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Nonlinear representation theory of equivariant CNNs on homogeneous spaces using group morphology

Jesús Angulo *

Mines Paris, PSL University,
CMM-Centre de Morphologie Mathématique, France
jesus.angulo@mines-paristech.fr

Abstract. This paper deals with a nonlinear theory of equivariant convolutional neural networks (CNNs) on homogeneous spaces under the action of a group. Many groups of image transforms fit this framework.

The purpose of our work is to have a universal equivariant representation of nonlinear maps between image features which is based on mathematical morphology operators for groups. In particular, we combine some powerful results of universal representation of nonlinear mappings with the equivariance properties of morphological group operators.

The approach considered here is significantly different from other theories of representation of equivariant CNNs. On the one hand, it is founded on results from lattice theory and other hand, it deals with the universal representation of nonlinear maps, which can involve in a unified framework (linear) convolutions, activation functions and other nonlinear layers.

Keywords: deep learning ; mathematical morphology ; group morphology ; equivariant CNN; nonlinear representation theory.

1 Introduction

Figure 1 depicts the typical operators (layers) in a convolutional neural network (CNN) block formalized by the nonlinear map Ψ . Let us consider a group \mathbb{G} , now \mathbb{G} -CNN means that Ψ is equivariant to the group action T_g , $g \in \mathbb{G}$; i.e., $\Psi(T_g f) = T'_g \Psi(f)$. We consider that feature maps in these networks represent functions (or image fields) $f(x)$ on a homogeneous space and the layers are equivariant maps between spaces of functions, i.e., $[T_g f](x) = [f \circ g^{-1}](x) = f(g^{-1}x)$.

The theoretical study and implementation of equivariant CNNs and more generally of equivariance in deep learning is an active area with many contributions. General theory of equivariant CNNs on homogeneous spaces [5] or on compact groups [12] considers mainly the generalization of convolution to groups and how any linear equivariant layer can be represented by combinations of equivariant group convolutions. The nonlinearity aspects of deep learning are secondary

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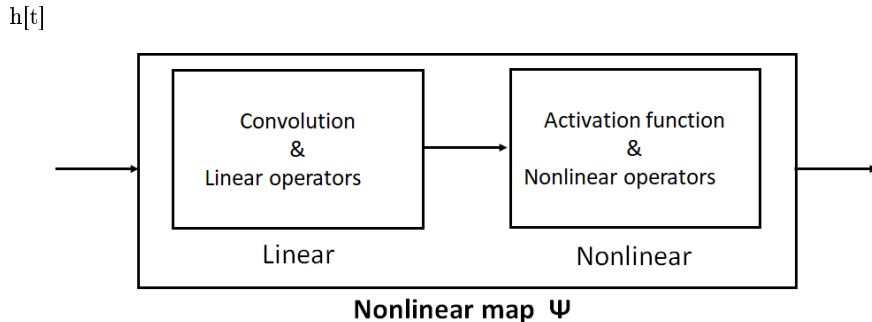


Fig. 1. Typical operators (layers) in a CNN block.

in this kind of approach and basically limited to point-wise nonlinearities. We propose to address in this paper the representation of nonlinear equivariant layers using group morphology and general morphological representation theory.

The interest of morphological scale-spaces in the context of equivariant deep learning has been explored previously from the perspective of Hamilton–Jacobi PDEs on groups [6,24]. This work proposes an initial contribution on the interest of group morphological operators for equivariant deep learning sketched in [1]. More precisely, our approach is a representation theory for nonlinear equivariant deep learning operators Ψ based on ordered structures (lattice algebra, tropical semirings) which will bring up the relevance of group morphological operators. However note that we are not dealing with the implementation of group morphological operators.

The main references of group morphology are the work done by Roerdink in [20,22], see also his papers on the commutative group case [19] or on the group of transformations for the camera projective model [21]. Another interesting theoretical contribution, in particular to the case of the affine group was proposed by Maragos [15]. The case of abelian groups was studied in detail in these references too [8,10]. Another related work which can be of interest for the reader is the definition of Minkowski product, instead of the sum, of two sets on the complex plane [7]. To the best of our knowledge, group morphology has been mainly used to solve problems on robotics, i.e., finding obstacles and free space, and on symmetry detection, both applications in [13].

Organisation of the paper. The rest of the paper is organized as follows. Section 2 provides a review of the main theoretical results of universal representation of nonlinear (increasing) operators using mathematical morphology. An introduction to group morphology on homogeneous spaces is given in Section 3. The main contribution of our work is considered on Section 4 with a first general result on the morphological representation of equivariant CNNs. Some perspectives in Section 5 close the paper.

2 Universal representation of nonlinear operators

In this section, we review the fundamental results for the morphological representation of nonlinear operators based on lattice algebra. We focus in particular on the case of increasing operators and provide just the references for the extension to non-increasing mappings.

2.1 Characterization of increasing operators on a complete lattice

Let us start by the most general result for the representation of any *increasing* nonlinear operator Ψ in a *lattice* \mathcal{L} with *partial order* \leq , i.e., $X, Y \in \mathcal{L}$, $X \leq Y \iff \Psi(X) \leq \Psi(Y)$.

Theorem 1 (Serra (1988) [23], Heijmans & Ronse (1990) [8]). *Let us consider a complete lattice \mathcal{L} and an increasing operator $\Psi : \mathcal{L} \rightarrow \mathcal{L}$, which preserves the greatest element \top ; i.e., satisfies $\Psi(\top) = \top$.*

Then Ψ is the supremum of a non-empty set of erosions \mathcal{E} :

$$\Psi = \bigvee_{\varepsilon \in \mathcal{E}} \varepsilon.$$

If the operator preserves the smallest element \perp ; i.e., $\Psi(\perp) = \perp$, the operator Ψ can be written as an infimum of dilations from a set \mathcal{D} :

$$\Psi = \bigwedge_{\delta \in \mathcal{D}} \delta.$$

That abstract universal representation theorem in terms of erosions and dilations can be instantiated in a particular case useful as starting point for this paper. Let us consider a translation equivariant¹ (TE) increasing operator Ψ . The domain of the functions considered here is either $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, with the additional condition that we consider only closed subsets of E . We focus first on the set operator case applied on $\mathcal{P}(E)$ and then that for functions $f : E \rightarrow \bar{\mathbb{R}}$. Here are the main results from Matheron [17] and Maragos [16].

Kernel and basis representation of TE increasing set operators. The kernel of the TE operator Ψ is defined as the following collection of input sets [17]:

$$\text{Ker}(\Psi) = \{A \subseteq E : \mathbf{0} \in \Psi(A)\},$$

where $\mathbf{0}$ denotes the origin of E . In the following, we use the classic definitions of Minkowski sum $X \oplus B$ and Minkowski difference $X \ominus B$ of sets $X, B \in \mathcal{P}(E)$.

¹ In the classic literature of image processing and mathematical morphology, it is used the term translation-invariant (TI) filters and operators.

Theorem 2 (Matheron (1975) [17]). Consider set operators on $\mathcal{P}(E)$. Let $\Psi : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a TE increasing set operator. Then

$$\Psi(X) = \bigcup_{A \in \text{Ker}(\Psi)} X \ominus A = \bigcap_{B \in \text{Ker}(\bar{\Psi})} X \oplus \check{B}.$$

where the dual set operator is $\bar{\Psi}(X) = [\Psi(X^c)]^c$ and \check{B} is the transpose structuring element.

The kernel of Ψ is a partially ordered set under set inclusion which has an infinity number of elements. In practice, by the property of absorption of erosion, that means that the erosion by B contains the erosions by any other kernel set larger than B and it is the only one required when taking the supremum of erosions. The morphological basis of Ψ is defined as the minimal kernel sets [16]:

$$\text{Bas}(\Psi) = \{M \in \text{Ker}(\Psi) : [A \in \text{Ker}(\Psi) \text{ and } A \subseteq M] \implies A = M\}.$$

A sufficient condition for the existence of $\text{Bas}(\Psi)$ is for Ψ to be an upper semi-continuous operator. We also consider closed sets on $\mathcal{P}(E)$.

Theorem 3 (Maragos (1989) [16]). Let $\Psi : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a TE, increasing and upper semi-continuous set operator. Then

$$\Psi(X) = \bigcup_{M \in \text{Bas}(\Psi)} X \ominus M = \bigcap_{N \in \text{Bas}(\bar{\Psi})} X \oplus \check{N}.$$

Kernel and basis representation of TE increasing operators on functions. Previous set theory was extended [16] to the case of mappings on functions $\Psi(f)$ and therefore useful for signal or gray-scale image operators. We focus on the case of closed functions f , i.e., its epigraph is a closed set. In that case, the dual operator is $\bar{\Psi}(f) = -\Psi(-f)$ and the transpose function is $\check{f}(x) = f(-x)$. Let

$$\text{Ker}(\Psi) = \{f : \Psi(f)(\mathbf{0}) \geq 0\}$$

be the kernel of operator Ψ . As for the TE set operators, a basis can be obtained from the kernel functions as its minimal elements with respect to the partial order \leq , i.e.,

$$\text{Bas}(\Psi) = \{g \in \text{Ker}(\Psi) : [f \in \text{Ker}(\Psi) \text{ and } f \leq g] \implies f = g\}.$$

This collection of functions can uniquely represents the operator Ψ .

Theorem 4 (Maragos (1989) [16]). Consider an upper semi-continuous operator Ψ acting on an upper semi-continuous function f . Let $\text{Bas}(\Psi) = \{g_i\}_{i \in I}$ be its basis and $\text{Bas}(\bar{\Psi}) = \{h_j\}_{j \in J}$ the basis of the dual operator. If Ψ is a TE and increasing operator then it can be represented as

$$\Psi(f)(x) = \sup_{i \in I} [(f \ominus g_i)(x)] = \sup_{i \in I} \left[\inf_{y \in \mathbb{R}^n} \{f(x+y) - g_i(y)\} \right] \quad (1)$$

$$= \inf_{j \in J} [(f \oplus \check{h}_j)(x)] = \inf_{j \in J} \left[\sup_{y \in \mathbb{R}^n} \{f(x-y) + \check{h}_j(y)\} \right] \quad (2)$$

The converse is true: given a collection of functions $\mathcal{B} = \{g_i\}_{i \in I}$ such that all elements of it are minimal in (\mathcal{B}, \leq) , the operator $\Psi(f) = \sup_{i \in I} \{f \ominus g_i\}$ is a TE increasing operator whose basis is equal to \mathcal{B} .

For some operators, the basis can be very large (potentially infinity) and even if the above theorem represents exactly the operator by using a full expansion of all erosions, we can obtain an approximation based on smaller collections or truncated bases $\mathcal{B} \subset \text{Bas}(\Psi)$ and $\bar{\mathcal{B}} \subset \text{Bas}(\bar{\Psi})$. Then, from the operators $\Psi_l(f) = \sup_{g \in \mathcal{B}} \{f \ominus g\}$ and $\Psi_u(f) = \inf_{h \in \bar{\mathcal{B}}} \{f \oplus h\}$ the original Ψ is bounded from below and above, i.e., $\Psi_l(f) \leq \Psi(f) \leq \Psi_u(f)$. Note also that in the case of a non minimal representation by a subset of the kernel functions larger than the basis, one just gets a redundant still satisfactory representation.

2.2 Extension to non-increasing mappings

The extension of this theory to TE *non necessarily increasing mappings* was introduced by Bannon and Barrera in [3]. It involves a supremum of a basis of operators combining an erosion and an anti-dilation. Other than that additional level of complexity of the underlying operators, the results are structurally similar to those that we discussed above.

3 Morphological group equivariant operators on homogeneous spaces

We revisit in this section the main results on Roerdink group morphology [20,22].

3.1 Group morphological operators for Boolean lattices

Let us consider E is now a homogeneous space under a group \mathbb{G} acting transitively on E . The object space of interest is the Boolean lattice $\mathcal{P}(E)$ of all subsets of E .

The strategy to introduce the group operators on $\mathcal{P}(E)$ will consist in

1. defining Minkowski operators on $\mathcal{P}(\mathbb{G})$, then
2. using a lifting of subsets of E to subsets of \mathbb{G} , apply these operators, and finally
3. projecting the corresponding result back to the original space E .

Dilation and erosion on $\mathcal{P}(\mathbb{G})$. A mapping $\Psi : \mathcal{P}(\mathbb{G}) \rightarrow \mathcal{P}(\mathbb{G})$ is called \mathbb{G} -left-equivariant when, for all $g \in \mathbb{G}$, $\Psi(gG) = g\Psi(G)$, $\forall G \in \mathcal{P}(\mathbb{G})$. And similarly, a \mathbb{G} -right-equivariant implies for all $\forall G \in \mathcal{P}(\mathbb{G})$, $\Psi(Gg) = \Psi(G)g$.

The dilation and erosion on $\mathcal{P}(\mathbb{G})$ will be defined as the \mathbb{G} -equivariant mappings commuting with unions and intersections respectively. Let H be a fixed

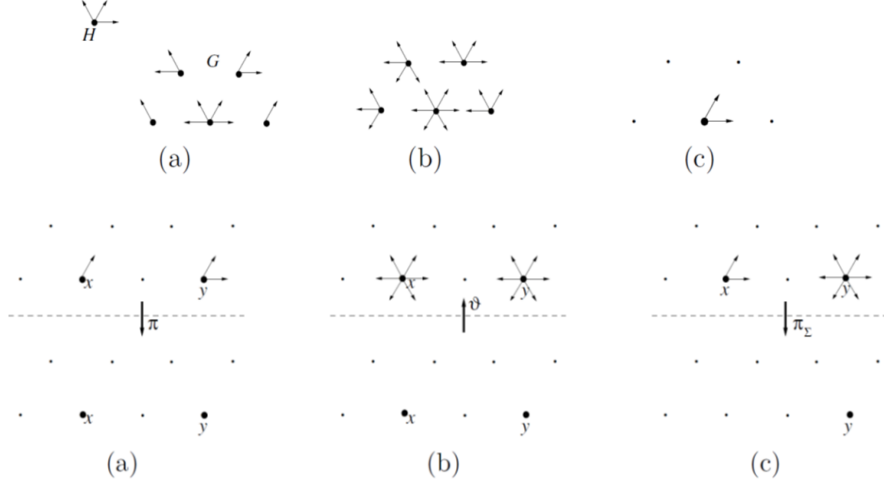


Fig. 2. Top, morphological operations on the motion group $SE(2)$: (a) group set G and group structuring element H , (b) dilation of G by H and (c) erosion of G by H . Bottom, illustration of the actions of operators ϑ (a), π (b) and (c) π_Σ on the rotation-translation group $SE(2)$. Extracted from [22].

subset of \mathbb{G} , called the group structuring element, we define the \mathbb{G} -left-equivariant dilation and erosion of G by H as

$$\delta_H^l(G) = G \oplus_{\mathbb{G}}^l H = \bigcup_{h \in H} Gh = \bigcup_{g \in G} gH = \{k \in \mathbb{G} : (k\check{H}) \cap G \neq \emptyset\}, \quad (3)$$

$$\varepsilon_H^l(G) = G \ominus_{\mathbb{G}}^l H = \bigcap_{h \in H} Gh^{-1} = \{g \in \mathbb{G} : gH \subseteq G\}, \quad (4)$$

where $gH = \{gh : h \in H\}$, $Hg = \{hg : h \in H\}$.

An example of dilation and erosion for the rotation-translation group $SE(2)$ is depicted in Figure 2-top.

Proposition 1 (Roerdink (2000) [22]). *The pair $(\delta_H^l, \varepsilon_H^l)$ forms an adjunction and all \mathbb{G} -left-equivariant adjunctions on $\mathcal{P}(\mathbb{G})$ are of this form.*

The duality by complement is given by the fact that $(G \oplus_{\mathbb{G}}^l H)^c = G^{-1} \ominus_{\mathbb{G}}^l H^{-1}$. Because of the non-commutativity of the set product $G \oplus_{\mathbb{G}}^l H$, it is possible to introduce \mathbb{G} -right-equivariant dilation and erosion.

Lifting and projections operators. We remind that work on the case of \mathbb{G} is acting transitively on E . Let the origin ω be an arbitrary point of E . The lifting operator $\vartheta : \mathcal{P}(E) \rightarrow \mathcal{P}(\mathbb{G})$ is the mapping defined for any subset $X \in \mathcal{P}(E)$ as

$$\vartheta(X) = \{g \in \mathbb{G} : g\omega \in X\}, \quad (5)$$

associates to X all group elements which map the origin ω to an element of X . The projection operator $\pi : \mathcal{P}(\mathbb{G}) \rightarrow \mathcal{P}(E)$ for any $G \in \mathcal{P}(\mathbb{G})$ as

$$\pi(G) = \{g\omega : g \in \mathbb{G}\}, \quad (6)$$

maps to each subset G of \mathbb{G} the collection of points $g\omega \in E$, where g ranges over G . The main benefit of creating these maps is that they translate the group action on X into multiplication in \mathbb{G} .

The stabilizer–projection operator $\pi_\Sigma : \mathcal{P}(\mathbb{G}) \rightarrow \mathcal{P}(E)$ first extracts the cosets and then carries out the projection π :

$$\pi_\Sigma = \pi \varepsilon_\Sigma^l(G),$$

where the erosion by the stabilizer σ ; i.e., $\varepsilon_\Sigma^l(G) = G \ominus_{\mathbb{G}}^l$ has the property $\varepsilon_\Sigma^l(G) = \varepsilon_\Sigma^l(\varepsilon_\Sigma^l(G)) = \delta_\Sigma^l(\varepsilon_\Sigma^l(G))$ which implies that $\varepsilon_\Sigma^l(G)$ is an idempotent operator in $\mathcal{P}(\mathbb{G}) \rightarrow \mathcal{P}(\mathbb{G})$ providing the invariant elements to Σ .

All the operators ϑ , π , π_Σ are increasing and \mathbb{G} -equivariant. It is obvious that $\pi\vartheta = \text{Id}_{\mathcal{P}(E)}$ and $\pi_\Sigma\vartheta = \text{Id}_{\mathcal{P}(E)}$.

The illustration of the actions of these operators for the example of rotation-translation group $SE(2)$ is given in Figure 2-bottom.

\mathbb{G} -equivariant dilation and erosion on $\mathcal{P}(E)$.

A \mathbb{G} -equivariant operator Ψ on $\mathcal{P}(E)$ can be constructed by using the group operator $\tilde{\Psi}$ according to the following commuting diagram:

$$\begin{array}{ccc} \mathcal{P}(\mathbb{G}) & \xrightarrow{\tilde{\Psi}} & \mathcal{P}(\mathbb{G}) \\ \uparrow \vartheta & & \downarrow \pi \\ \mathcal{P}(E) & \xrightarrow{\Psi} & \mathcal{P}(E) \end{array}$$

Let us consider in particular the \mathbb{G} -equivariant dilation and erosion on $\mathcal{P}(E)$. For any set X and structuring element B , $X, B \in \mathcal{P}(E)$:

$$\delta_B^{\mathbb{G}}(X) = \pi [\vartheta(X) \oplus_{\mathbb{G}}^l \vartheta(B)] = \bigcup_{g \in \vartheta(X)} gB, \quad (7)$$

$$\varepsilon_B^{\mathbb{G}}(X) = \pi_\Sigma [\vartheta(X) \ominus_{\mathbb{G}}^l \vartheta(B)] = \bigcap_{g \in \vartheta(X^c)} g\hat{B}^*, \quad (8)$$

with $\hat{Y}^* = (\pi(\vartheta(Y)))^c$.

3.2 \mathbb{G} -equivariant dilation and erosion on \mathcal{L}

The generalization to non-Boolean lattices and particular the case of numerical functions is based on the notion of sup-generating families of a lattice. A subset l of a complete lattice \mathcal{L} is called sup-generating if every element of \mathcal{L} can be written as a supremum of elements of l .

For every $X \in \mathcal{L}$, let $l(X) = \{x \in l : x \leq X\}$ and $X = \bigvee l(X)$. Let \mathcal{L} be a complete lattice with an automorphism group \mathbb{G} and a sup-generating subset l such that:

1. l is \mathbb{G} -equivariant; i.e., for every $g \in \mathbb{G}$ and $x \in l$, $gx \in l$;
2. \mathbb{G} is transitive on l : for every $x, y \in l$ there exists at least one $g \in \mathbb{G}$ such that $gx = y$.

In that case, the construction of operators follows the commuting diagram:

$$\begin{array}{ccc}
 \mathcal{P}(\mathbb{G}) & \xrightarrow{\tilde{\Psi}} & \mathcal{P}(\mathbb{G}) \\
 \uparrow \vartheta & & \downarrow \pi \\
 \mathcal{P}(l) & \xrightarrow{\tilde{\Psi}} & \mathcal{P}(l) \\
 \uparrow l & & \downarrow \vee \\
 \mathcal{L} & \xrightarrow{\Psi} & \mathcal{L}
 \end{array}$$

The lattice of numerical functions has a natural sup-generating family given by the impulse functions $f_{x,t}$, $x \in E$, $t \in \overline{\mathbb{R}}$ defined by

$$f_{x,t}(y) = \begin{cases} t, & y = x \\ -\infty, & y \neq x \end{cases}$$

For the complete results on that case, the reader is invited to [22]. In the framework of this paper, we propose to introduce numerical group operators as group convolutions in $(\max, +)$ -algebra which could be represented by the impulse functions.

4 Morphological representation of equivariant CNNs on homogeneous spaces

Let us come back to the diagram of Fig. 1. The network Ψ as a whole is \mathbb{G} -equivariant if all its layers are \mathbb{G} -equivariant. One can consider group convolution for the linear components of Ψ and to characterize the nonlinear ones using a group morphological representation. Or alternatively, to represent the network Ψ as a nonlinear map of \mathbb{G} -equivariant erosions (or dilations). As we mentioned above, for the sake of simplicity of this paper we assume Ψ is increasing.

4.1 From group convolution to group dilations/erosions for functions

Given a compact group \mathbb{G} , the group convolution (or more precisely, “correlation”) layer between \mathbb{G} -feature maps in $L_2(\mathbb{G})$ with kernel k is given by [4]

$$(f \star_{\mathbb{G}} k)(g) = \int_{\mathbb{G}} f(h)k(g^{-1}h)dh, \quad (9)$$

where dh is the left Haar measure on \mathbb{G} . Note that the feature map $f \in \mathcal{F}(\mathbb{G}, \mathbb{R})$ has been lifted to \mathbb{G} . A typical example is the group $SE(2)$:

$$(f \star_{SE(2)} k)(x, \theta) = \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} f(x', \theta')k(R_{\theta}^{-1}(x' - x), \theta' - \theta) dx' d\theta'.$$

Let us consider the counterpart group convolution in tropical semirings. We work on the set of functions, or \mathbb{G} -feature maps, $f : \mathbb{G} \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, +\infty\}$, where instead of square integrability we need upper (or lower) semi-continuity on \mathbb{G} . The \mathbb{G} -equivariant max-plus dilation and adjoint erosion of function f by the structuring function b , with $f, b \in \mathcal{F}(\mathbb{G}, \bar{\mathbb{R}})$, are defined as: $\forall g \in \mathbb{G}$

$$(f \oplus_{\mathbb{G}} b)(g) = \sup_{h \in \mathbb{G}} \{f(h) + b(gh^{-1})\}, \quad (10)$$

$$(f \ominus_{\mathbb{G}} b)(g) = \inf_{h \in \mathbb{G}} \{f(h) - b(g^{-1}h)\}. \quad (11)$$

For readers interested on the relationship between morphological operators and PDE models, we note that similar operators appears in the context of Hamilton–Jacobi equations and its viscosity solutions as the Hopf–Lax formula for the Heisenberg group [14] or the Carnot group [2], relevant in control theory.

Because of the combination of weighting and nonlinearities which are used in deep learning, which can be viewed from a morphological perspective [25], we can propose more general equivariant morphological operators. The \mathbb{G} -equivariant max-times-plus dilation and and erosion of function f by the pair of structuring functions $\{a, b\}$, $a(g) > 0$, $\forall g \in \mathbb{G}$, $f, a, b \in \mathcal{F}(\mathbb{G}, \bar{\mathbb{R}})$, are defined as

$$(f \oplus_{\mathbb{G}} \{a, b\})(g) = \sup_{h \in \mathbb{G}} \{a(gh^{-1})f(h) + b(gh^{-1})\}, \quad (12)$$

$$(f \ominus_{\mathbb{G}} \{a, b\})(g) = \inf_{h \in \mathbb{G}} \left\{ \frac{1}{a(g^{-1}h)} (f(h) - b(g^{-1}h)) \right\}. \quad (13)$$

These operators are the generalization to group morphology of the H-operators dilation and erosion studied by Heijmans in [9].

4.2 Representation of increasing \mathbb{G} -equivariant operators

Consider an increasing \mathbb{G} -equivariant group operator

$$\tilde{\Psi} : \mathcal{F}(\mathbb{G}, \bar{\mathbb{R}}) \rightarrow \mathcal{F}(\mathbb{G}, \bar{\mathbb{R}})$$

The kernel of the operator $\tilde{\Psi}$ is given by:

$$\text{Ker}(\tilde{\Psi}) = \left\{ b : \tilde{\Psi}(b)(\omega) \geq \perp \right\}, \quad b \in \mathcal{F}(\mathbb{G}, \bar{\mathbb{R}})$$

and the corresponding morphological minimal basis of $\tilde{\Psi}$ is obtained from the kernel functions as its minimal elements with respect to the partial order \leq , i.e.,

$$\text{Bas}(\tilde{\Psi}) = \left\{ b' \in \text{Ker}(\tilde{\Psi}) : [b \in \text{Ker}(\tilde{\Psi}) \text{ and } b \leq b'] \implies b = b' \right\}$$

This collection of functions can uniquely represent the $\tilde{\Psi}$ operator as follows.

Theorem 5. Consider a group operator $\tilde{\Psi}$ acting on an upper semi-continuous function f and satisfying $\tilde{\Psi}(\top) = \top$. Let $\text{Bas}(\tilde{\Psi}) = \{b_i\}_{i \in I}$ be its basis. If $\tilde{\Psi}$ is a \mathbb{G} -equivariant and increasing operator then it can be represented as a supremum of \mathbb{G} -equivariant erosions. Every \mathbb{G} -equivariant increasing operator $\tilde{\Psi}$ satisfying $\tilde{\Psi}(\perp) = \perp$ can be written as an infimum of \mathbb{G} -equivariant dilations. We have therefore the alternative representations:

$$\tilde{\Psi}(f)(g) = \sup_{i \in I} [(f \ominus_{\mathbb{G}} b_i)(g)] = \sup_{i \in I} \inf_{h \in \mathbb{G}} \{f(h) - b_i(g^{-1}h)\} \quad (14)$$

$$= \inf_{j \in J} [(f \oplus_{\mathbb{G}} b_j)(g)] = \inf_{j \in J} \sup_{h \in \mathbb{G}} \{f(h) + b_j(gh^{-1})\} \quad (15)$$

The converse is true.

Proof. This proof follows the same line as the proof in [9](Proposition 7.1). Let us prove the expression of the infimum of dilations. Then the second follows by duality.

Let $\tilde{\Psi}(f)(g) = \perp$ if $f(g) = \perp$, and let \mathcal{D} be the set of all dilations $f \oplus_{\mathbb{G}} b = \delta_b^{\mathbb{G}}(f)$ which dominate $\tilde{\Psi}$, that is $\delta_b^{\mathbb{G}} \geq \tilde{\Psi}$. It is clear that $\inf \mathcal{D} \geq \tilde{\Psi}$.

To prove the reverse inequality, it suffices to show that for every $f \in \mathcal{F}(\mathbb{G}, \mathbb{R})$ there is a $\delta_b^{\mathbb{G}} \in \mathcal{D}$ such that

$$\tilde{\Psi}(f)(\omega) \geq \delta_b^{\mathbb{G}}(\omega), \quad (16)$$

where w is the origin. That yields

$$\tilde{\Psi}(f)(\omega) \geq (\inf \mathcal{D})(f)(\omega),$$

and therefore using the transitivity and the notation $f_{g^{-1}}(\omega) = f(g)$, we have that

$$\tilde{\Psi}(f)(g) = \tilde{\Psi}(f_{g^{-1}})(\omega) \geq (\inf \mathcal{D}) f_{g^{-1}}(\omega) = (\inf \mathcal{D})(f)(g).$$

To prove (16), take $f \in \mathcal{F}(\mathbb{G}, \overline{\mathbb{R}})$. For $h \in \mathbb{G}$, we define the mapping $d_h : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ by

$$d_h(t) = \begin{cases} -\infty, & \text{if } t = -\infty \\ \tilde{\Psi}(f)(\omega), & \text{if } -\infty < t \leq f(h^{-1}) \\ +\infty, & \text{if } t = f(h^{-1}). \end{cases}$$

We can prove that d_h is dilation in t . Let $\delta_b^{\mathbb{G}}$ be the \mathbb{G} -equivariant group dilation given by

$$\delta_b^{\mathbb{G}}(f)(g) = (f \oplus_{\mathbb{G}} b)(g) = \sup_{h \in \mathbb{G}} d_h(f(gh^{-1})),$$

with $d_h(t) = t + b(h)$, $t \in \overline{\mathbb{R}}$ being a dilation [9](Example 2.1).

It follows that for $g = \omega$ one has

$$\delta_b^{\mathbb{G}}(f)(\omega) = \sup_{h \in \mathbb{G}} d_h(f(h^{-1})) = \tilde{\Psi}(f)(\omega).$$

Next, it should be shown that $\delta_b^{\mathbb{G}} \in \mathcal{D}$, or in other words that $\tilde{\Psi} \leq \delta_b^{\mathbb{G}}$. Note that the later will be case only if

$$\tilde{\Psi}(f')(\omega) \leq \delta_b^{\mathbb{G}}(f')(0), \quad \forall f',$$

There are three possibilities for that

- i) $f' = \perp$ (trivial);
- ii) $\perp \neq f' \leq f$, then $\delta_b^{\mathbb{G}}(f')(0) = \sup_{h \in \mathbb{G}} d_h(f'(h^{-1})) = \tilde{\Psi}(f)(\omega) \geq \tilde{\Psi}(f')(\omega)$;
- iii) $f' > f$, then $\delta_b^{\mathbb{G}}(f')(0) = \infty \geq \tilde{\Psi}(f')(\omega)$.

We can now state the final result of the paper which provide the universal representation of \mathbb{G} -equivariant and increasing operators using group morphological dilations and erosions.

Theorem 6. *Let Ψ be an operator acting on upper semi-continuous functions $f : \mathcal{F}(E, \bar{\mathbb{R}}) \rightarrow \mathcal{F}(E, \bar{\mathbb{R}})$ on a homogeneous space E .*

If Ψ is a \mathbb{G} -equivariant and increasing operator then it can be represented as

$$\Psi(f)(x) = \pi \left[\sup_{i \in I} (\vartheta(f) \ominus_{\mathbb{G}} b_i) \right] = \pi \left[\sup_{i \in I} \inf_{h \in \mathbb{G}} \{ \vartheta(f)(h) - b_i(g^{-1}h) \} \right]. \quad (17)$$

The proof makes use of 14 and the construction of \mathbb{G} -equivariant operators using the paradigm of section 3.2.

5 Perspectives

The first step forward in order to complete the scope of our program is to consider the case of non-increasing equivariant operators. Technically there is no major challenge and the representation combining both supremum of erosions and infimum of dilations will provide a sound framework to explore innovative deep learning architectures.

A second element to be studied is the particular case of the morphological representation of group convolution by morphological group operators. That makes sense in the particular case of finite discrete operators and functions, see [11].

Obviously the final perspective for us is the practical implementation of group morphology and its interest on equivariant deep learning. We do believe morphological representations yield the general structure of equivariant layers learning the nonlinear components of CNNs and other neural networks, as the companions to the equivariant convolution layers. For initial results, see [18].

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