Lévy Processes and Extreme Value Theory

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Chapter Eight

Lévy Processes and Extreme Value Theory

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8.1 INTRODUCTION

There are fundamentally two different ways of viewing the uncertainty of financial asset prices in continuous time. The first assumes the principle of continuity, the second does not. According to the first view, following Bachelier (1900) legacy, price movements are modeled by continuous diffusion processes, such as, for instance, Brownian motion. According to the other view, following Mandelbrot's (1963) legacy, price movements are modeled by discontinuous processes, such as, for instance, Lévy processes. In this chapter, we develop on the relationships connecting the Lévy processes and extreme value theory (EVT).

We begin by defining the modeling alternative and the challenges contemporary finance has to tackle. Next, we present the link with EVT. A convenient way of thinking the modeling alternative for today's financial stakes is to come back to the history of science to exhibit the roots of the puzzle. The story begins in the eighteenth century: Leibniz and Newton, the inventors of differential calculus, stated that "Natura non facit saltus" (nature does not make jumps). In other words, in physics, the so-called principle of continuity states that change is continuous rather than discrete. This same principle underpinned the thoughts of Linné on the classification of species and later Charles Darwin's theory of evolution (1859). If we move to economics, we find Alfred Marshall's Principles of Economics (1890) in which the principle of continuity is assumed, allowing the use of differential calculus in economics and the subsequent development of neoclassical economic theory. As modern financial theory grew out of neoclassical economics, the same principle of continuity is assumed in the conceptual framework

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of financial modeling. As a surprising consequence of this assumption, one of the great success stories of modern financial theory was the valuation of derivatives. Celebrated examples include the formulas of Black et al. (1900) for valuing options, and the subsequent fundamental theorem of asset pricing that emerged from the work of Michael Harrison, Daniel Kreps, and Stanley Pliska between 1979 and 1981. These success stories rest on the principle of continuity.

In the twentieth century, both physics and genetics abrogated the principle of continuity. Quantum mechanics postulated discrete energy levels, while genetics took discontinuities into account. But economics—including modern financial theory—stood back from this intellectual revolution. An early attempt by Benoit Mandelbrot in 1962 to take explicit account of discontinuities on all scales in stock market prices led to huge controversies in the financial industry for almost 40 years; see (Walter 2009). But, by the 1980s, the academic consensus reaffirmed the principle of continuity—despite the repeated financial crises following the 1987 stock market crash. Many popular financial techniques, such as portfolio insurance or the calculation of capital requirements in the insurance industry, assume that (financial) nature does not make jumps and therefore promote continuity (see, however, Bertrand and Prigent (2015) for a recent account on the Constant proportion portfolio insurance (CPPI) method and the EVT approach). Most statistical descriptions of time series in finance assume continuity.

It follows that Brownian representation became the standard model, part and parcel of finance curricula across the globe. It was the dominant view in the financial industry itself; and it still underlies almost all prudential regulation frameworks: for instance, the so-called square-root-of-time-rule underlying the regulatory requirements (Basle III and Solvency II) for calculating minimum capital, which is a very narrow subset of time scaling rule of risk, comes directly from the hypothesis that returns are independent, stationary (identically distributed),and normal, that is, Brownian-based representation. Brownian motion has the very important property of independence and stationarity of increments. The processes with independent and stationary increments are called Lévy processes, which constitute the class of stochastic process having this property. Brownian motion is a specific Lévy process: it assumes continuity. Other Lévy processes do not.

Now let us turn our attention to the extreme values in the tails. The classic theory of extreme values distinguishes between small and large quantities. One of the difficulties often encountered in the practical application of this theory is the choice of a threshold beyond which a variation is considered large. One of the main goals of this chapter is to show how the Lévy process approach can be reconciled with that of extreme values. Another objective is to be able to bypass the dead-end of the mixture of diffusions and EVT approaches to tail modeling. An important and puzzling consequence of assuming continuous dynamics is the truncation of financial time series into the so-called "normal" periods and periods of "insanity" or "irrationality." Adopting continuity implies adding an extra component, precisely the EVT framework, to the continuous part of the model. But this *ad hoc* mechanism has a negative spillover: it implies considering crises as "irrational exceptions" and evacuating every possibility of detecting the frailty of

the market by describing jumps at any scale. In other words, this dichotomy leaves the profession unable to explain the transition from one period to another. For example, Alan Greenspan, former chairman of the US central bank, commented on the financial crisis of 2007–2008 with these words: "We can never anticipate all discontinuities" (Financial Times, March 16, 2008). This demonstrates the limits of the diffusion + EVT approach.

This chapter contains five parts. We start by recalling the main definitions of the EVT framework. Then, we move to a presentation of Lévy processes. Section 4 is dedicated to the study of stable Lévy processes. In Section 5, we examine two subclasses of semi-heavy-tailed Lévy processes that are based on tempered stable and generalized hyperbolic distributions. Section 8.6 deals with the relationships between Lévy processes and extreme value distributions.

8.2 EXTREME VALUE THEORY

We start with a study of the Fisher–Tippett theorem, then we examine the Generalized Jenkinson–von Mises distribution, and finally we concentrate on the maximum domains of attraction.

8.2.1 The Fisher–Tippett theorem

Let $\{X_1, X_2, \cdots, X_n\}$ be n independent, identically distributed (i.i.d.) random variables equipped with the cumulative distribution function (c.d.f) $F_X(x) = \Pr(X \leq x)$. Consider a sample that consists of the n realizations $\{x_1, \cdots, x_n\}$. We rank them by decreasing order, and introduce the following convention:

$$x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}.$$

The largest of these realizations, $x_{(1)}$, can be considered to represent the realization of a new random variable $X_{(1)}$. The same idea prevails for the other observations $x_{(k)}$. Thus, we introduce n new random variables with the convention

$$X_{(1)} = \max(X_1, \cdots, X_n),$$

which represents the random variable giving the largest value in a sample of size n. In the statistics literature, this quantity is also sometimes denoted by $X_{(1:n)}$.

Similarly,

$$X_{(n)} = \min(X_1, \cdots, X_n)$$

represents the random variable giving the smallest value observed in a sample of size n. More generally, $X_{(k)}$—or $X_{(k:n)}$ —is the random variable related to the k th value $x_{(k)}$ obtained among n realizations. These n new random variables can be ordered in the following way:

$$X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}.$$

We want to show how it is traditionally possible to determine the values that can be taken by maxima over a given observation period, that is, we want to characterize the probability law of maxima. Because extreme values are by definition rare, one needs to be able to extrapolate from past observed values to future hypothetical large values. EVT proposes a framework adapted to this extrapolation: it describes the asymptotic behavior of $F_{1:n}$ when n tends to plus infinity. For technical reasons, this is not the maximum $X_{(1:n)}$, which is the object of study, but the centered and reduced maximum $(X_{(1:n)} - b_n)/a_n$.

The Fisher–Tippett theorem provides the shape of the possible asymptotic cdfs of the centered reduced maximum of a sample of size n. We remain in the context of a set of n random variables $\{X_1, \cdots, X_n\}$ that are i.i.d, and that have the cdf $F_X(x)$. We stress that this theory requires the validation of the i.i.d. hypothesis in order to be applied in an adequate way.

THEOREM 8.1 (Laws of maxima: Fisher–Tippett (1928)) *Let us assume a sequence of n i.i.d. random variables* $\{X_1, \cdots, X_n\}$ *with the cdf* $F_X(x)$*. If there are two sequences* $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{b_n\}_{n\in\mathbb{N}}$ *with* $\forall n$, $a_n \in \mathbb{R}^{+*}$ *and* $b_n \in \mathbb{R}$*, and a nondegenerate limit cdf H such that*

$$\lim_{n\to\infty} \Pr\left(\frac{X_{(1:n)} - b_n}{a_n} \leq x\right) = H(x), \tag{8.1}$$

then H belongs to one of the three classes:

- *Fréchet.* $H(z) = \Phi_\alpha(z) = \exp\ (-z^{-\alpha})$ *if* $z > 0$*, 0 if* $z \leq 0$*,*
- *Weibull.* $H(z) = \Psi_\alpha(z) = \exp\ (-(-z)^{\alpha})$ *if* $z \leq 0$*, 1 if* $z > 0$*,*
- *Gumbel.* $H(z) = \Lambda(z) = \exp\ (-e^{-z})$*,* $z \in \mathbb{R}$

with $z = \frac{x-b}{a}$*, where* $a > 0$*,* $b \in \mathbb{R}$*, and* $\alpha > 0$*.*

The three classes of distributions denoted by Φ_α, Ψ_α, and Λ are called the laws of maxima. The Fréchet and Weibull laws are linked by the relation

$$\Phi_\alpha\left(-\frac{1}{z}\right) = \Psi_\alpha(z). \tag{8.2}$$

The parameters a and b are scale and localization parameters, whereas α is a shape parameter. These laws are stable with respect to maxima: for a sample of independent random variables $\{Z_1, \cdots, Z_n\}$ that satisfy one of these laws, and for $Z_{(1:n)}$, the maximum of these variables, we have

$$Z_{(1:n)} \overset{d}{=} a_n Z + b_n, \tag{8.3}$$

where Z is a random variable that has the same distribution as $\{Z_1, \cdots, Z_n\}$ and that is independent of these latter random variables.

In particular, we have

– Fréchet law: $Z_{(1:n)} \overset{d}{=} n^{1/\alpha} Z$,

– Weibull law: $Z_{(1:n)} \overset{d}{=} n^{-1/\alpha} Z$,

– Gumbel law: $Z_{(1:n)} \overset{d}{=} Z + \ln\ n$.

We illustrate the previous developments with a simple example. Suppose that $\mathbb{E}[Z] = 1$ and consider the Fréchet (with, e.g., $\alpha = \frac{1}{2}$) and Gumbel laws. We do not select the Weibull law because its support is bounded on the right (it is therefore of little use for financial problems). In the first case, we see that $\mathbb{E}[Z_{(1:n)}] = n^2$, while in the second case $\mathbb{E}[Z_{(1:n)}] \approx \ln\ n$. Written differently, with the Fréchet law the maximum increases much more quickly than with the Gumbel law, and consequently the same behavior is also observed for the contribution of the maximum within the sum.

There is a formal analogy between this theorem for the maximum (expressed through the formula (8.1)) and the central limit theorem for the mean, which we recall here:

$$\lim_{n\to\infty} \Pr\left(\frac{\bar{X}_n - \mathbb{E}[X]}{\sigma(X)/\sqrt{n}} \leq x\right) = \Phi(x), \tag{8.4}$$

where Φ is the cdf of the centered reduced Gaussian law.

This comparison between the two theorems, as well as the two formulas (8.1) and (8.4), allows us to interpret the parameters a_n and b_n. The parameter b_n corresponds to the mean $\mathbb{E}[X]$ in the central limit theorem and is therefore a localization parameter. The parameter a_n corresponds to $\sigma(X)/\sqrt{n}$ in the central limit theorem and is therefore a scale parameter. However, the parameter α has no equivalent in the central limit theorem. The elements of this analogy are displayed synthetically in Table 8.1, which displays a comparison between two laws of large numbers, respectively associated with the means and the extreme values. In the two cases, limit theorems and standard distributions exist. We note that, in the case of the α-stable distribution, $\mathbb{E}(X)$ and $\sigma(X)$ do not necessarily exist (case of Pareto-power laws). Indeed, when the first moments of distributions are not defined, we asymptotically obtain a stable distribution, where the coefficient α of the central limit theorem is identical to the inverse of the coefficient ξ of EVT. The two laws of large numbers focus on two distinct quantities: the fluctuation of means and the fluctuation of maxima. For more details, see (Leadbetter 2015).

8.2.2 Generalized Jenkinson–von Mises distribution

The expressions of the Fréchet, Weibull, and Gumbel distributions can be united in a general expression using the condensed representation of Jenkinson and von Mises (1954–1955), which defines a generalized distribution for the maxima H_ξ:

$$H_\xi(x) = \exp\left(-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right), \tag{8.5}$$

TABLE 8.1 Limit laws of means and maxima

Laws of large numbers	
Central limit theorems	Extreme value theorems
Convergence of empirical means	Convergence of empirical maxima
$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$	$X_{(n)} = \max(X_1, \cdots, X_n)$
$\Phi(x) = \lim_{n\to\infty} \Pr\left(\frac{\bar{X}_n - \mathbb{E}[X]}{\sigma(X)/\sqrt{n}} \le x\right)$	$H(x) = \lim_{n\to\infty} \Pr\left(\frac{X_{(n)} - b_n}{a_n} \le x\right)$
Centered reduced Gaussian distribution if $\mathbb{E}[X]$ and $\sigma(X)$ exist	Standardized distribution of extreme values
	Fréchet:
Gaussian density:	$H(x) = \exp\ (-x^{-\alpha})$
$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$	Weibull:
Gaussian characteristic exponent:	$H(x) = \exp\ (-(-x)^{\alpha})$
$\Psi_X(u) = -\frac{u^2}{2}$	Gumbel:
	$H(x) = \exp\ (-e^{-x})$
Moivre (1730)–Laplace (1812)	Fisher–Tippett (1928)
α-stable distributions if $\mathbb{E}[X]$ and/or $\sigma(X)$ do not exist	Generalized distribution of extreme values
$\Psi_X(u) = i\mu u - \gamma^{\alpha}\lvert u\rvert^{\alpha}\left(1 - i\beta\frac{u}{\lvert u\rvert}\operatorname{tg}\frac{\pi\alpha}{2}\right)$	$H_{\xi}(x) = \exp\left(-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right)$
Lévy (1937)	Jenkinson–von Mises (1954–1955)
Fluctuations of means	Fluctuations of maxima

for all x such that $1 + \xi(x - \mu)/\sigma > 0$.

The density function of this generalized law of maxima is

$$h_{\xi}(x) = \frac{1}{\sigma}\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1+\xi}{\xi}} \exp\left(-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right). \tag{8.6}$$

We can interpret the parameters μ and σ by comparing Eqs (8.1) and (8.5):

$$\lim_{n\to\infty} \Pr\left(\frac{X_{(n)} - b_n}{a_n} \le x\right) = \exp\left(-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right).$$

This equality shows that the parameters μ and σ are the limits of b_n and a_n when n is large. Therefore, $\mu \in \mathbb{R}$ is a localization parameter and $\sigma > 0$ is a scale parameter. Finally, the parameter $\xi \in \mathbb{R}$ models the shape of the distribution tail. It is therefore called the shape of the extreme value index. More precisely, three situations can be encountered:

1. $\xi > 0$, this is the Fréchet law. The distribution tail is thick and $\xi = \alpha^{-1} > 0$, where α is the tail index. The larger the ξ, the thicker the distribution tail.
2. $\xi < 0$, this is the Weibull law. The distribution tail is thin and $\xi = -\alpha^{-1} < 0$.

3. $\xi = 0$, this is the Gumbel law. The distribution tail decreases exponentially. This is a limit case between the two preceding situations, where $\xi \to 0$.

8.2.3 Maximum Domain of Attraction

Let us now reformulate the Fisher–Tippett theorem in order to introduce the concept of the maximum domain of attraction (MDA). Because

$$\Pr\left(\frac{X_{(1:n)} - b_n}{a_n} \leq x\right) = \Pr\left(X_{(1:n)} < a_n x + b_n\right),$$

we have, using the definition of $X_{(1:n)}$

$$\Pr\left(X_{(1:n)} < a_n x + b_n\right) = \left[F(a_n x + b_n)\right]^n.$$

Then, the Fisher–Tippett theorem states that

$$\lim_{n\to\infty} \left[F\left(a_n x + b_n\right)\right]^n = H(x). \tag{8.7}$$

We say that the Fisher–Tippett theorem defines a MDA for the cdf $F(\cdot)$, and we write $F \in \text{MDA}(H)$.

This result is equivalent for the largest values to what the central limit theorem is for the mean values. A probability distribution of the cdf F belongs to the MDA of H if the distribution of the renormalized maximum converges to H.

Which types of probability distribution belong to which maximum domains of attraction and what are the renormalization constants? The answer to this question depends on the variation of the cdf of the random variable. If a random variable varies regularly, then it belongs to the MDA of Fréchet; if this random variable varies rapidly, then it belongs to the MDA of Gumbel. We now examine these two cases.

In the first situation, $\bar{F}$ is a regular function of index $-\alpha$. Recall that f is a regular function of index α if, for all $x > 0$, we have

$$\lim_{t\to+\infty} \frac{f(tx)}{f(t)} = x^\alpha.$$

In this case, $F \in \text{MDA}(\Phi_\alpha)$: the MDA is that of the Fréchet law with $a_n = F^{-1}(1 - n^{-1})$ and $b_n = 0$. This result was obtained by Gnedenko in 1943. In order to verify that $F \in \text{MDA}(\Phi_\alpha)$, we check that

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}.$$

Assume, for instance, that F is the cdf of a Pareto distribution of type I, so that $F(x) = 1 - x^{-\alpha}$. Then

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t\to\infty} \frac{(tx)^{-\alpha}}{t^{-\alpha}} = x^{-\alpha}.$$

Thus, a Pareto distribution of type I belongs to the MDA of the Fréchet law. We can also check the Fisher–Tippett theorem by calculating F^n. We have here $a_n = F^{-1}(1 - n^{-1}) = n^{1/\alpha}$. This yields

$$\lim_{n\to\infty}[F(a_n x + b_n)]^n = \lim_{n\to\infty}[F(n^{1/\alpha}x)]^n = \lim_{n\to\infty}(1 - \frac{x^{-\alpha}}{n})^n,$$

so that

$$\lim_{n\to\infty}[F(a_n x + b_n)]^n = \exp\ (-x^{-\alpha}),$$

and we have recovered the Fréchet law.

The second situation is that in which the distributions belong to the MDA of the Gumbel law and have an infinite right endpoint. These distributions are of rapid variation. Recall that a function f is of rapid variation if it satisfies

$$\begin{cases} \lim_{t\to+\infty} \frac{f(tx)}{f(t)} = 0, & x > 1, \\ \lim_{t\to+\infty} \frac{f(tx)}{f(t)} = +\infty, & 0 < x < 1. \end{cases}$$

These results conclude this first section on EVT. We will come back later on to these asymptotic results. Before that, let us conduct a study of infinitely divisible distributions and Lévy processes.

8.3 INFINITE DIVISIBILITY AND LÉVY PROCESSES

We start by recalling the definition of a characteristic exponent. Then we study infinitely divisible distributions, before examining Lévy processes, defined as the class of processes with independent and stationary increments, and Lévy measures.

8.3.1 Characteristic Exponent

Recall that the characteristic function of a random variable Y is the Fourier transform of its density function, that is,

$$\Phi_Y(u) = \mathbb{E}\left[e^{iuY}\right] = \underbrace{\sum_{k=0}^{\infty} e^{iuk}\Pr(Y=k)}_{\text{discrete r.v.}} = \underbrace{\int_{-\infty}^{+\infty} e^{iux} f_Y(x)dx}_{\text{continuous r.v.}},$$

where $f_Y(x)$ is the density function. The characteristic function can be expressed in the following way:

$$\Phi_Y(u) = \exp\ \left(\Psi_Y(u)\right), \tag{8.8}$$

where $\Psi_Y(u)$ is called the characteristic exponent of Y.

In the case of Lévy processes, the exponent Ψ is proportional to time. This follows from a very important property of these processes: their marginal distributions are infinitely divisible. We now examine this property in greater detail.

8.3.2 Infinitely Divisible Distributions

Roughly speaking, the property of infinite divisibility states that it is possible to write a random variable as a sum of n identical independent random variables, for any given n. More precisely, for any Lévy process X and any positive integer n, it is clear that

$$X_n = X_1 + (X_2 - X_1) + \cdots + (X_n - X_{n-1})$$

and

$$\begin{aligned}\Phi_{X_n}(u) &= \mathbb{E}\left[e^{iuX_n}\right] \\ &= \mathbb{E}\left[e^{iu(X_1+(X_2-X_1)+\cdots+(X_n-X_{n-1}))}\right] \\ &= \mathbb{E}\left[e^{iuX_1} \times e^{iu(X_2-X_1)} \times \cdots \times e^{iu(X_n-X_{n-1})}\right].\end{aligned}$$

From the independence of the increments of X, it follows that

$$\Phi_{X_n}(u) = \mathbb{E}\left[e^{iuX_1}\right] \times \mathbb{E}\left[e^{iu(X_2-X_1)}\right] \times \cdots \times \mathbb{E}\left[e^{iu(X_n-X_{n-1})}\right].$$

The stationarity of the increments of X then implies that

$$\Phi_{X_n}(u) = \underbrace{\mathbb{E}\left[e^{iuX_1}\right] \times \mathbb{E}\left[e^{iuX_1}\right] \times \cdots \times \mathbb{E}\left[e^{iuX_1}\right]}_{n \quad \text{times}} = \left(\mathbb{E}\left[e^{iuX_1}\right]\right)^n.$$

Finally, we obtain

$$\Phi_{X_n}(u) = \left(\Phi_{X_1}(u)\right)^n. \tag{8.9}$$

This result can be generalized in the case of continuous time t to

$$\Phi_{X_t}(u) = \left(\Phi_{X_1}(u)\right)^t. \tag{8.10}$$

If we now consider characteristic exponents, we readily have

$$\exp\left(\Psi_{X_t}(u)\right) = \left(\exp\left(\Psi_{X_1}(u)\right)\right)^t = \exp\left(t\Psi_{X_1}(u)\right).$$

From this equation, we deduce the relation between the distributions of the process X at time 1 and at time t:

$$\Psi_{X_t}(u) = t\,\Psi_{X_1}(u). \tag{8.11}$$

Thus, the characteristic exponent of a Lévy process is equal to t times that of its underlying infinitely divisible distribution, which is the distribution of X_1.

8.3.3 Characteristic Exponent of Lévy Processes

Lévy processes are described by their characteristic exponent, which can, in general, be decomposed as follows:

$$\Psi_{X_t}(u) = \underbrace{\underbrace{t\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right) + \underbrace{\text{Gaussian law}}}_{\text{Diffusion}}}_{\text{Jump Component}} \underbrace{t\int_{|x|<1}\left(e^{iux} - 1 - iux\right)\nu(dx)}_{\text{small jumps}} + \underbrace{t\int_{|x|\geq 1}\left(e^{iux} - 1\right)\nu(dx)}_{\text{large jumps}}. \tag{8.12}$$

The three terms of this formula represent the characteristic exponents of the normal distribution (diffusive component), of a generalized (in the sense that the arrival rate of jumps may be infinite) compensated compound Poisson process, and of a compound Poisson process, respectively. The second term corresponds to modeling small jumps, that is (by arbitrary construction), those jumps of size less than 1 in absolute value, and the third one to large jumps.

The last two terms can be written together in the same integral using an indicator function:

$$\Psi_{X_t}(u) = t\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right) + t\int_{\mathbb{R}^*}\left(e^{iux} - 1 - iux\,\mathbf{1}_{|x|<1}(x)\right)\nu(dx). \tag{8.13}$$

This is the Lévy–Khintchine formula. The characteristic exponent is indeed proportional to time:

$$\begin{cases}\Psi_{X_1}(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}^*}\left(e^{iux} - 1 - iux\,\mathbf{1}_{|x|<1}(x)\right)\nu(dx),\\ \Psi_{X_t}(u) = t\,\Psi_{X_1}(u).\end{cases} \tag{8.14}$$

A Lévy process is therefore completely determined by the following three quantities:

1. The expectation μ of the diffusion component. This is the drift of the process.
2. The diffusion coefficient σ. This coefficient functions as a scale of the diffusive fluctuations.
3. The measure $\nu(dx)$, called the Lévy measure, determines the fluctuations of the jumps, as well as the skewness and the kurtosis of the process. Note that for the integral in Eq. (8.13) to be defined, we impose the following restriction:
$$\int_{\mathbb{R}}(x^2 \wedge 1)\nu(dx) < +\infty$$

.

TABLE 8.2 Comparison of characteristic exponents

Process/characteristic exponent	$\int_{-\infty}^{+\infty} \nu(dx)$
Compound Poisson: $t\int_{-\infty}^{+\infty}(e^{iux}-1)\nu(dx)$	Finite
Compensated compound Poisson: $t\int_{-\infty}^{+\infty}(e^{iux}-1-iux)\nu(dx)$	Finite
Lévy pure jump: $t\int_{\lvert x\rvert\geq 1}(e^{iux}-1)\nu(dx)+t\int_{\lvert x\rvert<1}(e^{iux}-1-iux)\nu(dx)$ $t\int_{\mathbb{R}^*}\left(e^{iux}-1-iux\,\mathbf{1}_{\lvert x\rvert<1}(x)\right)\nu(dx)$	Infinite
Lévy: $t\left(i\mu u-\frac{1}{2}\sigma^2u^2\right)+t\int_{\mathbb{R}^*}\left(e^{iux}-1-iux\,\mathbf{1}_{\lvert x\rvert<1}(x)\right)\nu(dx)$	Infinite

Table 8.2 gives an overview of the jump component in the Lévy–Khintchine formula of the Poisson processes we have seen so far, which are written in terms of their Lévy measure.

8.3.4 Lévy Measure

The important term of the characteristic exponent is the Lévy measure. To explain its meaning intuitively, we choose the example, given in the preceding paragraph, of a compound Poisson process with intensity λ(the Poisson parameter). We assume that this process has the density function $f_Y(x)$ for the jump size given by Y. The Lévy measure can then be written as

$$\nu(x) = \lambda\, f_Y(x). \tag{8.15}$$

Intuitively, the Lévy measure gives the average number of jumps per unit of time in terms of their size. We can therefore view the Lévy measure as the mathematical object that quantifies the arrival of jumps and their size. It can be used to create discontinuities in the paths of the stochastic processes that represent fluctuations on the stock market.

For the Poisson processes introduced previously, the average number λ of jumps per unit of time is finite. However, in a very general way, we can arbitrarily adjust both the frequency with which jumps occur and their size by using the Lévy measure. We call the average number of jumps per unit of time the intensity of the process (also called activity of the process in analogy to turbulence). Then, if we also know the density function that determines the size of the jumps, we can join them together:

$$\text{intensity} \times \text{density},$$

which is exactly the Lévy measure in the case when it is finite. Let us now come to the study of two important classes of Lévy processes. We start with the heavy-tailed ones.

8.4 HEAVY-TAILED LÉVY PROCESSES

This section is dedicated to the study of stable Lévy processes. We first examine stable distributions, and then stable processes, and finally stable characteristic exponents and measures.

8.4.1 Stable Distributions

Let X, Y, and Z be three i.i.d. random variables, and A, B, C, and D four real numbers (the first three being strictly positive). Then, if the relation

$$AX + BY \stackrel{d}{=} CZ + D$$

holds, the random variable X(and its independent copies) is said to be stable.

This relation can be generalized for all $n \geq 2$ as follows:

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} C_n X + D_n,$$

where $X_1, X_2, \ldots, X_n$, and X are i.i.d. random variables, and C_n and D_n are real numbers (the first of which is strictly positive). If this last relation holds, X(and its independent copies) is said to be stable.

Stable distributions are described by four parameters $(\alpha, \beta, \gamma, \mu)$, which correspond to the fatness of the tails, the asymmetry, the width, and the mean of these distributions, respectively (when this last quantity does not exist, μ is interpreted as a shift parameter). We will denote stable distributions by $\mathcal{S}(\alpha, \beta, \gamma, \mu)$ in the following.

8.4.2 Stable Processes

A stable process is a process whose underlying distributions, marginal or joint, are stable. We call a stable process whose marginal distributions are symmetric and normalized, an alpha-stable process such as

$$X_t - X_s \sim \mathcal{S}(\alpha, 0, |t-s|^{\frac{1}{\alpha}}, 0).$$

These symmetric and normalized alpha-stable distributions are often denoted by $S\alpha S$.

If X is an alpha-stable process, then it has the following property:

$$X(at) - X(as) \stackrel{d}{=} a^{1/\alpha} X(t) - X(s), \tag{8.16}$$

for all $\alpha \in]0, 2]$, where we use $\stackrel{d}{=}$ to denote equality in distribution. In this case, the stochastic process X is said to be self-similar with exponent $1/\alpha$, or $1/\alpha$-self-similar with stationary increments ($1/\alpha$- sssi), or also fractal. This

means that the distribution of the return at date at (hence $X(at)$), where a determines some multiple or fraction of t(e.g., a year-fraction, or a noninteger multiple of a year, if a year is the unit of time), is related to the distribution at date t by simple invariance of scale. Graphically, the dilation of the time axis is equivalent to a dilation of the axis of cumulative returns, up to normalization (and stressing the fact that this is a dilation of the distributions, and not of the paths).

8.4.3 Characteristic Function and Lévy Measure

To obtain the characteristic function of the marginal distribution of a stable process at a given time t, we take the t th power of the characteristic function of a reference stable distribution taken at time $t = 1$(see Eq. (8.10)). In the general case, the density function is not known in closed form and the characteristic function is used. For a stable random variable X, we have by definition

$$\Psi_X(u) = i\mu u + \int_{-\infty}^{0} \psi(u,x)\, \frac{C_-}{|x|^{1+\alpha}} dx + \int_{0}^{+\infty} \psi(u,x)\, \frac{C_+}{x^{1+\alpha}} dx \tag{8.17}$$

with

$$\psi(u,x) = e^{iux} - 1 - iux\, \mathbf{1}_{|x|<1}. \tag{8.18}$$

The parameter μ gives the drift, while the term α describes the form of the distribution and the force of the jumps affecting the paths. The tails of the distribution become fatter and fatter as α decreases. C_- and C_+ are two constants, called the tail amplitudes, which represent scale parameters for the distribution tails. In practice, C_- and C_+ give the order of magnitude of large fluctuations (negative or positive). Comparing Eq. (8.17) to the Lévy–Khintchine formula (8.13), we see that the Lévy measure of a stable distribution is

$$\nu(dx) = \frac{C_-}{|x|^{1+\alpha}} \mathbf{1}_{(-\infty,0)}(x)dx + \frac{C_+}{x^{1+\alpha}} \mathbf{1}_{(0,+\infty)}(x)dx \tag{8.19}$$

or, more intuitively,

$$\nu(x) = \begin{cases} \frac{C_-}{|x|^{1+\alpha}} & \text{if} \quad x < 0, \\ \\ \frac{C_+}{x^{1+\alpha}} & \text{if} \quad x > 0, \end{cases} \tag{8.20}$$

which is the equation also found in (Samorodnitsky and Taqqu 1994).

An important feature evident in this form is a decrease given by a power law, which is slow for large values of x compared to the fast decrease observed for exponential laws. This slow decrease of the distribution tails is a key characteristic of Pareto distributions. Here, the Lévy measure is defined by a Pareto type I distribution, or simply Pareto distribution.

8.5 SEMI-HEAVY-TAILED LÉVY PROCESSES

In these section, we examine two subclasses of semi-heavy-tailed Lévy processes that are based on tempered stable and generalized hyperbolic distributions.

8.5.1 Tempered Stable Distributions

To remedy the inconvenience of not having any moments for those Lévy processes whose measure is given by a power function built on the model

$$\nu(x) = \begin{cases} \frac{C_-}{|x|^{1+\alpha}} & \text{if} \quad x < 0, \\ \\ \frac{C_+}{x^{1+\alpha}} & \text{if} \quad x > 0, \end{cases}$$

it has been suggested to weight this measure by an exponential quantity in order to reduce large fluctuations, and therefore recover the moments. This idea corresponds to a set of Lévy processes whose marginal distributions are stable distributions truncated, or "tempered," by exponential functions, hence their name *tempered stable processes*. The distribution tails of these stable models, tempered by the truncation, are semi-light.

The first model to be built on this principle was that of (Koponen 1995), later taken up by (Bouchaud and Potters 2003) and again by (Boyarchenko and Levendorskii 2002), in which the Lévy measure is given by

$$\nu(x) = \begin{cases} \frac{C_-}{|x|^{1+\alpha}} \, e^{-a_- \, |x|} & \text{if} \quad x < 0, \\ \\ \frac{C_+}{x^{1+\alpha}} \, e^{-a_+ \, x} & \text{if} \quad x > 0. \end{cases} \tag{8.21}$$

Observe that the parameterization is asymmetric in the decay rate of large jumps, but symmetric in the parameter α, which is the same for up and down jumps. The Variance Gamma and CGMY (Carr, Geman, Madan, and Yor) models are well-known special cases of this model. In the CGMY model, the coefficients C_- and C_+ have been symmetrized.

This model has the following Lévy measure:

$$\nu(x) = \begin{cases} \frac{C}{|x|^{1+\alpha}} \, e^{-a_- \, |x|} & \text{if} \quad x < 0, \\ \\ \frac{C}{x^{1+\alpha}} \, e^{-a_+ \, x} & \text{if} \quad x > 0. \end{cases} \tag{8.22}$$

As before, interpreting the parameters exposes two types of risk: size and shape. The parameter C is a measure of size (affecting all moments), whereas α, a_-, and a_+ determine the shape (force of the jumps for α and asymmetry for a_- and a_+). When $a_- > a_+$, the distribution is curtailed on the left and stretched to the right (the left part is squashed by the exponential factor), and vice versa.

Finally, for asymmetrical distributions, a change in α will result in a change in the asymmetry. In other words, α has an indirect leverage effect on the asymmetry.

8.5.2 Generalized hyperbolic distributions

We first concentrate on the simple hyperbolic distribution by developing an intuitive understanding of what motivates the name "hyperbolic." If we draw the graph of a Gaussian density in a semilogarithmic chart (i.e., one in which the ordinate is given on a logarithmic scale), then the density function (with e^{-x^2}) will have the shape of a parabola (since we have $-x^2$ now). This parabola is characterized visually by the rapid fall (rapid decline) of the distribution's tails. If we try to obtain more drawn-out distribution tails, we can slow down the fall by replacing the parabola by a hyperbola. In this case, the distribution's density function will be of hyperbolic form in a semilogarithmic chart. This is the reason why distributions constructed in this manner are called hyperbolic. Their usefulness for the modeling of price movements on the stock markets stems from the slower decrease of their tails.

This intuition allows us to reach in a simple way the general complicated form of these distributions. In fact, the equation defining a hyperbola (with vertex on top in order to obtain a density) is easily obtained as

$$y(x) = -\alpha\sqrt{1 + x^2} + \beta x, \tag{8.23}$$

where α and β are two constants that determine the shape of the hyperbola. As x goes to $\pm\infty$, the asymptotes are the two lines with slope $-\alpha + \beta$(for $x \to +\infty$) and $\alpha + \beta$(for $x \to -\infty$). It follows directly from Eq. (8.23) that the general form of the density function of a hyperbolic distribution is given by

$$f(x) = a \ \exp\ \left(-\alpha\sqrt{1 + x^2} + \beta x\right), \tag{8.24}$$

where a is a normalization constant for the integral of the density function.

By considering the slopes of the two asymptotes, we obtain a direct interpretation of the parameters: $\alpha > 0$ and $0 < |\beta| < \alpha$ determine the shape of the distribution, affecting the fatness of the tails (for α) and the asymmetry (for β). Introducing a localization parameter denoted by μ and a dispersion (or scale) parameter denoted by γ, we obtain a more general version of the preceding equation:

$$f(x) = a' \ \exp\ \left(-\alpha\sqrt{\gamma^2 + (x - \mu)^2} + \beta(x - \mu)\right). \tag{8.25}$$

We must now define the normalization constant. Since the density must integrate to 1, we compute its integral and deduce the value of the normalization constant from it. In this way, we obtain the result that the density function of a hyperbolic distribution (H) is given by

$$f_{\mathrm{H}}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\gamma K_1\left(\gamma\sqrt{\alpha^2 - \beta^2}\right)} \ \exp\Big(\ -\alpha\ \sqrt{\gamma^2 + (x - \mu)^2} + \beta(x - \mu)\Big), \tag{8.26}$$

where $K_1(\cdot)$ is a modified Bessel function of the third kind of order 1.

We see that, like alpha-stable distributions, hyperbolic distributions are defined by four parameters: the kurtosis, the asymmetry, the dispersion, and the localization of the distribution. In other words, here too we have completed the size of the random movements (their dispersion, or scale: γ) with their form (fatness of tails and asymmetry: α and β). The localization parameter keeps the same meaning in all cases. Although different in their asymptotic behavior, all hyperbolic distributions retain a twofold notion of risk in their description of price movements on an exchange (size–form, or amplitude–structure).

The construction of the density function of a hyperbolic distribution can therefore be carried out in two steps.

1. By defining the hyperbolic part of the exponential function in its general form with four parameters $(\alpha, \beta, \gamma, \mu)$:

$$-\alpha \sqrt{\gamma^2 + (x-\mu)^2} + \beta(x-\mu).$$

2. By adjusting the exponential of the hyperbola by a normalization constant that depends on the parameters: $a(\alpha, \beta, \gamma)$. This constant involves modified Bessel functions denoted by $K_\lambda(\cdot)$.

We recall that Bessel functions are solutions to the second-order differential equation

$$x^2y'' + xy' + (x^2 - \lambda^2)y = 0.$$

The solutions to this equation are denoted by $J_\lambda(x)$ and $Y_\lambda(x)$, depending on whether they are defined at $x = 0$, and are called Bessel function of the first kind of order λ and Bessel function of the second kind of order λ, respectively.

If at present we consider the following modified differential equation

$$x^2y'' + xy' - (x^2 + \lambda^2)y = 0$$

then its solutions are called modified Bessel functions of the first and second kinds, denoted by $I_\lambda(x)$ and $K_\lambda(x)$.

We can generalize the density (8.26) by choosing an order λ different from 1 for the modified Bessel function to obtain a generalized version of the hyperbolic distribution. The Bessel function of the second kind of order λ is defined for all $x > 0$ by

$$K_\lambda(x) = \int_0^\infty y^{\lambda-1} e^{-\frac{1}{2}x(y+y^{-1})}\, dy.$$

- For $\lambda = 1$, we have the case K_1 of the simple hyperbolic distribution described above.

- For $\lambda = -1/2$, we obtain the density of the normal inverse Gaussian (NIG) distribution

$$f_{\text{NIG}}(x) = \frac{\alpha\gamma}{\pi} \frac{K_1(\alpha\sqrt{\gamma^2 + (x-\mu)^2})}{\sqrt{\gamma^2 + (x-\mu)^2}} \exp\left(\gamma\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right). \tag{8.27}$$

The dispersion parameter γ takes values between 0 and infinity. Starting from the simple hyperbolic distribution with density (8.26), it can be shown that

  - when $\gamma \to +\infty$ and $\beta \to 0$, then $\gamma/\sqrt{\alpha^2 - \beta^2} \to \sigma^2$, and the simple hyperbolic density converges toward a Gaussian one with mean μ and standard deviation σ;
  - when $\gamma \to 0$, the simple hyperbolic density converges toward an asymmetric Laplace density

$$f(x) = C \begin{cases} e^{-(\alpha-\beta)(|x|-\mu)} & \text{if} \quad x \geq \mu, \\ e^{-(\alpha+\beta)(|x|-\mu)} & \text{if} \quad x < \mu. \end{cases} \tag{8.28}$$

- For $\lambda \neq 1$, we have the case of the generalized hyperbolic distribution(GH), whose density function is given by

$$f_{\text{GH}}(x) = a(\gamma^2 + (x-\mu)^2)^{\left(\lambda - \frac{1}{2}\right)/2} K_{\lambda - \frac{1}{2}}\left(\alpha\sqrt{\gamma^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)} \tag{8.29}$$

with normalizing constant

$$a = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\gamma^{\lambda} K_{\lambda}\left(\gamma\sqrt{\alpha^2 - \beta^2}\right)}.$$

8.6 LÉVY PROCESSES AND EXTREME VALUES

The infinitely divisible distributions whose underlying Lévy measure density is the product of an exponential function by a power function belong to the MDA of the Gumbel law. The asymptotic behavior of these distributions is studied in detail in (Albin and Sundén 2009). Consider, for instance, the infinitely divisible distributions having the Lévy measure density

$$\nu(x) = Cx^{\rho}e^{-\alpha x}. \tag{8.30}$$

It appears that both the distribution densities and the cdfs of these laws have, up to a constant, a mixed exponential-power form. It can also be noted that the distributions constructed from such Lévy measure densities (in particular the Variance Gamma, CGMY, or generalized hyperbolic distributions) possess cdfs that can be written as

$$F(x) = 1 - e^{-\int_{-\infty}^{x} c(t)\, dt},$$

where $\lim_{t\to+\infty} c(t) = \alpha$. This representation is that of a von Mises function. It readily implies that these distributions belong to the MDA of the Gumbel law.

Let us conclude on the behavior of distribution tails:

1. The MDA of the Fréchet law contains all distributions having stretched or heavy tails, such as power functions.
2. The MDA of the Gumbel law contains all thin and medium-tailed distributions. Examples of such distributions are the exponential distributions with semi-heavy tails thicker than the Gaussian, such as the distributions resulting from mixing a power function with an exponential function. The previous section gave a representative sample of such distributions.
3. The MDA of the Weibull law contains all distributions with a support bounded to the right. These distributions are said to be short-tailed.

These results are summarized in Table 8.3. It appears that most distributions of interest to the field of finance belong to the MDA of Gumbel, whereas distributions of interest to the field of insurance rather belong to the MDA of Fréchet.

To describe the behavior of the largest values of Lévy processes, it is sufficient to consider the Lévy measure. Indeed, this measure directly yields the asymptotic pattern of the density and cdf of any marginal X_t. Because Lévy processes have independent and stationary increments, considering a marginal distribution at time t, as we do in the next subsections, exhausts the discussion for the whole process.

8.6.1 CGMY and VG Distributions

We first consider the asymptotic behavior of the CGMY (see Carr, Geman, Madan, and Yor (2002) and Variance Gamma (see, e.g., Madan and Seneta (1990) or Madan and Milne (1991)) processes satisfying the relationship (8.30) with,

TABLE 8.3 Laws of maxima and maximum domain of attraction

Laws of maxima	Shape parameter	Basic distributions
Weibull	$\xi < 0$	Uniform
		Beta
Gumbel	$\xi = 0$	Gaussian
		Exponential
		Lognormal
		Gamma
		CGMY
Fréchet	$\xi > 0$	Cauchy
		Pareto
		α-Stable
		Student
		Loggamma

respectively, $\rho < -1$ and $\rho = -1$. For a process X of the CGMY type that has the Lévy measure density

$$\nu(x) = C\frac{e^{Gx}}{(-x)^{1+Y}}\mathbf{1}_{x<0} + C\frac{e^{-Mx}}{x^{1+Y}}\mathbf{1}_{x>0},$$

where $Y > 0$, (Albin and Sundén 2009) showed that, when u is large,

$$\Pr(X_t > u) \overset{+\infty}{\sim} \frac{Ct}{M}\ \mathbb{E}\left[e^{MX(t)}\right]\ \frac{e^{-Mu}}{u^{1+Y}}.$$

From the computation of the moment generating function $\mathbb{E}(e^{MX(t)})$, we deduce that

$$\Pr(X_t > u) \overset{+\infty}{\sim} \frac{Ct}{M}\quad e^{Ct\Gamma(-Y)}\left[(G+M)^Y - M^Y - G^Y\right]\frac{e^{-Mu}}{u^{1+Y}},$$

so that

$$\Pr(X_t > u) \overset{+\infty}{\sim} \tilde{C}(t)\ \frac{e^{-Mu}}{u^{1+Y}},$$

where

$$\tilde{C}(t) = \frac{Ct}{M}\quad e^{Ct\Gamma(-Y)\left[(G+M)^Y - M^Y - G^Y\right]}.$$

This confirms that the CGMY distribution tail is a mix of power and exponential functions, similar to the CGMY Lévy measure density.

In the case of a Variance Gamma process $Z = Z_1 + Z_2$ with the Lévy density

$$\nu(x) = C\frac{e^{Gx}}{(-x)}\mathbf{1}_{x<0} + C\frac{e^{-Mx}}{x}\mathbf{1}_{x>0},$$

where Z_1 is a gamma process describing the positive jumps and Z_2 is a gamma process describing the negative jumps, (Albin and Sundén 2009) showed that, when u is large

$$\Pr(X_t > u) \overset{+\infty}{\sim} \frac{M^{Ct-1}}{\Gamma(Ct)}\ \mathbb{E}\left[e^{MZ_2(t)}\right]\ \frac{e^{-Mu}}{u^{1-Ct}}.$$

From the computation of the moment-generating function $\mathbb{E}\left[e^{MZ_2(t)}\right]$ of the gamma process associated with negative jumps, we deduce that

$$\Pr(X_t > u) \overset{+\infty}{\sim} \frac{M^{Ct-1}}{\Gamma(Ct)}\quad \left(\frac{G}{G+M}\right)^{Ct}\ \frac{e^{-Mu}}{u^{1-Ct}},$$

so that

$$\Pr(X_t > u) \overset{+\infty}{\sim} \tilde{C}(t)\ \frac{e^{-Mu}}{u^{1-Ct}},$$

where

$$\tilde{C}(t) = \frac{M^{Ct-1}}{\Gamma(Ct)}\quad \left(\frac{G}{G+M}\right)^{Ct}.$$

Recall that

$$C = \frac{1}{v}, G = \left(\sqrt{\frac{\theta^2 v^2}{4} + \frac{\sigma^2 v}{2}} - \frac{\theta v}{2}\right)^{-1}, \; M = \left(\sqrt{\frac{\theta^2 v^2}{4} + \frac{\sigma^2 v}{2}} + \frac{\theta v}{2}\right)^{-1}.$$

Again, the distribution tail is a mix of power and exponential functions.

8.6.2 Alpha-Stable Distributions

In the case of an alpha-stable motion, described by the Lévy measure (8.19), we have, by extending Proposition 1.2.15 in (Samorodnitsky and Taqqu 1994) to an unconstrained time,

$$\Pr(X_t > u) \overset{+\infty}{\sim} \tilde{C}(t)\frac{1+\beta}{u^\alpha}, \tag{8.31}$$

and

$$\Pr(X_t < u) \overset{-\infty}{\sim} \tilde{C}(t)\frac{1-\beta}{(-u)^\alpha}, \tag{8.32}$$

where $\tilde{C}(t) = \frac{(1-\alpha)\quad \sigma^\alpha \quad t}{2\ \Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}$ in both situations, provided $\alpha \neq 1$.

8.6.3 Generalized Hyperbolic Distributions

Recall that generalized hyperbolic distributions (see, e.g., (Prause 1999)) have the density

$$f_{\text{HG}}(x) = a\frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\gamma^2+(x-\mu)^2}\right)e^{\beta(x-\mu)}}{\left(\gamma^2+(x-\mu)^2\right)^{\left(\frac{1}{2}-\lambda\right)/2}},$$

where a is defined by

$$a = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\gamma^\lambda\ K_\lambda\left(\gamma\sqrt{\alpha^2-\beta^2}\right)}.$$

From (Abramowitz and Stegun 1965), we have

$$K_v(z) \overset{\pm\infty}{\sim} \sqrt{\frac{\pi}{2z}}\quad e^{-z},$$

so that

$$f_{\text{HG}}(x) \overset{\pm\infty}{\sim} a\quad \frac{\sqrt{\frac{\pi}{2\alpha\sqrt{\gamma^2+(x-\mu)^2}}}\quad e^{-\alpha\sqrt{\gamma^2+(x-\mu)^2}}\quad e^{\beta(x-\mu)}}{\left(\gamma^2+(x-\mu)^2\right)^{\left(\frac{1}{2}-\lambda\right)2}}.$$

Because

$$\sqrt{\gamma^2+(x-\mu)^2} \overset{\pm\infty}{\sim} |x|,$$

we can write

$$f_{\mathrm{HG}}(x) \overset{\pm\infty}{\sim} a \frac{\sqrt{\frac{\pi}{2\alpha|x|}} e^{-\alpha|x|} e^{\beta(x-\mu)}}{|x|^{\frac{1}{2}-\lambda}},$$

so that, finally,

$$f_{\mathrm{HG}}(x) \overset{\pm\infty}{\sim} \tilde{C} \frac{e^{-\alpha|x|+\beta x}}{|x|^{1-\lambda}} \tag{8.33}$$

with

$$\tilde{C} = \frac{(\alpha^2-\beta^2)^{\lambda/2} e^{-\beta\mu}}{2\alpha^{\lambda}\gamma^{\lambda} K_{\lambda}\left(\gamma\sqrt{\alpha^2-\beta^2}\right)}.$$

This asymptotic expression corresponds to a mix of exponential and power functions. Consider the shape of the asymptotic density in the two particular cases below:

1. $\lambda = 1$: hyperbolic distribution. The asymptotic behavior of the density is

$$f_{\mathrm{H}}(x) \overset{\pm\infty}{\sim} \tilde{C}\ e^{-\alpha|x|+\beta x}$$

with

$$\tilde{C} = \frac{\sqrt{\alpha^2-\beta^2} e^{-\beta\mu}}{2\alpha\gamma K_1\left(\gamma\sqrt{\alpha^2-\beta^2}\right)}.$$

2. $\lambda = -1/2$: NIG distribution. The asymptotic behavior of the density is

$$f_{\mathrm{NGI}}(x) \overset{\pm\infty}{\sim} \tilde{C}\ \frac{e^{-\alpha|x|+\beta x}}{|x|^{\frac{3}{2}}}$$

with

$$\tilde{C} = \frac{\sqrt{\alpha\gamma} e^{-\beta\mu}}{2\left(\alpha^2-\beta^2\right)^{\frac{1}{4}} K_{-\frac{1}{2}}\left(\gamma\sqrt{\alpha^2-\beta^2}\right)}.$$

Compared to the alpha-stable distribution, whose tail is a power function, we observe here a faster asymptotic decrease due to the presence of an exponential function in the tail. The distribution tail remains, however, thicker than in the Gaussian case. Generalized hyperbolic distributions, similar to CGMY and VG distributions, show a medium asymptotic decrease and therefore represent a compromise between the Gaussian and alpha-stable distributions.

Table 8.4 concludes this chapter by presenting the main asymptotic behaviors of Lévy process marginals.

TABLE 8.4 Infinitely divisible distributions and asymptotic behavior

Basic distributions	Density function	Tail function	Laws of maxima
Gaussian	Exp. of parabola	Exp. of power	Gumbel
Hyperbolic	Exp. of hyperbola	Exp.-power mix	Gumbel
Trunc. α-stable		Exp.-power mix	Gumbel
α-stable		Power	Fréchet

8.7 CONCLUSION

This chapter started by recalling the important features of EVT. Then, it presented a synthetic account of infinite divisibility and Lévy processes. Further, it explained distinct studies of heavy-tailed and semi-heavy-tailed processes. Finally, it provided a recombination of all these elements and allowed us to understand the tail behaviors associated with the marginals of the most standard classes of Lévy processes.

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