2, \text{ or } i = k = 1, j = l = 3, \\ & \text{or } i = l = 2, j = k = 3 \\ -\frac{3}{2} & \text{if } i = l = 1, j = k = 2, \text{ or } i = l = 1, j = k = 3, \\ & \text{or } i = k = 2, j = l = 3 \\ \frac{1}{2} & \text{if } i = l = 2, 3, j = k = 4, \text{ or } i = l = 4, j = k = 2, 3, \\ & \text{or } i = k = 1, j = l = 4, \text{ or } i = k = 4, j = l = 1 \\ -\frac{1}{2} & \text{if } i = l = 1, j = k = 4, \text{ or } i = l = 4, j = k = 1, \\ & \text{or } i = k = 2, 3, j = l = 4, \text{ or } i = k = 4, j = l = 2, 3 \end{cases}$$

Proof. The Kulkarni–Nomizu product of two symmetric covariant 2-tensor h and l , can be obtained by:

$$\begin{aligned}
(h \otimes l)(w, x, y, z) &:= h(w, z)l(x, y) + h(x, y)l(w, z) \\
&\quad - h(w, y)l(x, z) - h(x, z)l(w, y)
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
W &= R_m - \frac{1}{2}R_{icci} \otimes k^+ + \frac{S}{12}k^+ \otimes k^+ \\
&= R_m - \frac{1}{2}R_{icci} \otimes k^+ - \frac{1}{4}k^+ \otimes k^+
\end{aligned}$$

where $R_m(X_1, X_2, X_3, X_4) = \langle R(X_1, X_2, X_3), X_4 \rangle_{k^+}$ for all $X_1, X_2, X_3, X_4 \in \mathfrak{X}(G_0)$ (see [7]), the result is followed. \square

Working in the (natural) coordinates system

$$(x_1, x_2, x_3, x_4) \equiv \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad (3)$$

we have a smooth frame

$$\begin{aligned} e_1^+ &= \frac{\sqrt{2}}{2} \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) \\ e_2^+ &= \frac{\sqrt{2}}{2} \left(x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) \\ e_3^+ &= \frac{\sqrt{2}}{2} \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right) \\ e_4^+ &= \frac{\sqrt{2}}{2} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \right) \end{aligned}$$

and its corresponding dual coframe

$$\begin{aligned} (e_1^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_3dx_1 + x_4dx_2 - x_1dx_3 - x_2dx_4) \\ (e_2^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (-x_3dx_1 + x_4dx_2 + x_1dx_3 - x_2dx_4) \\ (e_3^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_4dx_1 + x_3dx_2 - x_2dx_3 - x_1dx_4) \\ (e_4^+)^* &= \frac{1}{\sqrt{2}(x_1x_4 - x_2x_3)} (x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4) \end{aligned}$$

Then we have

Proposition 2. *In the coordinate system (3) the elements involved in Einstein's equation are written as*

$$\begin{aligned} k^+ &= - (e_1^+)^* (e_1^+)^* + (e_2^+)^* (e_3^+)^* + (e_3^+)^* (e_3^+)^* + (e_4^+)^* (e_4^+)^* \\ &= \frac{1}{2(x_1x_4 - x_2x_3)^2} (2x_4^2dx_1^2 - 4x_3x_4dx_1dx_2 - 4x_2x_4dx_1dx_3 \\ &\quad + 4x_2x_3dx_1dx_4 + 2x_3^2dx_2^2 + 4x_1x_4dx_2dx_3 - 4x_1x_3dx_2dx_4 \\ &\quad + 2x_2^2dx_3^2 - 4x_1x_2dx_3dx_4 + 2x_1^2dx_4^2) \end{aligned}$$

$$\begin{aligned}
R_{icci} = & \left(-\frac{x_1x_2}{2} + \frac{x_1x_4}{2} + \frac{x_2^2}{2} \right) dx_1^2 + \left(\frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_2}{2} \right) dx_1dx_2 \\
& + \left(\frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_4}{2} \right) dx_1dx_3 + \left(\frac{\sqrt{2}x_1}{2} + \frac{\sqrt{2}x_3}{2} \right) dx_1dx_4 \\
& + \left(\sqrt{2}x_1 \right) dx_2^2 + \left(-\frac{\sqrt{2}x_2}{2} + \frac{\sqrt{2}x_4}{2} \right) dx_2dx_3 \\
& + \left(\frac{\sqrt{2}x_2}{2} + \frac{\sqrt{2}x_3}{2} \right) dx_2dx_4 + \left(\sqrt{2}x_3 \right) dx_3^2 \\
& + \left(\frac{\sqrt{2}x_3}{2} - \frac{\sqrt{2}x_4}{2} \right) dx_3dx_4 + \left(\sqrt{2}x_4 \right) dx_4^2
\end{aligned}$$

1.1 Tidal Force and Jacobi Vector Fields

Definition 1. For a vector $0 \neq v \in T_\sigma(G_0)$ the tidal force operator $F_v : u^\perp \rightarrow v^\perp$ is given by $F_v(y) = R_{yv}v$.

The tidal force is a self-adjoint linear operator on u^\perp , and $\text{trace}(F_v) = -R_{icci}(v, v)$.

A direct calculation gives

Lemma 4. For $v_\sigma = \sum_{i=1}^4 f_i e_{i,\sigma}^+$, with $\sigma \in G_0$ we have

$$\begin{aligned}
\text{trace}(F_{v_\sigma}) &= -R_{icci}(v_\sigma, v_\sigma) \\
&= -R_{icci} \left(\sum_{i=1}^4 f_i e_{i,\sigma}^+, \sum_{j=1}^4 f_j e_{j,\sigma}^+ \right) \\
&= -\sum_{i,j=1}^4 f_i f_j R_{icci}(e_{i,\sigma}^+, e_{j,\sigma}^+) \\
&= -f_1^2 + f_2^2 + f_3^2
\end{aligned}$$

Definition 2. Jacobi Fields.

If $\tau(t)$ is a geodesic, its variations in (G_0, k^+) are given by vector fields Y solutions of the differential equation

$$\frac{D^2 Y}{dt^2} = R(Y, \tau', \tau') \tag{4}$$

where DY/dt denotes the affine covariant derivative relative to ∇ along τ .

Now, we will focus on the Jacobi equation in a Lie Group following [3].

Every vector field X on G_0 defines a map

$$\begin{aligned}\tilde{X} : G_0 &\rightarrow \mathfrak{g} \\ \sigma &\mapsto (\mathbf{L}_{\sigma^{-1}})_{*,\sigma} X_\sigma.\end{aligned}$$

Obviously, a vector field is left invariant if and only if the associated map is constant.

Given a curve $\sigma : [t_0, t_1] \rightarrow G_0$, every vector field Y on σ defines a curve in \mathfrak{g} :

$$\tilde{Y}(t) = (\mathbf{L}_{\sigma(t)^{-1}})_{*,\sigma(t)} Y(t)$$

and conversely, every curve in \mathfrak{g} defined on $[0, 1]$ determines a vector field on σ . We say that one is the **reflection** of the other and we write either $y^\sim = Y$ or $Y^\sim = y$.

Notice that $y(t) = (Y(t))^\sim$ is equivalent to $y(t)_{\sigma(t)}^+ = Y(t)$.

Theorem 2. *The reflection $y(t) = y_1 e_1^+ + y_2 e_2^+ + y_3 e_3^+ + y_4 e_4^+$ in \mathfrak{g} of the Jacobi field along a geodesic $\sigma : [0, 1] \rightarrow G$ with $\sigma(0) = \varepsilon$ and initial velocity $\dot{\sigma}(0) = ae_1 + be_2 + ce_3 + de_4$ has components*

$$\begin{aligned}y_1(t) &= A_0 t + A_1 e^{\alpha t} + A_2 e^{-\alpha t} \\ y_2(t) &= B_0 + B_1 e^{\alpha t} + A_2 e^{-\alpha t} \\ y_3(t) &= C_0 + C_1 e^{\alpha t} + C_2 e^{-\alpha t} \\ y_4(t) &= D_0 + D_1 t\end{aligned}$$

where A_i, B_i, C_i, D_i and α are constants that depend on a, b, c, d .

Proof. In fact, from the Jacobi equation, we obtain

$$\begin{aligned}y''_1 &= -\sqrt{2} cy'_2 + \sqrt{2} by'_3 \\ y''_2 &= -\sqrt{2} cy'_1 + \sqrt{2} ay'_3 \\ y''_3 &= \sqrt{2} by'_1 - \sqrt{2} ay'_2 \\ y''_4 &= 0\end{aligned}$$

And a direct calculation gives

$$\begin{aligned}
y_1(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left(2at(aC_1 - bC_2 - cC_3) + \right. \\
&\quad + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} (b^2C_1 + c^2C_1 - abC_2 + c\sqrt{-a^2+b^2+c^2}C_2 - acC_3 - b\sqrt{-a^2+b^2+c^2}C_3) + \\
&\quad \left. + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} (-b^2C_1 - c^2C_1 + abC_2 + c\sqrt{-a^2+b^2+c^2}C_2 + acC_3 - b\sqrt{-a^2+b^2+c^2}C_3) \right) \\
y_2(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left(-2bt(-aC_1 + bC_2 + cC_3) + \right. \\
&\quad + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} (abC_1 + c\sqrt{-a^2+b^2+c^2}C_1 - a^2C_2 + c^2C_2 - bcC_3 - a\sqrt{-a^2+b^2+c^2}C_3) + \\
&\quad \left. + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} (-abC_1 + c\sqrt{-a^2+b^2+c^2}C_1 + a^2C_2 - c^2C_2 + bcC_3 - a\sqrt{-a^2+b^2+c^2}C_3) \right) \\
y_3(t) &= \frac{1}{2(a^2 - b^2 - c^2)} \left(-2ct(-aC_1 + bC_2 + cC_3) + \right. \\
&\quad + \frac{e^{\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} (-acC_1 - b\sqrt{-a^2+b^2+c^2}C_1 + bcC_2 + a\sqrt{-a^2+b^2+c^2}C_2 + a^2C_3 - b^2C_3) + \\
&\quad \left. + \frac{e^{-\sqrt{2}\sqrt{-a^2+b^2+c^2}t}}{\sqrt{2}\sqrt{-a^2+b^2+c^2}} (acC_1 - b\sqrt{-a^2+b^2+c^2}C_1 - bcC_2 + a\sqrt{-a^2+b^2+c^2}C_2 - a^2C_3 + b^2C_3) \right) \\
y_4(t) &= C_4t + C_5
\end{aligned}$$

In the case of $a^2 = b^2 + c^2$ the components of Jacobi fields become

$$\begin{aligned}
y_1(t) &= C_1 + \frac{1}{3}(3t + b^2t^3 + c^2t^3)C_2 + \frac{1}{6}(-3\sqrt{2}ct^2 \pm 2b\sqrt{b^2 + c^2}t^3)C_4 \\
&\quad + \frac{1}{6}(3\sqrt{2}bt^2 \pm 2c\sqrt{b^2 + c^2}t^3)C_6 \\
y_2(t) &= \frac{1}{6}(-3\sqrt{2}ct^2 \mp 2b\sqrt{b^2 + c^2}t^3)C_2 + C_3 + \frac{1}{3}(3t - b^2t^3)C_4 \\
&\quad + \frac{1}{6}(\mp 3\sqrt{2}\sqrt{b^2 + c^2}t^2 - 2bct^3)C_6 \\
y_3(t) &= \frac{1}{6}(3\sqrt{2}bt^2 \mp 2c\sqrt{b^2 + c^2}t^3)C_2 + \frac{1}{6}(\mp 3\sqrt{2}\sqrt{b^2 + c^2}t^2 - 2bct^3)C_4 \\
&\quad + C_5 + \frac{1}{3}(3t - c^2t^3)C_6 \\
y_4(t) &= C_7t + C_8
\end{aligned}$$

□

1.2 Parallel transport in terms of reflections

The parallel transport is a tool that informs us about the holonomy of the Lorentzian manifold (G_0, k^+) .

A smooth vector field X along a smooth curve $\sigma : [a, b] \rightarrow G_0$ is said to be parallel if $\frac{DX}{dt} = 0$. The map $P_a^b : T_{\sigma(a)}(G_0) \rightarrow T_{\sigma(b)}(G_0)$ sending each $v_{\sigma(a)}$ to a parallel vector X on σ such that $X(a) = v_{\sigma(a)}$ is called **parallel transport along σ** .

In what follows we give an outline of the parallel transport starting at ε .

The integral curve of e_1^+ through ε is

$$\alpha(s) = \begin{pmatrix} \cos\left(\frac{\sqrt{2}}{2}s\right) & \sin\left(\frac{\sqrt{2}}{2}s\right) \\ -\sin\left(\frac{\sqrt{2}}{2}s\right) & \cos\left(\frac{\sqrt{2}}{2}s\right) \end{pmatrix}$$

Now, the lightlike vectors $u_\varepsilon = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the condition $a^2 + 2bc + d^2 = 0$, and the integral curves through ε are

$$\alpha(s) = e^{\frac{a+d}{2}s} \begin{pmatrix} \cos\left(\frac{\theta}{2}s\right) + \frac{(a-d)}{\theta} \sin\left(\frac{\theta}{2}s\right) & \frac{2b}{\theta} \sin\left(\frac{\theta}{2}s\right) \\ \frac{2c}{\theta} \sin\left(\frac{\theta}{2}s\right) & \cos\left(\frac{\theta}{2}s\right) - \frac{(a-d)}{\theta} \sin\left(\frac{\theta}{2}s\right) \end{pmatrix}$$

where $\theta = \text{Re}(\sqrt{2bc - 2ad})$. Moreover,

$$\begin{aligned} \det(\alpha(s)) &= e^{(a+d)s} \left(\cos\left(\frac{\theta}{2}s\right) + 2\left(\frac{ad-bc}{\theta^2}\right) \sin\left(\frac{\theta}{2}s\right) \right) \\ \text{trace}(\alpha(s)) &= e^{\frac{a+d}{2}s} \cos\left(\frac{\theta}{2}s\right) \end{aligned}$$

and the polynomial characteristic is

$$p(\lambda) = \lambda^2 + \text{trace}(\alpha(s))\lambda + \det(\alpha(s))$$

Let γ be a geodesic with initial direction x_0 starting in the unit of G_0 , if $x = \sum_{i=1}^4 x_i e_i$ is the reflection of γ' , the reflection y of the parallel transport of Y along γ is given by the solution of the following system of differential equation

$$\begin{aligned} y'_1 &= \frac{\sqrt{2}}{2} (y_3 x_2 - y_2 x_3) \\ y'_2 &= \frac{\sqrt{2}}{2} (y_3 x_1 - y_1 x_3) \\ y'_3 &= \frac{\sqrt{2}}{2} (y_1 x_2 - y_2 x_1) \\ y'_4 &= 0 \end{aligned}$$

1.3 Isometries of (G_0, k^+)

Recall that we have

Lemma 5. *The left and right multiplication are isometries of (G_0, k^+) , and consequently the inversion map and the map I_σ , defined by $I_\sigma(\tau) := \sigma\tau^{-1}\sigma$, for any $\sigma, \tau \in G_0$, are isometries.*

Note that I_σ inverse the geodesic, this means

$$I_\sigma(\gamma(s)) := \sigma\gamma(-s)\sigma$$

where $\gamma(s) = e^{su}$ is a geodesic with $\gamma(0) = I_\varepsilon \in G_0$ and $\gamma'(0) = u \in \mathfrak{g}$

In particular for $\sigma = \varepsilon$ and $u = e_1$

$$I_\varepsilon(\gamma(s)) = \begin{pmatrix} \cos\left(\frac{\sqrt{2}}{2}s\right) & -\sin\left(\frac{\sqrt{2}}{2}s\right) \\ \sin\left(\frac{\sqrt{2}}{2}s\right) & \cos\left(\frac{\sqrt{2}}{2}s\right) \end{pmatrix}$$

On the other hand, it is well-known that the metric k^+ can be lifted to the Lorentzian metric \tilde{k}^+ over the universal covering \tilde{G} such that the covering map p is a local isometry. For a better comprehension of the isometries of (G_0, k^+) it is convenient to study the isometries of the universal covering space (\tilde{G}, \tilde{k}^+) (for instance see Proposition 2.1 and Theorem 2.2 of [2]).

1.4 Causal Structure

Since (G_0, k^+) is quadratic, its causal structure is determined by the Minkowski space $(T_\varepsilon(G_0), k)$.

Determining the different types of vectors constitutes the first step to comprehend a spacetime model. We will find these types of vectors.

Note that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\varepsilon$

given a vector of lightlike (respectively timelike, spacelike) if it verifies $a^2 + 2bc + d^2 = 0$ (respectively $a^2 + 2bc + d^2 < 0$, $a^2 + 2bc + d^2 > 0$). *It is commonly accepted that the set of all lightlike vectors in $T_\sigma(G_0)$ is the lightcone at $\sigma \in G_0$.*

Let \mathcal{F} be the set of all timelike vectors in \mathfrak{g} . The timecone of \mathfrak{g} containing $u \in \mathfrak{g}$ is defined as $C(u) := \{v \in \mathcal{F} \mid k(u, v) < 0\}$, and the opposite timecone is $-C(u) := \{v \in \mathcal{F} \mid k(u, v) > 0\}$, so the timecone containing e_1 is

$$C(e_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c < b, a^2 + 2bc + d^2 < 0 \right\}$$

and the opposite timecone is

$$-C(e_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c > b, a^2 + 2bc + d^2 < 0 \right\}$$

Note that the function f from matrices 2×2 to real numbers defined as $f(a_{ij}) = a_{11}^2 + 2a_{12}a_{21} + a_{22}^2$ is a smooth map with constant rank 1, then $f^{-1}(0)$ is an embedded submanifold of dimension 4.

The following results are well known,

Lemma 6. *Timelike vectors u and v in a Lorentz vector space are in the same time-cone if and only if $k(u, v) < 0$. Furthermore, timecones are convex.*

2 Flat Affine Structure on $GL(2, \mathbb{R})_0$

Proposition 3. *Since the matrix multiplication is associative, the above Lie group G_0 is endowed with a bi-invariant flat affine structure ∇ i.e. a linear connection without torsion and curvature.*

The universal covering group of G_0 is the product manifold $\tilde{G} = \mathbb{R} \times \text{SDP}(2)$, where $\text{SDP}(2)$ are the symmetric definite positive matrices and the covering map is

$$\begin{aligned} p: \mathbb{R} \times \text{SDP}(2) &\rightarrow G_0 \\ (t, T) &\mapsto O_t T \end{aligned} \quad (5)$$

here O_t denotes an orthogonal matrix.

The product of G_0 rises to a product in \tilde{G} such that \tilde{G} is a Lie group and p is a homomorphism of Lie groups (see [4] pag. 53)

To find the multiplication in \tilde{G} , we must calculate the polar decomposition of $O_t T O_r R$, this is

$$\underbrace{O_t T O_r R (R O_{-r} T^2 O_r R)^{-\frac{1}{2}}}_{O_s} \underbrace{(R O_{-r} T^2 O_r R)^{\frac{1}{2}}}_S$$

Then the multiplication of two elements in \tilde{G} is given as follows:

$$(t, T) \cdot (r, R) = \left(s(t, r), (R O_{-r} T^2 O_r R)^{\frac{1}{2}} \right) \quad (6)$$

where $s(t, r) \in \mathbb{R}$ can be calculated using the orthogonal matrix of order 2×2

$$O_{s(t, r)} = O_t T O_r R (R O_{-r} T^2 O_r R)^{-\frac{1}{2}}$$

The multiplication (6) can be written as

$$(t, T) \cdot (r, R) = (\arctan(\theta) + t + r, O_{-\arctan(\theta) - t - r} T O_r R) \quad (7)$$

where

$$-\frac{\pi}{2} < \theta = \frac{\text{trace}(O_{-r} T O_r R \omega)}{\text{trace}(O_{-r} T O_r R)} < \frac{\pi}{2} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A direct calculation shows that (\tilde{G}, \cdot) is a Lie group.

Theorem 3. *The developed map of the flat affine manifold (G_0, ∇) in the Ehresmann's sense [5], following Koszul method [6], is given by*

$$\begin{aligned} Dev: \quad \tilde{G}_0 &\rightarrow \mathbb{R}^4 \\ (y_1, y_2, y_3, y_4) &\mapsto (y_1, y_2 - 1, y_3, y_4 - 1) \end{aligned}$$

Proof. Let (x_1, x_2, x_3, x_4) be the natural coordinates (3) of G_0 . We know that ∇ is locally isomorphic to usual connection ∇^0 on \mathbb{R}^4 , then its 1-form connection ω can be written in local coordinates $(x_1, x_2, x_3, x_4, (X)_{ij})$, like

$$\omega = (\omega_{ij}) = \left(\sum_{k=1}^4 Y_{ik} dX_{kj} \right), \quad \text{where } (Y_{ij}) = (X_{ij})^{-1}$$

Let (\tilde{G}, p) be the universal covering group of G_0 , where p is the covering map (5), and $\tilde{\mathfrak{g}}$ the Lie algebra of \tilde{G} .

We can lift the affine structure to \tilde{G} through $\tilde{\omega} := p^*\omega$. Now, we will compute the developed map of $(\tilde{G}, \tilde{\nabla})$. First, we find the vector fields Y on \tilde{G} such that $\tilde{\nabla}Y = 0$. In the local coordinates

$$\left(y_1, \begin{pmatrix} y_2 & y_3 \\ y_3 & y_4 \end{pmatrix} \right) \cong (y_1, y_2, y_3, y_4)$$

they are

$$Y = \sum_{i=1}^4 a_i \frac{\partial}{\partial y_i}, \quad \text{with } a_i \in \mathbb{R}$$

Second, we shall calculate the close $\tilde{\mathfrak{g}}$ -valued 1-form η on \tilde{G} such that $\tilde{\nabla}\eta = 0$ and $\eta(Y) = Y_{\tilde{\varepsilon}}$. A direct calculation shows that $\eta = (dy_1, dy_2, dy_3, dy_4)$

Finally, the developed map is given by $Dev(\tilde{\sigma}) = \int_C \eta$, where C is a curve joining $\tilde{\varepsilon} = (0, 1, 0, 1)$ and $\tilde{\sigma}$. Then

$$\begin{aligned} Dev : \quad \tilde{G}_0 &\rightarrow \mathbb{R}^4 \\ (y_1, y_2, y_3, y_4) &\mapsto (y_1, y_2 - 1, y_3, y_4 - 1) \end{aligned}$$

□

We have the following result (See [1])

Theorem 4. (G_0, ∇, k^+) is a local (in fact global) Hessian manifold.

Proof. A potential function for k^+ is

$$\begin{aligned} f : G_0 &\rightarrow \mathbb{R} \\ M &\mapsto \frac{1}{2} \text{trace}(M^2) \end{aligned}$$

In natural coordinates, the gradient has the expression

$$\begin{aligned} grad f &= \sum_{i=1}^4 \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \\ &= x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \end{aligned}$$

Hence

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

3 Python Computes

```
[1]: import numpy as np
from sympy import symbols, Matrix, simplify, sqrt, solve, Transpose,
Trace
```

3.1 k-symmetric Operators

```
[2]: # Defining metric k = Traza(AB) respect to Orthonormal basis
k = Matrix([[ -1, 0, 0, 0], [ 0, 1, 0, 0], [ 0, 0, 1, 0], [ 0, 0, 0, 1]])

# Defining variables
u11, u12, u13, u14, u21, u22, u23, u24, u31, u32, u33, u34, u41, u42,
u43, u44 = symbols('u11 u12 u13 u14 u21 u22 u23 u24 u31 u32 u33
u34 u41 u42 u43 u44')

# Matrix representation of an operator in the natural basis of GL_2
u=Matrix([[u11, u12, u13, u14], [u21, u22, u23, u24], [u31, u32, u33,
u34], [u41, u42, u43, u44]])
```

```
[3]: # Orthonormal basis respect to metric k=Traza(AB)
E1=Matrix([[0, sqrt(2)/2], [-sqrt(2)/2, 0]]) #Timelike
E2=Matrix([[0, sqrt(2)/2], [sqrt(2)/2, 0]]) #Spacelike
E3=Matrix([[sqrt(2)/2, 0], [0, -sqrt(2)/2]]) #Spacelike
E4=Matrix([[sqrt(2)/2, 0], [0, sqrt(2)/2]]) #Spacelike

#Compute k(u(a),b)=k(a,u(b)) on orthonormal basis
K0=Transpose(u)*k-k*u

solve([K0], u11, u12, u13, u14, u21, u22, u23, u24, u31, u32, u33,
u34, u41, u42, u43, u44, dict=True)
```

```
[3]: [{u12: -u21, u13: -u31, u14: -u41, u23: u32, u24: u42, u34: u43}]
```

```
[4]: # Matrix representation of the k-symmetric operator in the
      ↪ orthonormal basis
ku=Matrix([[u11,u12,u13,u14],[-u12,u22,u23,u24],
          ↪ [-u13,u23,u33,u34],[-u14,u24,u34,u44]])
print('k-symmetric operator in the orthonormal basis:')
ku
```

The k-symmetric operator on the orthonormal basis:

$$[4]: \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{bmatrix}$$

```
[5]: # Metrics given by the k-symmetrics operator in the orthonormal
      ↪ basis:
Transpose(ku)*k
```

$$[5]: \begin{bmatrix} -u_{11} & -u_{12} & -u_{13} & -u_{14} \\ -u_{12} & u_{22} & u_{23} & u_{24} \\ -u_{13} & u_{23} & u_{33} & u_{34} \\ -u_{14} & u_{24} & u_{34} & u_{44} \end{bmatrix}$$

3.2 Curvature

```
[6]: K0= Matrix([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
      #K1 = k is Levi-Civita
      K2= Matrix([[ -1,0,0,0],[0,-1,0,0],[0,0,1,0],[0,0,0,1]])
```

```

[7]: # Levi-Civita Christoffel Symbols
gamma = [[[0 for i in range(4)] for j in range(4)] for k in range(4)]

# Structure coefficients of Lie algebra C_{ij}^k=AE[i][j][k]
AE= [[[0,0,0,0],[0,0,sqrt(2),0],[0,-sqrt(2),0,0],[0,0,0,0]],
      [[0,0,-sqrt(2),0],[0,0,0,0,0,0],[-sqrt(2),0,0,0],[0,0,0,0]],
      [[0,sqrt(2),0,0],[sqrt(2),0,0,0],[0,0,0,0],[0,0,0,0]],
      [[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]]]

# Christoffel Symbols Computes
# Gamma_{ij}^k=\sum_{l,m,n}0.
↪5K^{kl}(-K_{jm}C_{il}^m-K[l,m]*C_{ji}^m+K_{im}*C_{lj}^m)
# K0
for i in range(4):
for j in range(4):
for k in range(4):
gamma[i][j][k]=0
for l in range(4):
for m in range(4):
gamma[i][j][k] += simplify(0.5 *
↪(K0[k,l]*(-K0[j,m]*AE[i][l][m]-K0[l,m]*AE[j][i][m]+K0[i,m]*
↪AE[l][j][m])))

print("The Christoffel Symbols are: ")
for i in range(4):
for j in range(4):
for k in range(4):
print("Gamma", i+1, j+1, k+1, "=", gamma[i][j][k])

```

```

[7]: The Christoffel Symbols are:
Gamma 1 1 1 = 0
Gamma 1 1 2 = 0
Gamma 1 1 3 = 0
Gamma 1 1 4 = 0
Gamma 1 2 1 = 0
Gamma 1 2 2 = 0
Gamma 1 2 3 = 1.5*sqrt(2)
Gamma 1 2 4 = 0
Gamma 1 3 1 = 0
Gamma 1 3 2 = -1.5*sqrt(2)
Gamma 1 3 3 = 0
Gamma 1 3 4 = 0
Gamma 1 4 1 = 0

```


Gamma 1 4 2 = 0
Gamma 1 4 3 = 0
Gamma 1 4 4 = 0
Gamma 2 1 1 = 0
Gamma 2 1 2 = 0
Gamma 2 1 3 = $0.5\sqrt{2}$
Gamma 2 1 4 = 0
Gamma 2 2 1 = 0
Gamma 2 2 2 = 0
Gamma 2 2 3 = 0
Gamma 2 2 4 = 0
Gamma 2 3 1 = $-0.5\sqrt{2}$
Gamma 2 3 2 = 0
Gamma 2 3 3 = 0
Gamma 2 3 4 = 0
Gamma 2 4 1 = 0
Gamma 2 4 2 = 0
Gamma 2 4 3 = 0
Gamma 2 4 4 = 0
Gamma 3 1 1 = 0
Gamma 3 1 2 = $-0.5\sqrt{2}$
Gamma 3 1 3 = 0
Gamma 3 1 4 = 0
Gamma 3 2 1 = $0.5\sqrt{2}$
Gamma 3 2 2 = 0
Gamma 3 2 3 = 0
Gamma 3 2 4 = 0
Gamma 3 3 1 = 0
Gamma 3 3 2 = 0
Gamma 3 3 3 = 0
Gamma 3 3 4 = 0
Gamma 3 4 1 = 0
Gamma 3 4 2 = 0
Gamma 3 4 3 = 0
Gamma 3 4 4 = 0
Gamma 4 1 1 = 0
Gamma 4 1 2 = 0
Gamma 4 1 3 = 0
Gamma 4 1 4 = 0
Gamma 4 2 1 = 0
Gamma 4 2 2 = 0
Gamma 4 2 3 = 0
Gamma 4 2 4 = 0
Gamma 4 3 1 = 0
Gamma 4 3 2 = 0

```
Gamma 4 3 3 = 0
Gamma 4 3 4 = 0
Gamma 4 4 1 = 0
Gamma 4 4 2 = 0
Gamma 4 4 3 = 0
Gamma 4 4 4 = 0
```

```
[8]: # Curvature of K_0
L1=Matrix([[0,0,0,0],[0,0,-3*sqrt(2)/2,0],[0,3*sqrt(2)/
↪2,0,0],[0,0,0,0]])
L2=Matrix([[0,0,-sqrt(2)/2,0],[0,0,0,0],[sqrt(2)/2,0,0,0],[0,0,0,0]])
L3=Matrix([[0,sqrt(2)/2,0,0],[-sqrt(2)/2,0,0,0],[0,0,0,0],[0,0,0,0]])

# R(e_1,e_3, )\neq 0
L3*L1-L1*L3+sqrt(2)*L2
```

```
[8]: 
$$\begin{bmatrix} 0 & 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{5}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```

```
[9]: # K2
for i in range(4):
for j in range(4):
for k in range(4):
gamma[i][j][k]=0
for l in range(4):
for m in range(4):
gamma[i][j][k] += simplify(0.5 *
↪(K2[k,l]*(-K2[j,m]*AE[i][l][m]-K2[l,m]*AE[j][i][m]+K2[i,m]*
↪AE[l][j][m])))

print("The Christoffel Symbols are: ")
for i in range(4):
for j in range(4):
for k in range(4):
print("Gamma", i+1, j+1, k+1, "=", gamma[i][j][k])
```

```
[9]: The Christoffel Symbols are:
Gamma 1 1 1 = 0
Gamma 1 1 2 = 0
Gamma 1 1 3 = 0
Gamma 1 1 4 = 0
```

Gamma 1 2 1 = 0
Gamma 1 2 2 = 0
Gamma 1 2 3 = $-0.5\sqrt{2}$
Gamma 1 2 4 = 0
Gamma 1 3 1 = 0
Gamma 1 3 2 = $-0.5\sqrt{2}$
Gamma 1 3 3 = 0
Gamma 1 3 4 = 0
Gamma 1 4 1 = 0
Gamma 1 4 2 = 0
Gamma 1 4 3 = 0
Gamma 1 4 4 = 0
Gamma 2 1 1 = 0
Gamma 2 1 2 = 0
Gamma 2 1 3 = $-1.5\sqrt{2}$
Gamma 2 1 4 = 0
Gamma 2 2 1 = 0
Gamma 2 2 2 = 0
Gamma 2 2 3 = 0
Gamma 2 2 4 = 0
Gamma 2 3 1 = $-1.5\sqrt{2}$
Gamma 2 3 2 = 0
Gamma 2 3 3 = 0
Gamma 2 3 4 = 0
Gamma 2 4 1 = 0
Gamma 2 4 2 = 0
Gamma 2 4 3 = 0
Gamma 2 4 4 = 0
Gamma 3 1 1 = 0
Gamma 3 1 2 = $0.5\sqrt{2}$
Gamma 3 1 3 = 0
Gamma 3 1 4 = 0
Gamma 3 2 1 = $-0.5\sqrt{2}$
Gamma 3 2 2 = 0
Gamma 3 2 3 = 0
Gamma 3 2 4 = 0
Gamma 3 3 1 = 0
Gamma 3 3 2 = 0
Gamma 3 3 3 = 0
Gamma 3 3 4 = 0
Gamma 3 4 1 = 0
Gamma 3 4 2 = 0
Gamma 3 4 3 = 0
Gamma 3 4 4 = 0
Gamma 4 1 1 = 0

```

Gamma 4 1 2 = 0
Gamma 4 1 3 = 0
Gamma 4 1 4 = 0
Gamma 4 2 1 = 0
Gamma 4 2 2 = 0
Gamma 4 2 3 = 0
Gamma 4 2 4 = 0
Gamma 4 3 1 = 0
Gamma 4 3 2 = 0
Gamma 4 3 3 = 0
Gamma 4 3 4 = 0
Gamma 4 4 1 = 0
Gamma 4 4 2 = 0
Gamma 4 4 3 = 0
Gamma 4 4 4 = 0

```

```

[10]: # Curvature of K_2
L21=Matrix([[0,0,0,0],[0,0,-sqrt(2)/2,0],[0,-sqrt(2)/
↪-2,0,0],[0,0,0,0]])
L22=Matrix([[0,0,-3*sqrt(2)/2,0],[0,0,0,0],[-3*sqrt(2)/
↪-2,0,0,0],[0,0,0,0]])
L23=Matrix([[0,-sqrt(2)/2,0,0],[sqrt(2)/2,0,0,0],[0,0,0,0],[0,0,0,0]])
# R(e_1, e_2, )\neq 0
L21*L22-L22*L21-sqrt(2)*L23

```

```

[10]: 
$$\begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$


```

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