The Extreme Value Problem in Finance: Comparing the Pragmatic Program with the Mandelbrot Program

Christian Walter

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Christian Walter [a*]

[a] *Fondation Maison des sciences de l'homme, Paris*

E-mail: christian.walter@msh-paris.fr

Abstract

This chapter gives an overview of the financial modelling of extreme values by using discontinuous stochastic Lévy processes. At least two distinct programmes using these processes are currently established in financial modelling: the first Mandelbrot programme based on stable Lévy processes and the alternative non-stable Lévy processes based approach. I term these two programmes the radical programme (RP) and the pragmatic programme (PP). At first, I use Sato's classification to contrast the two programmes. Next I adopt an historical perspective to present to the two programmes since 1960. The RP initiated huge controversies in the academic field because of the stable hypothesis. The PP began in the 1970s with explicitly renouncing the stable hypothesis. In the 1990s a new competitor appeared, called econophysics programme (EP). I show that, although the PP and the EP can be traced through separate lines in the academic fields, their shared the use of tempered stable processes and derive from their reliance on Mandelbrot's view. At the end, I suggest that Mandelbrot introduced the 'discontinuous turn' in financial modelling of extreme values.

Keywords: extreme values; Lévy processes; Mandelbrot; Financial modelling; Financial thought.

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1 The extreme value puzzle in financial modelling

The extreme value puzzle in finance has a long history (Longin, 1993; Fraga and Neves, 2015). As early as in the 1950s, one noticed that the price changes presented such phenomenon. For example, in a landmark paper published in the *Journal of the Royal Statistical Society*, Maurice Kendall wrote that the results from price data between 1883 and 1934 are such as "[t]he distributions are accordingly rather leptokurtic" (Kendall, 1953, 13). Seven years after, in the *Food Research Institute Studies*, Arnold Larson noticed that: "Examination of the pattern of occurrence of all price changes in excess of three standard deviations from zero (...) indicated that [there is] presence in the data of an *excessive number of extreme values*" (Larson, 1960, 224, our italics).

The mainstream view of the extreme values of distributions considered these data as irrelevant for financial modelling. For example, Granger and Orr (1972) asserted that "[i]f the long tails of empirical distributions are of concern to the time-series analyst or econometrician, it is natural to consider reducing the importance of these tails. The most obvious approach is to truncate the data". On the contrary, Benoît Mandelbrot viewed these extreme values as something extremely important for the understanding of market behaviour (Mandelbrot, 1962). But in the 1970s, the Gaussian distribution was the predominant tool used to describe the empirical distribution of returns and Brownian motion was just rediscovered by the founders of the 'modern' financial theory. Hence, at this time, it was not possible to take account of the tails of the distributions. I now elaborate this point, which is of a great importance for the history of financial thought, the origins of the multiple financial accidents since 1987 and the issue of extreme values I aim to address here.

I have argued that some of the key differences between the two approaches can be illuminated by reference to a familiar debate in philosophy over the principle of continuity (Le Courtois and Walter, 2014b). Although this philosophical debate may seem to be a scholastic preoccupation within a tight circle of specialists in philosophy of science, far from the financial stakes of modelling and with no impact on concrete financial practices, I argue, on the contrary, that the divergent positions about the mindset behind the price changes implicate entirely different views of what it is important to capture and how to model it. Let us emphasize this point.

In physics, the principle of continuity states that change is continuous rather than discrete. Leibniz

and Newton, the inventors of differential calculus, said "*Natura non facit saltus*" (nature does not make jumps). This same principle underpinned the thoughts of Linné on the classification of species and later Charles Darwin's theory of evolution (1859). In 1890, Alfred Marshall's *Principles of Economics* assumed the principle of continuity, allowing the use of differential calculus in economics and the subsequent development of neoclassical economic theory. In 1900, Louis Bachelier defends his thesis *Theory of speculation* in which price movements are modelled by Brownian motion (Bachelier, 1900). Modern financial theory grew out of neoclassical economics and naturally assumes the same principle of continuity. One of the great success stories of modern financial theory was the valuation of derivatives. Examples include the formulas of Fisher Black, Myron Scholes, and Robert Merton (1973) for valuing options, and the subsequent fundamental theorem of asset pricing that emerged from the work of Michael Harrison, Daniel Kreps, and Stanley Pliska between 1979 and 1981. These success stories rest on the principle of continuity. In the 20th century, both physics and genetics abrogated the principle of continuity. Quantum mechanics postulated discrete energy levels while genetics took discontinuities into account. But economics – including modern financial theory – stood back from this intellectual revolution.

The crux of my argument about the extreme value puzzle in financial modelling can be summarized in the following statement. There are two fundamentally different ways of viewing price changes in finance. One side of the debate, 'Leibnizian', takes the continuity as a cornerstone for financial modelling. In this view, following Bachelier's legacy, price movements are modelled by continuous diffusion processes. The contrary 'antiLeibnizian' position holds that discontinuity is crucial for grasping the true nature of price changes without intellectual cleavage. According to the second view, following Mandelbrot's legacy, price movements are modelled by discontinuous processes, as for instance Lévy processes. *Mandelbrot's pivotal move was to reconceptualize the discontinuity of price changes as empirical problem for financial modelling.*

With the presence of extreme value in financial time series, the Mandelbrot view might have been expected to start a new way of modelling price changes. But the early attempt by Mandelbrot to take explicit account of discontinuities on all scales in stock market prices led to huge controversies in the profession (Walter, 2009a). By the 1980s, the academic consensus reaffirmed the principle of continuity, despite the repeated financial crises following the 1987 stock market crash. The principle of

continuity was still predominant in the 1990s despite the growing evidence of extreme values in the tails of empirical distribution (Longin, 1996) mainly with the high-frequency data. Many popular financial techniques, such as portfolio insurance or the calculation of capital requirements in the insurance industry assumed that (financial) nature does not make jumps and therefore widely promoted continuity in the professional community (see however Bertrand and Prigent (2015) for a recent account of the CPPI method and EVT approach). Most statistical descriptions of time series in top journals in the field of finance assumed continuity. The Brownian representation still underlies almost all prudential regulation worldwide: for instance, the so-called square-root-of-time-rule underlying the regulatory requirements (Basle III and Solvency II) for calculating minimum capital is a very narrow subset of time scaling rule of risk, and comes directly from the hypothesis that returns follow a Brownian motion. I termed this Brownian mindset the 'Brownian virus' (Walter, 2009b), a notion inspired by Durkheim's writing (*Le suicide*, 1930), for suggesting a cognitive bias: a 'suicidogenic' school of thought creating an pathogenic environment (intellectual, social) that destroyed the prudent instincts of risk professionals and led to the financial meltdown of 2008.

In fact, one of the cognitive consequences of the Brownian virus is a negative spillover about the extreme value puzzle: the truncation of financial time series into "normal" periods (continuous market) and periods of "insanity" where markets are deemed "irrational" (extreme value periods). This dichotomy leaves the profession unable to explain the transition from one period to another. For example, in an editorial in the *Financial Times* (16.3.08), Alan Greenspan commented on the financial crisis of 2007-2008, "We can never anticipate all discontinuities in financial markets." For Greenspan, (financial) nature does not make jumps and extreme values are unpredictable outliers. This cognitive bias demonstrates the limits of risk management when considering the extreme value problem with a continuity Brownian based framework completed with an extreme value approach, and the need for a global discontinuous framework. If extreme value theory can efficiently help to choose a distribution for rare events (Longin, 2005), the problem of tackling discontinuities at all scales remains.

Brownian motion increments have the important property of being independent and identically distributed (hereafter IID). The processes with IID increments are called Lévy processes after the French mathematician Paul Lévy. Brownian motion is a specific Lévy process: it assumes continuity. Other Lévy processes don't. This chapter gives an overview of the financial modelling of extreme values and

discontinuities in the behaviour of stock market prices with Lévy processes. I present to the two main competitors for this stake: stable Lévy processes (Mandelbrot's first programme) and the non-stable Lévy processes based approach. I suggest that the non-stable Lévy based approach of discontinuities can be viewed as a 'pragmatic programme' launched in the 1970s against the Mandelbrot's first programme which I term 'radical programme'. I use Sato's classification to contrast the two programmes. Next I present the two competitors from an historical perspective. The pragmatic programme splitted in two branches: the heterodox financial modelling and the econophysics view. There are interesting parallels between them, which derive from their reliance on the radical programme.

The outline of the chapter is as follows. Section 2 introduces some fundamental notions from Lévy processes such as activity, variation and Lévy measure. Next I introduce Sato's classification to characterize the two programmes. Section 3 presents the Mandelbrot programme and discusses the related problems. Section 4 presents the pragmatic programme with is three stages: section 4.1 begins with mixed jump-diffusion processes in the 1970s; section 4.2 follows with infinite activity finite variation processes and infinite activity infinite variation processes in the 1990s. Section 5 concludes.

2 The Sato classification and the two programmes

This section presents in the simplest and most intuitive way possible the main characteristics of Lévy processes. Many books present a comprehensive view of these processes, as for example Bertoin (1996) and Sato (1999), for more details.

2.1 Lévy processes

To specify a Lévy process, there are two alternative routes: either to describe the marginal probability distribution of the process, i.e. the shape of the probability density function of the law, which describes the morphology of market uncertainty, considered from a static standpoint. Or to describe the Lévy measure, a mathematical object that captures the structure of the dynamics of jumps. The marginal probability distribution corresponds to a representation of uncertainty in the real world (here, the reality of the chance of the market, the reality of the stock price behaviour, the reality of the financial

Box 1. Characteristic function and characteristic exponent

The characteristic function of a random variable Y is the Fourier transform of its density function:

$$\Phi_Y(u) = \mathrm{E}\left[e^{iuY}\right] = \underbrace{\sum_{k=0}^{\infty} e^{iuk}\,\Pr(Y=k)}_{\text{discrete r.v.}} = \underbrace{\int_{-\infty}^{+\infty} e^{iux}\,f_Y(x)dx}_{\text{continue r.v.}}$$

where $f_Y(x)$ is the density function. The characteristic function can be written as:

$$\Phi_Y(u) = \exp\left(\Psi_Y(u)\right) \tag{1}$$

where $\Psi_Y(u)$ is the characteristic exponent of Y.

phenomenon), which can be used for real-world calibrations with market data; whereas the Lévy measure appears only in the transformed space of characteristic functions: the inverse Fourier transform of the probability density function. The characteristic functions can also be used as part of procedures for fitting probability distributions to samples of data. These two representations are equivalent for the specification of a Lévy process in the sense that knowing one of the functions always makes it possible to find the other. Both provide different insights for understanding the morphology of uncertainty in the financial world.

However, the two cannot be used indifferently. The probability density function does not always exist (closed form expression is not available); whereas the characteristic function of any infinitely divisible distribution always exists. Thus, for reasons of mathematical convenience, one uses the characteristic function and the characteristic exponent (see Box 1) to define in a simple way an infinitely divisible distribution and the Lévy processes corresponding to it. The characteristic function of a Lévy process has an equivalent meaning to the density function: it describes the *morphology of uncertainty* of the observed phenomenon.

The explicit form of the characteristic exponent of a stochastic process with IID increments was obtained in the most general case by Paul Lévy in 1934 and is the so-called Lévy-Khintchine formula (see Box 2). The Lévy measure was explicitly used in the models of the 1990s, whereas it was only

Box 2. Characteristic exponent and Lévy measure

The explicit form of the characteristic exponent of a random walk was obtained by Paul Lévy in 1934 for the processes with IID increments. This form is:

$$\Psi_{X_t}(u) = \underbrace{t\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right)}_{\text{DIFFUSION}} + \underbrace{t\int_{\mathbb{R}^*}\left(e^{iux} - 1 - iux\,\mathbf{1}_{|x|<1}(x)\right)\nu(dx)}_{\text{JUMPS}} \tag{2}$$

This is the Lévy-Khintchine formula, the interpretation of which is the following. The first term ('DIFFUSION') represents the normal distribution with the expectation μ and the standard deviation σ, the diffusive component of the stochastic process also termed the 'volatility' of the markets. The second term ('JUMPS') is the jump component of the stochastic process, which contains the Lévy measure $\nu(dx)$.

Heuristically, the Lévy measure provides the number of jumps per unit of time as a function of their size. It is the mathematical object which allows to quantify the jump arrival and the jump size. This is a key component of a Lévy process, which completely defines the structure of 'erraticity' of the market behaviour.

implicit in those of the 1970s, with the exception of Mandelbrot's model (1962, 1963), in which it appeared in an integrated form of the characteristic exponent.

A very important property of Lévy processes due to the IID property is that the characteristic exponent is proportional to the time duration: the marginal distributions of these processes are infinitely divisible (see Box 3). This means that a random variable can be understood as the sum of identical random variables, at any order. When modelling market uncertainty on whatever scale, one uses this property. The characteristic exponent at a given time t (uncertainty at time t) is easily obtained from the characteristic exponent at time 1 (uncertainty at time 1) following (see Box 3) the rule (8). This is one of the main attractions of Lévy processes, making them preferable to other types of model where the IID property does not hold.

2.2 Activity and variation of Lévy processes

One source of the extreme value puzzle is the interpretation of jumps in statistical descriptions.

2.2.1 Jumps? What jumps?

A stock price trajectory is by construction discontinuous; because it comprises jumps at all quote times. In the classic case of the Brownian representation of fluctuations, trajectories are continuous. The classic Brownian representation views quotes as points sampled in a continuous trajectory. In the case of a Brownian representation, quote jumps are proportional to the volatility of Brownian motion and hence it is not necessary to change representation. This statement correctly addresses the puzzle of jumps. In a given representation, are the points separated by distances that are consistent with the postulated model for paths? In the Brownian representation, are the observed jumps consistent with the diffusive nature of Brownian motion, or are they too large?

Strange as it may seem, this issue had not been tackled in finance literature until very recently. While normality tests have been well known for many years, it was not until the contributions of Aït-Sahalia and Jacod (2009) that appropriate tests for the detection of jumps were constructed, adding discontinuity tests to the classical toolbox of financial statistics.

Box 3. Characteristic exponent and infinitely divisible distributions

For any positive integer n, one has $X_n = X_1 + (X_2 - X_1) + \cdots + (X_n - X_{n-1})$ and therefore:

$$\begin{aligned} \Phi_{X_n}(u) &= \mathrm{E}\left[e^{iuX_n}\right] && (3) \\ &= \mathrm{E}\left[e^{iu(X_1+(X_2-X_1)+\cdots+(X_n-X_{n-1}))}\right] && (4) \\ &= \mathrm{E}\left[e^{iuX_1} \times e^{iu(X_2-X_1)} \times \cdots \times e^{iu(X_n-X_{n-1})}\right] && (5) \end{aligned}$$

If X is a Lévy process, the increments are IID. From the independence of the increments, it follows that:

$$\Phi_{X_n}(u) = \mathrm{E}\left[e^{iuX_1}\right] \times \mathrm{E}\left[e^{iu(X_2-X_1)}\right] \times \cdots \times \mathrm{E}\left[e^{iu(X_n-X_{n-1})}\right]$$

From the stationarity of the increments, it follows that:

$$\Phi_{X_n}(u) = \underbrace{\mathrm{E}\left[e^{iuX_1}\right] \times \mathrm{E}\left[e^{iuX_1}\right] \times \cdots \times \mathrm{E}\left[e^{iuX_1}\right]}_{n \text{ times}} = \left(\mathrm{E}\left[e^{iuX_1}\right]\right)^n$$

Finally one obtains:

$$\Phi_{X_n}(u) = \left(\Phi_{X_1}(u)\right)^n \quad (6)$$

This result can be generalized in the case of a continuous time t (real positive) to:

$$\Phi_{X_t}(u) = \left(\Phi_{X_1}(u)\right)^t \quad (7)$$

Now one considers the characteristic exponent. It follows from (7) that:

$$\exp\left(\Psi_{X_t}(u)\right) = \left(\exp\left(\Psi_{X_1}(u)\right)\right)^t = \exp\left(t\Psi_{X_1}(u)\right)$$

Hence the relationship between the characteristic exponent of the process X at time 1 and at time t is:

$$\Psi_{X_t}(u) = t \times \Psi_{X_1}(u) \quad (8)$$

This is the reason why the characteristic exponent of a Lévy process is equal to t times that of its underlying infinitely divisible distribution, which is in fact the distribution X_1.

2.2.2 Shaping the path irregularity in a IID framework with the Lévy measure

With a Lévy process, each trajectory is by definition discontinuous everywhere. Intuitively, the greater the number of jumps per time unit, the more the trajectory of the stochastic process will have a high degree of irregularity and the more erratic the random walk will be. Hence a stochastic process will be highly erratic if the average number of jumps occurring per unit of time is very large. The average number of jumps per unit of time defines the so-called 'intensity' of a Lévy process – also termed 'activity' by analogy with turbulence. The activity can be finite or infinite.

The rudimentary example of Box 4 shows that, in order to 'isolate' the activity of a Lévy process, it is sufficient to calculate the integral of the Lévy measure. This integral may be either finite or infinite. In all cases in which one constructs a compound Poisson process with another distribution, the number of jumps per unit of time (the occurrence rate of discontinuities) is finite and the resulting Lévy process is of finite activity. In this situation one can clearly separate the activity from the density. When the activity is finite, the product activity-density is the Lévy measure.

It makes sense to generalize this approach for moving from finite to infinite activity. Indeed there is no reason why the average number of small jumps per unit of time should stay finite. The advantage of generalizing in this way is that the very many small market movements can be taken into account. In the case of infinite activity, it is no longer possible to separate the activity λ from the density f. Both are "mixed" in the Lévy measure, which entirely shapes the morphology of the irregularity of the financial phenomenon. The activity is 'isolated' in the same way that in the simple previous example.

Let us consider now the average distance between two points of the process. The average distance can be finite or infinite (the mean may or may not exist). This idea of average distance corresponds to what is called the variation of a Lévy process. The variation may be finite or infinite. The variation is another feature of the morphology of financial uncertainty.

Let us summarize what has been presented so far. A Lévy process is fully defined by the specification of three quantities: the mean of the diffusive component (the trend of the process), the diffusion coefficient (the *scale of fluctuations*) and the Lévy measure (the *morphology of uncertainty*).The role of the Lévy measure is decisive. It contains all the information needed to characterize the trajectory of a Lévy process, apart from its tendency and its diffusive fluctuation scale ('volatility'). It is the quantity

Box 4. The Lévy measure in simple cases with Poisson processes

Let us consider a Poisson process with parameter λ: the average number of jumps per unit of time – the activity of the process – is simply λ. Let us continue with this simple example to get an intuitive idea of what the Lévy measure is. Whenever a jump in the Poisson process occurs, suppose that the magnitude Y of the jump is random with known density $f_Y(x)$. Let us note $\nu(x)$ the product:

$$\nu(x) = \lambda \times f_Y(x) \tag{9}$$

This product is the Lévy measure. One sees that the Lévy measure captures both the occurrence rate of discontinuities and their magnitude: the product (9) fully characterizes the jump structure of the process. One observes that the integral of the Lévy measure is equal to λ:

$$\int_{-\infty}^{+\infty} \nu(dx) = \lambda \times \underbrace{\int_{-\infty}^{+\infty} f_Y(x)dx}_{=1 \text{ (density)}} = \lambda = \text{activity}$$

Hence the integral of the Lévy measure provides the activity of the process.

Suppose that the distribution of jumps is normal with a mean μ_Y being the average size of jumps and a standard deviation σ_Y being the volatility of the size of jumps (a compound Poisson process with a normal distribution or 'compound Poisson-Normal' process). Hence, following (9), the Lévy measure is:

$$\nu(x) = \lambda \times \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_Y)^2}{2\sigma_Y^2}\right) \tag{10}$$

In the simple case of a compound Poisson process with any probability distribution Y with known density $f_Y(x)$, the Lévy measure is just the product:

$$\text{activity} \times \text{density}$$

that shapes the size of the tails of distribution, and the patterns of jumpy fluctuations.

$$\text{Characteristic exponent} = \text{Trend} + \text{Scale} + \text{Morphology}$$

The significance of the new approach adopted in the 1990s came precisely from this possibility of defining any market dynamics with irregularities at all scales by direct specification of the Lévy measure. Thus, the dynamic of stock prices being any Lévy process, the representation of market fluctuations shifted, in the 1990s, from exponentials of Brownian motions to exponentials of Lévy processes.

Another consideration also favoured Lévy processes. Lévy processes are semi-martingales. The work of Ross, Harrison and Pliska between 1976 and 1981 on arbitrage showed that the arbitraged prices of securities ought to be capable of being modelled by semi-martingales. Thus for these reasons applying both to financial modelling and to the technique of stochastic calculus, the Lévy processes disinterred in the early 1990s, after a decade of growing maturity in financial thinking around the theory of arbitrage and the usefulness of intrinsic market temporality, appeared extraordinarily well adapted to the new way of conceiving the modelling of arbitraged markets, whether in calendar time or market time. The match between the most modern finance (absence of arbitrage) and the development of working techniques on Lévy processes was pivotal for the introduction of these processes into financial research.

2.3 The two programmes in the light of Sato's classification

In brief, three alternatives exist for shaping financial uncertainty with Lévy process: either the activity is finite or infinite, or the variation is finite or infinite, and the variance can be finite or infinite. The Sato (1999) classification defines a process by its pair (activity, variation) according to the double criterion finite or infinite. In the pragmatic programme, the variance is finite, and in the radical programme (Mandelbrot's view) the variance is infinite. It appears that there are four types of stochastic processes depending on whether their activity and their variation are finite or infinite, and whether variance is finite or infinite. Table 1 below exhibits the Lévy processes in financial modelling following this double criterion, the variation being that of the jump part of the stochastic process.

In general, the models of the late 1990s and early 2000s used processes with infinite activity and

infinite variation but with finite variance. As an outlaw, the Mandelbrot model (1962) comprised infinite activity, infinite variation and infinite variance.

Table 1: The Sato classification, the variance issue and the financial modelling programmes

Activity	**Variation**	**Variance**	**Financial modelling**	**Programme**
Infinite	Infinite	Infinite	Mandelbrot (1962)	Radical
Finite	Finite	Finite	Press (1967), Merton (1976) Cox and Ross (1976)	Pragmatic stage 1
Infinite	Finite	Finite	Madan and Seneta (1990) Madan and Milne (1991)	Pragmatic stage 2
Infinite	Infinite	Finite	Eberlein and Keller (1995) Barndorff-Nielsen (1997) Eberlein Keller Prause (1998) Madan Carr Chang (1998) Prause (1999) Carr Geman Madan Yor* (2002, 2003) *(depends on exponent)	Pragmatic stage 3

3 Mandelbrot's programme: a fractal approach

The initial idea of discontinuities at any scale of the observation of markets behaviour came from Mandelbrot (1962, 1963). How to name financial markets that are 'continually discontinuous'? Mandelbrot felt that the name should reflect the fractured nature of the paths representing price changes. He coined the term "fractal" (from the Latin *fractus*, meaning fractured) to characterize discontinuities at all scales.

He developed his main ideas in a series of significant papers published in French. Some of these texts are translated in English, modified and reprinted in his 1997 *Fractals and Scaling in Finance*.

In this chapter, I term 'Mandelbrot's programme' the research programme described in these papers corresponding to Mandelbrot (1966, 1967, 1973a, 1973b, 1973c). The three main concepts of the first research programme for financial modelling are summarized in Walter (2015): fat tails, long range dependence and intrinsic time. A second strand of papers came after his 1997 book, corresponding to Mandelbrot (2001a, 2001b, 2001c), which build on the generalized multifractal model put forward in 1997, outside the IID framework. This model was the result of the Mandelbrot's coming back to finance after the 1987 crash. The three papers of 2001 echoed the three of 1973 and represent a second stage of the programme, moving from unifractals to multifractals. Ultimately, multifractal modelling allows a comprehensive view of discontinuities with bypassing the limitations of the first approach.

The idea of discontinuity in price variations at any scale is closely related to the scaling view of price fluctuations (the 'fractal description of markets'), a current of thought which is initially entangled with the 'chartist' approach to markets, before being adequately mathematicised with fractals. I now elaborate this point.

3.1 The fractal view of price behaviour

Stock market charts representing changes in the stock prices over a given period of time look like irregular patterns that seem to be reproduced and repeated in all scales of analysis. Rising periods follow periods of decline and the rises are punctuated with intermediate falling phases and falls are interspersed with partial rises, and this goes on until the quotation scale limit is reached.

This mixture of repetitive patterns of rising and falling waves at all scales was Ralph Elliott's (1938) intuition, to whom this idea occurred while observing the ebb and flow of tides on the sands of the seashore. From this, he coined a financial symbolization known as 'stock market waves' or 'Elliott's waves', which he subdivided into huge tides, normal waves and wavelets. In mathematical terms, the so-called Elliott wave principle presents a deterministic self-similar fractal description of stock markets with self-similar geometric patterns found on all scales of observations.

The fractals of Mandelbrot, though developed in a radically different intellectual context, fit in this understanding of stock market variations. It presents, like the common view with Elliott's waves, a method for disentangling the inextricable interlacing of stock markets moves at all scales. Fractals rep-

resented an adequate conceptualization allowing the translation of intuitions of technical analysts into rigorous mathematical representation, because this mathematics deals with two financial stylized facts: *discontinuity* and *scaling*. The notion of "roughness" addresses these facts by creating a strange nexus between two seemingly disparate cases: discontinuity and scaling. Random fractal curves adequately mimic stock market charts. In the following section, I elaborate on this.

3.2 Fractal modelling: discontinuity and scaling

Despite the promising results opened up with this new way of thinking financial modelling, the adventure of fractal modelling in finance does not display a smooth (continuous) history. It is more an eventful (discontinuous) progression of Mandelbrot's assumptions through the evolution of finance theory over forty years, from 1960 until 2000. In a paper summarizing the Mandelbrot's state of research programme in the 1980s, Mirowski (1995) observed that "the economics profession dropped the Mandelbrot hypothesis largely for reasons other than empirical adequacy and concise simplicity. [...] The only purpose of the negative studies was to refute Mandelbrot". In his admirable book about the development of financial economics, MacKenzie (2006) stressed this point by hypothesizing that Mandelbrot's model was viewed by the financial academic community as a probability 'monster'. Let us have a closer look on this point.

3.2.1 The extreme values and the Stable model

At the origin of the fractal modelling in finance is the "leptokurtic phenomenon", i.e. the presence of fat tails due to extreme values in the empirical distributions of returns. Mandelbrot's idea for suggesting the simplest generalization of Brownian motion that takes account of the extreme values was to put forward the simplest process, which was, in this sense, stable by addition, the alpha-stable motion. But the price to pay was the abandonment of finite variance because the variance of alpha-stable motion is infinite. This infiniteness of a crucial financial quantity which just arose in the new models for portfolio management (Markowitz, 1952, 1959) and option pricing (Black, Scholes and Merton, 1973) was seen as horrific by the academic mainstream of the 1970s. For example, one can find in a textbook that "many researchers find the conclusion of infinite variance unacceptable" (Taylor, 1986).

On the other hand, there was a lack of statistical tools to tackle the estimation of the parameters of stable distributions. For example, Fama (1965) wrote that:

> "economic models involving stable Paretian generating processes have developed more rapidly than the statistical theory of stable Paretian distributions. It is our hope that papers like this will arouse the interest of statisticians in exploring more fully the properties of these distributions"

In a reference textbook on one-dimensional stable distributions, Zolotarev (1986) echoes Fama by saying that "it can be said without exaggeration that the problem of constructing statistical estimators of stable laws entered into mathematical statistics due to the work of Mandelbrot".

An important point to be understood is the following. It is possible to tackle the extreme value puzzle with models other than alpha-stable motions, indeed an unlimited number of models. But if one wants to keep the IID hypothesis and have non-Gaussian tails with scaling property (Brownian motion) the only alternative is the alpha-stable motion. The controversies resulting from the leptokurtic phenomenon and extreme values of the distributions became entangled in the intrication of the static approach (Gaussian or non-Gaussian) and the dynamic approach (Brownian or non-Brownian). In the 1970s, the debates ignored the stochastic process issues and concentrated on the extreme values of the distributions[1].

3.2.2 The rejection of the Stable model

A review over forty years of searching for scaling laws in distributional properties of price variations (Walter, 2009a) exhibits a turbulent story with fierce controversies which stirred up the academic community with regard to the continuous / discontinuous debate. So strong was the opposition to Mandelbrot's hypothesis that any kind of alternative model was preferred to the idea of infinite variance as embedded in the alpha-stable motion proposed by Mandelbrot. For example, in a paper devoted to

[1]It is worth noting that this intrication is sometimes a source of confusion in the existing literature of historical financial thought, based on an analysis which does not distinguish between so-called "Lévy distributions" (actually stable distribution with Pareto tail) and Lévy processes (actually stochastic processes with IID increments). Here the semantics is misleading.

the statistical properties of exchange rates, Elie *et al.* (1993) wrote that "ARCH models allowed us to solve largely this problem of heavy tails of distributions while keeping a Gaussian framework which turns out more tractable than that of stable laws" (our translation).

Underpinned by the desire to reject fractality ("anything but Mandelbrot" or "ABM" because Mandelbrot was working at IBM, was the watchword of the pro-continuity approach activists), the debates shifted to the testing of the alpha-stability-under-addition-property. They ended with the empirical rejection of fractal modelling for distributional properties returns in the 1970s because the scale invariance principle was found too strong for adequately modelling price variations. The alpha-stable Lévy processes were abandoned by the mainstream.

3.2.3 The tempered stable family in econophysics

To remedy the inconvenience of not having any moments for the alpha-stable models of the Mandelbrot's programme, other models were developed with a truncation principle. In the alpha-stable models, the Lévy measure displays a power law which produces Paretian tails (see Box 5). This power law is precisely the origin of the non-existence of the moments when the Paretian exponent is less than 2. A simple way of avoiding this problem is to weight the Lévy measure by an exponential quantity in order to reduce large fluctuations and therefore recover the moments. This idea corresponds to a class of Lévy processes whose marginal distributions are truncated stable distributions, so-called "tempered stable" models. The stable distributions are truncated by exponential functions; hence the term tempered stable processes. The distribution tails of these models, tempered by the truncation, are semi-light.

In the 1990s, physicists began to propose such models combining truncated alpha-stable distributions with exponential tails (Mantegna and Stanley, 1994; Koponen, 1995; Bouchaud and Potters, 1997) and physicist research activity enters the financial modelling field. As Mantegna and Stanley (2000) noticed, "since 1990, a research community has begun to emerge". This new community baptised itself with the name "econophysics". Hence in the 1990s, research in financial modelling then split into two separate communities: that of financial academics – the mainstream – and that of physicists – the heterodox view known as "econophysics". Physicists continued along the way paved by Mandelbrot's model, working in particular with the scaling concept: as Mantegna and Stanley (2000) pointed out, financial academics were "trying to determine a characteristic scale for a problem that has no

characteristic scale".

While physicists launched this new strand of research, mathematical financial academics then moved to the development of Lévy processes, following the first jump-diffusion type models of the 1970s that were developed to tackle the discontinuity issue in the framework of the finiteness of the second moment. I term this mainstream strand of research the "pragmatic programme". The pragmatic programme opened the first period of model tinkering in financial modelling: a situation in which researchers, confronted with descriptive inadequacy, decide to 'repair' existing models with new data-driven approach. The story of jump processes represents an illustration of this data-driven model tinkering. I will now elaborate on the pragmatic programme.

4 The pragmatic programme: a data-driven approach

I begin this section with this excerpt from a paper by Applebaum (2004):

> "A sociologist investigating the behaviour of the probability community during the early 1990s would surely report an interesting phenomenon. Many of the best minds of this (or any other) generation began concentrating their research in the area of mathematical finance. The main reason for this can be summed up in two words: option pricing."

The Black-Scholes model is based on the assumption that returns from the underlying assets follow a diffusion-type process, in particular a geometric Brownian motion. A large number of empirical studies showed that this model was inadequate, partly because of the continuity assumption. For example, Merton (1976) admitted that "there is a prima facie case for the existence of jumps" and Cox and Ross (1976) agreed that "exploring alternative forms is useful to construct them as jump processes". Ball and Torous (1985) pointed out that "empirical evidence confirms the systematic mispricing of the Black-Scholes call option pricing model" and "the Merton model which explicitly admits jumps in the underlying security return process may potentially eliminate theses biases". The goal of the pragmatic programme was precisely to overcome these inadequacies by tackling the issue of discontinuities without accepting Mandelbrot's programme. For example, Carr *et al.* (2002) said that they "seek to replace this process with one that enjoys all of the fundamental properties of Brownian

Box 5. The Lévy measures of the radical and pragmatic programmes

The Stable model / radical programme proposed by Mandelbrot (1962) has the characteristic exponent:

$$\Psi_{X_t}(u) = t\left(i\mu u + \int_{-\infty}^{0} \psi(u,x)\,\frac{C_-}{|x|^{1+\alpha}}dx + \int_{0}^{+\infty} \psi(u,x)\,\frac{C_+}{x^{1+\alpha}}dx\right) \tag{11}$$

with:

$$\psi(u,x) = e^{iux} - 1 - iux\,\mathbf{1}_{|x|<1}(x)$$

From (2) and (11), it follows that the Lévy measure of the α-stable motion is:

$$\nu(dx) = \frac{C_-}{|x|^{1+\alpha}}\mathbf{1}_{(-\infty,0)}(x)dx + \frac{C_+}{x^{1+\alpha}}\mathbf{1}_{(0,+\infty)}(x)dx \tag{12}$$

or in a more intuitive form:

$$\nu(x) = \begin{cases} \frac{C_-}{|x|^{1+\alpha}} & \text{if} \quad x<0 \\ \\ \frac{C_+}{x^{1+\alpha}} & \text{if} \quad x>0 \end{cases} \tag{13}$$

This last form (13) exhibits a power Paretian law in the Lévy measure. This precisely allows to understand the Mandelbrot's intuition : *the search for Paretian tails for solving the extreme value issue*. A tale of fat tails, he named later "the power of power laws" (Mandelbrot, 2004, 13).

	Lévy measure for:			
Model	$x<0$	$x>0$	**Programme**	Field
Mandelbrot (1962)	$\frac{C_-}{\lvert x\rvert^{1+\alpha}}$	$\frac{C_+}{x^{1+\alpha}}$	RADICAL	Finance
Koponen (1995)	$\frac{C_-}{\lvert x\rvert^{1+\alpha}}\,e^{-a_-\lvert x\rvert}$	$\frac{C_+}{x^{1+\alpha}}\,e^{-a_+ x}$	PRAGMATIC	Econophysics
Carr *et al.* (2002)	$\frac{C}{\lvert x\rvert^{1+\alpha}}\,e^{-a_-\lvert x\rvert}$	$\frac{C}{x^{1+\alpha}}\,e^{-a_+ x}$	PRAGMATIC	Finance

In the Koponen (1995) and Carr *et al.* (2002) models, one sees that the stable distributions are truncated by exponential functions. These models are named tempered stable processes. The distribution tails of these models, because of tempered by the truncation, are semi-light. The parametrization is asymmetric in the decay rate of large jumps.

motion, except for pathwise continuity and scaling, but that permits a richer array of variation in higher moment structure, especially at shorter horizons". This will be achieved with a "non-Gaussian Merton-Black-Scholes Theory" (Boyarchenko and Levendorskii, 2002), which gained official recognition.

This section provides a brief overview of the pragmatic programme by following the evolution of the modelling of jump processes, from the rediscovery of Poisson's law in finance by S. James Press in 1967 through to the Lévy infinite activity processes of the 2000s. It came in two major stages. First, with the rediscovery of Poisson's law in the late 1960s, a jump component was added to the diffusion process (Brownian motion): this superposition of jump and diffusion processes opened the period of hybrid models known as jump diffusion-processes (1970-1990), which state that prices undergo large jumps followed by small continuous movements. These models were initiated by Press (1967) and Merton (1976). It is a simple case of Lévy process with finite activity and finite variation in the jump component. This is the first stage of the pragmatic programme. Then, in the second period, the diffusive component was removed leaving only the jump component, moving to Lévy processes keeping finite variation in the jump component but with infinite activity. This is the second stage of the pragmatic programme. The third stage of the pragmatic programme corresponds to the infiniteness of the variation, itself divided into two subgroups, according to the finiteness of infiniteness of variance.

In contrast to these pragmatic programmes, the radical programme proposed by Mandelbrot in 1962 had both infinite activity, infinite variation in the jump component and infinite variance. It was – for this reason – a heterodox view. A convenient way to grasp the conceptual difference between the framework of Mandelbrot's first representation (1962) and that of Press' (1967) successors is to consider the intuition underlying the modelling of trajectory discontinuities by jump-diffusion processes: the invalidation of the stability-under-addition-property, one of the cornerstones of Mandelbrot's models, precisely the scaling view of markets (fractal nature) embedded in the stability-under-addition-property.

This first approach to jump-diffusion processes, initially limited to Lévy processes with finite activity and finite variation, was generalized and fully developed only in the 1990s: the second life of Lévy processes belongs to the late twentieth century.

4.1 The Jump-diffusion models in the 1970s

The emergence of the pragmatic programme was prepared for a long time by research around Poisson's law and process. At first, I present the rediscovery of this law in finance. Next I present the two first stages of the pragmatic programme.

4.1.1 The rediscovery of Poisson's law in financial modelling

The issue of the explicit modelling of jumps (discontinuities) was well-known to insurance companies as early as 1903. In the context of managing their contracts, insurance companies had used the Poisson process to model the assessment of claims in non-life insurance. Lundberg's thesis of 1903 on insurance risk theory was the equivalent of Bachelier's theory of risk quantification in finance: Bachelier's (1900) Brownian model corresponded to Lundberg's (1903) Poisson model. Subsequently Harald Cramer and the Stockholm school introduced Lundberg's ideas into the theory of random processes, resulting in the so-called Cramer-Lundberg actuarial model.

If the Gaussian and Brownian motion constituted the mathematical basis of classical financial modelling, Poisson's law and process were their counterparts in traditional actuarial models. Brownian motion and the Poisson process are two examples of simple Lévy processes. When researchers tried to model the discontinuity of stock paths with a non-stable approach, this law and these processes emerge as the most "natural" candidates for the production of heavy tailed distributions, since Poisson's law precisely creates these tails. The Poisson framework appeared as the first response of financial economics mainstream to Mandelbrot's programme.

Thus in 1967 the Cramer-Lundberg actuarial model made its entry into finance. In that year, five years after Mandelbrot, to tackle the jumpy nature of the price process, Press' proposition provided, for the first time in financial modelling, a non-stable generalization of Bachelier's model, by complementing the Brownian continuous diffusive component with a discontinuous Poisson component (Press, 1967). This innovation was able to produce a representation of the morphology of static uncertainty with a non-Gaussian distribution tail, a tail resulting from the introduction of the Poisson law.

Poisson's formula enables us to determine the probability of the occurrence of infrequent events (sometimes called rare events), provided that we know the constant average frequency at which these

events occur. This frequency is described by the parameter of the Poisson distribution. Imagine, for example, that we consider a trajectory discontinuity as a jump. One easily sees to what extent the Poisson process is applicable in financial modelling: this process includes moments of jumps, the amplitude of which then simply has to be modelled. The combination of a Poisson process (for periods of jumps) and any law of distribution (for the size of jumps) produces what is called a compound Poisson process, that is to say, a process where the jumps occur at times coming from a simple Poisson process and have a determinate size. The choice of the probability law of the size fitting the possible values of this amplitude will then constitute the second stage of modelling.

If one chooses a Gaussian distribution to model the size of jumps, one will obtain a structure combining a Poisson process and a Gaussian distribution, also called the normal compound Poisson process. But it is possible to choose any probability law for the distribution of the size of jumps, such as a power law, a Gamma distribution, a Pareto law, and so on. Any distribution can be arbitrarily used for modelling the amplitude of discontinuities, coupled with the Poisson counting process. This linkage will then produce a compound Poisson process with these other laws (exponential Poisson, Gamma Poisson, etc.). It is this insight that underlies the representation of market discontinuities by jump processes.

4.1.2 The mixed jump-diffusion processes

However, the Poisson component is not sufficient to model all market changes since, with this pure Poisson representation, nothing happens between two jumps: the market remains inert, except when it jumps. It is therefore necessary to supplement it with another model. In the 1960s and 1970s, the only way to model this change in the market, perceived as "smoother", between two jumps was to opt for a Brownian motion. That's why one added a Brownian component to the Poisson component, and this linear combination of a compound Poisson process and Brownian motion corresponds precisely to Press' (1967) model. This model is thus presented as a simple juxtaposition of a process producing a very large number of small stock market fluctuations (Brownian motion) and a process of producing a small number of market discontinuities (the normal compound Poisson process). These two basic building blocks processes are completely separate ("orthogonal"). Thus it is a mixed process involving diffusion and jumps, termed mixed jump-diffusion.

As the increments of the mixed jump-diffusion processes are IID, mixed jump-diffusion processes are Lévy processes. These are special cases of general non-stable Lévy process. The mixed Press model thus represents the first introduction of non-stable Lévy processes into finance. These processes had already been highlighted by Samuelson in 1965, echoing the work of Mandelbrot, but without giving rise to an explicit use, since Samuelson preferred returning to the usual Brownian motion model.

The values of the Poisson parameter λ (average number of jumps per unit of time) allow us to localize the mixed jump-diffusion processes in relation to the Bachelier and Mandelbrot models. By characterising these two models by the number of jumps occurring during the evolution of market prices, i.e. by the Poisson parameter, the value of $\lambda = 0$ (no jumps) leads back to the Bachelier model, and the value of $\lambda = \infty$ (infinite number of jumps) leads to the Mandelbrot model. Between these two values (0 and infinity), any finite value of λ results in a finite number of jumps between two given quotes. There are an infinite number of possible mixed jump-diffusion processes, all filling the range between the Bachelier and Mandelbrot representations. The Press model thus represented an intermediate solution between Bachelier and Mandelbrot.

In the first stage of the pragmatic programme, the market dynamics of a given stock resulted simultaneously from frequent small movements, forming the continuous part of its trajectory and resulting from the Brownian diffusive component of the process, and from less frequent sudden movements forming the discontinuities of its trajectory, stemming from the Poisson component of the process. As Merton (1976) said, "the total change in the stock price is posited to be the composition of two types of changes: diffusion and jumps. The natural prototype process for the continuous component of the stock price change is a Wiener process, so the prototype for the jump component is a 'Poisson-driven' process". Again Cox and Ross (1976) stated that "in contrast to the diffusion process, the jump process [introduced] follows a deterministic movement upon which are superimposed discrete jumps".

From a financial standpoint, the mixed jump-diffusion processes modelled the fluctuation risk of any asset in terms of two dimensions: (classic) volatility risk corresponding to the Brownian diffusive component and a (new) jump risk corresponding to the Poisson component. This innovation was important because it indicated to professionals that usual risk diversification on the basis solely of the volatility dimension was not sufficient to protect against adverse stock market fluctuations. The market risk of any asset was therefore at least two-dimensional. The second component of risk, or

Box 6. Financial modelling with jump-diffusion processes (1967-1976)

The classical models of market behaviour implement the sole Brownian representation, i.e. the diffusive part of the Lévy processes. Let X being the cumulative return of a given asset. In these first jump-diffusion processes, the cumulative return X results simultaneously from a large number of small variations (Brownian part) and a small number of large variations (Poissonian part). In theses simple cases of Lévy processes, the activity and the variation remain finite.

The characteristic exponent of market dynamics in the classical Brownian models is:

$$\Psi_{X_t}(u) = t\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right) \tag{14}$$

Adding a Poissonian part to this Brownian component leads to:

$$\Psi_{X_t}(u) = t\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right) + t\,\lambda(\Phi_Y(u) - 1) \tag{15}$$

where λ is the Poisson parameter. The sizes of the jumps are independent and their stationary distribution Y has a characteristic function Φ_Y.

jump risk, was soon seen to be non-diversifiable, as became apparent from the work undertaken on the equity valuation models (Jarrow and Rosenfeld, 1984) and on the term structure of interest rates (Ahn and Thompson, 1988). This jump component creates a specific uncertainty as regards the risk usually measured by volatility.

The impossibility of perfect hedging for this type of risk was no doubt an obstacle to the widespread use of these mixed jump-diffusion processes in financial engineering for some fifteen years. Note that the use of alpha-stable motions also implied the need to take into consideration a second dimension of risk, namely jump risk. Because this second dimension of risk was not taken up in financial circles, it can be assumed that the professional community was not sufficiently mature in the 1970s to manage financial products with two risk dimensions.

4.2 Pure jump models in the 1990s

The rebirth of the random walk model in finance is due to the rediscovery of two important characteristics of Lévy processes. First, in order to describe the jumping behaviour of various asset prices and interest rates, it became clear that the use of Lévy processes with infinite activity was sufficient. Hence, it was no longer necessary to build superpositions of jump and diffusion process (Brownian motion) in price dynamics equations, namely what was called jump-diffusion processes (special case of very simple Lévy processes) in the 1970s. Second, it was rediscovered that any Lévy process has an interesting relation to the Brownian motion, considering the morphology of uncertainty. Using a subordinator process for measuring time that increases with a randomly varying speed, any Lévy process in calendar time (physical time) can be written as Brownian motion measured in a time distorted by the pace of trading. The randomly increasing time has been interpreted as an operational time or a trading time reflecting the market activity. The fact that a Lévy process can capture the time change of the markets opened a new strand of research about the nature of intrinsic time in markets. At the end, the random walk model is released from the prison of the Brownian representation in calendar time in which it was trapped, and becomes a powerful tool for financial modelling using these two characteristics that foster the understanding of market price behaviour: infinite activity and the distortion of time. I now elaborate on this.

4.2.1 Financial modelling with infinite activity

The separation between the two sources of market movements – the Brownian source, forming the continuous part of the trajectories, and the Poisson source, creating discontinuities – was simple and convenient, but limited the possibilities for modelling. Moreover, as one has seen, even those changes perceived as continuous (between two jumps) could be represented differently, since share quotes are by definition discontinuous, with the tick defining the smallest time interval between two quotes. The notion of discontinuity is essential for modelling stock market variations. In other words, the intrinsic bumpiness of the financial phenomenon did not require the diffusive Brownian part of models to be retained. It was necessary only to be able to account variously for a very large number of very small jumps (ticks), a large number of larger jumps, and a very small number of very large jumps (market discontinuities), to obtain a relevant model of stock market functioning. The probabilistic representation of market fluctuations did not ultimately entail the use of the Brownian diffusive component.

This idea slowly made its way into the academic community, up to the early 1990s. The diffusive part of probabilistic representations had been needed for the modelling of the small movements only in the case of finite activity: the finite activity of the process required the addition of another component. But as soon as it was admitted that infinite activity was possible, the usefulness of the diffusive component disappeared and a pure jump process seemed to be sufficient to represent the entire stock market phenomenon, i.e. its bumpiness at all scales. The argument is well described in the paper by Peter Carr, Hélyette Geman, Dilip Madan and Marc Yor published in 2002:

> "The rationale usually given for describing asset returns as jump-diffusions is that diffusions capture frequent small moves, while jumps capture rare large moves. Given the ability of infinite activity jump processes to capture both frequent small moves and rare large moves, the question arises as to whether it is necessary to employ a diffusion component when modelling asset returns"

These studies and those that followed mark the turning point in the modelling of jumps processes in finance, confirming their disembeddedness from Brownian representation, even if complemented by compound Poisson processes as in the case of the mixed jump diffusion processes of the 1970s.

Let us summarize. By adopting a representation of market fluctuations using an infinite activity

Lévy process, it appeared possible in the 2000s to manage without any diffusive component. The structure of trajectory discontinuities (the morphology of the bumpiness of the stock market phenomenon) is fully characterized by Lévy's measure. Compared to the mixed jump-diffusion processes that followed the path opened up by Press and Merton, this new representation of small market fluctuations was instead situated in the tradition of normal compound Poisson-type pure jump processes, as proposed by Cox and Ross in 1976 for evaluating options in markets with trajectory discontinuities. In jump-diffusion processes, jumps are considered as rare events. In Lévy processes with infinite activity, jumps are always present at any scale of the fluctuations.

4.2.2 The generalized hyperbolic family

The first studies focussing on general non-stable Lévy processes had been explored in a completely different context in Denmark and Germany, namely studies of sandstorms. Geophysicists like Ole Barndorff-Nielsen and Ernst Eberlein worked on a family of distributions called hyperbolic distributions. One of the arguments given from the beginning in favour of applying these distributions to finance was they were not stable. In this vein, Eberlein and Keller (1995) write that "real stock-price paths change drastically if one looks at them on different time scales". The hyperbolic distributions are infinitely divisible and can therefore be used to construct Lévy processes by specifying the underlying marginal distribution. But the hyperbolic distributions are not stable. In other words, if the underlying distribution is hyperbolic at one given scale, then this does not imply that it will remain this way at any other scale. Hence a numerical computation will be useful for going from one scale to any other scale. The paper by Eberlein and Keller (1995), which introduces the class of hyperbolic distributions – and as a consequence hyperbolic Lévy motions as driving processes for financial modelling – was the first used for analysing and modelling financial data. The hyperbolic model was next intensively examined by Eberlein, Keller and Prause (1998). Unlike previous work, the papers on these distributions aimed to fit the data; in other words, these distributions represent an "application-driven" approach, like an inflexion point in the pragmatic programme: "these distributions seem to be tailor-made to describe the statistical behaviour of asset returns" (Eberlein and Prause, 1998).

An intuitive understanding of what motivated the term "hyperbolic" and its fruitfulness in finance is the following. Let us consider the graph of a Gaussian density in a semi-logarithmic graph, i.e. a graph

where one axis is plotted on a logarithmic scale. One will find a parabola because of the square power of the variable. This parabola is characterized by a rapid fall of the distribution tails. But empirical semi-log graphs of empirical returns at any scale exhibit a hyperbola, contrary to the parabola of the Gaussian density. This is the reason why these distributions are called hyperbolic. An heuristically bottom-up building of an hyperbolic distribution is given in Le Courtois and Walter (2014b). The usefulness for the modelling of price changes stems from the slower decrease of their tails. The hyperbola fits the empirical data better. Like alpha-stable distributions, hyperbolic distributions are defined by four parameters: localisation, asymmetry, dispersion and kurtosis of the distribution. Like alpha-stable distributions, hyperbolic distribution can characterize the risk of any stochastic change with two dimensions: their size (the parameter of dispersion, or scale parameter), and their form (the fatness of the tails and asymmetry). But, unlike alpha-stable distributions, hyperbolic distributions have all their moments. Hence these distributions modelled both the skewness and leptokurtic features encountered in empirical distributions from the real financial world rather well, without running into the perceived inconvenience of alpha-stable distributions of Mandelbrot's programme. The capacity of these processes to model in an extremely powerful way all trajectory irregularities, while not retaining the property of stability by addition, the cornerstone of the first stage of Mandelbrot's programme, made the family of Lévy processes a serious candidate for the probabilistic representation of market fluctuations.

Another interesting feature of hyperbolic distributions is the limiting case, when the dispersion parameter takes values between 0 and infinity. The two limit cases correspond to the two Laplace laws: Gaussian (Laplace's second law, of 1778) and double exponential (Laplace's first law, of 1774). This shows Laplace's two laws as limit laws of hyperbolic distributions.

Hyperbolic distribution is a subclass of the generalized hyperbolic distribution introduced by Barndorff-Nielsen (1977) for the study of particle size in wind-blown sand deposits. The generalized version of the hyperbolic distributions allows other distributions to be obtained depending of the value of the generalization parameter λ. For example, the hyperbolic distribution corresponds to $\lambda = 1$. For $\lambda = 0.5$, one obtains the density of the normal inverse Gaussian distribution (NIG). The normal inverse Gaussian distribution is obtained by mixing normal and inverse Gaussian (IG) distributions. Barndorff-Nielsen moved into finance in 1995. In his papers, Barndorff-Nielsen (1995, 1997) used the normal inverse Gaussian (NIG) distributions. As for the hyperbolic distribution, the NIG is a subclass of the generalized

hyperbolic (GH) distributions. Next, the generalized hyperbolic (GH) distribution, which generates the generalized hyperbolic Lévy processes, was systematically analysed by Eberlein and Prause (1998) and Prause (1999). The first applications to the valuation of derivatives appeared and the pragmatic programme succeeded to price options.

4.2.3 The tempered stable family in finance and the link with econophysics

The symmetric Variance Gamma model was introduced by Madan and Seneta (1990) to generalize the Black-Scholes formula in the case of the evaluation of options. The main impetus for constructing this process concerned a practical market problem: finding a suitable model for the so-called volatility "smile" or "smirk" phenomenon. It was extended to incorporate skewness by Madan and Milne (1991) and Madan, Carr and Chang (1998) to become the so-called Variance Gamma model (VG). The terminology is due to the fact that the variance follows a Gamma distribution. The CGMY process of Carr, Geman, Madan and Yor (2002) generalizes the Variance Gamma process by adding a parameter permitting finite or infinite activity and finite or infinite variation.

The Variance Gamma model of Madan, Carr and Chang (1998) and the CGMY model of Carr, Geman, Madan and Yor (2002) are special cases of the Koponen (1995) model. Here there is an overlap with the physicist's approach: the academic territory of financial modelling is complex and overlapping. The pragmatic programme (PP) and the econophysics programme (EP) develop similar readings of the Mandelbrot view they shared with the tempered stable family. Hence the financial field of modelling extreme values is not simply divided into two camps: mainstream finance (moving to pragmatic programme) and econophysics. Despite their separate lines in the academic fields, the EP and the PP derive from their reliance on Mandelbrot's view: two offshoots of what I suggest to call the 'discontinuous turn' in financial modelling, introduced by Mandelbrot in 1962.

5 Conclusion

In this chapter, I have presented the two competitive programmes for solving the extreme value puzzle in financial modelling with Lévy processes: the radical (Mandelbrot) programme (RP) and the pragmatic programme (PP). Both programmes investigate the ability of Lévy processes to capture the extreme

Table 2: Examples of Lévy processes in financial modelling

	Financial models	**Type of Lévy process**
1900	Bachelier	Brownian motion
1962	Mandelbrot	Stable motion
1976	Merton	Brownian motion and Poisson component
1995	Eberlein and Keller	Hyperbolic motion
1997	Barndorff-Nielsen	Generalized Hyperbolic motion
1998	Madan, Carr, Chang	Variance Gamma process
2001	Kotz, Kozubowski, Podgorski	Laplace process
2002	Carr, Geman, Madan, Yor	Generalized Variance Gamma process
2014	Le Courtois and Walter	Generalized Laplace process

values of price changes. The Mandelbrot programme contributed to a better understanding of the discontinuous nature of price change, but the first Mandelbrot's models initially based on stable motions were not accepted by the mainstream financial academics community. I have argued that some of the key points of the academic debates can be illuminated by reference to a familiar debate in philosophy over the principle of continuity. One side of the debate, 'Leibnizian', takes the continuity as cornerstone for financial modelling and hence splits the market regimes between 'normal' periods and extreme values, seen as irrational periods. The contrary 'antiLeibnizian' position holds that Mandelbrot's view is crucial for grasping the true nature of price changes without intellectual cleavage. I have argued that in the 1970s, the mainstream view of price changes made specific assumptions to defend the mathematical tractability of the financial modelling based on continuous diffusion models, by using a compound *ad hoc* approach, which gained highly recognition in the 1980s: the pragmatic programme. I have explained the successes of the pragmatic programme in the 1990s by showing that an inflexion point appears due to a twofold phenomenon. Firstly, a reorientation of the mathematical financial research due to European academics who put forward the fruitfulness of infinite activity of the Lévy processes in case of pure jumps models. Secondly, the emergence of a challenger for mathematical finance

field: econophysics. At the end, although the pragmatic (financial) programme and the econophysics programme can be traced through separate lines in the academic fields, the two programmes derive from their reliance on Mandelbrot's view, and Mandelbrot can be regarded as the pivotal figure for a 'discontinuous turn' in financial modelling.

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