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Online and Offline Robust Multivariate Linear Regression

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Abstract

We consider the robust estimation of the parameters of multivariate Gaussian linear regression models. To this aim we consider robust version of the usual (Mahalanobis) least-square criterion, with or without Ridge regularization. We introduce two methods each considered contrast: (i) online stochastic gradient descent algorithms and their averaged versions and (ii) offline fix-point algorithms. Under weak assumptions, we prove the asymptotic normality of the resulting estimates. Because the variance matrix of the noise is usually unknown, we propose to plug a robust estimate of it in the Mahalanobis-based stochastic gradient descent algorithms. We show, on synthetic data, the dramatic gain in terms of robustness of the proposed estimates as compared to the classical least-square ones. We also show the computational efficiency of the online versions of the proposed algorithms. All the proposed algorithms are implemented in the R package `RobRegression` available on CRAN¹.

Keywords: Stochastic Optimization; Robust statistics; Multivariate Linear Model

1 Introduction

In this paper, we consider the classical multivariate Gaussian linear model defined for any couple of random vectors $(X, Y) \in \mathbb{R}^p \times \mathbb{R}^q$ (for some $p, q \in \mathbb{N}$, with $q > 1$) by

$$Y = \beta^* X + \epsilon \quad (1)$$

with $\beta^* \in \mathcal{M}_{q,p}(\mathbb{R})$. In addition, the noise vector ϵ and the explanatory vector X are supposed to be independent and $\epsilon \sim \mathcal{N}_q(0, \Sigma)$ where $\Sigma \in \mathcal{M}_{q,q}(\mathbb{R})$ is a positive definite matrix. A usual way for estimating the parameter β^* is to see it as the solution of the following (convex) stochastic optimization problem

$$\beta^* = \arg \min_{\beta \in \mathcal{M}_{q,p}(\mathbb{R})} \mathbb{E} \left[\|Y - \beta X\|^2 \right], \quad (2)$$

where $\|\cdot\|$ stands for the ℓ_2 -norm. Then, several methods can be used to estimate it. A most usual one is to consider the empirical function generated by the sample before approximating its minimizer, *i.e.* the M -estimate (Van der Vaart, 2000; Maronna et al., 2006), with the help of usual convex optimisation methods (Boyd and Vandenberghe, 2004). Although these methods are known to be efficient, they may encounter computational troubles when dealing with large samples of data lying in high dimensional spaces and arriving sequentially. To overcome this, stochastic gradient algorithms (Robbins and Monro, 1951) have become hegemonic and are deeply studied (see, among others, Duflo, 1997; Pelletier, 1998; Bach and Moulines, 2013; Gadat and Panloup, 2017). Indeed, they can deal sequentially with the data and so, with low computational costs per iteration. In addition, their averaged versions are known, under regularity assumptions, to be asymptotically efficient (Ruppert, 1988; Polyak and Juditsky, 1992; Pelletier, 2000).

Moreover, in order to account for the variance structure of the noise, one can prefer to consider the Mahalanobis norm between Y and its prediction. More precisely, one can see β^* as

$$\beta^* = \arg \min_{\beta \in \mathcal{M}_{q,p}(\mathbb{R})} \mathbb{E} \left[\|Y - \beta X\|_{\Sigma^{-1}}^2 \right], \quad (3)$$

where $\|y\|_{\Sigma^{-1}} = \sqrt{y^T \Sigma^{-1} y}$ denotes the Mahalanobis norm of $y \in \mathbb{R}^q$. The main difficulty here is that Σ is generally unknown. Σ^{-1} can be estimated, for instance, by using (an adaptation of) the Graphical Lasso as in (Yuan and Lin,

¹<https://cran.r-project.org/package=RobRegression>

2007; Friedman et al., 2007; Banerjee et al., 2008; Rothman et al., 2010) or by using the empirical covariance matrix of the residuals assuming that Σ has a particular structure as in (Perrot-Dockès et al., 2018, 2022).

Anyway, all these methods are very sensitive to the possible presence of outliers, which is due to the fact that they minimize squared errors. To reduce this sensitivity, one can consider the minimization of the ℓ^1 -loss (see Van der Vaart (2000); Maronna et al. (2006); Pesme and Flammarion (2020) for the univariate case), *i.e.* one can see β^* as

$$\beta^* = \arg \min_{\beta \in \mathcal{M}_{q,p}(\mathbb{R})} \mathbb{E} [\|Y - \beta X\|] \quad \text{or} \quad \beta^* = \arg \min_{\beta \in \mathcal{M}_{q,p}(\mathbb{R})} \mathbb{E} [\|Y - \beta X\|_{\Sigma^{-1}}]. \quad (4)$$

A first objective of this paper is to propose robust methods based on the minimization of these respective functions. Then, one step is to propose iterative and recursive estimates of the minimizers of these two functions (when Σ is known). In this paper, we first propose an iterative method which consists in a fixed-point algorithm that presents similarities with the well-known Weiszfeld algorithm (Weiszfeld, 1937; Vardi and Zhang, 2000). Then we propose new stochastic gradient algorithms and their averaged versions (Ruppert, 1988; Polyak and Juditsky, 1992) before establishing convergence results such that rate of convergence, asymptotic normality and giving uniform bound of the mean quadratic error. As the variance matrix Σ is usually unknown, a second objective of this paper is to propose a methodology to build estimates based on the Mahalanobis norm. This consists in obtaining a first (iterative or recursive) robust estimate of β^* (based on the Euclidean norm). This estimate will enable us to propose a robust estimate of Σ based on the recent work of Godichon-Baggioni and Robin (2024), before injecting it for the robust estimation of β^* based on the Mahalanobis norm. Furthermore, we consider regularized versions of the previous estimates by using the Ridge regression estimator (Hoerl and Kennard, 1970) for dealing in particular with the collinearity in a high dimensional design matrix.

The paper is organized as follows. In Section 2, we present our framework for robust regression by introducing the criteria to minimize. Section 3 is dedicated to the online estimates of β^* based on (weighted averaged) stochastic gradient algorithms denoted '*Online OLS*' for the version based on the ℓ_2 -norm and '*Online WLS*' for the version based on the Mahalanobis norm. Section 4 is devoted to the offline estimates of β^* based on fixed-point algorithms denoted '*Offline OLS*' for the version based on the ℓ_2 -norm and '*Offline WLS*' for the version based on the Mahalanobis norm. For both online and offline estimates, we establish theoretical results given the rate of convergence of these robust estimates and their asymptotic normality under weak assumptions. In Section 5, we propose a regularized version of the previous estimates by using an adapted Ridge regression. In Section 6, we propose the practical use of the proposed estimates when Σ is unknown. We present in Section 7, a simulation study in order to support our theoretical results. The proofs of our main theoretical results are postponed in Section 8. All the proposed methods are available in the R package `RobRegression` accessible on CRAN².

2 Framework for robust regression

Let us first recall that the proposed robust alternative for estimating the parameter β^* in the multivariate Gaussian linear model defined by (1) is based on the fact that, as soon as the distribution of ϵ is symmetric, one can rewrite

$$\beta^* = \arg \min_{\beta \in \mathcal{M}_{q,p}(\mathbb{R})} G(\beta) \quad \text{with} \quad G(\beta) := \mathbb{E} [\|Y - \beta X\|]. \quad (5)$$

Indeed, first note that $G : \mathcal{M}_{q,p}(\mathbb{R}) \rightarrow \mathbb{R}_+$ is a convex function. Second, the function G is differentiable and its gradient for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$ is defined by

$$\nabla G(\beta) = -\mathbb{E} \left[\frac{Y - \beta X}{\|Y - \beta X\|} X^T \right].$$

Since the distribution of ϵ is symmetric and since ϵ and X are independent,

$$\nabla G(\beta^*) = -\mathbb{E} \left[\frac{Y - \beta^* X}{\|Y - \beta^* X\|} X^T \right] = -\mathbb{E} \left[\frac{\epsilon}{\|\epsilon\|} X^T \right] = -\mathbb{E} \left[\frac{\epsilon}{\|\epsilon\|} \right] \mathbb{E} [X^T] = 0,$$

and β^* is so a minimizer of the functional G .

²<https://cran.r-project.org/package=RobRegression>

In addition, considering the use of the Mahalanobis norm leads to

$$\beta^* = \arg \min_{\beta \in \mathcal{M}_{q,p}(\mathbb{R})} G_{\Sigma}(\beta) \quad \text{with} \quad G_{\Sigma}(\beta) := \mathbb{E} [\|Y - \beta X\|_{\Sigma^{-1}}]. \quad (6)$$

Indeed, since the distribution of $\Sigma^{-1/2}\epsilon$ is symmetric and since X and $\Sigma^{-1/2}\epsilon$ are independent, it comes that

$$\nabla G_{\Sigma}(\beta^*) = -\mathbb{E} \left[\Sigma^{-1} \frac{Y - \beta^* X}{\|Y - \beta^* X\|_{\Sigma^{-1}}} X^T \right] = -\mathbb{E} \left[\Sigma^{-1} \frac{\epsilon}{\|\epsilon\|_{\Sigma^{-1}}} X^T \right] = -\Sigma^{-1/2} \mathbb{E} \left[\frac{\Sigma^{-1/2}\epsilon}{\|\Sigma^{-1/2}\epsilon\|} \right] \mathbb{E} [X^T] = 0,$$

by observing that $\|\epsilon\|_{\Sigma^{-1}} = \|\Sigma^{-1/2}\epsilon\|$.

In the following, we shall focus on the estimation of β^* through the minimization of G and G_{Σ} . Moreover, for both cases, we will introduce iterative algorithms (consisting in fixed-point algorithms) and online algorithms (consisting in averaged stochastic gradient algorithms).

Furthermore, we will also consider regularized versions of the previous estimates by naturally considering a Ridge penalization adapted to the functions G and G_{Σ} (where the considered norm is not a squared norm). Then, we shall consider the following corresponding penalized functions:

$$G_{\lambda}(\beta) := \mathbb{E} [\|Y - \beta X\|] + \lambda \|\beta\|_F \quad \text{and} \quad G_{\Sigma,\lambda}(\beta) := \mathbb{E} [\|Y - \beta X\|_{\Sigma^{-1}}] + \lambda \|\beta\|_F, \quad (7)$$

where $\lambda > 0$ is a parameter to calibrate that controls the amount of shrinkage. As soon as these functions are convex, continuous and diverge if $\|\beta\|_F$ goes to infinity, they admit a minimizer. In addition, supposing that this minimizer is different from 0, it is unique and denoted β_{λ}^* . Note that we have added a penalty term consistent with the seminal criterion to minimize, namely $\lambda \|\beta\|_F$ (instead of $\lambda \|\beta\|_F^2$).

3 Online estimation of β^*

The objective is to apply usual convergence results for stochastic algorithms (Robbins and Monro, 1951; Polyak and Juditsky, 1992; Pelletier, 1998, 2000; Godichon-Baggioni, 2021, 2019a; Gadat and Panloup, 2017; Bach and Moulines, 2013). In this aim, let us consider the following diffeomorphism:

$$\varphi : \begin{array}{ccc} \mathcal{M}_{q,p}(\mathbb{R}) & \longrightarrow & \mathbb{R}^{pq} \\ \beta = (\beta_{i,j})_{1 \leq i \leq q, 1 \leq j \leq p} & \longmapsto & \varphi(\beta) = (\beta_{1,1}, \dots, \beta_{1,p}, \dots, \beta_{q,p})^T, \end{array}$$

which will be useful in the proof as well as for establishing the asymptotic normality of the estimates.

3.1 Online estimation of β^* based on the minimization of G

Let us recall that the function G , defined by (5), to be minimized is defined for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$ by $G(\beta) = \mathbb{E} [g(X, Y, \beta)]$ where $g(X, Y, \beta) := \|Y - \beta X\|$. Then, as soon as $g(X, Y, \cdot)$ is differentiable with $\nabla g(X, Y, \beta) = -\frac{Y - \beta X}{\|Y - \beta X\|} X^T$, the stochastic gradient algorithm is defined recursively for all $n \geq 0$ by

$$\beta_{n+1} = \beta_n + \gamma_{n+1} \frac{Y_{n+1} - \beta_n X_{n+1}}{\|Y_{n+1} - \beta_n X_{n+1}\|} X_{n+1}^T, \quad (8)$$

where $\gamma_n = c_{\gamma} n^{-\gamma}$ with $c_{\gamma} > 0$ and $\gamma \in (1/2, 1)$ (Robbins and Monro, 1951). Observe that it is well known that stochastic gradient algorithms do not achieve the asymptotic efficiency. In order to accelerate the convergence, we should so consider its weighted averaged version (see (Ruppert, 1988; Polyak and Juditsky, 1992; Mokedem and Pelletier, 2011; Boyer and Godichon-Baggioni, 2020)) named 'Online OLS' and defined recursively, with $w \geq 0$, for all $n \geq 0$ by

$$\bar{\beta}_{n+1} = \bar{\beta}_n + \frac{\log(n+1)^w}{\sum_{k=0}^n \log(k+1)^w} (\beta_{n+1} - \bar{\beta}_n). \quad (9)$$

In particular, note that

$$\bar{\beta}_n = \frac{1}{\sum_{k=0}^n \log(k+1)^w} \sum_{k=0}^n \log(k+1)^w \beta_k$$

and the classical averaging consists in taking $w = 0$. Taking $w > 0$ effectively decreases the weight of earlier iterates in comparison to the later ones and consequently it enables to avoid initialization issues. In practice we will set the averaging power to $w = 2$. The following theorem gives the rate of convergence of the robust estimates defined by (8) and (9) under weak assumptions.

Theorem 1. *Let β_n and $\bar{\beta}_n$ be defined respectively by (8) and (9). Assume that $q \geq 3$, and that X admits a moment of order $\max\{3, 2 + 2\eta\}$ with $\eta > \frac{1}{\gamma} - 1$. Assume also that the matrix $\mathbb{E}[XX^T]$ is positive definite. Then,*

$$\|\beta_n - \beta^*\|_F^2 = O\left(\frac{\ln n}{n^\gamma}\right) \quad a.s. \quad \text{and} \quad \|\bar{\beta}_n - \beta^*\|_F^2 = O\left(\frac{\ln n}{n}\right) \quad a.s..$$

Moreover,

$$\sqrt{n}(\varphi(\bar{\beta}_n) - \varphi(\beta^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}_{pq}\left(0, \mathbb{E}\left[\frac{1}{\|\epsilon\|} (I_q - V_\epsilon)\right]^{-1} \mathbb{E}[V_\epsilon] \mathbb{E}\left[\frac{1}{\|\epsilon\|} (I_q - V_\epsilon)\right]^{-1} \otimes (\mathbb{E}[XX^T])^{-1}\right),$$

where $V_\epsilon := \frac{\epsilon \epsilon^T}{\|\epsilon\| \|\epsilon\|}$ and \otimes denotes the Kronecker product.

If we suppose also that X admits a moment of order 4, then there exist positive constants C_0 and C_1 such that for all $n \geq 1$

$$\mathbb{E}\left[\|\beta_n - \beta\|_F^2\right] \leq \frac{C_0}{n^\gamma} \quad \text{and} \quad \mathbb{E}\left[\|\bar{\beta}_n - \beta\|_F^2\right] \leq \frac{C_1}{n}.$$

Remark. Note that the constants can be explicitly given (see Godichon-Baggioni (2021)), but this is not the purpose of this paper. In addition, observe that the $q \times q$ matrix

$$\mathbb{E}\left[\frac{1}{\|\epsilon\|} (I_q - V_\epsilon)\right]^{-1} \mathbb{E}[V_\epsilon] \mathbb{E}\left[\frac{1}{\|\epsilon\|} (I_q - V_\epsilon)\right]^{-1}$$

would be the asymptotic variance of recursive estimates of the median of the residual ϵ if we had access to the sequence (ϵ_i) (see Cardot et al. (2013)).

3.2 Online estimation of β^* based on the minimization of G_Σ

Let us recall that the function G_Σ , defined by (6), to be minimized is defined for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$ by $G_\Sigma(\beta) = \mathbb{E}[g_\Sigma(X, Y, \beta)]$ where $g_\Sigma(X, Y, \beta) := \|Y - \beta X\|_{\Sigma^{-1}}$. Then, as soon as $g_\Sigma(X, Y, \cdot)$ is differentiable with $\nabla g_\Sigma(X, Y, \beta) = -\Sigma^{-1} \frac{Y - \beta X}{\|Y - \beta X\|} X^T$, the stochastic gradient algorithm and its weighted averaged version are respectively and recursively defined, for all $n \geq 0$, by

$$\beta_{n+1, \Sigma} = \beta_{n, \Sigma} + \gamma_{n+1} \Sigma^{-1} \frac{Y_{n+1} - \beta_{n, \Sigma} X_{n+1}}{\|Y_{n+1} - \beta_{n, \Sigma} X_{n+1}\|_{\Sigma^{-1}}} X_{n+1}^T, \quad (10)$$

$$\bar{\beta}_{n+1, \Sigma} = \bar{\beta}_{n, \Sigma} + \frac{\log(n+1)^w}{\sum_{k=0}^n \log(k+1)^w} (\beta_{n+1, \Sigma} - \bar{\beta}_{n, \Sigma}), \quad (11)$$

where $\gamma_n = c_\gamma n^{-\gamma}$ with $c_\gamma > 0$ and $\gamma \in (1/2, 1)$. The following theorem gives the rates of convergence of these estimates. In the following, we refer to (11) as 'Online WLS'.

Theorem 2. *Let $\beta_{n, \Sigma}$ and $\bar{\beta}_{n, \Sigma}$ be defined respectively by (10) and (11). Assume that $q \geq 3$, and that X admits a moment of order $\max\{3, 2 + 2\eta\}$ with $\eta > \frac{1}{\gamma} - 1$. Assume also that the matrix $\mathbb{E}[XX^T]$ is positive definite. Then,*

$$\|\beta_{n, \Sigma} - \beta^*\|_F^2 = O\left(\frac{\ln n}{n^\gamma}\right) \quad a.s. \quad \text{and} \quad \|\bar{\beta}_{n, \Sigma} - \beta^*\|_F^2 = O\left(\frac{\ln n}{n}\right) \quad a.s.$$

Moreover,

$$\sqrt{n}(\varphi(\bar{\beta}_{n, \Sigma}) - \varphi(\beta^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{2q}{(q-1)^2} \frac{\Gamma\left(\frac{q}{2}\right)^2}{\Gamma\left(\frac{q-1}{2}\right)^2} \Sigma \otimes (\mathbb{E}[XX^T])^{-1}\right),$$

where Γ is the Gamma function. If we suppose also that X admits a moment of order 4, then there exist positive constants $C_{0, \Sigma}$ and $C_{1, \Sigma}$ such that for all $n \geq 1$

$$\mathbb{E}\left[\|\beta_{n, \Sigma} - \beta^*\|_F^2\right] \leq \frac{C_{0, \Sigma}}{n^\gamma} \quad \text{and} \quad \mathbb{E}\left[\|\bar{\beta}_{n, \Sigma} - \beta^*\|_F^2\right] \leq \frac{C_{1, \Sigma}}{n}.$$

3.3 A comparison of these two approaches

It is known that the naive OLS and naive WLS yields the same estimates: $\arg \min_{\beta} \sum_{i=1}^n \|Y_i - \beta X_i\|^2 = \arg \min_{\beta} \sum_{i=1}^n \|Y_i - \beta X_i\|_{\Sigma^{-1}}^2$ (see Mardia et al., 1979, Chapter 5), which means there is no statistical advantage to consider naive WLS rather than naive OLS. Still, this equality does not hold in presence of (Ridge) regularization.

As for the robust counterparts, the robust OLS and WLS estimates are different and yield different asymptotic variance, as shown by Theorems 1 and 2. Unfortunately, because the asymptotic variance of the robust online OLS estimate given in Theorem 1 has no close form, we can not provide a systematic comparison of the two asymptotic variances. Namely, we can not prove that the difference between the OLS asymptotic variance matrix and the WLS asymptotic variance matrix is positive definite, which would imply a systematic interest of robust WLS as compared to robust OLS.

We can only report empirical observations. In parallel to the simulation study presented in Section 7, we simulated a large number of variance matrix Σ and computed the asymptotic variances given by Theorems 1 and 2. For all of them the difference between the OLS and the WLS variances turned out to be positive definite.

Finally, it should be noted that in the case where the function G_{Σ} is considered, the appearance of Σ^{-1} in stochastic gradient algorithms can be, to some extent, linked to adaptive algorithms (Duchi et al., 2011; Leluc and Portier, 2020). The latter are more effective when the problem is ill-conditioned. In our situation, this would correspond to the case where Σ is strongly correlated.

4 Offline estimation of β^*

4.1 A fixed-point algorithm based on the minimization of G

In order to propose a new estimate of β^* , we first show that finding the zero of the gradient of the function G defined by (5) leads to find a fixed point. Indeed, if for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$, $\mathbb{E} \left[\frac{XX^T}{\|Y - \beta X\|} \right]$ is positive definite, we observe that

$$\nabla G(\beta) = 0 \Leftrightarrow \beta \mathbb{E} \left[\frac{XX^T}{\|Y - \beta X\|} \right] = \mathbb{E} \left[\frac{YX^T}{\|Y - \beta X\|} \right] \Leftrightarrow \beta = \mathbb{E} \left[\frac{YX^T}{\|Y - \beta X\|} \right] \left(\mathbb{E} \left[\frac{XX^T}{\|Y - \beta X\|} \right] \right)^{-1}.$$

Then, given a sample $\mathcal{D}_n = ((X_1, Y_1), \dots, (X_n, Y_n))$ of (X, Y) and assuming that for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$, $\sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta X_i\|}$ is positive definite, the fixed-point algorithm is defined iteratively, for all $t \geq 0$, by

$$\beta_{n,t+1} = \sum_{i=1}^n \frac{Y_i X_i^T}{\|Y_i - \beta_{n,t} X_i\|} \left(\sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t} X_i\|} \right)^{-1} =: T_n(\beta_{n,t}). \quad (12)$$

By defining G_n the empirical function generated by the sample \mathcal{D}_n of G , i.e. for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$G_n(\beta) := \frac{1}{n} \sum_{i=1}^n \|Y_i - \beta X_i\|, \quad (13)$$

the fixed-point algorithm can be rewritten as a gradient descent as follows:

$$\beta_{n,t+1} = \beta_{n,t} + \frac{1}{n} \sum_{i=1}^n \frac{(Y_i - \beta_{n,t} X_i) X_i^T}{\|Y_i - \beta_{n,t} X_i\|} \left(\frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t} X_i\|} \right)^{-1} = \beta_{n,t} - \nabla G_n(\beta_{n,t}) L_{n,t}^{-1}, \quad (14)$$

with a step sequence "on the right" $L_{n,t}^{-1}$ where $L_{n,t} := \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t} X_i\|}$. In the following, $\hat{\beta}_n$ denotes the minimizer of G_n .

Theorem 3. Assume that $\sum_{i=1}^n X_i X_i^T$ is positive definite almost surely. Then,

$$\beta_{n,t} \xrightarrow[t \rightarrow +\infty]{} \hat{\beta}_n.$$

The proof of Theorem 3 is given in Section 8. The following result gives the asymptotic normality of the iterative estimate.

Corollary 1. Assume that X admits a moment of order 2 and that $\mathbb{E}[XX^T]$ is positive definite. Then,

$$\sqrt{n}(\varphi(\beta_{n,t}) - \varphi(\beta^*)) \xrightarrow[n,t \rightarrow +\infty]{\mathcal{L}} \mathcal{N}_{pq} \left(0, \mathbb{E} \left[\frac{1}{\|\epsilon\|} (I_q - V_\epsilon) \right]^{-1} \mathbb{E}[V_\epsilon] \mathbb{E} \left[\frac{1}{\|\epsilon\|} (I_q - V_\epsilon) \right]^{-1} \otimes (\mathbb{E}[XX^T])^{-1} \right).$$

Corollary 1 is a direct application of Theorem 3 coupled with Theorems 1 and 4 of Niemi (1992). Observe that this corollary yields the same asymptotic normality as the recursive estimates, but with larger computational complexity.

4.2 A fixed-point algorithm based on the minimization of G_Σ

Similarly as in Section 4.1, we first show that finding the zero of the gradient of G_Σ defined by (6) leads to find a fixed point. Indeed, assuming that for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$, $\mathbb{E} \left[\frac{XX^T}{\|Y - \beta X\|_{\Sigma^{-1}}} \right]$ is positive definite, we observe that

$$\nabla G_\Sigma(\theta) = 0 \Leftrightarrow -\Sigma^{-1} \mathbb{E} \left[\frac{Y - \beta X}{\|Y - \beta X\|_{\Sigma^{-1}}} X^T \right] = 0 \Leftrightarrow \beta = \mathbb{E} \left[\frac{YX^T}{\|Y - \beta X\|_{\Sigma^{-1}}} \right] \left(\mathbb{E} \left[\frac{XX^T}{\|Y - \beta X\|_{\Sigma^{-1}}} \right] \right)^{-1}.$$

Hence, the fixed-point estimate is given iteratively, for all $t \geq 0$, by

$$\beta_{n,t+1,\Sigma} = \sum_{i=1}^n \frac{Y_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma} X_i\|_{\Sigma^{-1}}} \left(\sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma} X_i\|_{\Sigma^{-1}}} \right)^{-1} =: T_{n,\Sigma}(\beta_{n,t,\Sigma}), \quad (15)$$

which can also be written as a gradient algorithm with respect to the empirical function

$$G_{n,\Sigma}(\beta) := \frac{1}{n} \sum_{i=1}^n \|Y_i - \beta X_i\|_{\Sigma^{-1}}.$$

In particular,

$$\beta_{n,t+1,\Sigma} = \beta_{n,t,\Sigma} + \Sigma \frac{1}{n} \sum_{i=1}^n \frac{\Sigma^{-1} (Y_i - \beta_{n,t,\Sigma} X_i)}{\|Y_i - \beta_{n,t,\Sigma} X_i\|_{\Sigma^{-1}}} L_{n,t,\Sigma}^{-1} = \beta_{n,t,\Sigma} - \Sigma \nabla G_{n,\Sigma}(\beta_{n,t,\Sigma}) L_{n,t,\Sigma}^{-1}, \quad (16)$$

with $L_{n,t,\Sigma} := \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma} X_i\|_{\Sigma^{-1}}}$. In the following, $\hat{\beta}_{n,\Sigma}$ denotes the minimizer of $G_{n,\Sigma}$.

Theorem 4. Assume that $\sum_{i=1}^n X_i X_i^T$ is positive definite almost surely. Then,

$$\beta_{n,t,\Sigma} \xrightarrow[t \rightarrow +\infty]{} \hat{\beta}_{n,\Sigma}.$$

The proof of Theorem 4 is given in Section 8. The following result gives the asymptotic normality of the iterative estimate.

Corollary 2. Assume that X admits a moment of order 2 and that $\mathbb{E}[XX^T]$ is positive definite. Then,

$$\sqrt{n}(\varphi(\bar{\beta}_{n,\Sigma}) - \varphi(\beta^*)) \xrightarrow[n,t \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0, \frac{2q}{(q-1)^2} \frac{\Gamma(\frac{q}{2})^2}{\Gamma(\frac{q-1}{2})^2} \Sigma \otimes (\mathbb{E}[XX^T])^{-1} \right).$$

Corollary 2 is a direct application of Theorem 4 coupled with Theorems 1 and 4 of Niemi (1992).

5 The robust regularized regression

We propose in this section the regularized version of the previous estimates. In particular, we only focus on the estimates built in Section 3.2 and Section 4.2. Note that similar results can be derived for the estimates proposed in Section 3.1 and Section 4.1.

5.1 Online estimation of β^* based on the minimization of $G_{\Sigma,\lambda}$

Recalling that $G_{\Sigma,\lambda}$ is defined by $G_{\Sigma,\lambda}(\beta) = \mathbb{E}[g_{\Sigma,\lambda}(X, Y, \beta)]$ where $g_{\Sigma,\lambda}(X, Y, \beta) := \|Y - \beta X\|_{\Sigma^{-1}} + \lambda \|\beta\|_F$ (see (7)). Then, as soon as $g_{\Sigma,\lambda}(X, Y, \cdot)$ is differentiable with $\nabla g_{\Sigma,\lambda}(X, Y, \beta) = -\Sigma^{-1} \frac{Y - \beta X}{\|Y - \beta X\|_{\Sigma^{-1}}} X^T + \lambda \frac{\beta}{\|\beta\|_F} \mathbf{1}_{\beta \neq 0}$. Then, the stochastic gradient algorithm and its weighted averaged version are defined recursively for all $n \geq 0$ by

$$\beta_{n+1,\Sigma,\lambda} = \beta_{n,\Sigma,\lambda} + \gamma_{n+1} \left(\Sigma^{-1} \frac{Y_{n+1} - \beta_{n,\Sigma,\lambda} X_{n+1}}{\|Y_{n+1} - \beta_{n,\Sigma,\lambda} X_{n+1}\|_{\Sigma^{-1}}} X_{n+1}^T - \lambda \frac{\beta_{n,\Sigma,\lambda}}{\|\beta_{n,\Sigma,\lambda}\|_F} \mathbf{1}_{\beta_{n,\Sigma,\lambda} \neq 0} \right), \quad (17)$$

$$\bar{\beta}_{n+1,\Sigma,\lambda} = \bar{\beta}_{n,\Sigma,\lambda} + \frac{\log(n+1)^w}{\sum_{k=0}^n \log(k+1)^w} (\beta_{n+1,\Sigma,\lambda} - \bar{\beta}_{n,\Sigma,\lambda}). \quad (18)$$

The following result gives the rate of convergence of the estimates.

Theorem 5. *Suppose that X admits a moment of order $\max\{3, 2 + 2\eta\}$ with $\eta > \frac{1}{\gamma} - 1$, and that $q \geq 3$. Suppose also that $\beta_\lambda^* \neq 0$, then*

$$\|\beta_{n,\Sigma,\lambda} - \beta_\lambda^*\|^2 = O\left(\frac{\ln n}{n^\gamma}\right) \quad a.s. \quad \text{and} \quad \|\bar{\beta}_{n,\Sigma,\lambda} - \beta_\lambda^*\|^2 = O\left(\frac{\ln n}{n}\right) \quad a.s.$$

In addition,

$$\sqrt{n}(\varphi(\bar{\beta}_{n,\Sigma,\lambda}) - \varphi(\beta_\lambda^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, H_{\Sigma,\lambda}^{-1} V_{\Sigma,\lambda} H_{\Sigma,\lambda}^{-1})$$

where $H_{\Sigma,\lambda} := \nabla^2 \tilde{G}_{\Sigma,\lambda}(\varphi(\beta_\lambda^*))$ and $V_{\Sigma,\lambda} = \mathbb{E} \left[\nabla \tilde{g}_{\Sigma,\lambda}(X, Y, \beta_\lambda^*) \nabla \tilde{g}_{\Sigma,\lambda}(X, Y, \beta_\lambda^*)^T \right]$.

5.2 Fixed-point algorithm based on the minimization of $G_{\Sigma,\lambda}$

Observe that in the penalized case, we did not succeed in writing the solution as an "easy" fix point. Any, following the scheme of proof of Theorems 3 and 4, this leads to an iterative conditioned gradient algorithm defined for all $t \geq 0$ by

$$\varphi(\beta_{n,t+1,\Sigma,\lambda}) = \varphi(\beta_{n,t,\Sigma,\lambda}) + \left(\Sigma^{-1} \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma,\lambda} X_i\|_{\Sigma^{-1}}} + \frac{\lambda \mathbf{1}_{\beta_{n,t,\Sigma,\lambda} \neq 0} I_{pq}}{\|\beta_{n,t,\Sigma,\lambda}\|_F} \right)^{-1} \varphi \left(\frac{1}{n} \sum_{i=1}^n \frac{(Y_i - \beta_{n,t,\Sigma,\lambda} X_i) X_i^T}{\|Y_i - \beta_{n,t,\Sigma,\lambda} X_i\|_{\Sigma^{-1}}} \right). \quad (19)$$

Indeed, here again, considering the functional

$$G_{n,\Sigma,\lambda}(\beta) := \frac{1}{n} \sum_{i=1}^n \|Y_i - \beta X_i\|_{\Sigma^{-1}} + \lambda \|\beta\|_F,$$

and denoting by $\nabla \tilde{G}_{n,\Sigma,\lambda}$ its gradient in the canonical basis of \mathbb{R}^{pq} , one can rewrite algorithm (19) as

$$\varphi(\beta_{n,t+1,\Sigma,\lambda}) = \varphi(\beta_{n,t,\Sigma,\lambda}) + \underbrace{\left(\Sigma^{-1} \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma,\lambda} X_i\|_{\Sigma^{-1}}} + \frac{\lambda \mathbf{1}_{\beta_{n,t,\Sigma,\lambda} \neq 0} I_{pq}}{\|\beta_{n,t,\Sigma,\lambda}\|_F} \right)^{-1} \nabla \tilde{G}_{n,\Sigma,\lambda}(\varphi(\beta_{n,t,\Sigma,\lambda}))}_{=: T_{n,\varphi,\Sigma,\lambda}(\beta_{n,t,\Sigma,\lambda})}. \quad (20)$$

In the sequel, let us denote by $\hat{\beta}_{n,\Sigma,\lambda}$ the minimizer of $G_{n,\Sigma,\lambda}$.

Theorem 6. *One has*

$$\beta_{n,t,\Sigma,\lambda} \xrightarrow[n \rightarrow +\infty]{} \hat{\beta}_{n,\Sigma,\lambda}.$$

In addition, if X admits a moment of order 2,

$$\sqrt{n}(\varphi(\beta_{n,t,\Sigma,\lambda}) - \varphi(\beta_\lambda^*)) \xrightarrow[n,t \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, H_{\Sigma,\lambda}^{-1} V_{\Sigma,\lambda} H_{\Sigma,\lambda}^{-1})$$

6 Estimation in practice when Σ is unknown

In the previous sections, we have assumed that Σ is known. We propose here to explain how we can use the proposed estimates in the case Σ is unknown in practice. Given a sample $(X_1, Y_1), \dots, (X_N, Y_N)$, the algorithm when Σ is unknown consists to first estimate β^* with the help of $\hat{\beta} = \bar{\beta}_N$ or $\hat{\beta} = \beta_T$ (where T is the number of iterations) given by (9) or (12). Then, one can estimate the Median Covariation Matrix of ϵ (see Kraus and Panaretos (2012) and Cardot and Godichon-Baggioni (2015)). More precisely, the MCM of ϵ is defined by

$$V^* = \arg \min_V \mathbb{E} \left[\left\| \epsilon \epsilon^T - V \right\|_F \right] \quad i.e \quad V^* = \arg \min_V \mathbb{E} \left[\left\| (Y - \beta^* X)(Y - \beta^* X)^T - V \right\|_F \right].$$

As soon as ϵ is a Gaussian vector, it is well known (Kraus and Panaretos, 2012) that the MCM and the variance Σ have the same eigenvectors. Then, in order to construct a robust estimate of the variance, a first aim is to estimate the MCM of ϵ . In this aim, let us consider the empirical function generated by the sample, and since β^* is unknown, the aim is to minimize

$$G_{MCM,N}(V) = \frac{1}{N} \sum_{i=1}^N \left\| (Y_i - \hat{\beta} X_i)(Y_i - \hat{\beta} X_i)^T - V \right\|_F,$$

which leads to the Weiszfeld algorithm given by Weiszfeld (1937) or an averaged stochastic gradient algorithm (Cardot et al., 2013; Cardot and Godichon-Baggioni, 2015). These two algorithms are precisely described in Appendix. We will denote by V_N the chosen estimate. In addition, one can calculate the eigenvalues $\delta_N = (\delta_{1,N}, \dots, \delta_{q,N})$ (and the associated matrix of eigenvectors P_N) of V_N . Moreover, denoting by δ (resp. λ) the eigenvalues of V^* (resp. Σ), one has (Kraus and Panaretos, 2012)

$$\lambda = \delta \mathbb{E} [h(\delta, \lambda, U)] \oslash \mathbb{E} [U^2 h(\delta, \lambda, U)], \quad (21)$$

where \oslash is the division coordinate per coordinate, $U = (U_1, \dots, U_q) \sim \mathcal{N}(0, I_q)$ and

$$h(\delta, \lambda, U) = \left(\left(\sum_{j=1}^q (\delta_j - \lambda_j U_j^2)^2 + 2 \sum_{1 \leq j < k \leq q} \lambda_j \lambda_k U_j^2 U_k^2 \right)^{-1/2} \right).$$

This can also be written as

$$\delta \mathbb{E} [h(\delta, \lambda, U)] - \lambda \cdot \mathbb{E} [U^2 h(\delta, \lambda, U)] = 0.$$

where \cdot is the multiplication coordinate per coordinate. Thus, one obtains a robust estimate $\delta_N = (\delta_{1,N}, \dots, \delta_{q,N})$ of the eigenvalues of the variance Σ , using either a Fix-Point algorithm, a "gradient" algorithm or a Weighted Averaged Robbins Monro algorithm (see Godichon-Baggioni and Robin (2024)). Then, the robust estimate of the variance and its inverse are given by

$$\Sigma_N = P_N \delta_N P_N^T \quad \text{and} \quad \Sigma_N^{-1} = P_N \delta_N^{-1} P_N^T$$

where P_N is the matrix composed of the normalized eigenvectors of the MCM and $\delta_N^{-1} = (\delta_{1,N}^{-1}, \dots, \delta_{q,N}^{-1})$. Thus, one can replace Σ^{-1} in (10) or (15) by Σ_N^{-1} to obtain last estimates. This leads, to the weighted averaged stochastic gradient algorithm defined recursively for all $n \leq N - 1$ by

$$\beta_{n+1,*} = \beta_{n,*} + \gamma_{n+1} \left(\Sigma_N^{-1} \frac{Y_{n+1} - \beta_{n,*} X_{n+1}}{\|Y_{n+1} - \beta_{n,*} X_{n+1}\|_{\Sigma_N^{-1}}} X_{n+1}^T - \lambda \frac{\beta_{n,*}}{\|\beta_{n,*}\|_F} \mathbf{1}_{\beta_{n,*} \neq 0} \right) \quad (22)$$

$$\bar{\beta}_{n+1,*} = \bar{\beta}_{n,*} + \frac{\log(n+1)^w}{\sum_{k=0}^n \log(k+1)^w} (\beta_{n+1,*} - \bar{\beta}_{n,*}) \quad (23)$$

or the fix point algorithm defined iteratively for all $t \geq 0$ by

$$\beta_{n,t+1,\Sigma_N} = T_{n,\Sigma_N}(\beta_{n,t,\Sigma_N}) = \sum_{i=1}^N \frac{Y_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma_N} X_i\|_{\Sigma_N^{-1}}} \left(\sum_{i=1}^N \frac{X_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma_N} X_i\|_{\Sigma_N^{-1}}} \right)^{-1}$$

if $\lambda = 0$ and the iterative algorithm

$$\varphi(\beta_{n,t+1,\Sigma_N,\lambda}) = \varphi(\beta_{n,t,\Sigma_N,\lambda}) + \left(\Sigma_N^{-1} \oslash \frac{1}{N} \sum_{i=1}^N \frac{X_i X_i^T}{\|Y_i - \beta_{n,t,\Sigma_N,\lambda} X_i\|_{\Sigma_N^{-1}}} + \frac{\lambda \mathbf{1}_{\beta_{n,t,\Sigma_N,\lambda} \neq 0} I_{pq}}{\|\beta_{n,t,\Sigma_N,\lambda}\|_F} \right)^{-1} \nabla \tilde{G}_{N,\Sigma_N,\lambda}(\varphi(\beta_{n,t,\Sigma_N,\lambda})).$$

if $\lambda > 0$.

7 Simulation study

We now present a simulation study to assess the performances of the proposed estimation procedures. The main goal of the study is to evaluate the gain of using robust estimators as opposed to classical estimators, in presence of a varying fraction of outliers. In this perspective, we considered two typical tasks that can be addressed with Model (1): multivariate regression and classification via a linear discriminant analysis.

7.1 Multivariate regression

Simulation design. We fixed the number of variables to $p = 5$ for the explanatory variables in X and to $q = 20$ for the response variables in Y . The $q \times p$ regression coefficients β were sampled as iid standard normal and fixed once for all. We also fixed the covariance matrix Σ (see Appendix B). We considered data sets with $n = 100, 1000$ and 10000 observations. We also considered increasing proportions f of outliers, namely: $f = 0, 2, 3, 5, 9, 16, 28$ and 50% .

For each dimension n , we sampled the $n \times p$ entries of the matrix X as iid normal and fixed them once for all. Then, for each fraction f , we sampled $B = 100$ $n \times q$ response matrices Y according to Model (1) in which a fraction $1 - f$ of the residual terms ϵ_i were sampled from a multivariate normal $\mathcal{N}_q(0, \Sigma)$, whereas the remaining fraction f were sampled as iid Student with variance 1 and degree of freedom 1.

We carried the inference on each simulated response matrix Y , using the following procedures:

Naive OLS refers to the classical ordinary least-square estimate of β : $\widehat{\beta} = \arg \min_{\beta} \sum_{i=1}^n \|Y_i - \beta X_i\|^2$;

Offline OLS refers to its robust counterpart, resulting from the fixpoint (offline) algorithm described in Section 4.1;

Online OLS refers to robust estimation resulting from the online algorithm described in Section 3.1.

Naive, Offline and Online WLS are defined similarly: '*Naive WLS*' refers to the minimizer of $\sum_{i=1}^n \|Y_i - \beta X_i\|_{\widehat{\Sigma}^{-1}}^2$ (where $\widehat{\Sigma}$ is the empirical residual covariance matrix), '*Offline WLS*' refers to the fixpoint algorithm from Section 4.2 and '*Online WLS*' to the algorithm from Section 3.2.

Ridge: For each procedure, we also considered a Ridge regularization of the parameter β , as described in Section 5. Namely, we added a penalty term consistent with each respective criterion, that is $\lambda \|\beta\|^2$ for the naive procedures and $\lambda \|\beta\|$ for the robust ones. In each case, the penalty coefficient λ was determined via 10-fold cross-validation.

Results. We now present the results obtained with the different procedures in terms of estimation accuracy, prediction accuracy, outlier detection and computational time. We use here the generic notation $\widehat{\beta}$ and $\widehat{\Sigma}$ for the estimates obtained with a given procedure.

Estimation accuracy. Figure 1 (top) displays the distribution of the mean squared error for the estimation of β ($MSE(\beta) = \|\widehat{\beta} - \beta\|^2 / (pq)$) for a sample size of $n = 1000$. As expected, the $MSE(\beta)$ of the naive estimates dramatically increases as the proportion of outliers increases, whereas these of both the offline and online robust estimates remain stable. The Ridge regularization (same figure, bottom) tends to limit the effect of outliers on the naive estimator. The same pattern is observed for the WLS estimators (see Appendix B, Figure 7).

In Appendix B, Figure 8 displays the mean squared error for the estimation of Σ ($MSE(\Sigma) = \|\widehat{\Sigma} - \Sigma\|^2 / q^2$) for the same sample size. Again, the accuracy of naive estimates dramatically deteriorates as the proportion of outliers increases, whereas these of both the offline and online robust estimates remain stable. The Ridge regularization (which acts on β) has almost no effect on the quality of the respective estimates of Σ .

Effect of the sample size. We also studied the effect of the sample size n on the accuracy of the estimates. We only considered the regression coefficients β . Figure 2 shows the gain resulting from the use of robust estimates as opposed to the naive one, for the OLS contrast. As expected, this gain increases when the fraction of outliers is large ($f = 28\%$), but also increases when the sample size increases. Again, the same pattern is observed for the WLS contrast (see Appendix B, Figure 9).

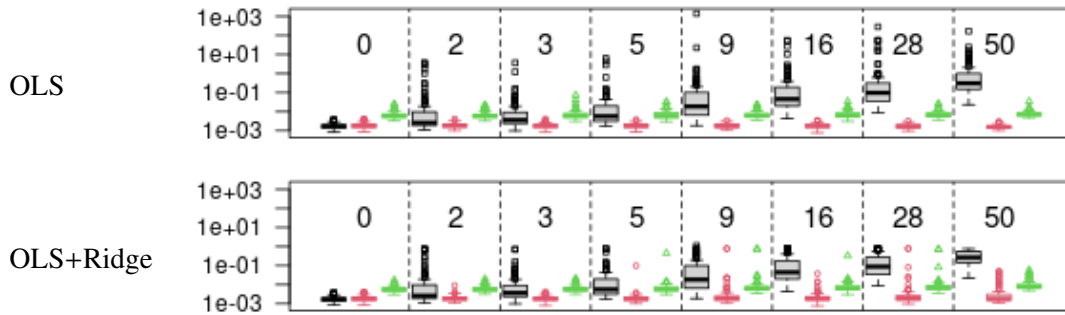


Figure 1: Mean squared error for the estimation of the regression coefficients β ($MSE(\beta)$) for OLS estimates without or with Ridge regularization for a sample size of $n = 1000$. The y-axis is log-scaled. Legend: \square = Naive, \circ = Offline, \triangle = Online. Number in each box = fraction f of outliers (in %).

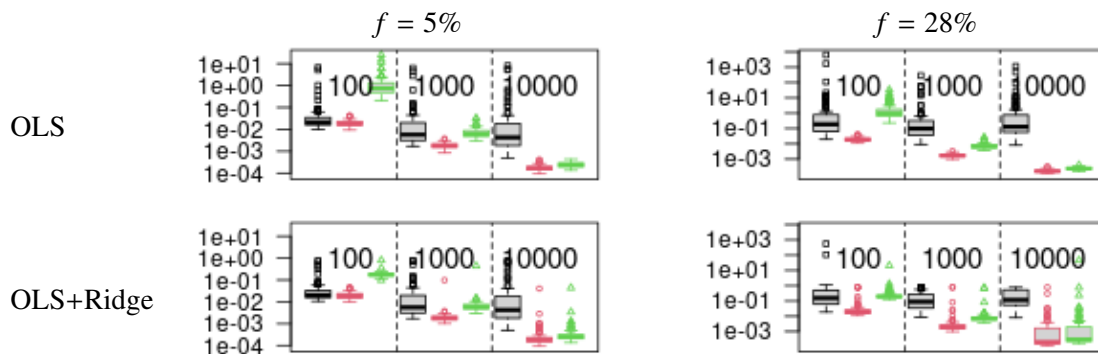


Figure 2: Mean squared error for the estimation of the regression coefficients β ($MSE(\beta)$) for OLS estimates with or without Ridge regularization for a fraction of outliers of $f = 5\%$ (left) and $f = 28\%$ (right). The y-axis is log-scaled. Legend: \square = Naive, \circ = Offline, \triangle = Online. Number in each box = sample size n .

Prediction accuracy. We then considered the precision of the prediction of the response variables in Y , measured with $MSE(Y) = \|\widehat{\beta}X - Y\|^2 / (nq)$, and focusing on non-outlier observations. Overall, the prediction accuracy inherits this of the regression coefficients β , so the behavior of $MSE(Y)$ is similar to this of $MSE(\beta)$. We devote Figure 10, which illustrates this point, to Appendix B.

Outlier detection. Based on the prediction \widehat{Y}_{ij} obtained for each observation $1 \leq i \leq n$ and response variable $1 \leq j \leq q$, we defined the estimated residual $\widehat{\epsilon}_{ij} = Y_{ij} - \widehat{Y}_{ij}$, based on which one may aim at deciding if observation i is an outlier or not. More specifically, based on the estimate $\widehat{\Sigma}$, and denoting by $\widehat{\epsilon}_i = [\widehat{\epsilon}_{ij}]_{1 \leq j \leq p}$ the vector of residuals for observation i , we used its $\widehat{\Sigma}^{-1}$ ℓ_2 -norm $\|\widehat{\epsilon}_i\|_{\widehat{\Sigma}^{-1}}^2$ as a score, which is expected to be small for non-outliers, and large for outliers.

Figure 3 gives the distribution of the area under the ROC curve (AUC) for the classification of outliers/non-outliers based on the proposed score for OLS estimates. Again, the performances of the naive procedure decreases as the fraction of outlier increases, although staying quite good up to an intermediate fraction of outliers ($f \approx 10 - 15\%$), whereas both robust procedures are not affected by their presence. Interestingly, the Ridge regularization does not improve the performances. Here also, OLS and WLS procedure behave similarly (see Appendix B, Figure 11).

OLS versus WLS. We then compared the performances of the robust OLS and WLS estimations, in terms of accuracy for the estimation of the regression coefficients β .

We computed the asymptotic variance given in Theorems 1 and 2 for the simulated covariance matrix X and the variance matrix Σ described in Appendix B and deduced the ratio between the two asymptotic variances $\mathbb{V}(\widehat{\beta}_{jk,\Sigma}) / \mathbb{V}(\widehat{\beta}_{jk})$ of each of the $p \times q = 100$ regression coefficients β_{jk} . These ratios turn out to be very homogeneous, with a mean of 95.7% and a standard deviation of 0.4%, indicating that using robust WLS rather than robust OLS induces a reduction

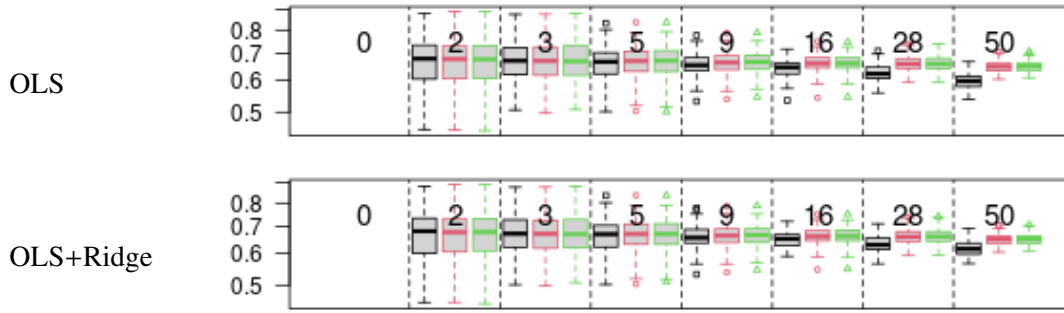


Figure 3: AUC for the detection of outliers for the OLS estimates with or without Ridge regularization for a sample size of $n = 1000$. Same legend as Figure 1. The box $f = 0$ is empty as no outlier exists.

of about 5% of the variance. As for the difference between the two un-normalized covariance matrices $n(\mathbb{V}(\widehat{\beta}) - \mathbb{V}(\widehat{\beta}_{\Sigma}))$, all its eigenvalues are verified to be non negative, ranging from about 300 to about 10^4 .

Figure 4 compares the $MSE(\beta)$ obtained with the OLS and WLS contrasts with the offline and online algorithms, without Ridge regularization. We observe that resorting to WLS does not really improve the accuracy of the offline estimates, and may even deteriorate it when the fraction of outliers is high. On the contrary, the online algorithm takes advantage of the WLS contrast to provide more precise estimates of β . The gain gets even higher as outliers get more numerous (up to a fraction of $f = 50\%$). A similar pattern is observed when using the Ridge regularization, as shown in Appendix B, Figure 12.

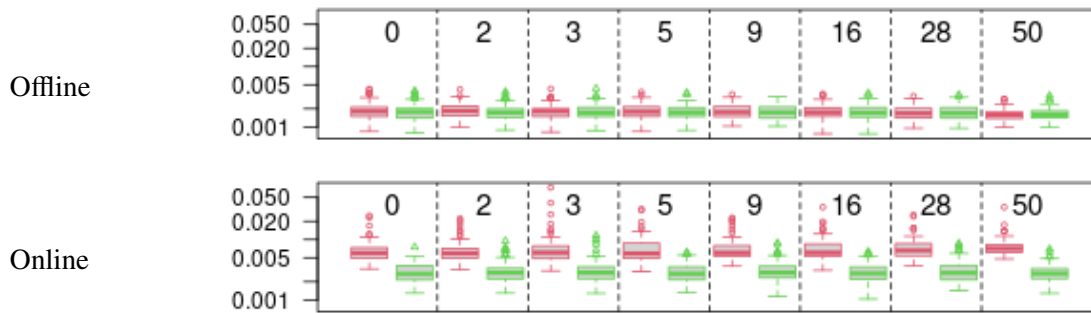


Figure 4: Comparison of the mean squared error for the estimation of the regression coefficients β ($MSE(\beta)$) of OLS and WLS estimates for Offline and Online procedures without Ridge regularization for a sample size of $n = 1000$. Legend: \circ = OLS, \triangle = WLS. Number in each box = fraction f of outliers (in %).

We also compared the empirical variance of the robust OLS and WLS estimates of the regression coefficients with their asymptotic counterparts, given by Theorems 1 and 2. Figure 5 presents this comparison for various sample sizes n and both the offline and online procedures. We observe that the offline procedure almost the asymptotic variance already for $n = 100$, whereas the offline procedure gets closer to it, but does still not reach it for $n = 10000$. Observe that Figure 5 compares variances whereas Figure 4 presents mean square errors.

Computation time. We recorded the mean CPU time for each procedure under each simulation configuration. The computational times does not significantly vary with the proportion of outliers (not shown) and is simply multiplied by a constant when using the Ridge regularization, due the fixed number of cross-validation steps.

Table 1 reports the results for a fraction of $f = 28\%$ outliers, without regularization. We observe the stability of the naive procedure, as opposed to the stochastic robust ones. More importantly, we observe the substantial gain of resorting to the online procedure, as opposed to offline (especially for the OLS contrast) when n increases.

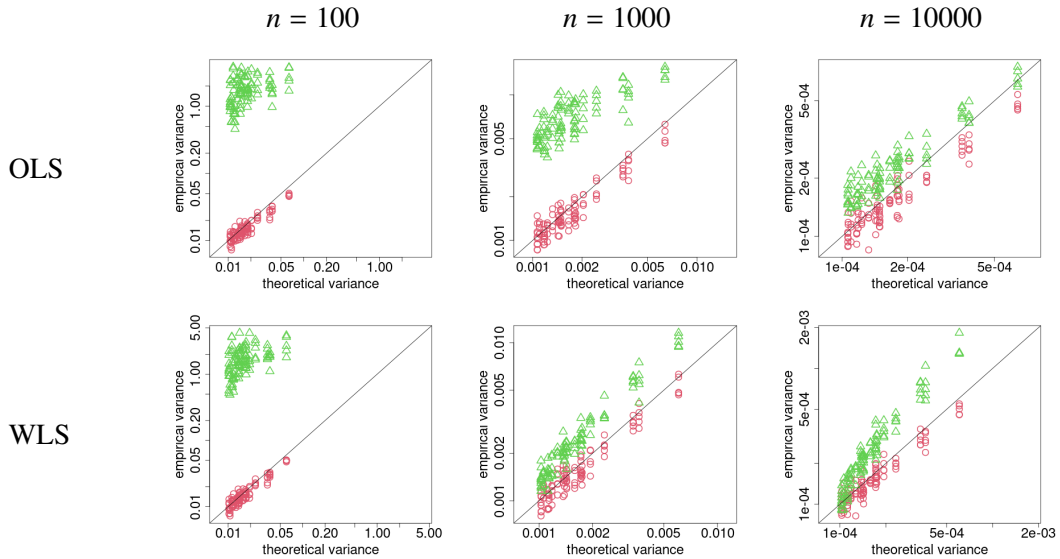


Figure 5: Comparison of the empirical variances of the $p \times q = 100$ regression coefficients estimates $\widehat{\beta}_{jk}$ with the asymptotic variance, with a fraction of 16% of outliers. Top: robust OLS procedure, bottom: Robust WLS procedure. Left: $n = 100$, center: $n = 1000$, right: $n = 10000$. Legend: \circ = offline, \triangle = online.

	OLS			WLS		
	Naive	Offline	Online	Naive	Offline	Online
$n = 100$	0.021	0.012	0.002	0.024	0.094	0.047
$n = 1000$	0.021	0.090	0.010	0.029	0.830	0.482
$n = 10000$	0.029	0.952	0.098	0.094	7.571	4.775

Table 1: Mean CPU time in seconds for the OLS and WLS estimation procedures without Ridge regularization.

7.2 Linear discriminant analysis

In the classification setting, the matrix X corresponds to the 0/1 matrix classifying each of the n observations into one of the q classes. Then, the regression coefficients β simply give the mean of each response variable in each class and Σ the variance matrix of the response variables. This setting corresponds to *linear* discriminant analysis (LDA) because Σ is supposed to be the same in all classes, as opposed to *quadratic* discriminant analysis (see Mardia et al., 1979, Chapters 11 and 12).

Simulation design. We used a design similar to this of Godichon-Baggioni and Robin (2024). We considered $p = 3$ classes and the same dimension $q = 20$ for the response vector. We used a fixed covariance matrix Σ . All entries of the $n \times p$ matrix X from Model (1) are 0's, but the first, second and third of the respective first, second and third column, respectively, which are filled with 1's. This amounts to consider three classes with equal size. Denoting by 1_q (resp. 0_q) the q -dimensional vector made of 1's (resp. 0's), we defined the mean vectors $\mu_1 = -\mu 1_q$, $\mu_2 = 0_q$ and $\mu_3 = +\mu 1_q$ in classes 1, 2, and 3 respectively. The matrix β from Model (1) is hence defined as $\beta^T = [\mu_1 \ \mu_2 \ \mu_3] = \mu[-1_q \ 0_q \ 1_q]$ and the parameter μ controls the difficulty of the classification problem. As for the outliers, we replace the $\mathcal{N}(0, \Sigma)$ distribution of the noise vector ϵ from Model (1) with a uniform distribution over the hypercube $(-20, 20)^q$.

Results. Based on the estimates $\widehat{\beta}$ (that is: $\widehat{\mu}_1, \dots, \widehat{\mu}_p$) and $\widehat{\Sigma}$, one may classify any observation i into to the class $k \in \llbracket p \rrbracket$ with nearest center, according to the ℓ_2 norm $\|Y_i - \widehat{\mu}_k\|_{\widehat{\Sigma}^{-1}}$. We compared the classification performances resulting from the different OLS estimation procedure, in terms of adjusted rand index (ARI). We focused on the non-outliers, as the classification of outliers is meaningless. The results are gathered in Figure 1.

Obviously, all procedures perform better when μ increases, as the classification task becomes easier. Apart from this main effect, the behavior is somewhat similar to Figure 3: the ARI of the robust methods remain stable, up to

$f = 50%$, whereas this of the naive procedure decreases when f increases. Still, the ARI remains high for the naive procedure even for a large fraction f , when μ is large, that is when the class are well distinct.

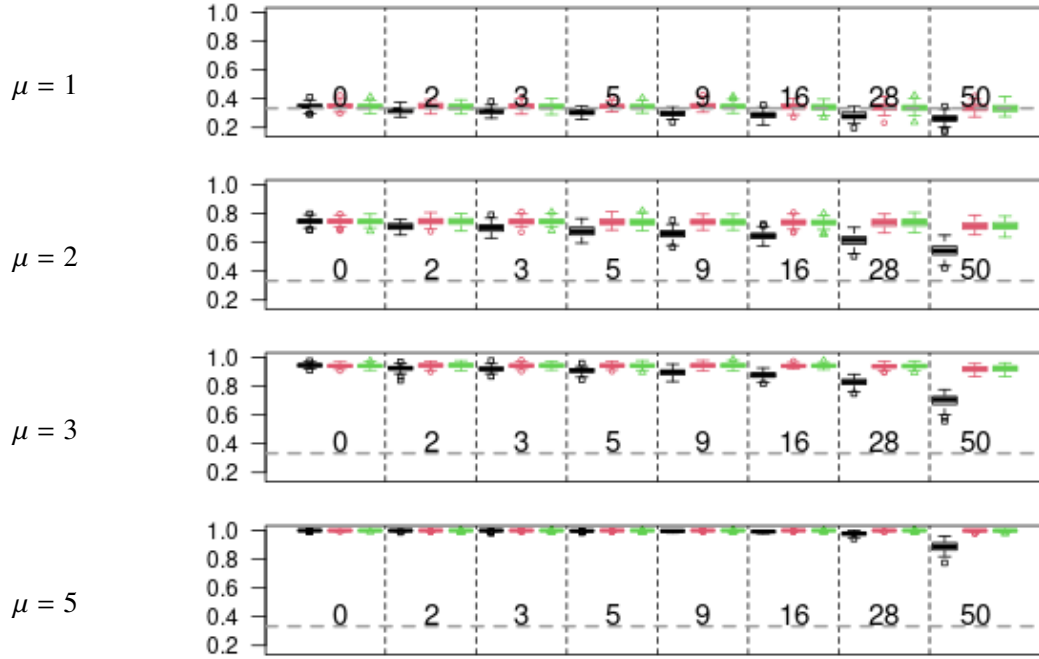


Figure 6: Adjusted rand index (ARI) for the OLS procedures. Same legend as Figure 1. Grey dashed line: $1/p = 1/3$ threshold.

8 Proofs

8.1 Proof of Theorem 1

First, let us vectorizing the algorithm. In this aim, denoting by $Y[j]$, $X[i]$ and $\beta[j, i]$ the coordinates, let us remark that one can rewrite rewrite G as

$$G(\beta) = \mathbb{E} \left[\sqrt{\sum_{j=1}^q \left(Y[j] - \sum_{i=1}^p \beta[j, i] X[i] \right)^2} \right]$$

and its gradient, in the basis of \mathbb{R}^{pq} , is given by

$$\nabla \tilde{G}(\varphi(\beta)) = -\mathbb{E} \left[\frac{1}{\|Y - \beta X\|} \underbrace{\begin{pmatrix} (Y[1] - \sum_{i=1}^p \beta[1, i] X[i]) X \\ \vdots \\ (Y[q] - \sum_{i=1}^p \beta[q, i] X[i]) X \end{pmatrix}}_{=: \nabla \tilde{g}(X, Y, \varphi(\beta))} \right]$$

In other word

$$\nabla \tilde{G}(\varphi(\beta)) = -\mathbb{E} \left[\frac{1}{\|Y - \beta X\|} (Y - \beta X) \odot X \right]$$

where for all $z \in \mathbb{R}^q$,

$$z \odot X = \begin{pmatrix} z[1]X \\ \vdots \\ z[q]X \end{pmatrix}$$

Observe that $\nabla \tilde{G}(\beta) = \varphi(\nabla G(\beta))$, so that regarding the stochastic gradient algorithm given by (8) or the stochastic gradient algorithm given by $\nabla \tilde{G}(\cdot)$ is equivalent; i.e considering the stochastic gradient algorithm given by

$$\beta_{n+1,\varphi} = \beta_{n,\varphi} + \gamma_{n+1} \nabla \tilde{g}(X_{n+1}, Y_{n+1}, \beta_{n,\varphi}),$$

and its weighted averaged version denoted $\bar{\beta}_{n,\varphi}$, one has $\beta_{n,\varphi} = \varphi(\beta_n)$ and $\bar{\beta}_{n,\varphi} = \varphi(\bar{\beta}_n)$. Then, the aim is to verify that $\beta_{n,\varphi}$ satisfies assumptions in Godichon-Baggioni (2021) among others.

Bounding the gradient: As soon as X admits a moment of order $2a$ for some $a \geq 1$, one has for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$\mathbb{E} \left[\|\nabla \tilde{g}(X, Y, \varphi(\beta))\|^{2a} \right] \leq q^a \mathbb{E} \left[\|X\|^{2a} \right]$$

and Assumption **(A1)** in Godichon-Baggioni (2021) is satisfied if X admits a moment of order 4.

Bounding the Hessian: Thanks to Lemma 8, the functional G is twice differentiable and its Hessian is of the form

$$\nabla^2 \tilde{G}(\varphi(\beta)) = \mathbb{E} \left[\frac{1}{\|Y - \beta X\|} (A - B(\beta)) \right]$$

with $A = \text{Diag}(XX^T, \dots, XX^T)$ and $B(\beta) = (B_{j,k}(\beta))$ where $B_{j,k}(\beta)_{j,k=1,\dots,q} \in \mathcal{M}_{q,q}(\mathbb{R})$ with

$$B_{j,k}(\beta) = \frac{(Y[j] - \sum_{i=1}^p X[j]\beta[j, i]) (Y[k] - \sum_{i=1}^p X[k]\beta[k, i])}{\|Y - \beta X\|} XX^T.$$

In other word, one can rewrite the Hessian as

$$\nabla^2 \tilde{G}(\varphi(\beta)) = \mathbb{E} \left[\frac{1}{\|Y - \beta X\|} \left(I_q - \frac{(Y - \beta X)(Y - \beta X)^T}{\|Y - \beta X\| \|Y - \beta X\|} \right) \otimes XX^T \right]$$

where \otimes is the Kronecker product. Observe that the Hessian is so at least semi-definite positive. Thanks to Lemma 8, one has

$$\begin{aligned} \|\nabla^2 \tilde{G}(\varphi(\beta))\|_F &\leq \left\| \mathbb{E} \left[\mathbb{E} \left[\frac{1}{\|Y - \beta X\|} \left(I_q - \frac{(Y - \beta X)(Y - \beta X)^T}{\|Y - \beta X\| \|Y - \beta X\|} \right) \otimes XX^T \mid X \right] \right] \right\|_F \\ &\leq \left(1 + \frac{\lambda_{\max}^{q/2}}{q 2^{q/2-1} \Gamma(q/2)} \frac{1}{q-1} \right) \mathbb{E} \left[\|I_q \otimes XX^T\|_F \right] \\ &= \left(1 + \frac{\lambda_{\max}^{q/2}}{\sqrt{q} 2^{q/2-1} \Gamma(q/2)} \frac{1}{q-1} \right) \mathbb{E} \left[\|X\|^2 \right] \end{aligned}$$

and the Hessian is so continuous, uniformly bounded, and satisfy Assumption **(A3)** in Godichon-Baggioni (2021). In addition, one has

$$\nabla^2 \tilde{G}(\varphi(\beta^*)) = \mathbb{E} \left[\frac{1}{\|\epsilon\|} \left(I_q - \frac{\epsilon}{\|\epsilon\|} \left(\frac{\epsilon}{\|\epsilon\|} \right)^T \right) \otimes XX^T \right].$$

Since ϵ and X are independent,

$$\nabla^2 \tilde{G}(\varphi(\beta^*)) = \mathbb{E} \left[\frac{1}{\|\epsilon\|} \left(I_q - \frac{\epsilon}{\|\epsilon\|} \left(\frac{\epsilon}{\|\epsilon\|} \right)^T \right) \right] \otimes \mathbb{E} [XX^T]$$

which is positive since Σ and $\mathbb{E} [XX^T]$ are positive. Assumption **(A2)** in Godichon-Baggioni (2021) is so satisfied. Let us now prove that the Hessian is Lipschitz. First notice that for all $\beta, \beta' \in \mathcal{M}_{q,p}(\mathbb{R})$, one has

$$\begin{aligned} \|\nabla^2 \tilde{G}(\varphi(\beta)) - \nabla^2 \tilde{G}(\varphi(\beta'))\|_F &\leq \mathbb{E} \left[\left\| \frac{1}{\|Y - \beta X\|} - \frac{1}{\|Y - \beta' X\|} \right\| \|I_q \otimes XX^T\|_F \right] =: R_1 \\ &\quad + \underbrace{\mathbb{E} \left[\frac{1}{\|Y - \beta' X\|} \left\| \frac{(Y - \beta X)(Y - \beta X)^T}{\|Y - \beta X\| \|Y - \beta X\|} - \frac{(Y - \beta' X)(Y - \beta' X)^T}{\|Y - \beta' X\| \|Y - \beta' X\|} \right\| \|X\|^2 \right]}_{=: R_2} \end{aligned}$$

First observe that the gradient of the function $\beta \mapsto \frac{1}{\|Y - \beta X\|}$ is given by $\frac{Y - \beta X}{\|Y - \beta X\|^3} \odot X$. Then, thanks to Cauchy-Schwarz inequality coupled with Lemma 8,

$$R_1 \leq \int_0^1 \mathbb{E} \left[\frac{1}{\|Y - (\beta + t(\beta' - \beta))X\|^2} \sqrt{q} \|X\|^3 \right] dt \|\beta - \beta'\| \leq \left(1 + \frac{\lambda_{\max}^{q/2}}{\sqrt{q} 2^{q/2} \Gamma(q/2)} \frac{1}{q-2} \right) \mathbb{E} [\|X\|^3] \|\beta - \beta'\|.$$

Second, observe that (see Cardot et al. (2012) page 25)

$$\left\| \frac{Y - \beta X}{\|Y - \beta X\|} - \frac{Y - \beta' X}{\|Y - \beta' X\|} \right\| \leq \frac{\|\beta X - \beta' X\|}{\|Y - \beta' X\|} \leq \frac{\|X\|}{\|Y - \beta' X\|} \|\beta - \beta'\|_F.$$

Then, since $aa^T - bb^T = (a - b)a^T - b(b^T - a^T)$ and using Cauchy Schwarz inequality,

$$\begin{aligned} R_2 &\leq \mathbb{E} \left[\mathbb{E} \left[\left(\frac{1}{\|Y - \beta' X\|^2} + \frac{1}{\|Y - \beta X\| \|Y - \beta' X\|} \right) |X| \right] \|X\|^3 \right] \|\beta - \beta'\|_F \\ &\leq 2 \left(1 + \frac{\lambda_{\max}^{q/2}}{\sqrt{q} 2^{q/2} \Gamma(q/2)} \frac{1}{q-2} \right) \mathbb{E} [\|X\|^3] \|\beta - \beta'\|_F \end{aligned}$$

Then,

$$\|\nabla^2 \tilde{G}(\varphi(\beta)) - \nabla^2 \tilde{G}(\varphi(\beta'))\|_F \leq 3 \left(1 + \frac{\sqrt{q} \lambda_{\max}^{q/2}}{2^{q/2} \Gamma(q/2)} \frac{1}{q-2} \right) \mathbb{E} [\|X\|^3] \|\beta - \beta'\|_F$$

and Assumption **(A5)** in Godichon-Baggioni (2021) is so satisfied.

Convergence of the estimates: Assumptions in Godichon-Baggioni (2019b, 2021) are fulfilled and it comes

$$\begin{aligned} \|\beta_n - \beta^*\|_F^2 &= \|\beta_{n,\varphi} - \varphi(\beta^*)\|^2 = O\left(\frac{\ln n}{n^\gamma}\right) \quad a.s \\ \|\bar{\beta}_n - \beta^*\|_F^2 &= \|\bar{\beta}_{n,\varphi} - \varphi(\beta^*)\|^2 = O\left(\frac{\ln n}{n}\right) \quad a.s \end{aligned}$$

In addition, one has

$$\sqrt{n}(\varphi(\bar{\beta}_n) - \varphi(\beta^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, H^{-1} V H^{-1})$$

where $H = \mathbb{E} \left[\frac{1}{\|\epsilon\|} \left(I_q - \frac{\epsilon}{\|\epsilon\|} \left(\frac{\epsilon}{\|\epsilon\|} \right)^T \right) \right] \otimes \mathbb{E} [XX^T]$ and since ϵ and X are independent,

$$\begin{aligned} V &= \mathbb{E} \left[\nabla \tilde{g}(X, Y, \beta^*) \nabla \tilde{g}(X, Y, \beta^*)^T \right] = \mathbb{E} \left[\frac{1}{\|Y - \beta^* X\|^2} (Y - \beta^* X)(Y - \beta^* X)^T \otimes XX^T \right] \\ &= \mathbb{E} \left[\frac{\epsilon}{\|\epsilon\|} \frac{\epsilon^T}{\|\epsilon\|} \right] \otimes \mathbb{E} [XX^T]. \end{aligned}$$

Then, since $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, it comes

$$\sqrt{n}(\varphi(\bar{\beta}_n) - \varphi(\beta^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0, \mathbb{E} \left[\frac{1}{\|\epsilon\|} \left(I_q - \frac{\epsilon}{\|\epsilon\|} \frac{\epsilon^T}{\|\epsilon\|} \right) \right]^{-1} \mathbb{E} \left[\frac{\epsilon}{\|\epsilon\|} \frac{\epsilon^T}{\|\epsilon\|} \right] \mathbb{E} \left[\frac{1}{\|\epsilon\|} \left(I_q - \frac{\epsilon}{\|\epsilon\|} \frac{\epsilon^T}{\|\epsilon\|} \right) \right]^{-1} \otimes (\mathbb{E} [XX^T])^{-1} \right).$$

Finally, thanks to Godichon-Baggioni (2021), there are positive constants C_0, C_1 such that for all $n \geq 1$

$$\begin{aligned} \|\beta_n - \beta^*\|_F^2 &= \|\beta_{n,\varphi} - \varphi(\beta)\|^2 \leq \frac{C_0}{n^\gamma} \\ \|\bar{\beta}_n - \beta^*\|_F^2 &= \|\bar{\beta}_{n,\varphi} - \varphi(\beta)\|^2 \leq \frac{C_1}{n}. \end{aligned}$$

8.2 Proof of Theorem 2

Writing in the basis of \mathbb{R}^{pq} : Vectorizing β , the gradient in the basis of \mathbb{R}^{pq} can be written as

$$\nabla \tilde{G}_\Sigma(\varphi(\beta)) = -\mathbb{E} \left[\underbrace{\frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} (\Sigma^{-1}(Y - \beta X) \odot X)}_{=: \nabla \tilde{g}_\Sigma(X, Y, \beta)} \right].$$

Hence, here again, the stochastic gradient given by (10) and the one derived from $\nabla \tilde{G}_\Sigma$ are equivalent. Indeed, considering the stochastic gradient algorithm defined recursively for all $n \geq 0$ by

$$\beta_{n+1, \Sigma, \varphi} = \beta_{n, \Sigma, \varphi} + \gamma_{n+1} \nabla \tilde{g}_\Sigma(X_{n+1}, Y_{n+1}, \beta_{n, \Sigma, \varphi})$$

and denoting by $\bar{\beta}_{n, \Sigma, \varphi}$ its weighted averaged version, one has $\beta_{n, \Sigma, \varphi} = \varphi(\beta_{n, \Sigma})$ and $\bar{\beta}_{n, \Sigma, \varphi} = \varphi(\bar{\beta}_{n, \Sigma})$. Then, as in the proof of Theorem 1, the aim is to check that assumptions in Godichon-Baggioni (2021) (among others) are satisfied.

Bounding the gradient: As soon as X admits a moment of order $2a$ for some $a \geq 1$, one has for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$

$$\mathbb{E} \left[\|\nabla \tilde{g}_\Sigma(X, Y, \varphi(\beta))\|^{2a} \right] \leq \frac{q}{\sqrt{\lambda_{\min}(\Sigma)}} \mathbb{E} \left[\|X\|^{2a} \right]$$

and Assumption (A1) in Godichon-Baggioni (2021) is so satisfied.

Bounding the Hessian The Hessian is given by

$$\nabla^2 \tilde{G}_\Sigma(\varphi(\beta)) = \mathbb{E} \left[\frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} \left(\Sigma^{-1} - \frac{\Sigma^{-1}(Y - \beta X)(Y - \beta X)^T \Sigma^{-1}}{\|Y - \beta X\|_{\Sigma^{-1}}^2} \right) \otimes XX^T \right].$$

Moreover, as in Theorem 1, one has with the help of Lemma 9,

$$\begin{aligned} \|\nabla^2 \tilde{G}_\Sigma(\varphi(\beta))\|_F &\leq \mathbb{E} \left[\mathbb{E} \left[\frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} |X| \right] \|\Sigma^{-1}\|_F \|X\|^2 \right] \\ &\leq \left(1 + \frac{1}{q 2^{q/2-1} \Gamma(q/2)} \frac{1}{q-1} \right) \|\Sigma^{-1}\|_F \mathbb{E} \left[\|X\|^2 \right] \end{aligned}$$

and the Hessian is so continuous and satisfies Assumption (A3) in Godichon-Baggioni (2021). In addition, denoting by $\tilde{\epsilon} = \Sigma^{-1/2} \epsilon$, observe that

$$H_\Sigma := \nabla^2 \tilde{G}(\varphi(\beta^*)) = \mathbb{E} \left[\frac{1}{\|\tilde{\epsilon}\|} \left(\Sigma^{-1} - \Sigma^{-1/2} \frac{\tilde{\epsilon} \tilde{\epsilon}^T}{\|\tilde{\epsilon}\|} \Sigma^{-1/2} \right) \otimes XX^T \right].$$

Then, since X is independent from $\tilde{\epsilon}$, and remarking that $\frac{\tilde{\epsilon}}{\|\tilde{\epsilon}\|}$ follows an uniform law on the sphere independent from $\|\tilde{\epsilon}\|$, it comes

$$H_\Sigma := \nabla^2 \tilde{G}(\varphi(\beta^*)) = \mathbb{E} \left[\frac{1}{\|\tilde{\epsilon}\|} \right] \left(\Sigma^{-1} - \Sigma^{-1/2} \mathbb{E} \left[\frac{\tilde{\epsilon} \tilde{\epsilon}^T}{\|\tilde{\epsilon}\|} \right] \Sigma^{-1/2} \right) \otimes \mathbb{E} [XX^T]$$

leading to

$$H_\Sigma = \mathbb{E} \left[\frac{1}{\|\mathcal{N}(0, I_q)\|} \right] \frac{q-1}{q} \Sigma^{-1} \otimes \mathbb{E} [XX^T]$$

which is positive definite as soon as $\mathbb{E} [XX^T]$ is. Then, Assumption (A2) in Godichon-Baggioni (2021) is so satisfied.

Note that, for $q > 1$, if $Z \sim \mathcal{N}(0, I_q)$, $U = \|Z\| \sim \chi(q)$ (chi-distribution) with pdf: $f_q(u) = \frac{u^{q-1} e^{-u^2/2}}{2^{q/2-1} \Gamma(\frac{q}{2})}$ for $u \geq 0$. Hence,

$$\mathbb{E} \left[\frac{1}{\|\mathcal{N}(0, I_q)\|} \right] = \mathbb{E} \left(\frac{1}{U} \right) = \int_0^\infty \frac{1}{u} f_q(u) du = \int_0^\infty \frac{2^{(q-1)/2-1} \Gamma(\frac{q-1}{2})}{2^{q/2-1} \Gamma(\frac{q}{2})} f_{q-1}(u) du = \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{q}{2})}.$$

The Hessian is Lipschitz: Let us now prove that the Hessian is Lipschitz. Note that for all $\beta, \beta' \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$\begin{aligned} \|\nabla^2 \tilde{G}_\Sigma(\varphi(\beta)) - \nabla^2 \tilde{G}_\Sigma(\varphi(\beta'))\|_F &\leq \mathbb{E} \left[\left\| \frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} - \frac{1}{\|Y - \beta' X\|_{\Sigma^{-1}}} \right\| \|\Sigma^{-1}\|_F \|X\|^2 \right] =: R'_1 \\ &+ \mathbb{E} \left[\frac{1}{\|Y - \beta' X\|_{\Sigma^{-1}}} \left\| \frac{\Sigma^{-1}(Y - \beta X)(Y - \beta X)^T \Sigma^{-1}}{\|Y - \beta X\|_{\Sigma^{-1}}^2} - \frac{\Sigma^{-1}(Y - \beta' X)(Y - \beta' X)^T \Sigma^{-1}}{\|Y - \beta' X\|_{\Sigma^{-1}}^2} \right\| \|X\|^2 \right] =: R'_2. \end{aligned}$$

Analogously to the proof of Theorem 1, the gradient of the functional $\beta \rightarrow \frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}}$ is given by $\frac{\Sigma^{-1}(Y - \beta X)}{\|Y - \beta X\|_{\Sigma^{-1}}^3} \odot X$, leading to

$$R'_1 \leq \int_0^1 \mathbb{E} \left[\frac{\|\Sigma^{-1/2}\|_F}{\|Y - (\beta + t(\beta' - \beta))X\|_{\Sigma^{-1}}^2} \|\Sigma^{-1}\|_F \|X\|^3 \right] dt \leq \left(1 + \frac{1}{q2^{q/2}\Gamma(q/2)} \frac{1}{q-2} \right) \|\Sigma^{-1/2}\|_F \|\Sigma^{-1}\|_F \mathbb{E} [\|X\|^3]$$

In addition, since for all $\beta, \beta' \in \mathcal{M}_{q,p}(\mathbb{R})$

$$\begin{aligned} \left\| \frac{\Sigma^{-1}(Y - \beta X)}{\|Y - \beta X\|_{\Sigma^{-1}}} - \frac{\Sigma^{-1}(Y - \beta' X)}{\|Y - \beta' X\|_{\Sigma^{-1}}} \right\| &\leq \left\| \Sigma^{-1/2} \frac{\Sigma^{-1/2}(Y - \beta X)}{\|\Sigma^{-1/2}(Y - \beta X)\|} - \frac{\Sigma^{-1}(Y - \beta' X)}{\|\Sigma^{-1/2}(Y - \beta' X)\|} \right\| \\ &\leq \|\Sigma^{-1/2}\|_F \frac{\|\Sigma^{-1/2}(\beta X - \beta' X)\|}{\|Y - \beta' X\|_{\Sigma^{-1}}} \leq \frac{\|\Sigma^{-1/2}\|_F^2 \|X\|}{\|Y - \beta' X\|_{\Sigma^{-1}}} \|\beta - \beta'\|_F \end{aligned}$$

Then, since $aa^T - bb^T = (a - b)a^T - b(b^T - a^T)$ and using Cauchy Schwarz inequality,

$$\begin{aligned} R'_2 &\leq \|\Sigma^{-1/2}\|_F^3 \mathbb{E} \left[\mathbb{E} \left[\left(\frac{1}{\|Y - \beta' X\|_{\Sigma^{-1}}^2} + \frac{1}{\|Y - \beta X\|_{\Sigma^{-1}} \|Y - \beta' X\|_{\Sigma^{-1}}} \right) \|X\| \|X\|^3 \right] \|\beta - \beta'\|_F \right] \\ &\leq \left(1 + \frac{1}{q2^{q/2}\Gamma(q/2)} \frac{1}{q-2} \right) \|\Sigma^{-1/2}\|_F^3 \mathbb{E} [\|X\|^3] \|\beta - \beta'\|_F \end{aligned}$$

Then, since $\|\Sigma^{-1/2}\|_F \|\Sigma^{-1}\|_F \leq \|\Sigma^{-1/2}\|_F^3$

$$\|\nabla^2 \tilde{G}_\Sigma(\varphi(\beta)) - \nabla^2 \tilde{G}_\Sigma(\varphi(\beta'))\|_F \leq 3 \left(1 + \frac{1}{q2^{q/2}\Gamma(q/2)} \frac{1}{q-2} \right) \|\Sigma^{-1/2}\|_F^3 \mathbb{E} [\|X\|^3] \|\beta - \beta'\|_F$$

and Assumption **(A5)** in Godichon-Baggioni (2021) is so satisfied.

Convergence of the estimates: Assumptions in Godichon-Baggioni (2019b, 2021) are fulfilled and it comes

$$\begin{aligned} \|\beta_{n,\Sigma} - \beta^*\|_F^2 &= \|\beta_{n,\varphi} - \varphi(\beta^*)\|_F^2 = O\left(\frac{\ln n}{n^\gamma}\right) \quad a.s \\ \|\bar{\beta}_{n,\Sigma} - \beta^*\|_F^2 &= \|\bar{\beta}_{n,\varphi} - \varphi(\beta^*)\|_F^2 = O\left(\frac{\ln n}{n}\right) \quad a.s \end{aligned}$$

In addition, one has

$$\sqrt{n}(\varphi(\bar{\beta}_{n,\Sigma}) - \varphi(\beta^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, H_\Sigma^{-1} V_\Sigma H_\Sigma^{-1})$$

where $H_\Sigma := \nabla^2 \tilde{G}_\Sigma(\varphi(\beta^*)) = \frac{1}{\sqrt{2}} \frac{(q-1)\Gamma(\frac{q-1}{2})}{q\Gamma(\frac{q}{2})} \Sigma^{-1} \otimes \mathbb{E}[XX^T]$ and

$$\begin{aligned} V_\Sigma &= \mathbb{E} \left[\nabla \tilde{g}_\Sigma(X, Y, \beta^*) \nabla \tilde{g}_\Sigma(X, Y, \beta^*)^T \right] = \mathbb{E} \left[\frac{1}{\|Y - \beta^* X\|_{\Sigma^{-1}}^2} \Sigma^{-1} (Y - \beta^* X)(Y - \beta^* X)^T \Sigma^{-1} \otimes XX^T \right] \\ &= \mathbb{E} \left[\frac{1}{\|\epsilon\|_{\Sigma^{-1}}^2} \Sigma^{-1} \epsilon \epsilon^T \Sigma^{-1} \otimes XX^T \right]. \end{aligned}$$

Then, since ϵ and X are independent, and since $\frac{\Sigma^{-1/2}\epsilon}{\|\epsilon\|_{\Sigma^{-1}}} \sim \mathcal{U}(\mathcal{S}^q)$ is independent from $\|\epsilon\|_{\Sigma^{-1}}$, it comes

$$V_{\Sigma} = \Sigma^{-1/2} \mathbb{E} \left[\frac{\Sigma^{-1/2}\epsilon}{\|\epsilon\|_{\Sigma^{-1}}} \left(\frac{\Sigma^{-1/2}\epsilon}{\|\epsilon\|_{\Sigma^{-1}}} \right)^T \right] \Sigma^{-1/2} \otimes \mathbb{E} [XX^T] = \frac{1}{q} \Sigma^{-1} \otimes \mathbb{E} [XX^T]$$

Then, since $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, it comes

$$\sqrt{n} (\varphi(\bar{\beta}_{n,\Sigma}) - \varphi(\beta^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0, \frac{2q}{(q-1)^2} \frac{\Gamma(\frac{q}{2})^2}{\Gamma(\frac{q-1}{2})^2} \Sigma \otimes (\mathbb{E} [XX^T])^{-1} \right).$$

Finally, thanks to Godichon-Baggioni (2021), there are positive constants $C_{0,\Sigma}, C_{1,\Sigma}$ such that for all $n \geq 1$

$$\begin{aligned} \|\beta_{n,\Sigma} - \beta^*\|_F^2 &= \|\beta_{n,\Sigma,\varphi} - \varphi(\beta)\|_F^2 \leq \frac{C_{0,\Sigma}}{n^\gamma} \\ \|\bar{\beta}_{n,\Sigma} - \beta^*\|_F^2 &= \|\bar{\beta}_{n,\Sigma,\varphi} - \varphi(\beta)\|_F^2 \leq \frac{C_{1,\Sigma}}{n}. \end{aligned}$$

8.3 Proof of Theorem 3

In all the sequel, for the sake of simplicity, we denote $\beta_t := \beta_{n,t}$ for all $t \geq 0$. Let us first give a useful result on the Hessian of G_n .

Lemma 1. *Suppose that $\sum_{i=1}^n X_i X_i^T$ is positive, then for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$, the matrix $\sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta X_i\|}$ is positive.*

Proof of Lemma 1. Observe that $\sum_{i=1}^n X_i X_i^T$ positive implies necessary that there are p indices i_1, \dots, i_p such that $\{X_{i_1}, \dots, X_{i_p}\}$ are linearly independent. Then, denoting by \succcurlyeq the order relation between symmetric matrices, one has

$$\sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta X_i\|} \succcurlyeq \sum_{j=1}^p \frac{X_{i_j} X_{i_j}^T}{\|Y_{i_j} - \beta X_{i_j}\|} \succcurlyeq \sum_{j=1}^p \frac{X_{i_j} X_{i_j}^T}{\|Y_{i_j}\| + \|\beta\|_F \|X_{i_j}\|} > 0.$$

□

Let us now follow the scheme of proof in Beck and Sabach (2015). In this aim, let us first prove the following Lemma (which can be seen as a translation of Lemma 5.1 in Beck and Sabach (2015)):

Lemma 2. *For all $\beta' \in \mathcal{M}_{q,p}(\mathbb{R})$,*

$$2G_n(T_n(\beta')) \leq 2G_n(\beta') + 2 \langle \nabla G_n(\beta'), T_n(\beta') - \beta' \rangle_F + (\varphi(T_n(\beta')) - \varphi(\beta'))^T L_n(\beta') (\varphi(T_n(\beta')) - \varphi(\beta')).$$

Proof of Lemma 2. Let us consider the functional $h_n : \mathcal{M}_{q,p}(\mathbb{R}) \times \mathcal{M}_{q,p}(\mathbb{R}) \rightarrow \mathbb{R}_+$ defined for all β, β' by

$$h_n(\beta, \beta') = \frac{1}{n} \sum_{i=1}^n \frac{\|Y_i - \beta X_i\|^2}{\|Y_i - \beta' X_i\|}.$$

Then, the functional $h_n(\cdot, \beta')$ is quadratic, and writting it in the canonical basis of \mathbb{R}^{pq} , the Hessian is given by $2L_n(\beta')$ with $L_n(\beta') := I_q \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta' X_i\|}$. Then, denoting by $\nabla_{\beta} h_n(\cdot, \cdot)$ the gradient of h_n with respect to the first coordinate argument, and with the help of a Taylor's expansion, it comes

$$h_n(\beta, \beta') = h_n(\beta', \beta') + \langle \nabla_{\beta} h_n(\beta', \beta'), \beta - \beta' \rangle + (\varphi(\beta) - \varphi(\beta'))^T L_n(\beta') (\varphi(\beta) - \varphi(\beta'))$$

Observe that $h(\beta', \beta') = G_n(\beta')$ and that $\nabla_{\beta} h_n(\beta', \beta') = 2\nabla G_n(\beta')$, so that

$$h_n(\beta, \beta') = G_n(\beta') + 2 \langle \nabla G_n(\beta'), \beta - \beta' \rangle_F + (\varphi(\beta) - \varphi(\beta'))^T L_n(\beta') (\varphi(\beta) - \varphi(\beta'))$$

Then, taking $\beta = T_n(\beta')$, it comes

$$h_n(T_n(\beta'), \beta') = G_n(\beta') + 2 \langle \nabla G_n(\beta'), T_n(\beta') - \beta' \rangle_F + (\varphi(T_n(\beta')) - \varphi(\beta'))^T L_n(\beta') (\varphi(T_n(\beta')) - \varphi(\beta')).$$

In addition, since for all $a \in \mathbb{R}$ and $b > 0$ one has $\frac{a^2}{b} \geq 2a - b$, it comes that $h_n(\beta, \beta') \geq 2G_n(\beta) - G_n(\beta')$. Then

$$2G_n(T_n(\beta')) \leq 2G_n(\beta') + 2 \langle \nabla G_n(\beta'), T_n(\beta') - \beta' \rangle_F + (\varphi(T_n(\beta')) - \varphi(\beta'))^T L_n(\beta') (\varphi(T_n(\beta')) - \varphi(\beta')).$$

□

Let us now prove the following Lemma (which can be seen as a translation of Lemma 5.2 in Beck and Sabach (2015)):

Lemma 3. For all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$G_n(\beta_{t+1}) \leq G_n(\beta) + \frac{1}{2} (\varphi(\beta_t) - \varphi(\beta))^T L_n(\beta_t) (\varphi(\beta_t) - \varphi(\beta)) - \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta)).$$

Proof of Lemma 3. With the help of Lemma 2, one has taking $\beta' = \beta_t$,

$$G_n(\beta_{t+1}) \leq G_n(\beta_t) + \langle \nabla G_n(\beta_t), \beta_{t+1} - \beta_t \rangle_F + \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta_t))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta_t)).$$

In addition, since G_n is convex, one has for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$, $G_n(\beta_t) \leq G_n(\beta) + \langle \nabla G_n(\beta_t), \beta_t - \beta \rangle_F$, leading to

$$\begin{aligned} G_n(\beta_{t+1}) &\leq G_n(\beta) + \langle \nabla G_n(\beta_t), \beta_t - \beta \rangle_F + \langle \nabla G_n(\beta_t), \beta_{t+1} - \beta_t \rangle_F \\ &\quad + \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta_t))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta_t)) \\ &= G_n(\beta) + \langle \nabla G_n(\beta_t), \beta_{t+1} - \beta \rangle_F + \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta_t))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta_t)). \end{aligned}$$

In addition, remark that equality (14) can be written as

$$\nabla G_n(\beta_t) = -(\beta_{t+1} - \beta_t) \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_t X_i\|}$$

which can be written, in the basis of \mathbb{R}^{pq} as

$$\nabla \tilde{G}_n(\varphi(\beta_t)) = -L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta_t))$$

Then, following Beck and Sabach (2015),

$$\begin{aligned} -\langle \nabla G_n(\beta_t), \beta_{t+1} - \beta \rangle_F &= (\varphi(\beta_{t+1}) - \varphi(\beta_t))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta)) \\ &= \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta)) + \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta_t))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta)) \\ &\quad - \frac{1}{2} (\varphi(\beta_t) - \varphi(\beta))^T L_n(\beta_t) (\varphi(\beta_t) - \varphi(\beta)). \end{aligned}$$

Then, for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$G_n(\beta_{t+1}) \leq G_n(\beta) + \frac{1}{2} (\varphi(\beta_t) - \varphi(\beta))^T L_n(\beta_t) (\varphi(\beta_t) - \varphi(\beta)) - \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta)).$$

□

We can now prove Theorem 3.

Proof of Theorem 3. First, with the help of Lemma 3, it comes taking $\beta = \beta_t$

$$G_n(\beta_{t+1}) \leq G_n(\beta_t) - \frac{1}{2} (\varphi(\beta_{t+1}) - \varphi(\beta_t))^T L_n(\beta_t) (\varphi(\beta_{t+1}) - \varphi(\beta_t)). \quad (24)$$

and since $L_n(\beta_t)$ is positive, it comes that the sequence $(G_n(\beta_t))_{t \geq 0}$ is decreasing, and since G_n is lower bounded by 0, it means that it converges to a non negative constant G_∞ . Now, since the functional G_n is strictly convex (as $\frac{1}{n} \sum_{i=1}^n X_i X_i^T$ is assumed to be positive), it means that there are a positive constant r_0 and a rank t_0 such that for all $t \geq t_0$, $\|\beta_t\|_F \leq r_0$, and in a particular case, the sequence $(\beta_t)_t$ is bounded, and let us denote by \mathcal{K} a compact containing this sequence. Observe that on this compact, $L_n(\cdot)$ is uniformly lower bounded by a positive matrix $L_{\mathcal{K}}$. Let us now consider a converging subsequence β_{u_t} of (β_t) , and let us denote by β_u its limit. Then, since $G(\beta_t)$ converges to G_∞ , inequality (24) can be written as

$$\begin{aligned} &0 \frac{1}{2} \lambda_{\min}(L_{\mathcal{K}}) \|\beta_{u_t} - \beta_u\|_F^2 \\ &\leq \frac{1}{2} (\varphi(\beta_{u_t}) - \varphi(\beta_u))^T L_n(\beta_t) (\varphi(\beta_{u_t}) - \varphi(\beta_u)) \leq G_n(\beta_{u_t}) - G_n(\beta_{u_{t+1}}) \xrightarrow{t \rightarrow +\infty} 0 \end{aligned}$$

Then, since $\lambda_{\min}(L_{\mathcal{K}}) > 0$, $\|T_n(\beta_{u_t}) - \beta_{u_t}\|$ converges to 0, and β_u is so the fixed point of T_n . In addition,

$$\lim_{t \rightarrow \infty} G_n(\beta_t) = \lim_{t \rightarrow \infty} G_n(\beta_{u_t}) = G_n(\hat{\beta}_n)$$

where $\hat{\beta}_n$ is the minimizer of G_n . Then, thanks to Lemma 1, G_n is locally strongly convex, which implies that β_t converges to $\hat{\beta}_n$. \square

8.4 Proof of Theorem 4

In all the sequel, for the sake of simplicity, we denote $\beta_{t,\Sigma} := \beta_{n,t,\Sigma}$ for all $t \geq 0$. The aim is to adapt the proof of Theorem 3 in this context. First, observe that Lemma 1 can be directly derivated. We now prove the analogue of Lemma 2 and 3.

Lemma 4. For all $\beta' \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$2G_{n,\Sigma}(T_{n,\Sigma}(\beta')) \leq 2G_{n,\Sigma}(\beta') + 2 \langle \nabla G_{n,\Sigma}(\beta'), T_{n,\Sigma}(\beta') - \beta' \rangle_F + (\varphi(T_{n,\Sigma}(\beta')) - \varphi(\beta'))^T L_{n,\Sigma}(\beta') (\varphi(T_{n,\Sigma}(\beta')) - \varphi(\beta')).$$

Proof of Lemma 4. Let us consider the functional $h_{n,\Sigma} : \mathcal{M}_{q,p}(\mathbb{R}) \times \mathcal{M}_{q,p}(\mathbb{R}) \rightarrow \mathbb{R}_+$ defined for all β, β' by

$$h_{n,\Sigma}(\beta, \beta') = \frac{1}{n} \sum_{i=1}^n \frac{\|Y_i - \beta X_i\|_{\Sigma^{-1}}^2}{\|Y_i - \beta' X_i\|_{\Sigma^{-1}}}.$$

Then, the functional $h_{n,\Sigma}(\cdot, \beta')$ is quadratic, and writing it in the canonical basis of \mathbb{R}^{pq} , the Hessian is given by $2L_{n,\Sigma}(\beta')$ with $L_{n,\Sigma}(\beta') := \Sigma^{-1} \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta' X_i\|_{\Sigma^{-1}}}$. Then, with analogous developpement as in the proof of Lemma 2, for all $\beta, \beta' \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$h_{n,\Sigma}(\beta, \beta') = G_{n,\Sigma}(\beta') + 2 \langle \nabla G_{n,\Sigma}(\beta'), \beta - \beta' \rangle_F + (\varphi(\beta) - \varphi(\beta'))^T L_{n,\Sigma}(\beta') (\varphi(\beta) - \varphi(\beta'))$$

and taking $\beta = T_{n,\Sigma}(\beta')$, it comes

$$2G_{n,\Sigma}(T_{n,\Sigma}(\beta')) \leq 2G_{n,\Sigma}(\beta') + 2 \langle \nabla G_{n,\Sigma}(\beta'), T_{n,\Sigma}(\beta') - \beta' \rangle_F + (\varphi(T_{n,\Sigma}(\beta')) - \varphi(\beta'))^T L_{n,\Sigma}(\beta') (\varphi(T_{n,\Sigma}(\beta')) - \varphi(\beta')).$$

\square

We can now translate Lemma 3 in this context.

Lemma 5. For any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$\begin{aligned} G_{n,\Sigma}(\beta_{t+1,\Sigma}) &\leq G_{n,\Sigma}(\beta) + \frac{1}{2} (\varphi(\beta_{t,\Sigma}) - \varphi(\beta))^T L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t,\Sigma}) - \varphi(\beta)) \\ &\quad - \frac{1}{2} (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta))^T L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta)). \end{aligned}$$

Proof of Lemma 5. With the help of Lemma 2 and as in the proof of Lemma 3, one has for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$G_{n,\Sigma}(\beta_{t+1,\Sigma}) = G_{n,\Sigma}(\beta) + \langle \nabla G_{n,\Sigma}(\beta_{t,\Sigma}), \beta_{t+1,\Sigma} - \beta \rangle_F + \frac{1}{2} (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta_{t,\Sigma}))^T L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta_{t,\Sigma})).$$

In addition, remark that equality (16) can be written as

$$\nabla G_{n,\Sigma}(\beta_{t,\Sigma}) = -\Sigma^{-1} (\beta_{t+1,\Sigma} - \beta_{t,\Sigma}) \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - \beta_{t,\Sigma} X_i\|_{\Sigma^{-1}}}$$

which can be written, in the basis of \mathbb{R}^{pq} as

$$\nabla \tilde{G}_{n,\Sigma}(\varphi(\beta_{t,\Sigma})) = -L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta_{t,\Sigma}))$$

Then, for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$\begin{aligned} G_{n,\Sigma}(\beta_{t+1,\Sigma}) &\leq G_{n,\Sigma}(\beta) + \frac{1}{2} (\varphi(\beta_{t,\Sigma}) - \varphi(\beta))^T L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t,\Sigma}) - \varphi(\beta)) \\ &\quad - \frac{1}{2} (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta))^T L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta)). \end{aligned}$$

\square

Thus, the proof of Theorem 4 is now the same as the one of Theorem 3.

8.5 Proof of Theorem 5

Writing in the basis of \mathbb{R}^{pq} : Vectorizing β , the gradient in the basis of \mathbb{R}^{pq} can be written as

$$\nabla \tilde{G}_{\Sigma, \lambda}(\varphi(\beta)) = -\mathbb{E} \left[\underbrace{\frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} (\Sigma^{-1}(Y - \beta X)) \odot X - \lambda \frac{\beta}{\|\beta\|_F} \mathbf{1}_{\beta \neq 0}}_{=: \nabla \tilde{g}_{\Sigma, \lambda}(X, Y, \beta)} \right].$$

Hence, here again, the stochastic gradient and the one derived from $\nabla \tilde{G}_{\Sigma, \lambda}$ are equivalent. Indeed, considering the stochastic gradient algorithm defined recursively for all $n \geq 0$ by

$$\beta_{n+1, \Sigma, \lambda, \varphi} = \beta_{n, \Sigma, \lambda, \varphi} + \gamma_{n+1} \left(\nabla \tilde{g}_{\Sigma} (X_{n+1}, Y_{n+1}, \beta_{n, \Sigma, \lambda, \varphi}) - \lambda \frac{\beta_{n, \Sigma, \lambda, \varphi}}{\|\beta_{n, \Sigma, \lambda, \varphi}\|_F} \mathbf{1}_{\beta_{n, \Sigma, \lambda, \varphi} \neq 0} \right)$$

and denoting by $\bar{\beta}_{n, \Sigma, \lambda, \varphi}$ its weighted averaged version, one has $\beta_{n, \Sigma, \lambda, \varphi} = \varphi(\beta_{n, \Sigma, \lambda})$ and $\bar{\beta}_{n, \Sigma, \lambda, \varphi} = \varphi(\bar{\beta}_{n, \Sigma, \lambda})$. Be careful that here, the functional $G_{\Sigma, \lambda}$ is not twice continuously differentiable, and the Hessian, when it exists, is not uniformly bounded. We so refer to Pelletier (1998) and Pelletier (2000) for asymptotic results.

Bounding the gradient: As soon as X admits a moment of order $2a$ for some $a \geq 1$, one has for any $\beta \in \mathcal{M}_{q, p}(\mathbb{R})$

$$\mathbb{E} \left[\|\nabla \tilde{g}_{\Sigma, \lambda}(X, Y, \varphi(\beta))\|^{2a} \right] \leq 2^{2a-1} \frac{q}{\sqrt{\lambda_{\min}(\Sigma)}} \mathbb{E} [\|X\|^{2a}] + 2^{2a-1}.$$

Bounding the Hessian on a neighborhood of β_{λ}^* . The Hessian is given for all $\beta \neq 0$ by

$$\nabla^2 \tilde{G}_{\Sigma}(\varphi(\beta)) = \mathbb{E} \left[\frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} \left(\Sigma^{-1} - \frac{\Sigma^{-1}(Y - \beta X)(Y - \beta X)^T \Sigma^{-1}}{\|Y - \beta X\|_{\Sigma^{-1}}^2} \right) \otimes XX^T \right] + \frac{1}{\|\beta\|_F} \left(I_{pq} - \frac{\varphi(\beta)\varphi(\beta)^T}{\|\beta\|_F^2} \right). \quad (25)$$

Moreover, as in Theorem 1, one has with the help of Lemma 9,

$$\begin{aligned} \|\nabla^2 \tilde{G}_{\Sigma, \lambda}(\varphi(\beta))\|_F &\leq \mathbb{E} \left[\mathbb{E} \left[\frac{1}{\|Y - \beta X\|_{\Sigma^{-1}}} |X| \|\Sigma^{-1}\|_F \|X\|^2 \right] \right] + \frac{pq}{\|\beta\|_F} \\ &\leq \left(1 + \frac{1}{q^{2q/2-1} \Gamma(q/2)} \frac{1}{q-1} \right) \|\Sigma^{-1}\|_F \mathbb{E} [\|X\|^2] + \frac{pq}{\|\beta\|_F} \end{aligned}$$

and the Hessian is so continuous and bounded on a neighborhood of $\beta^* \neq 0$. Observe that it is unfortunately not possible here to get a nice writing of $H_{\Sigma, \lambda} = \nabla^2 \tilde{G}_{\Sigma, \lambda}(\beta_{\lambda}^*)$.

The Hessian is Lipschitz on a neighborhood of β_{λ}^* : Let us now prove that the Hessian is locally Lipschitz. Let $\|\beta_{\lambda}^*\|_F > \epsilon > 0$ and V_{ϵ} a neighborhood of β_{λ}^* such that for all $\beta \in V_{\epsilon}$, $\|\beta\|_F \geq \epsilon/2$. Then, for all $\beta, \beta' \in V_{\epsilon}$,

$$\|\nabla^2 \tilde{G}_{\Sigma, \lambda}(\varphi(\beta)) - \nabla^2 \tilde{G}_{\Sigma, \lambda}(\varphi(\beta'))\|_F \leq \|\nabla^2 \tilde{G}_{\Sigma}(\varphi(\beta)) - \nabla^2 \tilde{G}_{\Sigma}(\varphi(\beta'))\|_F + \left| \frac{1}{\|\beta\|_F} - \frac{1}{\|\beta'\|_F} \right| + \left| \frac{\beta}{\|\beta\|_F} - \frac{\beta'}{\|\beta'\|_F} \right|.$$

Furthermore,

$$\left| \frac{1}{\|\beta\|_F} - \frac{1}{\|\beta'\|_F} \right| \leq \frac{\|\beta - \beta'\|_F}{\|\beta\|_F \|\beta'\|_F} \leq \frac{4}{\epsilon^2} \|\beta - \beta'\|_F$$

In a same way

$$\left| \frac{\beta}{\|\beta\|_F} - \frac{\beta'}{\|\beta'\|_F} \right| \leq \frac{\|\beta - \beta'\|_F}{\|\beta'\|_F} \leq \frac{\epsilon}{2} \|\beta - \beta'\|_F.$$

Then, the Hessian is $3 \left(1 + \frac{1}{q^{2q/2-1} \Gamma(q/2)} \frac{1}{q-2} \right) \|\Sigma^{-1/2}\|_F^3 \mathbb{E} [\|X\|^3] + \frac{4}{\epsilon^2} + \frac{2}{\epsilon}$ Lipschitz on V_{ϵ} .

Convergence of the estimates: Assumptions in Godichon-Baggioni (2019b, 2021) are fulfilled and it comes

$$\|\beta_{n,\Sigma} - \beta^*\|_F^2 = \|\beta_{n,\varphi} - \varphi(\beta^*)\|^2 = O\left(\frac{\ln n}{n^\gamma}\right) \quad a.s$$

$$\|\bar{\beta}_{n,\Sigma} - \beta^*\|_F^2 = \|\bar{\beta}_{n,\varphi} - \varphi(\beta^*)\|^2 = O\left(\frac{\ln n}{n}\right) \quad a.s$$

In addition, one has

$$\sqrt{n}(\varphi(\bar{\beta}_{n,\Sigma,\lambda}) - \varphi(\beta_\lambda^*)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, H_{\Sigma,\lambda}^{-1} V_{\Sigma,\lambda} H_{\Sigma,\lambda}^{-1})$$

where $H_{\Sigma,\lambda} := \nabla^2 \tilde{G}_{\Sigma,\lambda}(\varphi(\beta_\lambda^*))$ and $V_{\Sigma,\lambda} = \mathbb{E}[\nabla \tilde{g}_{\Sigma,\lambda}(X, Y, \beta^*) \nabla \tilde{g}_{\Sigma,\lambda}(X, Y, \beta^*)^T]$.

8.6 Proof of Theorem 6

In all the sequel, for the sake of simplicity, we denote $\beta_{t,\Sigma,\lambda} := \beta_{n,t,\Sigma,\lambda}$ for all $t \geq 0$. The aim is to adapt the proof of Theorem 3 in this context, i.e to translate and prove the lemma. First, observe that Lemma 1 can be directly derivated.

Lemma 6. For all $\beta' \in \mathcal{M}_{q,p}(\mathbb{R}) \setminus \{0\}$,

$$2G_{n,\Sigma,\lambda}(T_{n,\varphi,\Sigma,\lambda}(\beta')) \leq 2G_{n,\Sigma,\lambda}(\beta') + 2 \langle \nabla \tilde{G}_{n,\Sigma,\lambda}(\beta'), T_{n,\varphi,\Sigma,\lambda}(\beta') - \varphi(\beta') \rangle + (T_{n,\varphi,\Sigma,\lambda}(\beta') - \varphi(\beta'))^T L_{n,\Sigma,\lambda}(\beta') (T_{n,\varphi,\Sigma,\lambda}(\beta') - \varphi(\beta')).$$

with $L_{n,\Sigma,\lambda}(\beta') := \Sigma^{-1} \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - X_i \beta'\|_{\Sigma^{-1}}} + \frac{\lambda}{\beta'} I_{pq}$.

Proof of Lemma 6. Let us consider the functional $h_{n,\Sigma} : \mathcal{M}_{q,p}(\mathbb{R}) \times \mathcal{M}_{q,p}(\mathbb{R}) \rightarrow \mathbb{R}_+$ defined for all β, β' with $\beta' \neq 0$ by

$$h_{n,\Sigma,\lambda}(\beta, \beta') = \frac{1}{n} \sum_{i=1}^n \frac{\|Y_i - \beta X_i\|_{\Sigma^{-1}}^2}{\|Y_i - \beta' X_i\|_{\Sigma^{-1}}} + \lambda \frac{\|\beta\|_F}{\|\beta'\|_F}.$$

Then, the functional $h_{n,\Sigma,\lambda}(\cdot, \beta')$ is quadratic, and writting it in the canonical basis of \mathbb{R}^{pq} , the Hessian is given by $2L_{n,\Sigma,\lambda}(\beta')$ with $L_{n,\Sigma,\lambda}(\beta') := \Sigma^{-1} \otimes \frac{1}{n} \sum_{i=1}^n \frac{X_i X_i^T}{\|Y_i - X_i \beta'\|_{\Sigma^{-1}}} + \frac{\lambda}{\beta'} I_{pq}$. Then, with analogous development as in the proof of Lemma 2, for all $\beta, \beta' \in \mathcal{M}_{q,p}(\mathbb{R})$ with $\beta' \neq 0$,

$$h_{n,\Sigma,\lambda}(\beta, \beta') = G_{n,\Sigma,\lambda}(\beta') + 2 \langle \nabla G_{n,\Sigma,\lambda}(\beta'), \beta - \beta' \rangle_F + (\varphi(\beta) - \varphi(\beta'))^T L_{n,\Sigma,\lambda}(\beta') (\varphi(\beta) - \varphi(\beta'))$$

and taking $\beta = T_{n,\varphi,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})$, it comes

$$2G_{n,\Sigma,\lambda}(T_{n,\varphi,\Sigma,\lambda}(\beta')) \leq 2G_{n,\Sigma,\lambda}(\beta') + 2 \langle \nabla \tilde{G}_{n,\Sigma,\lambda}(\beta'), T_{n,\varphi,\Sigma,\lambda}(\beta') - \varphi(\beta') \rangle + (T_{n,\varphi,\Sigma,\lambda}(\beta') - \varphi(\beta'))^T L_{n,\Sigma,\lambda}(\beta') (T_{n,\varphi,\Sigma,\lambda}(\beta') - \varphi(\beta')).$$

□

We can now translate Lemma 3 in this context.

Lemma 7. For any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$G_{n,\Sigma}(\beta_{t+1,\Sigma,\lambda}) \leq G_{n,\Sigma,\lambda}(\beta) + \frac{1}{2} (\varphi(\beta_{t,\Sigma,\lambda}) - \varphi(\beta))^T L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda}) (\varphi(\beta_{t,\Sigma,\lambda}) - \varphi(\beta)) - \frac{1}{2} (\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta))^T L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda}) (\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta)).$$

Proof of Lemma 7. With the help of Lemma 2 and as in the proof of Lemma 3, one has for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$G_{n,\Sigma}(\beta_{t+1,\Sigma}) = G_{n,\Sigma}(\beta) + \langle \nabla G_{n,\Sigma}(\beta_{t,\Sigma}), \beta_{t+1,\Sigma} - \beta \rangle_F + \frac{1}{2} (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta_{t,\Sigma}))^T L_{n,\Sigma}(\beta_{t,\Sigma}) (\varphi(\beta_{t+1,\Sigma}) - \varphi(\beta_{t,\Sigma})).$$

In addition, remark that equality (20) can be written as

$$\nabla \tilde{G}_{n,\Sigma,\lambda}(\varphi(\beta_{t,\Sigma})) = L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})(\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta_{t,\Sigma,\lambda}))$$

Then, for any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$,

$$\begin{aligned} G_{n,\Sigma,\lambda}(\beta_{t+1,\Sigma,\lambda}) &\leq G_{n,\Sigma,\lambda}(\beta) + \frac{1}{2}(\varphi(\beta_{t,\Sigma,\lambda}) - \varphi(\beta))^T L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})(\varphi(\beta_{t,\Sigma,\lambda}) - \varphi(\beta)) \\ &\quad - \frac{1}{2}(\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta))^T L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})(\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta)). \end{aligned}$$

□

Thus, as in the proof of Theorem 3, taking $\beta = \beta_{t,\Sigma,\lambda}$, it comes

$$G_{n,\Sigma,\lambda}(\beta_{t+1,\Sigma,\lambda}) \leq G_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda}) - \frac{1}{2}(\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta_{t,\Sigma,\lambda}))^T L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})(\varphi(\beta_{t+1,\Sigma,\lambda}) - \varphi(\beta_{t,\Sigma,\lambda})) \quad (26)$$

Then, $(G_{n,\Sigma,\lambda}(\beta_{t+1,\Sigma,\lambda}))_t$ is a decreasing sequence and converges to a non negative constant $G_{\infty,\Sigma,\lambda}$. Here again, $G_{n,\Sigma,\lambda}$ is strictly convex and this means that $\beta_{t,\Sigma,\lambda}$ belongs to a compact \mathcal{K} . In addition, since ϵ is Gaussian, observe that for all t , $\beta_{t,\Sigma,\lambda} \neq 0$ almost surely. Then, for any t , $L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})$ is positive, and $\lambda_{\min}(L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})) \geq \frac{\lambda}{\|\beta_{t,\Sigma,\lambda}\|}$. Then, denoting by $D_{\mathcal{K}} = \sup\{\|\beta\|, \beta \in \mathcal{K}\}$, it comes that for any t , $\lambda_{\min}(L_{n,\Sigma,\lambda}(\beta_{t,\Sigma,\lambda})) \geq \frac{\lambda}{D_{\mathcal{K}}}$. Then, one can conclude as in the proof of Theorem 3.

8.7 Two useful lemma

8.7.1 The lemma

The following lemma ensures that the functional G is twice continuously differentiable.

Lemma 8. *Suppose $q \geq 3$, then for any $a < q$, and for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$*

$$\mathbb{E} \left[\frac{1}{\|Y - \beta X\|^a} |X| \right] \leq 1 + \frac{\lambda_{\max}^{q/2}}{q2^{q/2-1}\Gamma(q/2)} \frac{a}{q-a}.$$

where $\lambda_{\max} = \lambda_{\max}(\Sigma)$.

In order to ensure that the functional G_{Σ} is twice differentiable, let us translate Lemma 8 as follows:

Lemma 9. *Suppose $q \geq 3$, then for any $a < q$, and for all $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$*

$$\mathbb{E} \left[\frac{1}{\|\epsilon - X\beta\|_{\Sigma^{-1}}^a} |X| \right] \leq 1 + \frac{1}{q2^{q/2-1}\Gamma(q/2)} \frac{a}{q-a}.$$

8.7.2 Proofs of Lemma 8 and 9

For any $\beta \in \mathcal{M}_{q,p}(\mathbb{R})$, one has since $q \geq 3$ and $a < q$ and since $\epsilon \sim \mathcal{N}(0, \Sigma)$,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{\|\epsilon - \beta X + \beta^* X\|_{\Sigma^{-1}}^a} |X| \right] &= \int_0^{+\infty} \mathbb{P} \left[\|\epsilon - \beta X + \beta^* X\|_{\Sigma^{-1}}^a \leq \frac{1}{t} |X| \right] dt \\ &\leq \int_0^{+\infty} \mathbb{P} \left[\|\epsilon\|_{\Sigma^{-1}}^a \leq \frac{1}{t} |X| \right] dt \\ &\leq \int_0^{+\infty} \mathbb{P} \left[\chi_q^2 \leq \frac{1}{t^{2/a}} |X| \right] dt \\ &\leq 1 + \frac{1}{\Gamma(q/2)} \int_1^{+\infty} \gamma(q/2, t^{-2/a}/2) dt \end{aligned}$$

where $\gamma(\cdot, \cdot)$ is the incomplete Gamma function. Then,

$$\begin{aligned}
\mathbb{E} \left[\frac{1}{\|\epsilon - \beta X + \beta^* X\|_{\Sigma^{-1}}^a} | X \right] &= 1 + \frac{1}{\Gamma(q/2)} \int_1^{+\infty} \int_0^{\frac{1}{2t^{2/a}}} u^{\frac{q}{2}-1} e^{-u} du \\
&\leq 1 + \frac{1}{\Gamma(q/2)} \int_1^{+\infty} \int_0^{\frac{1}{2t^{2/a}}} u^{\frac{q}{2}-1} du \\
&= 1 + \frac{1}{q2^{q/2-1}\Gamma(q/2)} \int_1^{+\infty} \frac{1}{t^{q/a}} dt \\
&\leq 1 + \frac{1}{2^{q/2-1}\Gamma(q/2)} \frac{a}{q-a}
\end{aligned}$$

which concludes the proof of Lemma 9. Analogously, since $\frac{1}{\lambda_{\min}} \|h\|_{\Sigma} \geq t\|h\| \geq \frac{1}{\lambda_{\max}} \|h\|_{\Sigma}$,

$$\begin{aligned}
\mathbb{E} \left[\frac{1}{\|\epsilon - \beta X + \beta^* X\|_{\Sigma^{-1}}^a} | X \right] &= \int_0^{+\infty} \mathbb{P} \left[\|\epsilon - \beta X + \beta^* X\|_{\Sigma^{-1}}^a \leq \frac{1}{t} | X \right] dt \\
&\leq \int_0^{+\infty} \mathbb{P} \left[\|\epsilon - \beta X + \beta^* X\|_{\Sigma^{-1}}^a \leq \frac{\lambda_{\max}^a}{t} | X \right] dt \\
&\leq \int_0^{+\infty} \mathbb{P} \left[\|\epsilon\|_{\Sigma^{-1}}^a \leq \frac{\lambda_{\max}^a}{t} | X \right] dt \\
&\leq 1 + \frac{1}{\Gamma(q/2)} \int_1^{+\infty} \gamma(q/2, \lambda_{\max} t^{-2/a}/2) dt \\
&= 1 + \frac{1}{\Gamma(q/2)} \int_1^{+\infty} \int_0^{\frac{\lambda_{\max}}{2t^{2/a}}} u^{\frac{q}{2}-1} e^{-u} du
\end{aligned}$$

where $\gamma(\cdot, \cdot)$ is the incomplete Gamma function. Then,

$$\begin{aligned}
\mathbb{E} \left[\frac{1}{\|\epsilon + \beta X - \beta^* X\|_{\Sigma^{-1}}^a} | X \right] &\leq 1 + \frac{1}{\Gamma(q/2)} \int_1^{+\infty} \int_0^{\frac{\lambda_{\max}}{t^{2/a}}} u^{\frac{q}{2}-1} du \\
&= 1 + \frac{\lambda_{\max}^{q/2}}{q2^{q/2-1}\Gamma(q/2)} \int_1^{+\infty} \frac{1}{t^{q/a}} dt \\
&\leq 1 + \frac{\lambda_{\max}^{q/2}}{q2^{q/2-1}\Gamma(q/2)} \frac{a}{q-a}
\end{aligned}$$

References

- Bach, F. and Moulines, E. (2013). Non-strongly-convex smooth stochastic approximation with convergence rate $o(1/n)$. In *Advances in neural information processing systems*, pages 773–781.
- Banerjee, O., El Ghaoui, L., and D’Aspremont, A. (2008). Model selection through sparse maximum likelihood estimation for multivariate gaussian or binary data. *Journal of Machine Learning Research*, 9:485 – 516.
- Beck, A. and Sabach, S. (2015). Weiszfeld’s method: Old and new results. *Journal of Optimization Theory and Applications*, 164:1–40.
- Boyd, S. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.
- Boyer, C. and Godichon-Baggioni, A. (2020). On the asymptotic rate of convergence of stochastic newton algorithms and their weighted averaged versions. *arXiv preprint arXiv:2011.09706*.
- Cardot, H., Cénac, P., and Monnez, J.-M. (2012). A fast and recursive algorithm for clustering large datasets with k -medians. *Computational Statistics and Data Analysis*, 56:1434–1449.
- Cardot, H., Cénac, P., and Zitt, P.-A. (2013). Efficient and fast estimation of the geometric median in Hilbert spaces with an averaged stochastic gradient algorithm. *Bernoulli*, 19(1):18–43.

- Cardot, H. and Godichon-Baggioni, A. (2015). Fast estimation of the median covariation matrix with application to online robust principal components analysis. *TEST*, pages 1–20.
- Duchi, J., Hazan, E., and Singer, Y. (2011). Adaptive subgradient methods for online learning and stochastic optimization. *Journal of machine learning research*, 12(7).
- Duflo, M. (1997). *Random iterative models*, volume 34 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. Translated from the 1990 French original by Stephen S. Wilson and revised by the author.
- Friedman, J., Hastie, T., and Tibshirani, R. (2007). Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441.
- Gadat, S. and Panloup, F. (2017). Optimal non-asymptotic bound of the ruppert-polyak averaging without strong convexity. *arXiv preprint arXiv:1709.03342*.
- Godichon-Baggioni, A. (2019a). Lp and almost sure rates of convergence of averaged stochastic gradient algorithms: locally strongly convex objective. *ESAIM: Probability and Statistics*, 23:841–873.
- Godichon-Baggioni, A. (2019b). Online estimation of the asymptotic variance for averaged stochastic gradient algorithms. *Journal of Statistical Planning and Inference*, 203:1–19.
- Godichon-Baggioni, A. (2021). Convergence in quadratic mean of averaged stochastic gradient algorithms without strong convexity nor bounded gradient. *arXiv preprint arXiv:2107.12058*.
- Godichon-Baggioni, A. and Robin, S. (2024). A robust model-based clustering based on the geometric median and the Median Covariation Matrix. *Statistics and computing*, pages 24–55.
- Hoerl, A. E. and Kennard, R. W. (1970). Ridge regression: Biased estimation for nonorthogonal problems. *Technometrics*, 12(1):55–67.
- Kraus, D. and Panaretos, V. M. (2012). Dispersion operators and resistant second-order functional data analysis. *Biometrika*, 99:813–832.
- Leluc, R. and Portier, F. (2020). Towards asymptotic optimality with conditioned stochastic gradient descent. *arXiv preprint arXiv:2006.02745*.
- Mardia, K., Kent, J., and Bibby, J. (1979). *Multivariate analysis*. Academic press.
- Maronna, R. A., Martin, R. D., and Yohai, V. J. (2006). *Robust statistics*. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester. Theory and methods.
- Mokkadem, A. and Pelletier, M. (2011). A generalization of the averaging procedure: The use of two-time-scale algorithms. *SIAM Journal on Control and Optimization*, 49(4):1523–1543.
- Niemiro, W. (1992). Asymptotics for m-estimators defined by convex minimization. *The Annals of Statistics*, pages 1514–1533.
- Pelletier, M. (1998). On the almost sure asymptotic behaviour of stochastic algorithms. *Stochastic processes and their applications*, 78(2):217–244.
- Pelletier, M. (2000). Asymptotic almost sure efficiency of averaged stochastic algorithms. *SIAM J. Control Optim.*, 39(1):49–72.
- Perrot-Dockès, M., Lévy-Leduc, C., and Rajjou, L. (2022). Estimation of large block structured covariance matrices: Application to ‘multi-omic’ approaches to study seed quality. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 71(1):119–147.
- Perrot-Dockès, M., Lévy-Leduc, C., Sansonnet, L., and Chiquet, J. (2018). Variable selection in multivariate linear models with high-dimensional covariance matrix estimation. *Journal of Multivariate Analysis*, 166:78–97.

- Pesme, S. and Flammarion, N. (2020). Online robust regression via sgd on the l1 loss. *Advances in Neural Information Processing Systems*, 33:2540–2552.
- Polyak, B. and Juditsky, A. (1992). Acceleration of stochastic approximation. *SIAM J. Control and Optimization*, 30:838–855.
- Robbins, H. and Monro, S. (1951). A stochastic approximation method. *The annals of mathematical statistics*, pages 400–407.
- Rothman, A., E., L., and Zhu, J. (2010). Sparse multivariate regression with covariance estimation. *Journal of Computational and Graphical Statistics*, 19(4):947–962.
- Ruppert, D. (1988). Efficient estimations from a slowly convergent robbins-monro process. Technical report, Cornell University Operations Research and Industrial Engineering.
- Van der Vaart, A. W. (2000). *Asymptotic statistics*, volume 3. Cambridge university press.
- Vardi, Y. and Zhang, C.-H. (2000). The multivariate L_1 -median and associated data depth. *Proc. Natl. Acad. Sci. USA*, 97(4):1423–1426.
- Weiszfeld, E. (1937). Sur le point pour lequel la somme des distances de n points donnés est minimum. *Tohoku Math. J.*, 43(355-386):2.
- Yuan, M. and Lin, Y. (2007). Model selection and estimation in the Gaussian graphical model. *Biometrika*, 94(1):19–35.

A Estimating the Median Covariation Matrix

A.1 Without estimating the Median

Let us recall that the aim is to estimate

$$G_{MCM,N}(V) = \frac{1}{N} \sum_{i=1}^n \left\| (Y_i - \bar{\beta}_N X_i)(Y_i - \bar{\beta}_N X_i)^T - V \right\|_F,$$

Weiszfeld algorithm: For all $t \geq 0$

$$V_{t+1} = \frac{\sum_{i=1}^N \frac{(Y_i - \bar{\beta}_N X_i)(Y_i - \bar{\beta}_N X_i)^T}{\left\| (Y_i - \bar{\beta}_N X_i)(Y_i - \bar{\beta}_N X_i)^T - V_t \right\|_F}}{\sum_{i=1}^N \frac{1}{\left\| (Y_i - \bar{\beta}_N X_i)(Y_i - \bar{\beta}_N X_i)^T - V_t \right\|_F}}$$

Averaged stochastic gradient algorithm : For all $n \leq N - 1$,

$$V_{n+1} = V_n + \gamma_{n+1} \frac{(Y_{n+1} - \bar{\beta}_N X_{n+1})(Y_{n+1} - \bar{\beta}_N X_{n+1})^T - V_n}{\left\| (Y_{n+1} - \bar{\beta}_N X_{n+1})(Y_{n+1} - \bar{\beta}_N X_{n+1})^T - V_n \right\|_F}$$

$$\bar{V}_{n+1} = \bar{V}_n + \frac{1}{n+1} (V_{n+1} - \bar{V}_n)$$

A.2 With the estimates of the Median

Observe that previous estimates consider that $\bar{\beta}_N$ is an unbiased estimate of β^* in the sense that $\text{med}(Y - X\bar{\beta}_N)$, which cannot be verified theoretically and which is probably wrong. Then, one can consider these following alternatives (Cardot and Godichon-Baggioni, 2015).

Wesizfeld algorithms (estimating the median): For all $t \geq 0$,

$$m_{t+1} = \frac{\sum_{i=1}^N \frac{Y_i - \bar{\beta}_N X_i}{\|Y_i - \bar{\beta}_N X_i - m_t\|}}{\sum_{i=1}^N \frac{1}{\|Y_i - \bar{\beta}_N X_i - m_t\|}}$$

Weiszfeld algorithm (estimating the MCM): Considering the estimate of the median m_T for some integer T , for all $t \geq 0$

$$V_{t+1} = \frac{\sum_{i=1}^N \frac{(Y_i - \bar{\beta}_N X_i - m_T)^T (Y_i - \bar{\beta}_N X_i - m_T)}{\|(Y_i - \bar{\beta}_N X_i - m_T)(Y_i - \bar{\beta}_N X_i - m_T)^T - V_t\|_F}}{\sum_{i=1}^N \frac{1}{\|(Y_i - \bar{\beta}_N X_i - m_T)(Y_i - \bar{\beta}_N X_i - m_T)^T - V_t\|_F}}$$

Averaged stochastic gradient algorithms: For all $n \leq N - 1$,

$$m_{n+1} = m_n + \gamma_{n+1} \frac{Y_{n+1} - \bar{\beta}_N X_{n+1} - m_n}{\|Y_{n+1} - \bar{\beta}_N X_{n+1} - m_n\|}$$

$$\bar{m}_{n+1} = \bar{m}_n + \frac{1}{n+1} (m_{n+1} - \bar{m}_n)$$

$$V_{n+1} = V_n + \gamma_{n+1} \frac{(Y_{n+1} - \bar{\beta}_N X_{n+1})(Y_{n+1} - \bar{\beta}_N X_{n+1})^T - V_n}{\|(Y_{n+1} - \bar{\beta}_N X_{n+1})(Y_{n+1} - \bar{\beta}_N X_{n+1})^T - V_n\|_F}$$

$$\bar{V}_{n+1} = \bar{V}_n + \frac{1}{n+1} (V_{n+1} - \bar{V}_n)$$

B Additional simulation results

Covariance matrix.

The $q \times q$ covariance matrix Σ was build as $\Sigma = uu^T + \sigma^2 I_q$, where u is a q -dimensional vector with iid $\mathcal{N}(0, 1)$ coordinates and $\sigma = 1$. This structure was chosen so to induce a fairly strong correlation between the responses, to highlight the difference between OLS and WLS estimates.

Estimation accuracy.

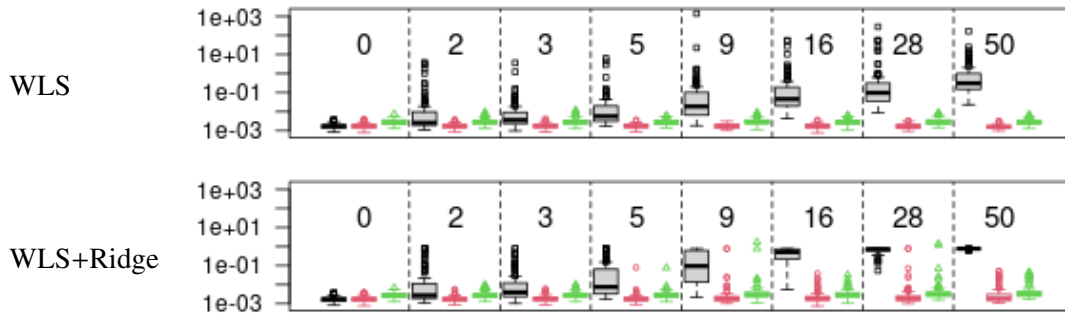


Figure 7: Mean squared error for the estimation of the regression coefficients β for WLS estimates without or with Ridge regularization. Legend: \square = Naive, \circ = Offline, \triangle = Online. Top of each panel = fraction f of outliers (in %).

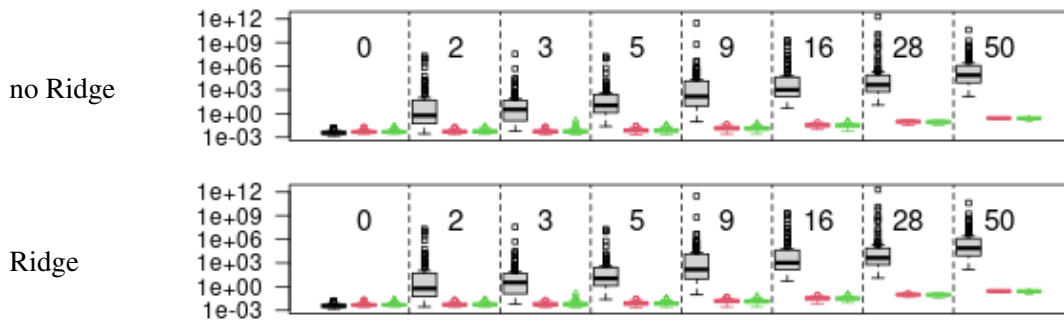


Figure 8: Mean squared error for the estimation of the covariance matrix Σ ($MSE(\Sigma)$) without or with Ridge regularization. The y-axis is log-scaled. Same legend as Figure 1.

Effect of the sample size.

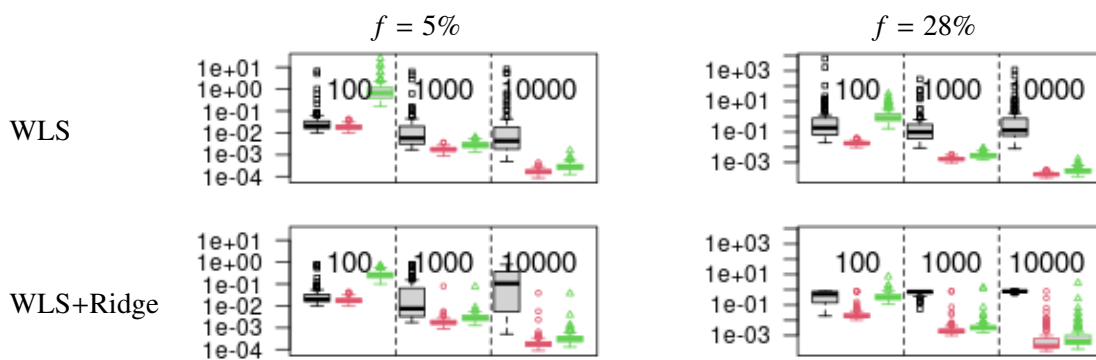


Figure 9: Mean squared error for the estimation of the regression coefficients β ($MSE(\beta)$) for WLS estimates with or without Ridge regularization for a fraction of outliers of $f = 5\%$ (left) and $f = 28\%$ (right). The y-axis is log-scaled. Same legend as Figure 2.

Prediction accuracy.

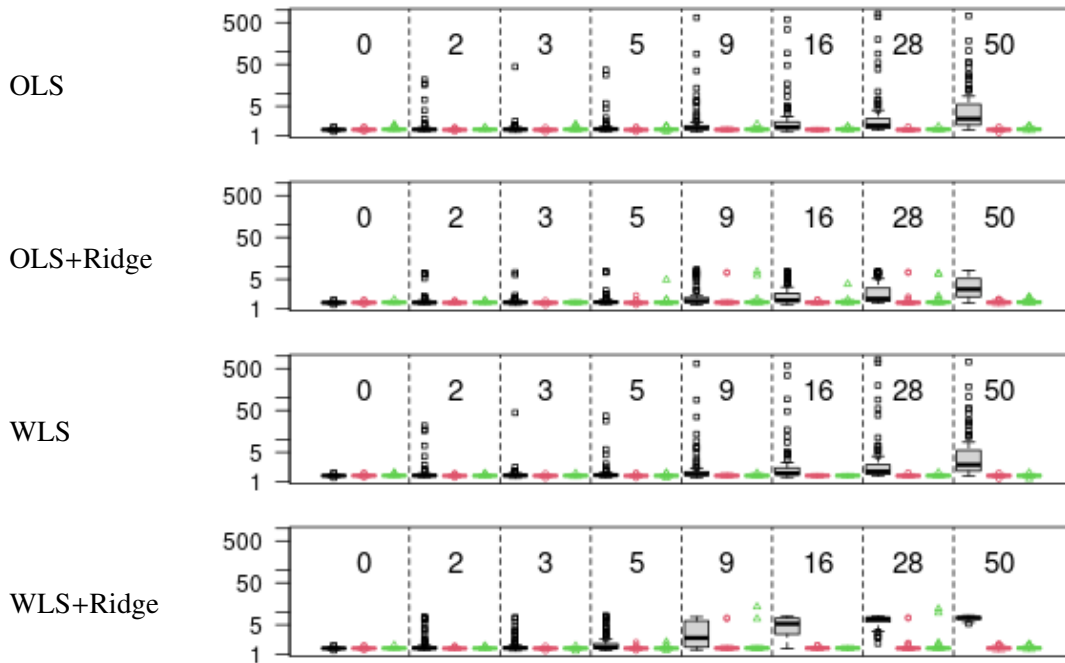


Figure 10: Mean squared error for the predicted value $\widehat{\beta}X$ as compared to the true response Y ($MSE(Y)$) for the OLS and WLS estimates with or without Ridge regularization for a sample size of $n = 1000$. Same legend as Figure 1.

Outlier detection.

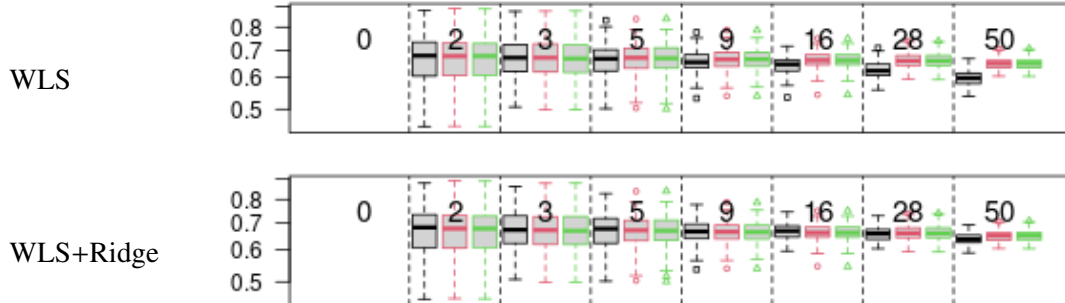


Figure 11: AUC for the detection of outliers for the WLS estimates with or without Ridge regularization for a sample size of $n = 1000$. Same legend as Figure 1.

OLS versus WLS.

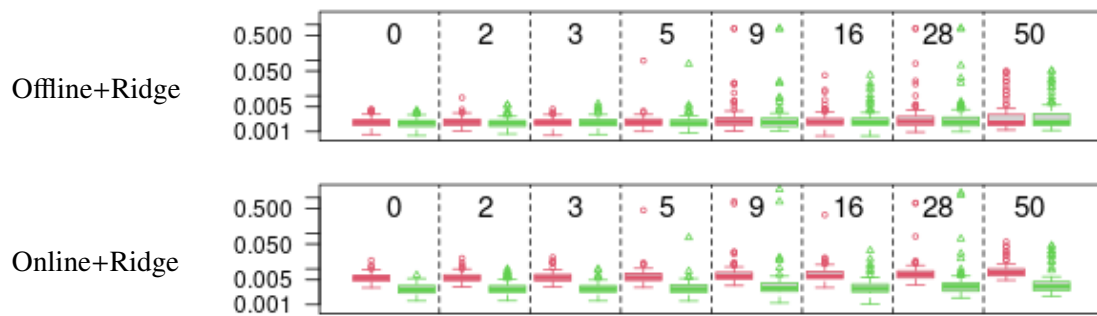


Figure 12: Comparison of the mean squared error for the estimation of the regression coefficients β ($MSE(\beta)$) of OLS and WLS estimates for Offline and Online procedures with Ridge regularization for a sample size of $n = 1000$. Legend: \circ = Offline, \triangle = Online. Number in each box = fraction f of outliers (in %).