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TWISTED DIFFERENTIALS AND LEE CLASSES OF LOCALLY CONFORMALLY SYMPLECTIC COMPLEX SURFACES

VESTISLAV APOSTOLOV AND GEORGES DLOUSSKY

To the memory of Marco Brunella, whose work inspired us, and underline the importance of the theory of PSH functions for the classification problem.

ABSTRACT. We study the set of deRham classes of Lee 1-forms of the locally conformally symplectic (LCS) structures taming the complex structure of a compact complex surface in the Kodaira class VII, and show that the existence of non-trivial upper/lower bounds with respect to the degree function correspond respectively to the existence of certain negative/non-negative PSH functions on the universal cover. We use this to prove that the set of Lee deRham classes of taming LCS is connected, as well as to obtain an explicit negative upper bound for this set on the hyperbolic Kato surfaces. This leads to a complete description of the sets of Lee classes on the known examples of class VII complex surfaces, and to a new obstruction to the existence of bi-hermitian structures on the hyperbolic Kato surfaces of the intermediate type. Our results also reveal a link between bounds of the set of Lee classes and non-trivial logarithmic holomorphic 1-forms with values in a flat holomorphic line bundle.

1. INTRODUCTION

The classification, up to biholomorphisms, of the compact complex manifolds of complex dimension 2 was reduced by the seminal works of Enriques and Kodaira to understanding the minimal models of 10 classes of complex surfaces [8, Tab.10]. The description of these minimal models is essentially optimal in the case when the complex surfaces admit Kähler metrics, a purely topological condition equivalent to the first Betti number being even [16, 43, 57]. On the other hand, in the non-Kähler case, there is a still open classification conjecture regarding the minimal complex surfaces in the class VII, i.e. those having the first Betti number equal to 1 and negative Kodaira dimension. It is now known [22, 40, 47, 58] that such a surface must be given by a list of *known examples* if one can show that, when the second Betti number is positive, the surface admits a *global spherical shell* (GSS). The existence of such a GSS is often referred to as the *GSS Conjecture* (see [47] and the references therein). Whereas some structure results about the existence of compatible hermitian metrics, and especially locally conformally Kähler hermitian metrics [9, 12, 13, 34, 31], on the complex surfaces in the Kodaira class VII were obtained by assuming the GSS Conjecture, our thesis in this paper is that getting such structure results, and studying the geometric invariants

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associated to them, for possibly unknown surfaces is a promising angle of attack of the classification conjecture.

In the above general vein, this paper is a continuation of our previous work [5, 6], in which we showed that *any* non-Kähler compact complex surface $S = (M, J)$ admits hermitian metrics induced by locally conformally symplectic (LCS) 2-forms ω which tame the underlying complex structure J .

Recall that a LCS structure ω is a non-degenerate 2-form on M , which is closed in the twisted sense that

$$d_\alpha \omega := d\omega - \alpha \wedge \omega = 0,$$

where α is a closed 1-form on M , called the *Lee form* of ω . The taming condition means that the $(1, 1)$ -part $F := (\omega)^{1,1}$ of ω with respect to J is positive-definite, i.e. is the fundamental form of a hermitian structure (g, J) on S . This definition is tailored to conformal geometry: if $\tilde{\omega} = e^{-f}\omega$ for some smooth function f , then $\tilde{\omega}$ is a taming LCS structure on S with Lee form $\tilde{\alpha} = \alpha + df$, and induced hermitian metric $\tilde{g} = e^{-f}g$.

Given a compact complex surface S , we introduce the following subset of the first deRham cohomology group $H_{dR}^1(S, \mathbb{R})$:

$$\mathcal{T}(S) := \{[\alpha] \in H_{dR}^1(S, \mathbb{R}) \mid \alpha \text{ is a Lee form of a taming LCS structure on } S\},$$

which is a natural complex geometric invariant of S .

A central fact, established in [5], is that $\mathcal{T}(S)$ is non-empty, i.e. *any compact complex surface admits taming LCS structures*. This contrasts the well-known obstruction to the existence of taming symplectic structures, which forces $b_1(S)$ to be even [33, 43] and S to be Kähler [16, 43]. In this latter case, it is also observed in [5] that $\mathcal{T}(S) = \{0\}$ reduces to a single trivial class.

Thus motivated, we have initiated in [5, 6] a systematic investigation of the set $\mathcal{T}(S)$ on the *non-Kähler* compact complex surfaces. This naturally extends the related invariant studied in [9, 12, 13, 61, 48] in the case of *locally conformally Kähler hermitian metrics*, i.e. when ω is a LCS structure which not only tames the underlying complex structure but is also compatible with it.

The question of determining $\mathcal{T}(S)$ is particularly interesting and yet difficult to answer in the case when $b_1(S) = 1$ and the $H^0(S, K_S^m) = 0, \forall m \geq 1$, i.e. on the complex surfaces in the Kodaira class VII. To state our results, we first order $H_{dR}^1(S, \mathbb{R})$ by the degree function with respect to some Gauduchon metric g on S , i.e. for $a, b \in H_{dR}^1(S, \mathbb{R})$ we let \mathcal{L}_a and \mathcal{L}_b be the corresponding flat holomorphic line bundles associated to the flat real line bundles determined via a and b , and say that $a \leq b$ (resp. $a < b$) if $\deg_g(\mathcal{L}_a) \leq \deg_g(\mathcal{L}_b)$ (resp. if $\deg_g(\mathcal{L}_a) < \deg_g(\mathcal{L}_b)$). It is well-known [60] that the relation $a \leq b$ (resp. $a < b$) is independent of the choice of Gauduchon metric. In these terms, we have shown in [5] that

$$\mathcal{T}(S) \subseteq (-\infty, 0).$$

Furthermore, in [6] we proved that $\mathcal{T}(S)$ is a non-empty *open* subset, unless S is obtained by blowing up Inoue–Bombieri surface in which case $\mathcal{T}(S) = \{a\}$ is a single point.

In this paper, we improve the above statements and establish the following general structure result:

Theorem 1.1. *Let S be a compact complex surface in the Kodaira class VII. Then $\mathcal{T}(S)$ is either a single point, in which case S is obtained from an Inoue–Bombieri surface by blowing points, or else $\mathcal{T}(S)$ is an open interval*

$$\mathcal{T}(S) = (d_0, b_0) \subseteq (-\infty, 0).$$

The main new idea in the proof of Theorem 1.1 is a relation between the existence of upper and lower bounds for the set $\mathcal{T}(S) \subset (-\infty, 0)$ with the existence of, respectively, negative and positive currents of degree 0 on S , satisfying a twisted PSH condition, see Propositions 3.7 and 3.8 below. This new observation can be loosely stated as follows:

Proposition 1.2. *(see Proposition 3.7 and Lemma 3.11) Suppose S is a compact complex surface in the Kodaira class VII and denote by \hat{S} the minimal \mathbb{Z} -cover of S such that $H^1(\hat{S}, \mathbb{R}) = \{0\}$, see Definition 3.10.*

- (a) *There exists a deRham class $b < 0$ such that $\mathcal{T}(S) \subset (-\infty, b)$ if and only if \hat{S} admits a negative strictly PSH function which is multiplicative automorphic with respect to the action of the cyclic generator of the covering subgroup in $\text{Aut}(\hat{S})$, with constant of automorphy $C > 1$.*
- (b) *There exists a deRham class $d < 0$ such that $\mathcal{T}(S) \subset (d, 0)$ if and only if \hat{S} admits a non-negative strictly PSH function which is multiplicative automorphic with respect to the action of the cyclic generator, with constant of automorphy $C > 1$.*

The proof of Proposition 1.2 and the derivation of Theorem 1.1 rely on relatively elementary methods which come in the form of theory of linear second order elliptic differential operators, and applications of Hahn-Banach’s theorem as in [37].

Specializing to the *known* examples of class VII complex surfaces, Proposition 1.2 and Theorem 1.1 can be used to obtain the following sharp classification result:

Theorem 1.3. *Suppose S is a compact complex surface obtained from a known minimal complex surface in the Kodaira class VII, by blowing up points.*

- (a) *If S is obtained by blowing up points of an Inoue–Bombieri surface, then $\mathcal{T}(S) = \{a\} \in (-\infty, 0)$ consists of a single point;*
- (b) *If S is obtained by blowing up points of a Hopf surface, then $\mathcal{T}(S) = (-\infty, 0)$;*
- (c) *If S is obtained by blowing up points of a hyperbolic Kato surface, then $\mathcal{T}(S) = (-\infty, b)$ for some $b < 0$;*
- (d) *If S is obtained by blowing up points of an Enoki surface, then $\mathcal{T}(S) = (-\infty, 0)$.*

As we have mentioned, (a) follows from [6] whereas (b) is established in [5, 61], using the constructions of [9, 34]. The non-boundedness from below of $\mathcal{T}(S)$ in the cases (c) and (d) is due to Brunella [12]. To the best of our knowledge, (c) provides the first known constraint on $\mathcal{T}(S)$, other than the inclusion $\mathcal{T}(S) \subset (-\infty, 0)$ established in [5]. In this case, our result is new even for the set of Lee classes of the locally conformally Kähler metrics, and in fact it partially answers questions raised in [12, Rem.9]. Similarly, (d) provides the first proof that $\mathcal{T}(S) = (-\infty, 0)$ on the Enoki surfaces. It will be interesting to know whether or not in the cases (b)-(d) of the above theorem, $\mathcal{T}(S)$ coincides with the set of Lee classes of the locally conformally Kähler metrics on S .

The existence of PSH functions on \hat{S} as in Proposition 1.2 was studied by Brunella [14, 15] who showed that if such a PSH function has also *analytic singularities*, then S must be a known complex surface (see also Section 5 below). We thus expect that when

$\mathcal{T}(S) \subsetneq (-\infty, 0)$ in Theorem 1.1, one would be able to reduce the situation to the one studied by Brunella, and propose the following, where hyperbolic Kato surfaces are by definition Kato surfaces of intermediate type or Inoue-Hirzebruch surfaces:

Conjecture A. *Suppose S a compact complex surface in the Kodaira class VII.*

- (a) *If there exists a deRham class $b \in (-\infty, 0) \subset H_{dR}^1(S, \mathbb{R})$ such that $\mathcal{T}(S) \subset (-\infty, b)$, then S is obtained from either an Inoue–Bombieri or a hyperbolic Kato surface by blowing up points.*
- (b) *If there exists a de Rham class $d \in (-\infty, 0) \subset H_{dR}^1(S, \mathbb{R})$ such that $\mathcal{T}(S) \subset (d, 0)$, then S is obtained from an Inoue–Bombieri surface by blowing up points.*

Of course, the above will be an immediate corollary of Theorem 1.3 if one assumes the Global Spherical Shell Conjecture [45], but we would rather think about it as a plausible intermediate step in deriving a general classification.

To illustrate the usefulness of Theorem 1.3, recall that [45] the minimal complex surfaces with a GSS and second Betti number equal to k are all diffeomorphic to $M_k := (S^1 \times S^3) \#_k \overline{\mathbb{C}P^2}$. The underlying smooth manifold M_k then admits three types of complex structures with a GSS, described in the cases (b), (c) and (d) of Theorem 1.3. As the taming condition is open in the C^∞ topology of almost complex structures, we then get from Theorem 1.3 the following stability result for hyperbolic Kato surfaces under the action of the diffeomorphism group:

Corollary 1.4. *Suppose $S = (M_k, J)$ is a hyperbolic Kato surface with $k = b_2(S)$, and $(\phi_i) \subset \text{Diff}_0(M_k)$ is a sequence of diffeomorphisms in the connected component of the identity, such that $\phi_i(J)$ converges in C^∞ to an integrable complex structure J_∞ on M_k . If $S_\infty := (M_k, J_\infty)$ admits a GSS, then it must be a hyperbolic Kato surface.*

The above corollary is potentially useful in the application of analytic methods, such as J. Streets’ theory of the pluriclosed flow [55]. By the results in [26], Corollary 1.4 holds true in the setting of *analytic degenerations* of J .

We also find an *explicit* upper bound b for $\mathcal{T}(S)$ on the hyperbolic Kato surfaces, by relating the corresponding deRham class b with the existence of a non-zero logarithmic holomorphic 1-form with coefficients in the flat holomorphic line bundle determined by b (see Section 4.3 and Lemma 6.6). As another application of Theorem 1.3, we return to our initial motivation of studying LCS structures on class VII surfaces [3], which stems from the classification of *bi-hermitian structures* on these complex surfaces [3, 31, 56]. We give now a new obstruction to the existence of bi-hermitian structures on Kato surfaces.

Theorem 1.5. *(see Theorem 4.6) There are intermediate type hyperbolic Kato surfaces S , satisfying $H^0(S, K_S^{-1} \otimes \mathcal{L}_b) \neq 0$ where \mathcal{L}_b is the flat holomorphic line bundle associated to a deRham class $b \in (-\infty, 0)$, but which do not admit any bi-hermitian structure.*

Our explicit construction on the hyperbolic Kato surfaces leads us to investigate in the final Section 6.1 of the paper the general problem of existence of logarithmic 1-forms with values in a flat line bundle. We recall that any such form determines a holomorphic foliation \mathcal{F} on S and, the *characteristic numbers* of a non-degenerate singularity p of \mathcal{F} are given by the two ratios of eigenvalues of the linear part of any local holomorphic vector field generating \mathcal{F} around p and vanishing at p . We prove the following result of independent interest.

Theorem 1.6. *Suppose S a minimal compact complex surface in the Kodaira class VII, which admits a non-zero holomorphic logarithmic 1-form θ with values in a flat holomorphic line bundle \mathcal{L}_b obtained as the complexification of a real flat line bundle corresponding to a deRham class $b \in H_{dR}^1(S, \mathbb{R})$. Suppose, furthermore, that $\deg_g(\mathcal{L}_b) \leq 0$ with respect to some (and hence any) Gauduchon metric g on S , and that the holomorphic foliation \mathcal{F} defined by θ has real negative characteristic numbers at any non-degenerate singular point situated on the pole of θ . Then b bounds $\mathcal{T}(S)$ and one of the following must hold:*

- (a) $b = 0$ and $\mathcal{T}(S) = (-\infty, 0)$. In this case S is either a Hopf surface or an Enoki surface.
- (b) $b < 0$ and $\mathcal{T}(S) \subseteq (-\infty, b)$. In this case S is a hyperbolic Kato surface.
- (c) $b < 0$ and $\mathcal{T}(S) = \{b\}$. In this case S is an Inoue–Bombieri surface.
- (d) $b < 0$ and $\mathcal{T}(S) \subseteq (b, 0)$.

All known minimal complex surface in the Kodaira class VII, except possibly some secondary Hopf surfaces, do appear in the cases (a)-(c) of the above statement whereas the case (d) should not occur if Conjecture A-(b) holds true. It is interesting to know whether or not in the case (b) we have equality $\mathcal{T}(S) = (-\infty, b)$.

2. TWISTED DIFFERENTIALS AND CURRENTS

Let $X = (M, J)$ be a compact complex manifold of complex dimension n , α a closed 1-form on M , representing a de Rham class $a := [\alpha] \in H_{dR}^1(M, \mathbb{R})$. We denote by $L_\alpha = M \times \mathbb{R}$ the topologically trivial real line bundle over M , endowed with the flat connection $\nabla^\alpha = d + \alpha$, and by $\mathcal{L}_\alpha = L_\alpha \otimes \mathbb{C}$ the corresponding topologically trivial flat holomorphic line bundle. This construction fits in the sequence

$$(1) \quad H_{dR}^1(M, \mathbb{R}) \xrightarrow{\text{exp}} H_{dR}^1(M, \mathbb{R}_+^*) \longrightarrow H_0^1(M, \mathbb{C}^*) \longrightarrow \text{Pic}^0(X),$$

where $H_0^1(M, \mathbb{C}^*)$ and $\text{Pic}^0(X)$ respectively denote the spaces of equivalent classes of flat and holomorphic *topologically trivial* complex line bundles. We shall slightly abusively denote by $L = L_a$ the isomorphism class of topologically trivial flat real bundles corresponding to $a \in H_{dR}^1(M, \mathbb{R})$, and by $L^* = L_{-a}$ its dual. The corresponding flat holomorphic line bundles over (M, J) will be denoted by $\mathcal{L} = L_a \otimes \mathbb{C}$ and $\mathcal{L}^* = L_{-a}$, respectively.

We denote by $\mathcal{E}^k(M, \mathbb{R})$ (resp. $\mathcal{E}^{p,q}(X, \mathbb{C})$) the sheaf of smooth real k -forms on M (resp. of complex-valued (p, q) -forms on X).

For any $\alpha \in a$, the differential operator $d_\alpha := d - \alpha \wedge \cdot$ on $\mathcal{E}^{k-1}(M, \mathbb{R})$ corresponds to the induced differential on $\mathcal{E}^k(M, L^*)$ via the flat connection $\nabla^{-\alpha} := d - \alpha$ on $L^* = L_{-a}$. Furthermore, d_α defines the Lichnerowicz–Novikov complex

$$(2) \quad \dots \xrightarrow{d_\alpha} \mathcal{E}^{k-1}(M, \mathbb{R}) \xrightarrow{d_\alpha} \mathcal{E}^k(M, \mathbb{R}) \xrightarrow{d_\alpha} \dots$$

which is isomorphic to the de Rham complex of differential forms with values in L^*

$$(3) \quad \dots \xrightarrow{d_{L^*}} \mathcal{E}^{k-1}(M, L^*) \xrightarrow{d_{L^*}} \mathcal{E}^k(M, L^*) \xrightarrow{d_{L^*}} \dots$$

This isomorphism can be made explicit by writing $\alpha|_{U_i} = df_i$ on an open covering $\mathfrak{U} = (U_i)$ of M : then, for any d_α -closed smooth form ω on M , $\omega_i|_{U_i} := e^{-f_i}\omega$ gives rise to a d_{L^*} -closed form with values in L^* .

In particular, we have an isomorphism of cohomology groups

$$H_\alpha^k(M, \mathbb{R}) \simeq H_{d_{L^*}}^k(M, L^*).$$

For the Dolbeault cohomology groups of X with values in the flat holomorphic line bundle \mathcal{L}^* , we have similarly

$$d_{\mathcal{L}^*} = \partial_{\mathcal{L}^*} + \bar{\partial}_{\mathcal{L}^*} \quad \text{and} \quad d_{\alpha} = \partial_{\alpha} + \bar{\partial}_{\alpha},$$

where

$$\partial_{\alpha} = \partial - \alpha^{1,0} \wedge \quad \text{and} \quad \bar{\partial}_{\alpha} = \bar{\partial} - \alpha^{0,1} \wedge,$$

thus giving rise to the isomorphisms

$$(4) \quad H_{\partial_{\alpha}}^{p,q}(X, \mathbb{C}) \simeq H_{\partial_{\mathcal{L}^*}}^{p,q}(X, \mathcal{L}^*).$$

We can further consider the Aeppli and the Bott–Chern cohomology groups of (p, q) -forms with values in \mathcal{L}^* , respectively denoted $H_A^{p,q}(X, \mathcal{L}^*)$ and $H_{BC}^{p,q}(X, \mathcal{L}^*)$, also seen as the spaces of $\partial_{\alpha}\bar{\partial}_{\alpha}$ -closed (p, q) -forms modulo $\text{im}(\partial_{\alpha}) \oplus \text{im}(\bar{\partial}_{\alpha})$ (resp. d_{α} -closed (p, q) -forms modulo $\partial_{\alpha}\bar{\partial}_{\alpha}$ -exact ones). The Hodge duality theory (see e.g. [1, Cor. 1.5]) yields in this case

Lemma 2.1. *Let $X = (M, J)$ be a compact complex manifold of complex dimension n and L a real flat line bundle. Then, the natural pairing through integration of wedge products of 2-forms with values in L and L^* respectively, gives rise to canonical isomorphisms*

$$\left(H_{d_L}^k(M, L) \right)^* \cong H_{d_{L^*}}^{2n-k}(M, L^*), \quad \left(H_{BC}^{p,q}(X, \mathcal{L}) \right)^* \cong H_A^{n-p, n-q}(X, \mathcal{L}^*),$$

where the upper $*$ denotes the dual vector space.

For any open subset $U \subset M$, we denote by $\mathcal{E}_c^k(U, \mathbb{R})$ and $\mathcal{E}_c^{p,q}(U, \mathbb{C})$ the respective spaces of smooth forms with compact support in U . The space $\mathcal{D}'_k(U, \mathbb{R})$ of currents of dimension k (and degree $2n - k$) is the dual topological space of $\mathcal{E}_c^k(U)$. Similarly, the space $\mathcal{D}'_{p,q}(U, \mathbb{C})$ of currents of bi-dimension (p, q) (and bi-degree $(n - p, n - q)$) is the dual topological space of $\mathcal{E}_c^{p,q}(U, \mathbb{C})$. We shall use the standard notation $\mathcal{D}'^r(U, \mathbb{R}) := \mathcal{D}'_{2n-r}(U, \mathbb{R})$ and $\mathcal{D}'^{r,s}(U, \mathbb{C}) := \mathcal{D}'_{n-r, n-s}(U, \mathbb{C})$ and denote by j the natural injections $j : \mathcal{E}^k(M, \mathbb{R}) \hookrightarrow \mathcal{D}'^k(M, \mathbb{R})$ and $j : \mathcal{E}^{p,q}(X, \mathbb{C}) \hookrightarrow \mathcal{D}'^{p,q}(X, \mathbb{C})$ obtained by the wedge product and integration.

The action of the operator $d_{\alpha} : \mathcal{D}'_k(M, \mathbb{R}) \mapsto \mathcal{D}'_{k-1}(M, \mathbb{R})$ on currents is defined by

$$(5) \quad (d_{\alpha}T)(\phi) := (-1)^{k-1}T(d_{\alpha}\phi), \quad \forall \phi \in \mathcal{E}^{k-1}(M, \mathbb{R}),$$

and similarly for the operators

$$\partial_{\alpha} : \mathcal{D}'_{p,q}(X, \mathbb{C}) \rightarrow \mathcal{D}'_{p-1,q}(X, \mathbb{C}), \quad \bar{\partial}_{\alpha} : \mathcal{D}'_{p,q}(X, \mathbb{C}) \rightarrow \mathcal{D}'_{p,q-1}(X, \mathbb{C}).$$

Notice that for any $(k - 1)$ -form ϕ and $(2n - k)$ -form ψ on M , we have

$$(6) \quad d(\phi \wedge \psi) = (d_{\alpha}\phi) \wedge \psi + (-1)^{k-1}\phi \wedge d_{-\alpha}\psi.$$

Thus, the sign convention in (5) is such that for any smooth $(2n - k)$ form ψ on M , giving rise to a current $T_{\psi} := j(\psi) \in \mathcal{D}'_k(M, \mathbb{R})$, we have

$$(7) \quad d_{\alpha}T_{\psi} = T_{(d_{-\alpha}\psi)}, \quad d_{-\alpha}T_{\psi} = -T_{(d_{\alpha}\psi)}.$$

Let $H_k^{\alpha}(M, \mathbb{R})$, $H_{p,q}^{BC,\alpha}(X, \mathbb{C})$, $H_{p,q}^{A,\alpha}(X, \mathbb{C})$ be the corresponding homology groups in terms of currents, defined respectively as d_{α} -closed modulo d_{α} -exact currents of degree

$2n - k$, d_α -closed modulo $\partial_\alpha \bar{\partial}_\alpha$ -exact currents of bi-degree $(n - p, n - q)$, and $\partial_\alpha \bar{\partial}_\alpha$ -closed currents of bi-degree $(n - p, n - q)$ modulo $\text{im}(\partial_\alpha) \oplus \text{im}(\bar{\partial}_\alpha)$. The injection j defined above gives rise to linear isomorphisms

$$\begin{aligned} H_{-\alpha}^{2n-k}(M, \mathbb{R}) &\cong H_k^\alpha(M, \mathbb{R}), \\ H_{BC, -\alpha}^{n-p, n-q}(X, \mathbb{C}) &\cong H_{p,q}^{BC, \alpha}(X, \mathbb{C}), \\ H_{A, -\alpha}^{n-p, n-q}(X, \mathbb{C}) &\cong H_{p,q}^{A, \alpha}(X, \mathbb{C}). \end{aligned}$$

Combined with Lemma 2.1, we have canonical isomorphisms

$$\begin{aligned} H_k^\alpha(M, \mathbb{R}) &\cong (H_\alpha^k(M, \mathbb{R}))^*, \\ H_{p,q}^{BC, \alpha}(X, \mathbb{C}) &\cong (H_{A, \alpha}^{p,q}(X, \mathbb{C}))^*, \\ H_{p,q}^{A, \alpha}(X, \mathbb{C}) &\cong (H_{BC, \alpha}^{p,q}(X, \mathbb{C}))^*. \end{aligned}$$

We use the usual notions of *positivity* of $T \in \mathcal{D}'_{p,p}(U, \mathbb{R})$, defined as follows

Definition 2.2. $T \in \mathcal{D}'_{p,p}(U, \mathbb{R})$ is a *weakly positive* (or *positive*) (p, p) -current ($T \geq 0$) if for each $\xi_1, \dots, \xi_p \in \mathcal{E}_c^{1,0}(U)$, $T(i\xi_1 \wedge \bar{\xi}_1 \wedge \dots \wedge i\xi_p \wedge \bar{\xi}_p) \geq 0$; the current T is called *strongly positive* if $T - j(\psi^{n-p})$ is a positive (p, p) -current, where ψ is a positive-definite smooth $(1, 1)$ -form on X .

3. TWISTED EXACT CURRENTS AND LCS STRUCTURES ON COMPLEX SURFACES

We now specialize to the case when $X = S$ is a compact complex surface with odd first Betti number $b_1(S)$, in which case we have the following result:

Theorem 3.1. [5] *Let $S = (M, J)$ be a compact complex surface with odd first Betti number, and \mathfrak{c} a conformal class of hermitian metrics on S . Then, there exists a non-zero de Rham class $a \in H_{dR}^1(M, \mathbb{R})$ such that for any metric $g \in \mathfrak{c}$, there exists a representative $\alpha \in a$ such that the fundamental 2-form F of g satisfies*

$$(8) \quad d_\alpha d_\alpha^c F = 0,$$

where $d_\alpha^c := Jd_\alpha J^{-1}$.

The following can be obtained by a simple and standard modification of the arguments in [37, 49].

Proposition 3.2. *The complex surface $S = (M, J)$ admits a $d_\alpha d_\alpha^c$ -closed positive definite 2-form F if and only if S admits no non-trivial positive current $T \geq 0$ which is $d_\alpha d_\alpha^c$ -exact.*

Proof. (\implies) If there exists $\tau \in \mathcal{D}'_4(S, \mathbb{R})$ such that $T = d_\alpha d_\alpha^c \tau$, then $T(F) = \tau(d_\alpha d_\alpha^c F) = 0$. As $F > 0$, it follows that $T = 0$.

(\impliedby) The closed mapping theorem asserts that the adjoint of a linear mapping with a closed range has a closed range (see e.g. [51], Ch.IV, Sec.7.7). As $H_{1,1}^{BC, \alpha}(S, \mathbb{C}) \cong H_{BC, -\alpha}^{1,1}(S, \mathbb{C}) \cong H_{BC}^{1,1}(S, \mathcal{L}^*)$ is finite dimensional, a similar argument as the one in [37, p. 174] shows that the subspace $\mathcal{C} = \{T = d_\alpha d_\alpha^c \tau, \tau \in \mathcal{D}'_4(S, \mathbb{R})\}$ is a closed subspace in $\mathcal{D}'_{1,1}(S, \mathbb{R})$. Let $\psi > 0$ be a positive definite $(1, 1)$ form and consider the weakly compact convex subset $\mathcal{K} = \{T \in \mathcal{D}'_{1,1}(S, \mathbb{R}) \mid T \geq 0, T(\psi) = 1\}$. We have $\mathcal{C} \cap \mathcal{K} = \emptyset$ by the assumption. The spaces $\mathcal{E}^{p,q}(S, \mathbb{C})$ and $\mathcal{D}'_{p,q}(S, \mathbb{C})$ being dual of each other [53, p. 75], Hahn–Banach separation theorem implies that there exists a smooth $(1, 1)$ -form F such that

$$F|_{\mathcal{C}} = 0 \text{ and } F|_{\mathcal{K}} > 0.$$

The first condition implies that for each $\tau \in \mathcal{D}'_4(S, \mathbb{R})$, $d_\alpha d_\alpha^c \tau(F) = \tau(d_\alpha d_\alpha^c F) = 0$, hence $d_\alpha d_\alpha^c F = 0$, whereas the second condition yields $F > 0$. \square

Furthermore, as shown in [5, Prop. 2.12 & Lemma 2.13], the relevance of the condition (8) is given by the following

Proposition 3.3. *On a compact complex surface S with $b_1(S) = 1$, Kodaira dimension $\text{kod}(S) = -\infty$, and whose minimal model has positive second Betti number, the following conditions are equivalent.*

- (i) $0 \neq a \in H_{dR}^1(S, \mathbb{R})$ is such that there exists a hermitian metric g on S and a representative $\alpha \in a$, so that fundamental 2-form F of g satisfies (8);
- (ii) $0 \neq a \in H_{dR}^1(S, \mathbb{R})$ is such that for any $\alpha \in a$, there exists a hermitian metric g on S whose fundamental 2-form F of g satisfies (8);
- (iii) $0 \neq a \in H_{dR}^1(S, \mathbb{R})$ is such that for any $\alpha \in a$, S admits a locally conformally symplectic 2-form ω which tames J and satisfies $d_\alpha \omega = 0$.

Remark 3.4. On any complex manifold $X = (M, J)$ endowed with a LCS structure ω taming J and with Lee form α , the fundamental 2-form $F = \omega^{1,1} > 0$ of the corresponding hermitian structure satisfies (8). The above Proposition provides a converse.

Definition 3.5. On a complex surface S we denote by $\mathcal{T}(S) \subset H_{dR}^1(S, \mathbb{R})$ the set of deRham classes of the Lee forms α of locally conformally symplectic 2-forms ω which tame the complex structure J .

By [5, Thm.1], on any compact complex surface $\mathcal{T}(S) \neq \emptyset$. Furthermore, when $b_1(S) = 1$, we can define an order on $H_{dR}^1(S, \mathbb{R})$ by the degree of a flat holomorphic line bundle \mathcal{L}_a corresponding to $a \in H_{dR}^1(S, \mathbb{R})$ via (1), with respect to some (and hence any [60]) Gauduchon metric on S . Then, [5, Prop. 4.3] yields

Proposition 3.6. *On a compact complex surface S with $b_1(S) = 1$ and $\text{kod}(S) = -\infty$ we have $\emptyset \neq \mathcal{T}(S) \subset (-\infty, 0)$.*

We are now going to establish the following refinement of Proposition 3.2 in the case of complex surfaces with $b_1(S) = 1$ and $\text{kod}(S) = -\infty$.

Proposition 3.7. *Let S be a compact complex surface with $b_1(S) = 1$ and Kodaira dimension $\text{kod}(S) = -\infty$.*

- (a) *Suppose $\beta \in b \in H_{dR}^1(S, \mathbb{R})$ is such that there exists a weakly negative current $\tau \leq 0$ of degree 0, with $T := d_\beta d_\beta^c \tau \geq 0$ and $T \neq 0$. Then $b \in (-\infty, 0)$ and $\mathcal{T}(S) \subset (-\infty, b)$.
Conversely, if $\mathcal{T}(S) \subset (-\infty, b)$ for some $b < 0$, then for each $\beta \in b$ there exists a weakly negative current $\tau \leq 0$ of degree 0, with $T := d_\beta d_\beta^c \tau \geq 0$ and $T \neq 0$.*
- (b) *Suppose $\delta \in d \in H_{dR}^1(S, \mathbb{R})$ is such that there exists a weakly positive current $\tau \geq 0$ of degree 0, with $T := d_\delta d_\delta^c \tau \geq 0$ and $T \neq 0$. Then either $d \in (0, \infty)$ or else $\mathcal{T}(S) \subset (d, 0)$.
Conversely, if $\mathcal{T}(S) \subset (d, 0)$ for some de Rham class $d < 0$, or if $d > 0$, then for each $\delta \in d$ there exists a weakly positive current $\tau \geq 0$ of degree 0, with $T := d_\delta d_\delta^c \tau \geq 0$ and $T \neq 0$.*

Proof. (a) We first notice that the condition $\tau \leq 0$ is conformally invariant and

$$d_\beta d_\beta^c (e^f \tau) = e^f (d_{\bar{\beta}} d_{\bar{\beta}}^c \tau)$$

with $\tilde{\beta} = \beta - df$. Thus, after making a conformal modification of T we can assume that $T = d_\beta d_\beta^c \tau \geq 0$, $T \neq 0$ for some weakly negative degree zero current τ and any given $\beta \in b$.

Let $a \in \mathcal{T}(S)$. By Remark 3.4, there is an $\alpha \in a$ and a hermitian metric g on S whose fundamental 2-form F satisfies (8). By changing g in its conformal class and α in its deRham class a , we can assume that g is a Gauduchon metric [32] on S . Writing $b = ta$ for some $t > 0$, we let $\beta := t\alpha \in b$. Denoting by $\theta_g = J\delta^g F$ be the Lee form of g , we compute as in [5, Lemma 2.5]:

$$\begin{aligned}
d_\beta d_\beta^c F &= \left(d(J\theta_g - J\beta) + (\theta_g - \beta) \wedge J(\theta_g - \beta) \right) \wedge F \\
&= \frac{1}{2} \left(\delta^g(\beta - \theta_g) + \langle (\beta - \theta_g), \beta \rangle_g \right) F \wedge F \\
(9) \quad &= \frac{1}{2} \left(\delta^g \beta + \langle (\beta - \theta_g), \beta \rangle_g \right) F \wedge F \\
&= \frac{t}{2} \left(\delta^g \alpha + \langle (t\alpha - \theta_g), \alpha \rangle_g \right) F \wedge F \\
&= \frac{(t-1)t}{2} |\alpha|_g^2 F \wedge F,
\end{aligned}$$

where for passing from the first line to the second we use the computation appearing in the proof of [5, Lemma 2.5], from the second line to the third we have used that g is Gauduchon (i.e. $\delta^g \theta_g = 0$) and for passing from the fourth line to the fifth we have used the identity

$$\delta^g \alpha + \langle \alpha - \theta_g, \alpha \rangle_g = 0$$

which follows from Lemma 2.5 in [5] and the fact that g is Gauduchon. Applying $T = d_\beta d_\beta^c \tau$ to F (with $\beta = t\alpha$), we get

$$(10) \quad 0 < T(F) = \frac{t(t-1)}{2} \tau \left(|\alpha|^2 F \wedge F \right).$$

As $\tau \left(|\alpha|^2 F \wedge F \right) \leq 0$ by assumption, we obtain $t(t-1) < 0$, i.e. $t \in (0, 1)$. By Proposition 3.6, $t > 0$ is equivalent to $b \in (-\infty, 0)$; furthermore, $\alpha = \beta/t < \beta$.

We now establish the other direction. We thus assume that S is a compact complex surface in the Kodaira class VII and $b \in (-\infty, 0) \subset H_{dR}^1(S, \mathbb{R})$ is such that $\mathcal{T}(S) \subset (-\infty, b)$. We are going to show that for any smooth closed 1-form $\beta \in b$, there exists a weakly negative current $\tau \leq 0$ of degree 0 on S , such that $T := d_\beta d_\beta^c \tau \geq 0$ and $T \neq 0$.

We first assume that the minimal model of S has non-zero second Betti number, i.e. we are under the hypothesis of Proposition 3.3. In this case, b equivalently bounds the de Rham classes a for which (8) holds for some hermitian metric on S . For $\beta \in b$ and a positive definite smooth (1, 1)-form ψ on S we consider the following sets

$$\begin{aligned}
\mathcal{P} &:= \{T \in \mathcal{D}'_{1,1}(S, \mathbb{R}) \mid T \geq 0\}, \\
\mathcal{K} &:= \{T \in \mathcal{D}'_{1,1}(S, \mathbb{R}) \mid T = d_\beta d_\beta^c \tau, \tau \leq 0, \tau(\psi \wedge \psi) = -1\}.
\end{aligned}$$

\mathcal{P} is a closed convex cone whereas \mathcal{K} is a weakly compact (by the Banach–Alaoglu theorem) and convex subset of $\mathcal{D}'_{1,1}(S, \mathbb{R})$. We want to show that $\mathcal{P} \cap \mathcal{K} \neq \emptyset$.

Suppose for contradiction that $\mathcal{P} \cap \mathcal{K} = \emptyset$. By the Hahn–Banach separation theorem [51, p.65], there exists a smooth (1, 1)-form \tilde{F} such that for any $T_1 \in \mathcal{P}$, $T_1(\tilde{F}) \geq 0$ and for any $T_2 \in \mathcal{K}$, $T_2(\tilde{F}) < 0$. The first condition is equivalent to \tilde{F} being semi-positive (i.e. $\tilde{F} \geq 0$) whereas the second condition is equivalent to $(d_\beta d_\beta^c \tilde{F}) > 0$ on S . As the

latter condition is open, we can assume without loss that $\tilde{F} > 0$ is positive-definite. Let f be a smooth function such that $F := e^f \tilde{F}$ is a Gauduchon metric, see [32]. We now apply Theorem 3.1. By the assumption and Proposition 3.3 there exist a real number $t \in (0, 1)$ and a smooth function h , such that $d_{\tilde{\alpha}} d_{\tilde{\alpha}}^c F = 0$ for the 1-form $\tilde{\alpha} = \frac{1}{t}(\beta + dh)$. Letting $\beta' = \beta + df = \tilde{\beta} + d(f - h)$, it follows from (9)

$$\begin{aligned} 0 < e^f (d_{\beta} d_{\beta}^c \tilde{F}) &= e^f (d_{\beta} d_{\beta}^c (e^{-f} F)) = d_{\beta'} d_{\beta'}^c F \\ &= \frac{1}{2} \left(\delta^g(\beta' - \theta_g) + \langle \beta' - \theta_g, \beta' \rangle \right) F \wedge F \\ &= \frac{1}{2} \left(\Delta^g(f - h) + \langle d(f - h), (2\tilde{\beta} - \theta_g) \rangle_g + |d(f - h)|_g^2 \right) F \wedge F \\ &\quad + d_{\tilde{\beta}} d_{\tilde{\beta}}^c F \\ &= \frac{1}{2} \left(\Delta^g(f - h) + \langle d(f - h), (2\tilde{\beta} - \theta_g) \rangle_g + |d(f - h)|_g^2 \right. \\ &\quad \left. + \frac{(t-1)}{t} |\tilde{\beta}|_g^2 \right) F \wedge F. \end{aligned}$$

It thus follows that everywhere on S ,

$$\Delta^g(f - h) + \langle d(f - h), (2\tilde{\beta} - \theta_g) \rangle_g + |d(f - h)|_g^2 > 0,$$

At a point where $f - h$ reaches its minimum $\Delta^g(f - h) \leq 0$ and $d(f - h) = 0$, which is impossible.

We now consider the case when the minimal model of S has zero second Betti number. By [58], S is obtained from either a Hopf surface or an Inoue–Bombieri surface by blowing up points. In the first case $\mathcal{T}(S) = (-\infty, 0)$ by [5, Thm.1.4], so there is nothing to prove. In the former case, $\mathcal{T}(S) = \{a\}$ by [6, Thm. 1.3]. This also corresponds to the case when there exists an $\alpha \in a$ such that $d_{-\alpha} d_{-\alpha}^c(\pm 1) = 0$ (see [6, Thm.4.1]). We can argue in this case (see the proof of Proposition 4.2 below) that for any $b > a$ there exists a $\beta \in b$ and a degree zero current $\tau < 0$ such that $d_{\beta} d_{\beta}^c \tau > 0$.

Part (b) follows by similar arguments. \square

The previous result can be complemented with the following useful observation.

Proposition 3.8. *Suppose $b \in (-\infty, 0) \subset H_{dR}^1(S, \mathbb{R})$ is such that there exists a closed 1-form $\beta \in b$ and a degree 0 current τ_{β} on S with $T := d_{\beta} d_{\beta}^c \tau_{\beta} \geq 0$, $T \neq 0$. Suppose, furthermore, that there exists $a \in \mathcal{T}(S)$ such that $a < b$. Then, τ_{β} is a negative current.*

Proof. It follows by the assumptions on $T = d_{\beta} d_{\beta}^c \tau_{\beta}$ that

$$(11) \quad (d_{\beta} d_{\beta}^c \tau_{\beta})(F) > 0$$

for any hermitian 2-form F . Letting

$$\tilde{\beta} := \beta + d\varphi, \quad \varphi \in C^{\infty}(S),$$

and using that

$$e^{\varphi} d_{\tilde{\beta}} d_{\tilde{\beta}}^c (e^{-\varphi} \tau_{\beta}) = d_{\beta} d_{\beta}^c \tau_{\beta},$$

(which follows from the identity $e^{-\varphi} d_{\tilde{\beta}} d_{\tilde{\beta}}^c e^{\varphi} F = d_{\beta} d_{\beta}^c F$ on forms) we see that $\tau_{\tilde{\beta}} := e^{-\varphi} \tau_{\beta}$ satisfies the assumption (11) with respect to $\tilde{\beta} \in b$. Furthermore, $\tau_{\tilde{\beta}}$ has the same sign as τ_{β} . This gives rise to a conformal class of degree 0 currents

$$[\tau_{\beta}] = \{\tau_{\beta}, \beta \in b\},$$

satisfying the assumption of the Proposition.

By [5], there exists a Gauduchon hermitian form F_0 and a closed 1-form $\alpha_0 \in a$ such that $d_{\alpha_0} d_{\alpha_0}^c F_0 = 0$. Let $F = e^f F_0$ and $\alpha = \alpha_0 + df$. We then have

$$d_{\alpha} d_{\alpha}^c F = e^f (d_{\alpha_0} d_{\alpha_0}^c (e^{-f} F)) = e^f (d_{\alpha_0} d_{\alpha_0}^c F_0) = 0.$$

Our assumption $a < b < 0$ yields that $b = t_0 a$, $t_0 \in (0, 1)$. Letting $\beta := t_0 \alpha$, we compute as in (9)

$$\begin{aligned} d_{\beta} d_{\beta}^c F &= \left(d(J\theta_g - J\beta) + (\theta_g - \beta) \wedge J(\theta_g - \beta) \right) \wedge F \\ &= \frac{1}{2} \left(\delta^g(\beta - \theta_g) + \langle (\beta - \theta_g), \beta \rangle_g \right) F \wedge F \\ &= \frac{t_0}{2} \left(\delta^g(\alpha - \theta_g) + \frac{(t_0 - 1)}{t_0} (\delta^g \theta_g) + \langle (t_0 \alpha - \theta_g), \alpha \rangle_g \right) F \wedge F \\ &= \frac{t_0}{2} \left(\delta^g(\alpha - \theta_g) + \langle \alpha - \theta_g, \alpha \rangle_g + \frac{(t_0 - 1)}{t_0} (\delta^g \theta_g) + (t_0 - 1) |\alpha|_g^2 \right) F \wedge F \\ &= \frac{(t_0 - 1)}{2} \left(\delta^g \theta_g + t_0 |\alpha|_g^2 \right) F \wedge F, \end{aligned}$$

where we have used the equivalence (which follows from the equality at the second line with $\alpha = \beta$)

$$\delta^g(\alpha - \theta_g) + \langle \alpha - \theta_g, \alpha \rangle_g = 0 \iff d_{\alpha} d_{\alpha}^c F = 0,$$

for passing to the last line. Using that the hermitian metric g is related to the Gauduchon metric by $g = e^f g_0$, so that $\theta_g = \theta_{g_0} + df$, and $\alpha = \alpha_0 + df$, we have (see e.g. [10, pp. 59])

$$\begin{aligned} \delta^g \theta_g &= e^{-f} \left(\delta^{g_0}(\theta_{g_0} + df) - \langle df, \theta_{g_0} + df \rangle_{g_0} \right) \\ &= e^{-f} \left(\Delta_{g_0} f - \langle df, \theta_{g_0} + df \rangle_{g_0} \right), \\ |\alpha|_g^2 &= e^{-f} \langle \alpha_0 + df, \alpha_0 + df \rangle_{g_0}. \end{aligned}$$

Hence

$$d_{\beta} d_{\beta}^c F = \frac{(t_0 - 1)}{2} \left(\Delta_{g_0} f + \langle 2t_0 \alpha_0 - \theta_{g_0}, df \rangle_{g_0} + (t_0 - 1) |df|_{g_0}^2 + t_0 |\alpha_0|_{g_0}^2 \right) e^f F_0 \wedge F_0.$$

Making the substitution $u := e^{(1-t_0)f}$, we conclude

$$(12) \quad d_{\beta} d_{\beta}^c F = -\frac{1}{2} \left(\Delta_{g_0} u + \langle 2t_0 \alpha_0 - \theta_{g_0}, du \rangle_{g_0} + t_0(1 - t_0) |\alpha_0|_{g_0}^2 u \right) e^{t_0 f} F_0 \wedge F_0.$$

Consider, for $0 < t_0 < 1$, the linear differential operator,

$$P(\varphi) := \Delta_{g_0} \varphi + \langle 2t_0 \alpha_0 - \theta_{g_0}, d\varphi \rangle_{g_0} + t_0(1 - t_0) |\alpha_0|_{g_0}^2 \varphi, \quad \varphi \in C^{\infty}(S).$$

As $\alpha_0 \neq 0$, by the strong maximum principle (see e.g. [28, p.349]) P has a trivial kernel. As the index of P is zero, P has also a trivial co-kernel. It then follows from standard linear elliptic theory that for any smooth function $w \in C^{\infty}(S)$, there exists a unique smooth function $u \in C^{\infty}(S)$ solving the PDE

$$(13) \quad P(u) = w.$$

We can thus choose $w \geq 0$, $w \not\equiv 0$ and denote by u the corresponding solution of (13). Evoking the strong maximum principle again, it follows that $u > 0$ on S . This shows

that there exists a smooth function f such that $F := e^f F_0$ satisfies

$$d_\beta d_\beta^c F = -\frac{1}{2} w e^{t_0 f} F_0 \wedge F_0, \quad w \geq 0.$$

where $\beta := t_0(\alpha_0 + df)$ and $\beta_0 := t_0 \alpha_0$. Considering the corresponding currents τ_β and τ_{β_0} in the conformal class, we have by (11):

$$0 < T(F) = d_\beta d_\beta^c \tau_\beta(F) = \tau_\beta(d_\beta d_\beta^c F) = -\frac{1}{2} \tau_\beta(e^{t_0 f} w F_0 \wedge F_0) = -\frac{1}{2} \tau_{\beta_0}(w F_0 \wedge F_0).$$

As this inequality holds for any $w \geq 0, w \neq 0$, we conclude that $\tau_{\beta_0} < 0$. \square

We have the following corollary of the above results.

Theorem 3.9. *Let S be a class VII surface. Then $\mathcal{T}(S)$ is connected.*

Proof. Recall that $\mathcal{T}(S) \neq \emptyset$ by [5]. If $\mathcal{T}(S)$ is not connected, we can find $a_1, a_2 \in \mathcal{T}(S)$ and $b \notin \mathcal{T}(S)$ with $a_1 < b < a_2$. If the minimal model S_0 of S has zero second Betti number, by [58] S is either a Hopf surface or an Inoue–Bombieri surface. In these cases the set $\mathcal{T}(S)$ is known and connected by the results in [5, 6]. We now assume $b_2(S_0) > 0$. By Propositions 3.3 and 3.2, there exists a zero degree current τ with $T := d_\beta d_\beta^c \tau \geq 0, \beta \in b$ and $T \neq 0$. By Proposition 3.8, $\tau \leq 0$ and by Proposition 3.7-(a), $a_2 < b$, a contradiction. \square

Proof of Theorem 1.1. This is an immediate corollary of [5, Thm.1.1, Prop.4.3] (which show that $\mathcal{T}(S)$ is a non-empty subset of $(-\infty, 0)$), [6, Thm.1.3] (which shows that $\mathcal{T}(S)$ is either open or a single point and characterizes the latter case) and Theorem 3.9. \square

Definition 3.10. The *minimal \mathbb{Z} -cover* of a complex surface S with $b_1(S) = 1$ is defined as follows. The torsion-free part $H_1(S, \mathbb{Z})^{\text{free}}$ of $H_1(S, \mathbb{Z})$ being \mathbb{Z} , S admits an infinite cyclic cover $\hat{\pi} : \hat{S} \rightarrow S$ whose fundamental group is the kernel of the morphism

$$\pi_1(S) \mapsto \left(\frac{\pi_1(S)}{[\pi_1(S), \pi_1(S)]} \right)^{\text{free}}.$$

By construction, $\hat{\pi}^*(H_{dR}^1(S, \mathbb{R})) = \{0\}$ and \hat{S} can be characterized as the smallest \mathbb{Z} -cover of S with that property.

We denote by γ the deck transformation on \hat{S} such that $S = \hat{S}/\langle \gamma \rangle$ and notice the following simple fact.

Lemma 3.11. *Let S be a compact complex surface with $b_1(S) = 1$ and \hat{S} be its minimal \mathbb{Z} cover with $S = \hat{S}/\langle \gamma \rangle$. Then S admits a closed 1-form β and a weakly negative (resp. weakly positive) current τ of degree 0 such that $T := d_\beta d_\beta^c \tau \geq 0$ if and only if \hat{S} admits a negative (resp. non-negative) locally integrable plurisubharmonic function \hat{u} , which is automorphic in the sense that*

$$(14) \quad \hat{u} \circ \gamma = C \hat{u}$$

for a positive constant C .

Proof. Suppose first β is a closed 1-form and τ a degree 0 current on S satisfying $T := d_\beta d_\beta^c \tau \geq 0$. We can think of τ as a function in $L_{loc}^1(S)$ and denote by $\hat{\tau}$ the lifted function to \hat{S} . This gives rise to a degree 0 current on \hat{S} such that $\gamma^*(\hat{\tau}) = \hat{\tau}$. Let $\hat{\beta} = \hat{\pi}^*(\beta)$ be the lift of the closed 1-form β . We then define $\hat{T} := d_{\hat{\beta}} d_{\hat{\beta}}^c \hat{\tau}$. It is not

hard to see that \hat{T} is a positive current on \hat{S} (as $T = d_\beta d_\beta^c \tau \geq 0$ on S). We can write $\hat{\beta} = d\hat{f}$ on \hat{S} for a smooth function satisfying

$$\hat{f} \circ \gamma - \hat{f} = c$$

for some real constant c . It then follows that the degree 0 current $e^{\hat{f}} \hat{\tau}$ on \hat{S} satisfies

$$dd^c(e^{\hat{f}} \hat{\tau}) = e^{\hat{f}}(d_{\hat{\beta}} d_{\hat{\beta}}^c \hat{\tau}) = e^{\hat{f}} \hat{T} \geq 0$$

It follows that (see e.g. [19, Ch.I, (5.5)]) $e^{\hat{f}} \hat{\tau}$ defines a negative (resp. non-positive) PSH function $\hat{u} \in L_{loc}^1(\hat{S})$ (where we have used the maximum principle to rule out zeroes of \hat{u} in the non-positive case) which has the property $\hat{u} \circ \gamma = e^c \hat{u}$.

Conversely, suppose \hat{u} is a negative (resp. non-negative) PSH function on \hat{S} satisfying $\hat{u} \circ \gamma = e^c \hat{u}$ for some real constant c . As $H_{dR}^1(S, \mathbb{R}) \neq \{0\}$, there exists a closed but not exact 1-form β whose lift on \hat{S} satisfies $\hat{\beta} = d\hat{f}$ for a smooth function \hat{f} such that $\hat{f} \circ \gamma = \hat{f} + c_0$, $c_0 \neq 0$. Multiplying \hat{f} by a suitable constant if necessary, we may suppose that $c_0 = c$. We then consider the degree zero current $\hat{\tau} := e^{-\hat{f}} \hat{u}$ on \hat{S} and check as before that

$$d_{\hat{\beta}} d_{\hat{\beta}}^c \hat{\tau} = e^{-\hat{f}} dd^c \hat{u} \geq 0,$$

where $\hat{\beta} = d\hat{f}$. Furthermore, as $e^{-\hat{f}} \hat{u}$ is γ -invariant and negative (resp. non-negative), there exists a degree zero weakly negative (resp. weakly positive) current τ on S such that $\hat{\tau}$ is the pull-back of τ to \hat{S} . Using that $\gamma^*(\hat{\beta}) = \hat{\beta}$ we conclude that τ satisfies $d_\beta d_\beta^c \tau \geq 0$ on S with respect to the induced 1-form β from $\hat{\beta}$. \square

Remark 3.12. For any closed 1-form α on S , the lift $\hat{\alpha}$ to \hat{S} is exact, and $\hat{\alpha} = d\hat{f}$ for a smooth function \hat{f} satisfying

$$\hat{f} \circ \gamma - \hat{f} = c$$

for some real constant c . The constant $c = c_a$ is independent of the choice of α in a given deRham class $a \in H_{dR}^1(S, \mathbb{R})$, being zero precisely when α is exact. It follows that the positive constant $C_a = e^{c_a}$ is 1 iff $a = 0$.

There is a 2-fold choice for the generator γ : we can replace γ by γ^{-1} , which changes C_a with $1/C_a$. To fix this ambiguity, in this paper we assume that the generator γ is chosen so as $C_a > 1$ iff $a < 0$. As a corollary of Proposition 3.7(a), the constant $C = C_b$ in Lemma 3.11 must satisfy $C > 1$ (this is equivalent to the inequality $b < 0$).

Furthermore, using that the function $f(x) = -(-x)^t$ is increasing and convex on $(-\infty, 0)$ for any real number $t \in (0, 1)$, we conclude that $-(-\hat{u})^t$ is also a negative PSH function on \hat{S} which satisfies (14) with constant $C^t < C$; it thus follows that if b satisfies the hypothesis of Proposition 3.7(a), then any $b' \in (b, 0)$ does too.

Remark 3.13. Related to Remark 3.12 is the following useful observation. On any compact complex surfaces satisfying $b_1(S) = 1$ and $\text{kod}(S) = -\infty$ we have the identification (see [42])

$$(15) \quad H_0^1(S, \mathbb{C}^*) = \text{Pic}^0(S) \cong \mathbb{C}^*,$$

compare with (1). We can then write for an isomorphism class of flat bundles $\mathcal{L} \in H_0^1(S, \mathbb{C}^*)$

$$\mathcal{L} = \mathcal{L}_\mu, \quad \mu \in \mathbb{C}^*.$$

In terms of the sequence (1), for any $a \in H_{dR}^1(S, \mathbb{R})$, the corresponding flat complex bundle $\mathcal{L}_a = L_a \otimes \mathbb{C}$ can be then written as \mathcal{L}_{C_a} , where $C_a > 0$ is the constant

introduced according to the convention in Remark 3.12. We thus have that $\mathcal{L}_\mu = \mathcal{L}_a$ with $a \in (-\infty, 0)$ iff $\mu \in \mathbb{R}, \mu > 1$, an identification that we will use throughout the paper.

4. EXAMPLES

In this section, we examine the existence of degree 0 currents as in Proposition 3.7 on the known surfaces in the Kodaira class VII (see [8]), i.e. the compact complex surfaces satisfying $b_1(S) = 1$ and $\text{kod}(S) = -\infty$. Notice that the existence of such currents is invariant under blow-up and blow-down of points, so we shall restrict attention to the minimal case. Minimal complex surfaces in the class VII are referred to as *class VII₀ complex surfaces*.

We first recall the rough classification of the known minimal complex surfaces of Kodaira class VII.

4.1. Known examples of class VII₀ complex surfaces.

Definition 4.1. A *known* minimal complex surface in the Kodaira class VII₀ is one of the following

- An Inoue–Bombieri surface [38] ($b_2(S) = 0$ and S admits no curves);
- A Hopf surface [42, II, section 10] and [41] ($b_2(S) = 0$ and S admits curves);
- A Kato surface [40, 20] ($b_2(S) > 0$ and S admits a global spherical shell).

Following [20, 15], we shall further distinguish between two different types of Kato surfaces

- Enoki surfaces [27], and
- hyperbolic Kato surfaces [15].

If $b_2(S) = 0$ there are no other surfaces than the Inoue–Bombieri and Hopf surfaces listed above [58]. The minimal surfaces in the class VII are said to belong in the class VII₀ whereas those with $b_2(S) > 0$ are said to belong to the class VII₀⁺. According to the still open GSS conjecture (proved in [59] when $b_2(S) = 1$) any complex surface in the class VII₀⁺ must be a Kato surface.

4.2. Definite currents on the known class VII₀ surfaces. We prove with the following

Proposition 4.2. *Suppose S is a known minimal surface of the Kodaira class VII₀. Then,*

- (a) *S admits a weakly negative current $\tau \leq 0$ with $T = d_\beta d_\beta^c \tau \geq 0$ and $T \neq 0$ for some closed 1-form β if and only if S is either an Inoue–Bombieri surface, or a hyperbolic Kato surface;*
- (b) *S admits a weakly positive current $\tau \geq 0$ with $T = d_\delta d_\delta^c \tau \geq 0$ and $T \neq 0$ for some $\delta \in d \in (-\infty, 0) \subset H_{dR}^1(S, \mathbb{R})$ if and only if S is an Inoue–Bombieri surface.*

Proof. We start with showing necessary part of (a): to this end, we need to prove that Hopf and Enoki surfaces do not admit a degree 0 current τ as in the case (a) of the Proposition. If a degree zero current $\tau \leq 0$ existed on a Hopf surface S , it would define a non-positive PSH function \hat{u} on the universal cover $\mathbb{C}^2 \setminus \{0\}$ by Lemma 3.11. As $\mathbb{C}^* \times \mathbb{C} \subset \mathbb{C}^2 \setminus \{0\}$ is covered by \mathbb{C}^2 , by Liouville theorem we must have that \hat{u} is constant, and thus $T = 0$, a contradiction. The case of Enoki surfaces can be

treated similarly: we notice that such a surface contains a cycle of rational curves whose complement has a covering biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ (see [27], [26, Cor. 1.20]).

We now show (b). To this end, we need to check that a Hopf or a Kato surface do not admit a degree 0 current as in the statement. We shall argue by absurd, using Proposition 3.7 (b). In the case of a Hopf surface S , we have by [5, Thm. 1.4] that $\mathcal{T}(S) = (-\infty, 0)$ so we conclude by Proposition 3.7 (b). Similarly, if S is a Kato surface, Brunella [12, 13] shows that $\mathcal{T}(S)$ contains an interval $(-\infty, c)$ for some $c < 0$, so we conclude again by Proposition 3.7 (b).

We finally address the existence part. Notice that Inoue–Bombieri surfaces admit a closed 1-form α with $d_{-\alpha}d_{-\alpha}^c(1) = 0$ and $a := [\alpha] \in (-\infty, 0) \subset H_{dR}^1(S, \mathbb{R})$ (see e.g. [6]). It follows that the integration over S is a positive degree 0 current on S which is $d_{\alpha}d_{\alpha}^c$ -closed. By the proof of Lemma 3.11, it gives rise to a strictly positive PH function $\hat{u} = e^{-f}$, where $\hat{\beta} = df$, on the minimal \mathbb{Z} -cover \hat{S} , which is automorphic with constant $C_0 > 1$. Taking $(\hat{u})^t, t > 1$ and $-(\hat{u})^t, t \in (0, 1)$ we see that for any $C_0 < C$ there exists a positive strictly PSH automorphic function on \hat{S} with automorphy constant C , and for any $1 < C' < C_0$, there exists a negative automorphic strictly PSH function on \hat{S} with automorphy constant C' . Equivalently, for any $\delta \in d < a = [\alpha]$, there exists a weakly positive degree 0 current on S satisfying the hypothesis of (b), and for any $\beta \in b > a$, there exists a current of degree 0 on S satisfying the hypothesis of (a). Proposition 3.7 is then consistent with the fact that in this case $\mathcal{T}(S) = \{a\}$, see [6]. The existence of negative PSH functions on the \mathbb{Z} -cover of a hyperbolic Kato surface is treated in detail in the next subsection. \square

Proof of Theorem 1.3. $\mathcal{T}(S)$ is a non-empty subset of $(-\infty, 0)$ by [5, Thm.1.1, Prop.4.3], see also Proposition 3.6 above. Part (a) is established in [6, Thm.1.3]; for part (b) see e.g. [5, Thm.1.4] and [61]. In the cases (c) and (d), $\mathcal{T}(S)$ is open by [6, Thm.1.3] and connected by Theorem 3.9. The lower bound follows from [12, 13] and the upper bounds in each case follow from Proposition 4.2 above. \square

Proof of Corollary 1.4. Let $\tilde{J}_i := \phi_i(J)$ be the integrable almost complex structure on M_k obtained by the pull-back with a diffeomorphism $\phi_i \in \text{Diff}_0(S)$, and denote by $S_i := (M_k, \tilde{J}_i) \cong S$ the corresponding complex surface. As ϕ_i acts trivially on $H_{dR}^1(M_k, \mathbb{R})$, we have by Theorem 1.3 $\mathcal{T}(S_i) = \mathcal{T}(S) = (-\infty, b) \subset H_{dR}^1(M_k, \mathbb{R})$. By assumption, \tilde{J}_i converges in C^∞ to a complex structure J_∞ : it then follows that any LCS structure ω taming J_∞ will also tame \tilde{J}_i for i large enough. This and $\mathcal{T}(S_i) = \mathcal{T}(S)$ show that $\tau(S_\infty) \subset \mathcal{T}(S) = (-\infty, b)$. Applying Theorem 1.3 again, we conclude that S_∞ must be a hyperbolic Kato surface. \square

4.3. An explicit upper bound of the Lee classes of a hyperbolic Kato surface.

In this section, we show the existence of an upper bound $b = b(S) < 0$ of $\mathcal{T}(S)$ given by Proposition 3.7 for a hyperbolic Kato surface S . Recall that a Kato surface S is a minimal compact complex surface in the class VII with $b_2(S) > 0$ and containing a GSS. Such a surface is obtained by the following general construction. Given

- a composition $\pi = \pi_0 \circ \cdots \circ \pi_{n-1} : W \rightarrow \mathbb{C}^2$ of $n = b_2(S)$ blow-ups over the origin of \mathbb{C}^2 , in such a way that π_0 blows up the origin of \mathbb{C}^2 , $C_0 = \pi_0^{-1}(0)$ and π_i blows up a point $O_{i-1} \in C_{i-1}$ for any $i \geq 1$, and
- a biholomorphism $\sigma : \bar{B} \rightarrow W$ from a neighbourhood of the closed ball to W such that $\sigma(\bar{B}) \subset \pi^{-1}(B)$ and $\sigma(0) \in C_{n-1}$,

we remove $\sigma(B)$ from $\pi^{-1}(\bar{B})$. We obtain an annulus with a boundary composed of two spheres $\pi^{-1}(S^3)$ and $\sigma(S^3)$ where $S^3 = \partial B$. The surface $S = S(\pi, \sigma)$ is obtained by glueing by $\sigma \circ \pi$ isomorphic neighbourhoods of these two spheres (see [20] for details). The resulting smooth manifold is diffeomorphic to $(S^1 \times S^3) \# k \overline{\mathbb{C}P^2}$ and the minimal \mathbb{Z} -cover \hat{S} of S (see Definition 3.10) coincides with the universal cover. Let

$$F = \pi \circ \sigma : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$$

be the associated germ. The order of creation of rational curves induced by the successive blow-ups produces a total order on the rational curves in \hat{S} . Given one of these curves, say C , it is possible to collapse all the curves $C' > C$ to a point \hat{O}_C of C . Doing this, C becomes an exceptional curve of the first kind and we denote by \hat{S}_C the resulting manifold and by $p_C : \hat{S} \rightarrow \hat{S}_C$ this collapsing. Notice that p_C is not proper, and induces by restriction an isomorphism

$$p_C : \hat{S} \setminus \bigcup_{C' > C} C' \rightarrow \hat{S}_C \setminus \{\hat{O}_C\}.$$

We shall use next the following fact taken from [20, Prop. 3.9]. Let $\gamma : \hat{S} \rightarrow \hat{S}$ be the generator of $\pi_1(S) \cong \mathbb{Z}$. Then, for any curve C in \hat{S} as above, we have the following commutative diagram

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\gamma} & \hat{S} \\ p_C \downarrow & & \downarrow p_C \\ \hat{S}_C & \xrightarrow{F_C} & \hat{S}_C \end{array}$$

Moreover, if \hat{O}_C is the critical value of p_C (i.e. the image of all the rational curves $C' > C$ with notations of [20]), we can identify the germ (\hat{S}_C, \hat{O}_C) with $(\mathbb{C}^2, 0)$ in such a way that the equation of C at \hat{O}_C is $C = \{z_2 = 0\}$.

Lemma 4.3. *Let S be a Kato surface defined by a germ*

$$F = \pi \circ \sigma : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$$

and $v_C : (\hat{S}_C, \hat{O}_C) \rightarrow \mathbb{R} \cup \{-\infty\}$ a germ of PSH function such that $F_C^*(v_C) = v_C + c$ for a constant $c \in \mathbb{R}$. Then, there exists on \hat{S} a unique PSH function $\hat{v} : \hat{S} \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying $\gamma^* \hat{v} = \hat{v} + c$, and such that if v_C is defined in a neighbourhood V of \hat{O}_C , then on $p_C^{-1}(V)$ we have

$$v_C \circ p_C = \hat{v}.$$

Proof. The inverse image $p_C^{-1}(V)$ contains the pseudo-concave end of the universal covering space \hat{S} (in the terminology of [20]). Notice that $\hat{S} = \bigcup_{i \in \mathbb{Z}} A_i$ where A_i is a copy of a fundamental domain isomorphic to an annulus $\overline{\Pi^{-1}(B)} \setminus \sigma(B)$. We have $\gamma(A_{i-1}) = A_i$ for every $i \in \mathbb{Z}$. If $A_i \subset p_C^{-1}(V)$ we define \hat{v} on A_{i-1} by $\hat{v}(x) := \hat{v}(\gamma(x)) - c$. \square

We now recall that the hyperbolic Kato surfaces are either Kato surfaces of intermediate type or Inoue–Hirzebruch surfaces which we will treat in the next two examples, respectively.

Example 4.4. Let S be a Kato surface of intermediate type, i.e. a Kato surface which contains a cycle of rational curves with branches. A defining germ of S is given by (see [29])

$$(16) \quad F(z_1, z_2) = (\lambda_1 z_1 z_2^s + P(z_2) + \lambda_2 z_2^{\frac{sk}{k-1}}, z_2^k),$$

where $\lambda_1 \in \mathbb{C}^*$, $\lambda_2 \in \mathbb{C}$, $s, k \in \mathbb{N}^*$, $k \geq 2$, $\frac{s}{k-1} \in \mathbb{N}$ when $\lambda_2 \neq 0$, and $P(z)$ is a certain polynomial with complex coefficients and $\deg(P) \leq s$. The maximal divisor D of S is defined by the equation $z_2 = 0$. For $|z_2| < 1$, the function

$$v(z_1, z_2) := -\log(-\log|z_2|)$$

is PSH, and satisfies

$$F^*v = v - \log(k).$$

By Lemma 4.3, v gives rise to a PSH function \hat{v} on \hat{S} , satisfying $\gamma^*\hat{v} = \hat{v} - \log(k)$. Define $\hat{u} := -\exp(-\hat{v}) = \log|z_2| < 0$. It follows that \hat{u} is negative PSH satisfying

$$\gamma^*\hat{u} = k\hat{u}.$$

The fact that $k > 1$ tells us that the choice for the generator is consistent with the convention of Remark 3.12. Furthermore, the support of \hat{u} is on $D = \{z_2 = 0\}$. It follows that in this case, the upper bound $b \in (-\infty, 0)$ of Proposition 3.7 is given by the deRham class corresponding to the topologically trivial flat line bundle defined by $k \in \mathbb{N}^*$.

Example 4.5. Let S be an Inoue–Hirzebruch Kato surface. In this case, the contraction germ defining S is given by (see [21, p.667])

$$F(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$$

where $A = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathbf{GL}(2, \mathbb{Z})$ with $\det A = \pm 1$. The characteristic polynomial

$$P_A(X) = X^2 - (p + s)X + \det A$$

has two eigenvalues λ_1, λ_2 satisfying $0 < |\lambda_2| < 1 < \lambda_1$, $\lambda_2 = \pm\lambda_1^{-1}$. Each eigenvalue λ_i ($i = 1, 2$) has an eigenvector (ξ_i, η_i) with $\xi_1 > 0, \eta_1 > 0$ and $\xi_2 > 0, \eta_2 < 0$. We next let $(\xi, \eta) := (\xi_1, \eta_1)$, $\lambda := \lambda_1 > 1$, and

$$G(z_1, z_2) := \xi \log|z_1| + \eta \log|z_2|.$$

We have that $G \circ F(z) = \lambda G(z)$ and $\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : G(z_1, z_2) < 0\}$ contains a neighbourhood of $0 \in \mathbb{C}^2$. We then define (on Ω) the PSH function $v(z) := -\log(-G(z))$ which satisfies $v \circ F = v + \log \lambda$ and hence, by Lemma 4.3, gives rise to a PSH function \hat{v} on \hat{S} satisfying $\gamma^*\hat{v} = \hat{v} + \log \lambda$. Letting $\hat{u} := -\exp(-\hat{v}) = G(z)$ we get a negative PSH function on \hat{S} satisfying

$$\gamma^*\hat{u} = \lambda\hat{u}.$$

It follows in this case too that the upper bound $b \in (-\infty, 0)$ of Proposition 3.7 corresponds to the constant $C = C_b = \lambda > 1$, see Remark 3.12.

4.4. An obstruction to the existence of bi-hermitian structures on intermediate Kato surfaces. By a *bi-hermitian structure* on a connected compact complex surface $S = (M, J)$ we mean a conformal class \mathfrak{c} of hermitian metrics such that there exists another integrable complex structure I on M which is positively oriented, \mathfrak{c} -orthogonal, and $I(x) \neq \pm J(x)$ at at least one point of S . The main interest in bi-hermitian structures comes from their link with generalized Kähler geometry revealed in [36]. When $b_1(S)$ is even, i.e. on a Kähler complex surface, it follows by the results in [7, 35] the existence of a bi-hermitian structure is equivalent to the condition $H^0(S, K_S^*) \neq 0$ where K_S stands for the canonical bundle and K_S^* denotes its dual. On the other hand, in the case $b_1(S)$ odd, the situation is not completely understood yet. It is known from [2] that in this case $b_1(S) = 1$, $\text{kod}(S) = -\infty$, and $H^0(S, K_S^* \otimes \mathcal{L}_a) \neq 0$ for some flat holomorphic line bundle \mathcal{L}_a corresponding to a class $a \in (-\infty, 0] \subset H_{dR}^1(S, \mathbb{R})$ via (1). It was observed in [23, 4] that the latter conditions imply that S must be obtained by blowing up points of either a Hopf surface or a Kato surface of index 1 (or NAC of index 1 in the terminology of [23]). Since the existence of bi-hermitian structures is stable under blow-ups preserving the index 1 NAC condition [18], one can concentrate on the minimal case, and one situation in which the existence of bi-hermitian structures remains unknown [31] is the case when S is a Kato surface of intermediate type, as described in Example 4.4 above. In this case, the necessary index 1 NAC condition is studied in detail in [23, 30, 50]. It is observed there that if $H^0(S, K_S^* \otimes \mathcal{L}) \neq 0$, then $\mathcal{L} = \mathcal{L}_\mu$ for a unique $\mu \in \mathbb{C}^*$, where we have used the identification of \mathcal{L} with a $\mu \in \mathbb{C}^*$ as in Remark 3.13. In terms of the germ (16), the index 1 NAC condition is equivalent to $\frac{s}{(k-1)} \in \mathbb{N}$ (see [29]). Indeed, in this case

$$\sigma := z_2^r \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \right), \quad r = \frac{s}{(k-1)} + 1,$$

defines a germ satisfying $F_*(\sigma) = (\lambda_1 k)\sigma$, so it defines a holomorphic section of $K_S^* \otimes \mathcal{L}_\mu$ with $\mu = \lambda_1 k$. In particular, as noticed in [50], $\mu \in \mathbb{R}$ and $\mu > 1$ are necessary conditions for an intermediate Kato surface S to admit a bi-hermitian structure. We now prove that they are not in general sufficient.

Theorem 4.6. *Suppose S is an intermediate Kato surface as described in Example 4.4, which admits a bi-hermitian structure. Then λ_1 in (16) must be a real number satisfying $\lambda_1 > 1$ or, equivalently, $H^0(S, K_S^* \otimes \mathcal{L}_\mu) \neq 0$ where μ is a positive real number satisfying $\mu > k$.*

Proof. By [23], \mathcal{L}_μ is the only flat line bundle on S such that $H^0(S, K_S^* \otimes \mathcal{L}_\mu) \neq 0$ and, furthermore, in this case $H^0(S, K_S^* \otimes \mathcal{L}_\mu) = \mathbb{C}$ and the support of the NAC divisor is the maximal curve of S (which is connected). It follows from [3, Prop. 2.4] that any bi-hermitian structure on S (if it exists) must be given by the construction of [3, Prop. 2.3]. In particular, J must be tamed by a locally conformally symplectic 2-form ω with Lee deRham class a and $\mathcal{L}_\mu = \mathcal{L}_a$ as explained above. The claim follows from Proposition 3.7 and Example 4.4. \square

5. CHARACTERIZATION OF SOME KNOWN CLASS VII SURFACES

Lemma 3.11 allows us to recast the results of M. Brunella [14, 15] which in turn provide a partial converse of Proposition 4.2(a).

Theorem 5.1. [14, 15] *Let S be a compact complex surface with $b_1(S) = 1$ and $\text{kod}(S) = -\infty$. Then the following conditions are equivalent*

- (i) S is obtained from either a hyperbolic Kato surface or a Bombieri–Inoue surface by blowing up points.
- (ii) S admits a weakly negative non zero current $\tau \leq 0$ of degree 0, and a smooth closed 1-form β such that

$$T := d_\beta d_\beta^c \tau \geq 0$$

and the support of T is either empty or is an analytic set Z of pure dimension 1 in S .

Proof. By Lemma 3.11, we can replace condition (ii) with the existence of non-positive PSH automorphic function on \hat{S} .

The direction (i) \implies (ii) can be established explicitly, see Section 4 for the case of a hyperbolic Kato surface.

To establish the direction (ii) \implies (i), we first notice that the support of T is empty iff $T = 0$ (and hence $\hat{T} = 0$). In this case, the PSH function \hat{u} on \hat{S} is pluriharmonic, and therefore is smooth and strictly negative. In particular, the constant C in Lemma 3.11 cannot be 1, i.e. $b := [\beta] \neq 0$ in $H_{dR}^1(S, \mathbb{R})$. The equality in (10) (and using that τ is a negative smooth function) now gives $\mathcal{T}(S) = \{b\}$. It follows that S cannot be a blow up of a Hopf surface (in which case $\mathcal{T}(S) = (-\infty, 0)$, see e.g. [5]), so the algebraic dimension of S is zero (see e.g. [8]). Thus, S must be a blow up of a Bombieri–Inoue surface by [14].

In the case when T is supported over a curve, $T \neq 0$, and by Proposition 3.7 we have $\mathcal{T}(S) \in (-\infty, b)$. Again, S cannot be a blow-up of a Hopf surface, and thus the algebraic dimension of S is zero. As \hat{S} admits a negative PSH function with analytic support over a curve, S must be a blow up of a hyperbolic Kato surface by [15]. \square

The following result was obtained in [6].

Theorem 5.2. [6] *Let S be a compact complex surface with $b_1(S) = 1$ and $\text{kod}(S) = -\infty$. Then the following conditions are equivalent*

- (i) $\mathcal{T}(S) = \{a\}$ consists of a single point;
- (ii) $\mathcal{T}(S)$ is a proper closed subset of $(-\infty, 0)$;
- (iii) S is obtained by blowing up points of an Inoue–Bombieri surface.

6. LEE CLASSES AND TWISTED LOGARITHMIC HOLOMORPHIC 1-FORMS.

Definition 6.1. Let X be a complex manifold, $D \subset X$ an effective divisor, and θ a holomorphic p -form on $X \setminus D$. We say that θ is a *logarithmic p -form* on X with logarithmic pole along D if θ and $d\theta$ have a pole of order at most one along D . We denote by $\Omega^p(\log D)$ the coherent sheaf of logarithmic p -forms on X with pole along D . We shall also consider logarithmic p -forms with values in a holomorphic line bundle \mathcal{L} , i.e. sections of $H^0(X, \Omega^p(\log D) \otimes \mathcal{L})$. In particular, an \mathcal{L} -valued logarithmic 1-form $\theta \in H^0(X, \Omega^1(\log D) \otimes \mathcal{L})$ is a $\bar{\partial}_{\mathcal{L}}$ -closed \mathcal{L} -valued $(1, 0)$ -form on $X \setminus D$ such that, around each point $p \in D$, there exists a neighbourhood U , holomorphic functions $f_j, g_j \in \mathcal{O}(U), j = 1, \dots, k$, a holomorphic 1-form $\theta_0 \in \Omega^1(U)$, and a holomorphic section $s \in H^0(U, \mathcal{L}|_U)$ with

$$(17) \quad \theta|_U = \left(\theta_0 + \sum_{j=1}^k g_j \frac{df_j}{f_j} \right) \otimes s,$$

where $\{f_j = 0\}$ are the reduced equations of the irreducible components of $D \cap U$.

We shall be interested in particular in *twisted* logarithmic 1-forms on a complex surface S , i.e. elements $\theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L})$ where $\mathcal{L} \in H_0^1(S, \mathbb{C}^*)$ is a *topologically trivial* flat \mathbb{C}^* -line bundle. Using the short exact sequence

$$\{0\} \rightarrow \mathbb{Z} \hookrightarrow \mathbb{C} \xrightarrow{\exp 2\pi i} \mathbb{C}^* \rightarrow \{1\},$$

it follows that the first morphism in

$$(18) \quad H_{dR}^1(M, \mathbb{C}) \xrightarrow{\exp} H_0^1(M, \mathbb{C}^*) \longrightarrow \text{Pic}^0(X),$$

is surjective, i.e. \mathcal{L} can be written as $\mathcal{L} = \mathcal{L}_\alpha$ for a closed *complex-valued* 1-form α . As explained in Section 2, θ can be equivalently viewed as a $\bar{\partial}_\alpha$ -closed $(1, 0)$ -form on $S \setminus D$, such that we can choose a neighbourhood U of $p \in D$ with $\alpha|_U = dh$ for a smooth complex valued function h , holomorphic 1-form θ_0 and holomorphic functions f_j, g_j on U with

$$(19) \quad e^h \theta|_U = \theta_0 + \sum_{j=1}^k g_j \frac{df_j}{f_j}$$

It follows from the above expression that θ gives rise to a $(1, 2)$ -current P_θ , defined by

$$(20) \quad P_\theta(\phi) := \int_{S \setminus D} \theta \wedge \phi, \quad \forall \phi \in \mathcal{E}^{1,2}(S, \mathbb{C}).$$

Furthermore, assuming that D is a simple normal crossing divisor, we compute (using (19) and the basic identity $i\partial\bar{\partial} \log |z|^2 = 4\pi\delta_0$)

$$(21) \quad T_\theta := \bar{\partial}_\alpha P_\theta = 2\pi i \text{Res}_\alpha(\theta) T_D$$

where T_D is the current of integration over D , and $\text{Res}_\alpha(\theta)$ is a complex valued function on D , called the *residue* of θ on D . The above computations show that on each component D_j of D defined by $\{f_j = 0\}$, we have

$$(22) \quad \text{Res}_\alpha(\theta)|_{D_j} = (e^{-h} g_j)|_{D_j}.$$

It follows from (21) and (22) that on D_j , $\text{Res}_\alpha(\theta)|_{D_j}$ satisfies $\bar{\partial}_{-\alpha} \left(\text{Res}_\alpha(\theta)|_{D_j} \right) = 0$, i.e. is an element of $H^0 \left(D_j, (\mathcal{L}_\alpha)|_{D_j} \right)$ (see (4)). If $\alpha = 0$ and D_j is compact, then $\text{Res}_0(\theta)|_{D_j} = a_j$ is a constant. If D_j is a rational curve, then the flat \mathbb{C}^* bundle $(\mathcal{L}_\alpha)|_{D_j}$ is trivial, and therefore $\text{Res}_\alpha(\theta)|_{D_j} = e^{h_j} a_j$ for some complex valued function h_j on D_j .

What will be important for us is that in the case when $\mathcal{L}_a = L_a \otimes \mathbb{C}$ is a flat line bundle corresponding to a *real* deRham class $a \in H_{dR}^1(S, \mathbb{R})$, there is a well-defined notion of positivity of the real and imaginary parts of T_θ , independent of the representative α of $a \in H_{dR}^1(S, \mathbb{R})$.

Definition 6.2. Let $\theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_a)$ be a logarithmic 1-form on S with values in the flat line bundle \mathcal{L}_a corresponding to a deRham class $a \in H_{dR}^1(S, \mathbb{R})$ via (1). We say that θ is of *positive type* if for some (and hence any) $\alpha \in a$, $\text{Res}_\alpha(\theta)$ is a real valued function which is positive on each component D_j of D . In particular, the current $\text{Im}(T_\theta)$ is weakly positive and supported on D .

We next specialize to class VII complex surfaces. We recall that in this case, we have (15), i.e. the second morphism in (18) is an isomorphism. For this reason we shall refer to the topologically trivial flat \mathbb{C}^* line bundle \mathcal{L} as a *flat bundle*. We tacitly use

the identification $H_0^1(S, \mathbb{C}^*) \cong \mathbb{C}^*$ and for a $\mu \in \mathbb{C}^*$ we write \mathcal{L}_μ for the corresponding flat \mathbb{C}^* bundle. Thus, $\mathcal{L}_\mu = \mathcal{L}_\alpha$ for some closed complex-valued 1-form α with α real iff $\mu > 0$. Furthermore $[\alpha] < 0$ iff $\mu > 1$, according to the convention in Remark 3.12.

6.1. Twisted logarithmic 1-forms on known class VII₀ complex surfaces. We start with Hopf surfaces.

Lemma 6.3. *Let S be a Hopf surface with $H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_b) \neq 0$, $b \in H_{dR}^1(S, \mathbb{R})$. Then $b \geq 0$, and, if S is a primary Hopf surface, the set of classes b such that the latter condition holds includes $b = 0$.*

Proof. A primary Hopf surface is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the cyclic group generated by a contraction of the type

$$(23) \quad F(z_1, z_2) = (\alpha z_1 + t z_2^m, \beta z_2), \quad 0 < |\alpha| < |\beta| < 1, \quad t(\alpha - \beta^m) = 0, \quad m \geq 1.$$

The secondary Hopf surfaces are obtained by taking a further quotient with a finite group, see [41]. To prove the statement, it is thus enough to assume that S is a primary Hopf surface given by (23)

Clearly, (23) preserves the 1-form $\frac{dz_2}{z_2}$, giving rise to a non-trivial section of $H^0(S, \Omega^1(\log D))$ with D being the elliptic curve $z_2 = 0$. We can thus realize the value $\mu = 1$, where we use the correspondence $\mathcal{L}_\mu = \mathcal{L}_b$ for $\mu \in \mathbb{R}, \mu > 0$ given by Remark 3.12. Notice that $z_2^k dz_2$, $k \in \mathbb{N}$ give rise to twisted holomorphic 1-forms (i.e. with $D = \emptyset$) with values in the flat bundle \mathcal{L}_{β^k} ; in the case when $\beta \in \mathbb{R}$ these give rise to infinitely many values of $\mu \in (0, 1)$.

Suppose now $\theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_\mu) \neq 0$ with $\mu > 1$. We shall derive that $\theta = 0$. The twisted logarithmic form θ is induced by a germ of logarithmic 1-forms (still denoted θ) on $(\mathbb{C}^2, 0)$, such that $F^*\theta = \mu\theta$. We consider the following 3 cases, depending of the form of (23):

- (i) Case $t \neq 0$. Then there exists only one elliptic curve on S , given by $\{z_2 = 0\}$, and θ has the form

$$\theta(z_1, z_2) = A(z_1, z_2)dz_1 + B(z_1, z_2)\frac{dz_2}{z_2},$$

where A, B are germs of holomorphic functions on $(\mathbb{C}^2, 0)$ satisfying

$$A(F(z))\alpha = \mu A(z) \quad A(F(z))t z_2^{m-1} + \frac{B(F(z))}{z_2} = \mu \frac{B(z)}{z_2}.$$

As $|\alpha|, |\beta| < 1$ and $\mu > 1$, it follows that $A = 0$; then the remaining equation is $B(F(z)) = \mu B(z)$, which yields $B = 0$.

- (ii) Case $t = 0$ and $\alpha^k \neq \beta^l$, $k, l \in \mathbb{N}$. In this case S admits only two elliptic curves [8, Prop.18.2], corresponding to $z_1 = 0$ and $z_2 = 0$. Therefore θ has the form

$$\theta(z_1, z_2) = A(z_1, z_2)\frac{dz_1}{z_1} + B(z_1, z_2)\frac{dz_2}{z_2},$$

where A, B are holomorphic and satisfy

$$A(F(z))\alpha = \mu A(z) \quad B(F(z))\beta = \mu B(z).$$

Using $0 < |\alpha| < |\beta| < 1 < \mu$, we derive again $A = B = 0$.

- (iii) Case $t = 0$ and $\alpha^k = \beta^l$ for $k, l \in \mathbb{N}$. In this case the Hopf surface S is elliptic fibered over $\mathbb{P}^1(\mathbb{C})$ by the meromorphic function $z_1^k z_2^{-l}$ [8, Prop.18.2]. There are no other curves in S than the fibers of this fibration, because the algebraic

dimension is $a(S) = 1$. Thus, any finite number of polar curves for θ have the equations

$$z_1^k - x_i z_2^l = 0, \quad i = 1, \dots, p.$$

Thus, we get

$$\theta = \sum_{i=1}^p A_i(z) \frac{d(z_1^k - x_i z_2^l)}{z_1^k - x_i z_2^l}.$$

The condition $F^*\theta = \mu\theta$ is then satisfied if and only if $A_i(F(z)) = \mu A_i(z)$, for $i = 1, \dots, p$. It is easy to see that $A_i = 0$, $i = 1, \dots, p$. □

Lemma 6.4. *Let S be an Inoue–Bombieri surfaces. There exists a unique flat vector bundle \mathcal{L}_μ with $\mu \in \mathbb{R}$, such that $H^0(S, \Omega^1 \otimes \mathcal{L}_\mu) \neq 0$. Furthermore, $\mu > 1$ and writing $\mathcal{L}_\mu = \mathcal{L}_a$ for $a \in H_{dR}^1(S, \mathbb{R})$, we have $\mathcal{T}(S) = \{a\}$.*

Proof. We use notations in [38]. Accordingly, any Inoue–Bombieri surface S is the quotient of $\mathbb{H} \times \mathbb{C}$ by a discrete subgroup G of automorphisms, and we denote by $w = w_1 + iw_2$, $w_2 > 0$ and $z = z_1 + iz_2$ the complex coordinates on \mathbb{H} and \mathbb{C} , respectively. We have the following three types of Inoue–Bombieri surfaces:

- S of type $S_M = \mathbb{H} \times \mathbb{C}/G_M$. Then the 1-form dw on $\mathbb{H} \times \mathbb{C}$ induces a twisted 1-form in $H^0(S_M, \Omega^1 \otimes \mathcal{L}_\mu)$ with μ real and $\mu > 1$. Notice that in this case, there is another twisted holomorphic 1-form, given by dz , i.e. $H^0(S, \Omega_S^1 \otimes \mathcal{L}_\beta) \neq 0$ with $\beta \notin \mathbb{R}$.
- S of type $S_{N,p,q,r;t}^{(+)} = \mathbb{H} \times \mathbb{C}/G_{N,p,q,r;t}^{(+)}$. Then the normal subgroup $\Gamma = \langle g_1, g_2, g_3 \rangle$ of $G_{N,p,q,r;t}^{(+)}$ gives rise to the covers

$$\mathbb{H} \times \mathbb{C} \rightarrow \mathbb{H} \times \mathbb{C}/\Gamma \rightarrow S_{N,p,q,r;t}^{(+)}$$

Then dw induces a holomorphic 1-form on $\mathbb{H} \times \mathbb{C}/\Gamma$ and a twisted holomorphic 1-form in $H^0(S_{N,p,q,r;t}^{(+)}, \Omega^1 \otimes \mathcal{L}_\mu)$, with $\mu > 1$.

- S of type $S_{N,p,q,r}^{(-)}$. We have a double covering

$$S_{N^2,p_1,q_1,r;0}^{(+)} \rightarrow S_{N,p,q,r}^{(-)}$$

In this case a twisted holomorphic 1-form in $H^0(S_{N^2,p_1,q_1,r;0}^{(+)}, \Omega^1 \otimes \mathcal{L}_{\mu^2})$ with $\mu > 1$, induces a twisted holomorphic 1-form in $H^0(S_{N,p,q,r}^{(-)}, \Omega^1 \otimes \mathcal{L}_\mu)$.

In [6, Lemma 4.4], it is shown that $\alpha = dw_2/w_2 = d \log(w_2)$ defines a closed 1-form on S , such that the deRham class $a = [\alpha]$ satisfies $\mathcal{T}(S) = \{a\}$. We notice that in the above constructions, we have $\mathcal{L}_\mu = \mathcal{L}_a$ in terms of (18).

We now argue that if $H^0(S, \Omega^1 \otimes \mathcal{L}_\mu) \neq 0$ and $\mu \in \mathbb{R}^*$, then μ is unique. Indeed, if we have $0 \neq \theta_1 = dw \in H^0(S, \Omega^1 \otimes \mathcal{L}_\mu)$ and $0 \neq \theta_2 \in H^0(S, \Omega^1 \otimes \mathcal{L}_\nu)$, $\nu \in \mathbb{R}^*$, $\nu \neq \mu$, then $\omega := \theta_1 \wedge \theta_2 \in H^0(S, K_S \otimes \mathcal{L}_{\mu\nu})$ is non-zero. Otherwise θ_1/θ_2 would define a non-constant meromorphic function on S , an absurd. As there are no curves on S (see e.g. [38] or [8]), we have that ω does not vanish on S , i.e. $K_S \otimes \mathcal{L}_{\mu\nu} = \mathcal{O}$. The flat factor $\mathcal{L}_{\mu\nu}$ such that the latter holds is unique (as S admits no curves). On the other hand, inspecting the action of G on the holomorphic 2-form $dw \wedge dz$ on $\mathbb{H} \times \mathbb{C}$ according to the three types of Inoue–Bombieri surfaces, we get

- for S_M , $K_S \otimes \mathcal{L}_{\mu\beta} = \mathcal{O}$ with $\mu\beta \in \mathbb{C}^* \setminus \mathbb{R}^*$, an absurd.
- for $S^{(+)}$, $K \otimes \mathcal{L}_\mu = \mathcal{O}$, therefore $\nu = 1$ an absurd as $H^0(S, \Omega^1) = 0$ (see [8]).

- for $S^{(-)}$, $K \otimes \mathcal{L}_\mu = \mathcal{O}$, therefore $\nu = 1$ an absurd as $H^0(S, \Omega^1) = 0$ (see [8]).

□

Lemma 6.5. *Let S be an Enoki surface. Then $H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_\mu) \neq 0$ if and only if either D is the maximal divisor and $\mu = 1$, or $D = \emptyset$ and $\mu \in \mathbb{C}^*$, $0 < |\mu| < 1$.*

Proof. Enoki surfaces are defined by a contraction

$$F(z_1, z_2) = \left(t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2 \right), \quad 0 < |t| < 1, \quad n = b_2(S).$$

The curve $\{z_2 = 0\}$ induces a cycle of rational curves in S . By [25, thm 5.5], on Enoki surfaces, there exists a unique foliation. As the logarithmic 1-form $\frac{dz_2}{z_2}$ defines a foliation on S there is no other and since $F^* \frac{dz_2}{z_2} = \frac{dz_2}{z_2}$, $\theta = \frac{dz_2}{z_2}$ may be seen as a section in $H^0(S, \Omega_S^1(\log D))$. Besides the line bundle $[D]$ is flat and $[D] \simeq \mathcal{L}_t$, $t \in \mathbb{C}^*$, $0 < |t| < 1$ and θ may be seen as 1-form in $H^0(S, \Omega_S^1 \otimes \mathcal{L}_t)$. □

Lemma 6.6. *Suppose S is a hyperbolic Kato surface. Then there exists a unique $b < 0$ such that $H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_b) \neq 0$. Furthermore, $b \in H_{dR}^1(S, \mathbb{R})$ is an upper bound of $\mathcal{T}(S)$.*

Proof. Consider first the case of an Inoue–Hirzebruch surface. In the notation of Example 4.5, one can check that

$$\theta := \xi \frac{dz_1}{z_1} + \eta \frac{dz_2}{z_2}$$

gives rise to a twisted logarithmic 1-form on S with values in the flat bundle is \mathcal{L}_{λ_1} with $\lambda_1 > 1$. The corresponding deRham class $b \in (-\infty, 0)$ is the explicit upper bound of the set $\mathcal{T}(S)$ found in Example 4.5.

Similarly, if S is a Kato surface of intermediate type, Example 4.4 shows that dz_2 gives rise to a twisted logarithmic 1-form with values in \mathcal{L}_k , $k \geq 2$. Again, $\mathcal{L}_k = \mathcal{L}_b$ where $b \in (-\infty, 0)$ is the explicit upper bound for $\mathcal{T}(S)$ identified in Example 4.4. By [25, section 2.2] there are exactly two foliations on an Inoue–Hirzebruch surface given by two twisted logarithmic 1-forms $\theta \in H^0(S, \Omega_S^1(\log D) \otimes \mathcal{L}_{\lambda_i})$, $i = 1, 2$, with $\lambda_1 \lambda_2 = \pm 1$. This last condition leads to unicity of $\mu > 1$, hence of $b < 0$. □

We summarize the above discussion in the following

Corollary 6.7. *Suppose S is a known minimal complex surface in the class VII, different than a secondary Hopf surface. Then S admits a flat bundle \mathcal{L}_b corresponding to a deRham class $b \in (-\infty, 0]$, such that $H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_b) \neq 0$. Furthermore, b is an upper bound of $\mathcal{T}(S)$.*

Proof. See Lemmas 6.3, 6.4, 6.5, 6.6. □

6.2. Classification results. In this section we do not assume that S is a known surface, and we are aiming to classify the minimal complex surfaces in the Kodaira class VII₀ which admit a non-trivial section

$$(24) \quad 0 \neq \theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_a) \neq 0,$$

for some divisor D and some flat line bundle \mathcal{L}_a associated to a real deRham class $a \in H_{dR}^1(S, \mathbb{R})$, or equivalently, via (18), to a flat line bundle \mathcal{L}_μ with μ real and positive. We notice that the twisted logarithmic 1-form θ defines a (possibly singular) holomorphic foliation \mathcal{F} on S . We recall the following basic

Definition 6.8. Let \mathcal{F} be a holomorphic foliation, defined in a local chart around a singularity p by a vector field $V(z_1, z_2)$, $V(p) = 0$. Denote by λ_1 and λ_2 the eigenvalues of $DV(p)$. The singularity (\mathcal{F}, p) is called *non-degenerate* if $\lambda_1\lambda_2 \neq 0$. We notice that at any non-degenerate singularity p , the numbers $\lambda(p) = \frac{\lambda_1}{\lambda_2}$ and $\lambda^{-1}(p) = \frac{\lambda_2}{\lambda_1}$ are invariants of (\mathcal{F}, p) , independent of the chosen vector field V , and are called *characteristic numbers* of the non-degenerate singularity of \mathcal{F} .

Our main result in this section is the following

Proposition 6.9. *Let S be a unknown minimal compact complex surface in the class VII_0 which satisfies the condition (24) for some divisor D and a flat line bundle $\mathcal{L}_b, b \in H_{dR}^1(S, \mathbb{R})$, and \mathcal{F} be the corresponding holomorphic foliation. Then*

- *D has one connected component, all of its irreducible components are either smooth rational curves or rational curves with a double point, D contains a cycle of rational curves and coincides with the maximal divisor of S .*
- *The intersection of two irreducible components of D is a non-degenerate singularity of \mathcal{F} with real characteristic numbers.*
- *If the characteristic numbers of \mathcal{F} at the intersections of irreducible components of D are all negative, then $b \notin \mathcal{T}(S)$, i.e. b bounds $\tau(S)$ by virtue of Theorem 1.1.*

The proof of the above result will occupy the remainder of the section and is divided on several steps, corresponding to the Lemmas below.

First we consider the case when the flat line bundle is trivial, i.e. $b = 0$ in (24).

Lemma 6.10. *Let S be a minimal compact complex surface in the class VII_0 . There exists a non-trivial logarithmic 1-form $\theta \in H^0(S, \Omega^1(\log D))$ if and only if S is a Hopf surface or an Enoki surface.*

Proof. By Lemmas 6.3 and 6.5 the primary Hopf surfaces and all Enoki surfaces admit non-trivial logarithmic 1-forms, i.e (24) holds with $\mathcal{L}_\mu = \mathcal{O}$.

Conversely, suppose $H^0(S, \Omega^1(\log D)) \neq 0$. As $H^0(S, \Omega^1) = 0$ for any surface in class VII_0 , $D \neq \emptyset$. If D has more than one connected components, it must have precisely two connected components and S must be an Inoue–Hirzebruch surface (see [24, Thm.4.22]). By [25], such a surface has only twisted logarithmic 1-forms, i.e. $H^0(S, \Omega^1(\log D)) = 0$. Thus, D is connected. If the intersection matrix of D is negative definite, D can be contracted onto a normal point x . Let $p : S \rightarrow S'$ be this contraction. Let V' be a Stein neighbourhood of x and $V = p^{-1}(V')$, $U' = V' \setminus \{x\}$, $U = V \setminus D$. By the theorem of Steenbrinck–Van Straten [54, Cor.1.4] the mapping induced by the differentiation

$$d : H^0(U', \Omega_{U'}^1) / H^0(V, \Omega_V^1) \rightarrow H^0(U', \Omega_{U'}^2) / H^0(V, \Omega_V^2(\log D))$$

is injective. Since $[d\omega] = 0$, we obtain a contradiction. Thus, the intersection matrix of D cannot be definite, and therefore there is a divisor A with support in D such that $A^2 = 0$. If $b_2(S) = 0$, it follows that S is a Hopf surface; if $b_2(S) > 0$, Enoki’s theorem [27] shows that S must be an affine bundle over an elliptic curve. Such a surface is an Enoki (Kato) surfaces, by the construction of generic contractions in [26]. \square

We next characterize the case when $D = \emptyset$ in (24). The argument works for any flat bundle $\mathcal{L} = \mathcal{L}_\alpha$, $[\alpha] \in H_{dR}^1(S, \mathbb{C})$.

Lemma 6.11. *Let S be a minimal complex surface in the class VII_0 . There exists a non-trivial twisted holomorphic 1-form $\theta \in H^0(S, \Omega^1 \otimes \mathcal{L})$ only if S is a Hopf surface, an Inoue–Bombieri surface, or an Enoki surface.*

Proof. If $b_2(S) = 0$, then S is either a Hopf surface or an Inoue–Bombieri surface by [58]. In these cases the existence of non-trivial twisted holomorphic 1-forms follows from the arguments in Lemmas 6.3 and 6.4. If $b_2(S) > 0$, the claim follows from [6, Thm. 5.6]. \square

From now on, we exclude the cases covered by Lemmas 6.10 and 6.11, and thus assume $D \neq \emptyset$ and $\mu \neq 1$.

Lemma 6.12. *Let S be a surface in the class VII_0^+ , which is not an Enoki surface, and suppose that S admits a non-trivial logarithmic 1-form $0 \neq \theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L})$ with pole along a divisor $D \neq \emptyset$ and values in a flat line bundle $\mathcal{L} = \mathcal{L}_\alpha$ for some closed complex-valued one form α . Then:*

- (a) $d_{-\alpha}\theta = 0$ on $S \setminus D$, i.e. $d_{\mathcal{L}}\theta$ is the zero section of $H^0(S, \Omega^2(\log D) \otimes \mathcal{L})$.
- (b) Each connected component of D contains a cycle of rational curves Γ . In particular, D has at most two connected components, in which case S is an Inoue–Hirzebruch surface.
- (c) Each connected component of D coincides with a connected component of the maximal divisor of S .
- (d) The analytic set of points where θ vanishes is either empty or composed of isolated points.

Proof. (a) If $\theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_\alpha)$, then $d_{-\alpha}\theta = \partial_{-\alpha}\theta \in H^0(S, \Omega^2(\log D) \otimes \mathcal{L}_\alpha)$ by (19). If $d_{-\alpha}\theta \neq 0$, it gives rise to an effective divisor Δ with $[\Delta] = K_S + [D] + \mathcal{L}$. Notice that $D - \Delta \neq 0$ because $K_S^2 = -b_2(S) \neq 0 = \mathcal{L}^2$. It follows that $D - \Delta$ is a numerically anticanonical divisor [23], and therefore S is a Kato surface by the results in that reference. On a Kato surface, twisted logarithmic 1-forms are classified in [25] and, by inspection, they are all $d_{\mathcal{L}}$ -closed. By the discussion in Section 2, this is equivalent to $d_{-\alpha}\theta = 0$, a contradiction. We thus conclude that $d_{-\alpha}\theta = 0$ on $S \setminus D$ for the chosen flat connection ∇^α on $\mathcal{L} = \mathcal{L}_\alpha$.

(b) Let D' be any connected component of the polar set D of θ . Then the intersection matrix is negative definite by [27]. We take a simply connected neighbourhood V' of D' , which is sufficiently small to be contractible onto a Stein space V with an isolated singular normal point $x \in V$. Let $\pi : V' \rightarrow V$ be this contraction, $D' = \pi^{-1}(x)$ be the exceptional divisor, and $U := V \setminus \{x\}$. If D' does not contain a cycle, D' is simply connected then the restriction of \mathcal{L} to V' is trivial. By a result of Steenbrink–van Straten [54, Cor. 1.4] the mapping induced by the differentiation

$$d : H^0(U, \Omega_U^1) / H^0(V', \Omega_{V'}^1) \rightarrow H^0(U, \Omega_U^2) / H^0(V', \Omega_{V'}^2(\log D'))$$

is injective, which gives a contradiction. If there are two cycles in D , we apply [45, Thm 8.1].

(c) Let C_1 be an irreducible component of D and suppose that there is a rational curve C_2 , which is not contained in D , and such that $C_1 \cap C_2 = \{p\}$. There are holomorphic coordinates in a neighbourhood of p with $C_i = \{z_i = 0\}$, $i = 1, 2$, and a locally defined smooth function h and holomorphic functions g_i , such that

$$e^h \theta = g_1 \frac{dz_1}{z_1} + g_2 dz_2.$$

Since

$$0 = d(e^h \theta) = \frac{\partial g_1}{\partial z_2} dz_2 \wedge \frac{dz_1}{z_1} + \frac{\partial g_2}{\partial z_1} dz_1 \wedge dz_2 = \left(z_1 \frac{\partial g_2}{\partial z_1} - \frac{\partial g_1}{\partial z_2} \right) \frac{dz_1}{z_1} \wedge dz_2$$

we see that g_1 must have the form

$$g_1(z_1, z_2) = \beta(z_1) + z_1 z_2 \gamma(z_1, z_2).$$

If $\beta(0) \neq 0$, then the restriction of θ to C_2 has a simple pole at p , which is impossible. Therefore $\beta(0) = 0$ and

$$e^h \theta = \left(\frac{\beta}{z_1} + z_2 \gamma \right) dz_1 + g_2 dz_2.$$

The above form has no pole along C_2 , a contradiction. Thus D contains the whole connected component of Γ .

(d) If there is a divisor $A \subset \{\theta = 0\}$, its connected component has an empty intersection with the pole and there are at least two connected components of curves. We conclude by [24]. \square

The above Lemma allows us also to exclude the Inoue–Hirzebruch surfaces, and assume from now on that D is a connected non-trivial divisor.

Lemma 6.13. *Let S be a minimal surface in the class VII_0^+ , which is not an Inoue–Hirzebruch or Enoki surface. Suppose that S admits a non-trivial logarithmic 1-form*

$$0 \neq \theta \in H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_\alpha), \quad [\alpha] \in H_{dR}^1(S, \mathbb{R}),$$

with pole along a non-trivial connected divisor $D = \sum_i D_i$. Let \mathcal{F} be the foliation defined by θ . Then

- (i) *The intersections of curves in D are non-degenerate singularities of \mathcal{F} with characteristic numbers $\lambda^{\pm 1}(p) \in \mathbb{R}^*$,*
- (ii) *If $D_i \cap D_j \neq \emptyset$, then $\left(\frac{\text{Res}_\alpha(\theta)_{\hat{D}_j}}{\text{Res}_\alpha(\theta)_{\hat{D}_i}} \right) \in \mathbb{R}^*$, see (22).*

Proof. Using a result of Nakamura [45, Thm. 10.2], under the hypotheses of the Lemma, we know that each component D_j is either smooth rational curve, or is a rational curve with a double-point singularity. Let $\hat{\pi} : \hat{S} \rightarrow S$ be the minimal \mathbb{Z} -cover, see Definition 3.10, and $\hat{D} = \hat{\pi}^{-1}(D)$, $\hat{\alpha} := \hat{\pi}^* \alpha$, $\hat{\alpha} = d\hat{h}$, $\hat{\theta} = \hat{\pi}^* \theta$. Each component \hat{D}_j is again a rational curve. In a local chart \hat{U} around $\hat{D}_1 \cap \hat{D}_2$ we have (see(19))

$$e^{\hat{h}} \hat{\theta} = \theta_0 + g_1(z) \frac{dz_1}{z_1} + g_2(z) \frac{dz_2}{z_2}$$

where θ_0, g_1, g_2 are holomorphic. As \hat{D}_i are compact curves, it follows that the restriction of $g_1(0, z_2)$ on \hat{D}_1 is a constant, say a_1 , and similarly, the restriction of $g_2(z_1, 0)$ to \hat{D}_2 is a constant a_2 . Therefore

$$\text{Res}_{\hat{\alpha}}(\hat{\theta})|_{z_1=0} = e^{-\hat{h}} g_1(0, z_2) = e^{-\hat{h}} a_1, \quad \text{Res}_{\hat{\alpha}}(\hat{\theta})|_{z_2=0} = e^{-\hat{h}} g_2(z_1, 0) = e^{-\hat{h}} a_2.$$

Notice that if $a_1 = 0$ (resp. $a_2 = 0$) then \hat{D}_1 (resp. \hat{D}_2) will not be a pole, a contradiction. It thus follows that the holomorphic foliation \mathcal{F} defined by θ has a non-degenerate singularity at $p \in D_1 \cap D_2$, with corresponding characteristic numbers $(\lambda, \lambda^{-1}) := (-\frac{a_1}{a_2}, -\frac{a_2}{a_1})$, and, moreover

$$\frac{a_1}{a_2} = \frac{\text{Res}_\alpha(\theta)_{\hat{D}_1}}{\text{Res}_\alpha(\theta)_{\hat{D}_2}}.$$

We next compute the Camacho–Sad indices CS of the foliation \mathcal{F} along the curves D_i of D (see [17]):

$$\begin{aligned}
(25) \quad CS(\mathcal{F}, D_2, D_2 \cap D_1) &:= \operatorname{Res}_{z_1=0} \left(\frac{\partial}{\partial z_2} \left(-\frac{z_2 g_1}{z_1 g_2} \right) \Big|_{D_2} \right) \\
&= \operatorname{Res}_{z_1=0} \left(-\frac{g_1(z_1, 0)}{z_1 g_2(z_1, 0)} \right) \\
&= -\frac{g_1(0, 0)}{g_2(0, 0)} = -\frac{a_1}{a_2} = -\frac{\operatorname{Res}_\alpha(\theta)_{\bar{D}_1}}{\operatorname{Res}_\alpha(\theta)_{\bar{D}_2}} \\
&= \lambda(p).
\end{aligned}$$

Recall that by [17], \mathcal{F} admits a *separatrix* through its non-degenerate singularity $p = (0, 0)$, i.e. there exists an \mathcal{F} -invariant complex analytic curve passing through $(0, 0)$. Looking at the classification of non-degenerate singularities of foliations (see e.g. [11, pp. 2–3]), the only possibility with $\lambda = \lambda(p) \notin \mathbb{R}$ are singularities in the Poincaré domain with two separatrices $\{z_i = 0\}$, $i = 1, 2$. The holonomy of each separatrix is generated by the contraction (or its inverse) $z_i \mapsto \mu z_i$ with $\mu = \exp(2\pi i \lambda^{\pm 1})$. As the holonomy group is a contraction in this case, we see that any leaf L accumulates on D and all intersection points p have complex non-real characteristic numbers $\lambda^{\pm 1}(p)$. Furthermore, any leaf L is closed in $S \setminus D$. In particular \bar{L} is not analytic. Under the hypotheses of the Lemma and Enoki theorem [27], the intersection matrix of the components D_i of D is negative definite. We can then contract D onto a normal singular point $f : S \rightarrow X$ by Grauert theorem (see e.g. [44, Thm 4.9]). Since the Hausdorff measure of dimension one of a point is zero, a theorem of Remmert–Stein–Shiffman [52] shows that $M = \overline{f(L)}$ is an analytic curve. But then \bar{L} is contained in the analytic curve $f^{-1}(M)$, a contradiction. We thus conclude $\lambda(p) \in \mathbb{R}$ and $\frac{\operatorname{Res}_\alpha(\theta)_{\bar{D}_i}}{\operatorname{Res}_\alpha(\theta)_{\bar{D}_j}} = -\lambda(p) \in \mathbb{R}$ (see (25)). \square

Lemma 6.14. *Under the hypotheses of Lemma 6.13, suppose that $\mathcal{L} = \mathcal{L}_\alpha$ for a real deRham class $[\alpha] \in H_{dR}^1(S, \mathbb{R})$ and that the characteristic numbers at the intersection of curves of D are negative reals. Then, we can multiply θ by a non-zero complex number such that the residues $\operatorname{Res}_\alpha(\theta)$ defined in (22) are real valued and positive, and the real part $\operatorname{Re}(P_\theta)$ of the current P_θ associated to θ via (20) is exact, i.e. satisfies $\operatorname{Re}(P_\theta) = d_\alpha \tau$ for a degree 0 current τ on S . Furthermore,*

$$d_\alpha d_\alpha^c \tau = 2\pi \operatorname{Res}_\alpha(\theta) T_D > 0,$$

and $[\alpha] \neq \mathcal{T}(S)$.

Proof. The argument at the beginning of the proof of Lemma 6.13 shows that $\operatorname{Res}_\alpha(\theta)|_{\bar{D}_j} \in H^0(D_j, \mathcal{L}_\alpha)$ is of the form $e^{-h_j} a_j$ where h_j is the restriction of a smooth function to D_j and $a_j \neq 0$ is a complex number. Let us fix one such component $D_j \subset D$. By Lemma 6.13, and using the connectedness of D , it follows that if we multiply θ by the non-zero constant $1/a_j$, we have that $\operatorname{Res}_\alpha(\theta)$ is *real-valued* and non-zero along any other component D_i , whence on D . Furthermore, if all the characteristic numbers of \mathcal{F} are negative, by (25), we may also assume that $\operatorname{Res}_\alpha(\theta) > 0$.

We now notice that the RHS of (21) is d_α -closed (as a current on S), which follows directly from the formula (19) and the fact that $(g_j)|_{D_j} = a_j$ are constants. It follows that $Q := \partial_\alpha P_\theta$ is a $\bar{\partial}_\alpha$ -closed current of bi-degree $(2, 0)$, and thus is of the form $Q = T_\psi$ for a $(2, 0)$ -form ψ on S with $\bar{\partial}_{-\alpha} \psi = 0$, i.e. $\psi \in H^0(S, K_S \otimes \mathcal{L}_\alpha)$. As S is in the class

VII₀⁺ (and \mathcal{L} is flat) $\psi = 0$ (see e.g. [23, Remark 2.21]), and thus $Q = 0$. We conclude that

$$d_\alpha P_\theta = \bar{\partial}_\alpha P_\theta = 2\pi i \operatorname{Res}_\alpha(\theta) T_D.$$

Taking real parts yields $d_\alpha R = 0$, where $R := \operatorname{Re}(P_\theta)$. Thus, R gives rise to a class $[R] \in H_1^\alpha(S, \mathbb{R}) \cong (H_\alpha^1(S, \mathbb{R}))^* \cong H_{-\alpha}^3(S, \mathbb{R})$, as discussed in Section 2. By Lemma 6.12-(b), S contains a cycle of rational curves and, therefore, is diffeomorphic to $(S^1 \times S^3) \# k \overline{\mathbb{C}P}^2$ by the results in [46]. It follows from [6, Theorem 1.5] that either $\mathcal{L} = \mathcal{O}$ is trivial, or else $[R] = 0$ and hence $R = d_\alpha \tau$. As we have assumed $[\alpha] \neq 0$, the first possibility is impossible. We thus conclude that $d_\alpha d_\alpha^c \tau = 2\pi \operatorname{Res}_\alpha(\theta) T_D > 0$ for a degree zero (real valued) current τ . The last claim then follows from Propositions 3.2 and 3.3. \square

Proof of Proposition 6.9. Direct consequence of Lemmas 6.12, 6.13 and 6.14. \square

Proof of Theorem 1.6. If $b = 0$, the claim in (a) follows from Lemma 6.10. We suppose from now on $b < 0$. If $b_2(S) = 0$, by [58] S is either a Hopf surface or an Inoue–Bombieri surface. The Hopf surfaces do not admit twisted logarithmic 1-forms with values in a flat line bundle \mathcal{L}_b with $b < 0$ by Lemma 6.3. The only non-zero twisted logarithmic 1-forms on Inoue–Bombieri surface have values in the flat bundle \mathcal{L}_a with $\mathcal{T}(S) = \{a\}$ by Lemma 6.4. We thus recover the case (c) of Theorem 1.6.

From now on we assume that $b_2(S) > 0$ and $b \neq 0$, i.e. the surface is in the class VII₀⁺. As the Enoki surfaces admit no logarithmic 1-forms with values in a non-trivial flat bundle [25, Thm 5.5], we can assume that S is different from an Enoki surface. We are thus in the setup of Lemma 6.12.

If D has two connected components, S is an Inoue–Hirzebruch surface by Lemma 6.12-(b). In this case, there are two flat holomorphic line bundles $\mathcal{L} = \mathcal{L}_{\mu_i}$, $i = 1, 2$ with $H^0(S, \Omega^1(\log D) \otimes \mathcal{L}_{\mu_i}) \neq 0$, corresponds to a real numbers $-1 < \mu_1 < 1$ or $\mu_2 > 1$ [25]. By Remark 3.13, $\mathcal{L}_{\mu_2} = \mathcal{L}_b$ with $b < 0$, and in this case, by Example 4.5 and Lemma 6.6, $\mathcal{T}(S) \subset (-\infty, b)$, i.e. we obtain the case (b) of the Theorem.

We now suppose that D has one connected component. In this case, by Lemma 6.14 we can assume without loss that $\operatorname{Res}_\alpha(\theta) > 0$ and $b \notin \mathcal{T}(S)$. Recall that $\mathcal{T}(S)$ is an open interval by Theorem 1.1. Suppose b is an upper bound of $\mathcal{T}(S)$. By Lemma 6.14 again, there exists a degree 0 current τ on S satisfying $T = d_\beta d_\beta^c \tau = 2\pi \operatorname{Res}_\beta(\theta) T_D > 0$, $T \neq 0$. We conclude by Proposition 3.8 that $\tau < 0$. By Theorem 5.1, S must be a hyperbolic Kato surface (we assume $b_2(S) > 0$ at this point) and furthermore it must be of intermediate type as we supposed that the maximal divisor D is connected. We recover again the case (b) of the Theorem. The only remaining case is that $\mathcal{T}(S) \subset (b, \infty)$, a situation which we recorded in the case (d) of the Theorem. \square

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