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Stability of partially congested travelling wave solutions for the extended Aw-Rascle system

Émile Deléage*, Muhammed Ali Mehmood†

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Abstract

We prove the non-linear stability of a class of travelling-wave solutions to the extended Aw-Rascle system with a singular offset function, which is formally equivalent to the compressible pressureless Navier-Stokes system with a singular viscosity. These solutions encode the effect of congestion by connecting a congested left state to an uncongested right state, and may also be viewed as approximations of solutions to the ‘hard-congestion model’. By using carefully weighted energy estimates we are able to prove the non-linear stability of viscous shock waves to the Aw-Rascle system under a small zero integral perturbation, which in particular extends previous results that do not handle the case where the viscosity is singular.

Keywords: Aw-Rascle system, compressible Navier-Stokes equations, singular limit, viscous shock waves, nonlinear stability.

MSC: 35Q35, 35L67, 35B25, 76T20.

1 Introduction

1.1 Presentation of the model

We study the following generalised Aw-Rascle system on the real line:

$$\begin{cases} \partial_t \rho + \partial_y(\rho u) = 0, \\ \partial_t(\rho w) + \partial_y(\rho u w) = 0. \end{cases} \quad (1)$$

Here, the quantities ρ represents the density while u and w respectively refer to the actual and desired velocities of agents. This system was originally coined in 2000 by Aw and Rascle [2] and has popularly been used to model the evolution of a system of interacting agents, such as the flow of vehicular traffic [21, 10, 18] or crowd dynamics [1]. The standard Aw-Rascle system is complemented by the relation $w = u + P(\rho)$, where $P = P(\rho)$ is known as the ‘offset’ function. In this paper, we consider the case where $w = u + \partial_y p_\epsilon(\rho)$, with p_ϵ being a singular function of the density depending on a parameter ϵ . More precisely, for $\epsilon > 0$ fixed, we take

$$w = u + \partial_y p_\epsilon(\rho) = u + \partial_y \left(\epsilon \left(\frac{\rho}{1 - \rho} \right)^\gamma \right), \quad \gamma \geq 1. \quad (2)$$

*Aix Marseille Univ., CNRS, Centrale Marseille, I2M UMR CNRS 7373, 13000 Marseille, France and Univ. Grenoble Alpes, CNRS, INRAE, IRD, Grenoble INP, IGE, 38000 Grenoble, France; emile.deleage@univ-amu.fr

†Imperial College London, London, United Kingdom; muhammed.mehmood21@imperial.ac.uk

Note that the singularity as ρ approaches 1 in (2) is physically significant, since it implies that the density of agents within the system may not exceed a maximal packing constraint, i.e. $\rho < 1$. In one-dimension it is interesting to note that the system may be formally rewritten as

$$\begin{cases} \partial_t \rho + \partial_y(\rho u) = 0, \\ \partial_t(\rho u) + \partial_y(\rho u^2) - \partial_y(\rho^2 p'_\epsilon(\rho) \partial_y u) = 0, \end{cases} \quad (3)$$

which resembles the one-dimensional compressible pressureless Navier-Stokes equations with a singular degenerate diffusion coefficient $\lambda_\epsilon := \rho^2 p'_\epsilon(\rho)$.

The system (3) with (2) was rigorously derived by Lefebvre-Lepot and Maury in [17] from a microscopic lubrication model, and describes the evolution of particles suspended in a viscous fluid that interact with each other via a lubricating force. The diffusion coefficient obtained by Lefebvre-Lepot and Maury is $\epsilon(1-\rho)^{-1}$, where ϵ is the viscosity of the interstitial fluid. The case $\gamma = 1$ has also proven to be physically relevant in applications. Indeed, the diffusion coefficient of dense granular suspensions was experimentally measured to behave like $(\phi_c - \phi)^{-2}$ as the solid volume fraction ϕ (which corresponds to the non-dimensionalised density) approaches the maximal volume fraction ϕ_c (see for instance [15] for a review).

Let us now go back to the momentum equation of (3). In the region where ρ is far from 1, the viscosity λ_ϵ vanishes uniformly as ϵ goes to 0 and (3) formally degenerates towards the pressureless gas system [3, 4]. On the other hand, in the congested region, ρ is very close to 1, and the singularity $(1-\rho)^{-(\gamma+1)}$ compensates the degeneracy in ϵ . The limit $\epsilon \rightarrow 0$ in model (3)-(2) (see [10]) is then viewed as a transition towards a granular suspension model, where the interacting force is governed by the contact between the solid grains. The Aw-Rascle system with this choice of offset function was investigated by the authors of [10] on a one-dimensional periodic domain, where the global well-posedness for fixed ϵ was studied, as well as the limit $\epsilon \rightarrow 0$, which is known as the ‘hard-congestion limit’. The authors demonstrated that up to a subsequence, solutions of (1) with (2) converge towards weak solutions of the ‘hard-congestion model’ (see also [18, 6]):

$$\begin{cases} \partial_t \rho + \partial_y(\rho u) = 0, & (4a) \\ \partial_t(\rho u + \partial_y \pi) + \partial_y((\rho u + \partial_y \pi)u) = 0, & (4b) \\ 0 \leq \rho \leq 1, (1-\rho)\pi = 0, \pi \geq 0. & (4c) \end{cases}$$

In this sense, the original system (1) may be referred to as an approximation of (4). System (4) is an example of a free-congested system, where the free phase refers to the region where $\rho < 1$ and the congested region is where $\rho = 1$. The potential π is obtained in the limit and can be viewed as a Lagrange multiplier associated with the incompressibility constraint $\partial_x u = 0$ which holds in the congested region. The existence of strong solutions to the system (4) is still not known and there are also no results on the stability of non-trivial solutions (i.e. where π is not identically 0). Note however that the existence of weak and measure-valued (duality) solutions to this model was recently obtained on the real line in [9]. We also refer to [8, 14, 16] and the references therein for further examples of the theoretical and numerical analysis pertaining to the Aw-Rascle system.

In the present work, we are concerned with the stability of a specific class of solutions to the system (1), namely the solutions (ρ, u) which are travelling waves that connect a congested left state ($\rho = 1$) to a non-congested right state ($\rho < 1$). Since systems (1) and (3) are equivalent (for sufficiently regular solutions), our task is closely related to the stability of travelling wave solutions to the compressible Navier-Stokes system, which has been studied by Dalibard and Perrin in [12]. There, the authors studied system (3) with the addition of a pressure and a constant viscosity coefficient $\mu > 0$. A similar study was also carried out in [22] by Vasseur and Yao. Both of these works make use of a new formulation of the system, which is obtained by introducing the ‘effective velocity’ w and rewriting the system in terms of (w, v) . The parabolic equation satisfied by v and the transport structure for w then allows for desirable energy estimates, which are carried out with the help of integrated variables (see Equation (13) below). Rewriting the system using the effective velocity has also proven to be advantageous when investigating the existence and uniqueness of weak/strong solutions to the compressible Navier-Stokes system with a density-dependent viscosity of the form $\mu(\rho) = \rho^\alpha$ for $\alpha > 0$ (see [7, 11] for example).

To the best of our knowledge, there are no known results concerning the stability of shock waves where the viscosity coefficient is singular. The interest in such a result is twofold. Firstly, the stability of strong

solutions to (1) is significant due to the equivalence with the compressible Navier-Stokes system (3), for which such a result does not exist in the literature as far as we know. On the other hand, a stability result for partially congested solutions would imply that the system (4) is also expected to be stable, which provides additional validity for the model (4) and further verifies the need for a rigid well-posedness theory. Note that in our case the presence of a singular, degenerate viscosity coefficient and the lack of pressure prevents us from using the arguments of [12, 22] to obtain the estimates required to prove the global existence and stability of travelling wave solutions to (1). Nonetheless, we demonstrate in this paper that through a careful choice of weighted energy estimates and the identification of good unknowns taking congestion into account, we can obtain results analogous to those of [12] for the Aw-Rascle system (1) with the singular offset function (2).

Let us now give an outline of the paper. In the next subsection we detail our main results, which are the existence of travelling wave solutions to System (1)-(2), and the nonlinear stability of these solutions. Then in Section 2 we introduce basic properties of travelling wave solutions, an integrated formulation of the system and some useful preliminary estimates. The bulk of our work goes into Section 3, where we obtain the well-posedness of the integrated system. Lastly, we prove the equivalence between the integrated system and the original system in Section 4.

1.2 Main results

We now mention our main results. Let the Lagrangian mass coordinate x be such that $dx = \rho dy - \rho u dt$, and $v := 1/\rho$ the specific volume. Then (3) becomes

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u - \partial_x(\phi_\epsilon(v)\partial_x u) = 0, \end{cases} \quad (5)$$

where

$$\phi_\epsilon(v) := \frac{p'_\epsilon(1/v)}{v^3} = \frac{\epsilon\gamma}{v(v-1)^{\gamma+1}}, \quad \text{such that} \quad \partial_y p_\epsilon(\rho) = -\phi_\epsilon(v)\partial_x v. \quad (6)$$

In these coordinates, it follows that

$$w = u + \partial_y p_\epsilon(\rho) = u - \phi_\epsilon(v)\partial_x v$$

is constant, i.e. solves $\partial_t w = 0$.

Remark 1.1. *All of our results remain true if we replace (2) by*

$$w = u + \partial_y \tilde{p}_\epsilon(\rho) = u + \partial_y \left(\frac{\epsilon f(\rho)}{(1-\rho)^\gamma} \right), \quad \gamma \geq 1,$$

where f is smooth on \mathbb{R}_+^* and such that $\gamma f(\rho) + (1-\rho)f'(\rho) > 0$, i.e. such that $\tilde{p}'_\epsilon(\rho) > 0$. In this case, one obtains that

$$\phi_\epsilon(v) = \epsilon v^{\gamma-3} \frac{(v-1)f'(1/v) + \gamma f(1/v)}{(v-1)^{\gamma+1}}. \quad (7)$$

In order to improve readability and without loss of generality, we will stick to the case $f(\rho) = \rho^\gamma$, for which the coefficient ϕ_ϵ can be written in the more compact form (6). This is also the form considered in [10]. All computations are similar (but heavier) in the general case (7).

Our first lemma establishes the existence of travelling wave solutions, and gives a quantitative description of the profile.

Proposition 1.2. *Let $1 = v_- < v_+$ and $u_- > u_+$ be real numbers. Then there exists a unique (up to a translation) travelling wave solution $(\mathbf{u}, \mathbf{v})(t, x) = (u_\epsilon, v_\epsilon)(\xi)$ of (5), complemented with the far field condition $(\mathbf{u}, \mathbf{v}) \rightarrow (u_\pm, v_\pm)$ as $\xi \rightarrow \pm\infty$, where $\xi := x - st$ and s is the shock speed which satisfies*

$$s = \frac{u_- - u_+}{v_+ - 1}. \quad (8)$$

The solution is a smooth monotone increasing function connecting 1 (at $-\infty$) to v_+ (at $+\infty$).

By setting $v_\epsilon(0) = (1 + v_+)/2$, one then has the following estimates:

- In the congested region ($\xi < 0$),

$$1 + \left(B - \frac{A_0}{\epsilon} \xi\right)^{-1/\gamma} \leq v_\epsilon(\xi) \leq 1 + \left(B - \frac{A_1}{\epsilon} \xi\right)^{-1/\gamma}, \quad (9)$$

where

$$A_0 := \frac{s(v_+ - 1)(v_+ + 1)}{2}, \quad A_1 := \frac{s(v_+ - 1)}{2}, \quad \text{and} \quad B := \left(\frac{2}{v_+ - 1}\right)^\gamma.$$

- In the free region $\xi > 0$,

$$v_+ - \frac{v_+ - 1}{2} \exp\left(-\frac{A_2}{\epsilon} \xi\right) \leq v_\epsilon(\xi) \leq v_+ - \frac{v_+ - 1}{2} \exp\left(-\frac{A_3}{\epsilon} \xi\right), \quad (10)$$

where

$$A_2 := \frac{s(v_+ + 1)(v_+ - 1)^{\gamma+1}}{2^{\gamma+2}\gamma} \quad \text{and} \quad A_3 := \frac{sv_+(v_+ - 1)^{\gamma+1}}{\gamma}.$$

It follows that v_ϵ converges to the shock wave $v(\xi) := \mathbf{1}_{\xi < 0} + v_+ \mathbf{1}_{\xi > 0}$ as $\epsilon \rightarrow 0$, and it holds

$$v_\epsilon(\xi) \leq \left[1 + \left(B - \frac{A_1}{\epsilon} \xi\right)^{-1/\gamma}\right] \mathbf{1}_{\xi < 0} + v_+ \mathbf{1}_{\xi \geq 0}. \quad (11)$$

With this in hand, we dedicate the rest of our effort towards studying the stability of the profiles (u_ϵ, v_ϵ) where $\epsilon \ll 1$. We first express System (5) in term of the unknowns v and $w = u - \phi_\epsilon(v)\partial_x v$:

$$\begin{cases} \partial_t w = 0, \\ \partial_t v - \partial_x w - \partial_x(\phi_\epsilon(v)\partial_x v) = 0. \end{cases} \quad (12)$$

In order to obtain effective energy estimates, we take inspiration from [12, 22] and choose to re-write this system in terms of the integrated variables. Suppose we have initial data (w_0, v_0) such that $(w_0 - (w_\epsilon)(0), v_0 - (v_\epsilon)(0)) \in (L_0^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^2$ where $L_0^1(\mathbb{R})$ is the subset of $L^1(\mathbb{R})$ consisting of zero mean functions. Then we define the integrated initial data as

$$W_0(x) := \int_{-\infty}^x (w_0(z) - w_\epsilon(z)) dz, \quad V_0(x) := \int_{-\infty}^x (v_0(z) - v_\epsilon(z)) dz.$$

Supposing that $(w - w_\epsilon, v - v_\epsilon)(t) \in L_0^1(\mathbb{R})$ holds true for positive times, we may also define the integrated variables

$$W(t, x) := \int_{-\infty}^x (w(t, z) - w_\epsilon(t, z)) dz, \quad V(t, x) := \int_{-\infty}^x (v(t, z) - v_\epsilon(t, z)) dz. \quad (13)$$

Integrating (12) between $-\infty$ and x formally, we see that (W, V) solves

$$\begin{cases} \partial_t W = 0, \\ \partial_t V - \partial_x W - \phi_\epsilon(v)\partial_x v + \phi_\epsilon(v_\epsilon)\partial_x v_\epsilon = 0, \\ (W, V)(0, \cdot) = (W_0, V_0). \end{cases} \quad (14)$$

Since W is constant in time, we will denote it by its initial value W_0 from now on. The following result pertains to this system.

Theorem 1.3 (Existence of a strong solution to the integrated system). *Let $T > 0$ and $\gamma \geq 1$. Assume that $(W_0, V_0) \in (H^2(\mathbb{R}))^2$ and define*

$$\eta_0 := \frac{\partial_x V_0}{v_\epsilon|_{t=0} - 1}. \quad (15)$$

Suppose that $\eta_0 \in H^1(\mathbb{R})$, $\sqrt{x}\partial_x^k W_0 \in L^2(\mathbb{R}_+)$ for $k = 0, 1, 2$, and that the following estimate holds:

$$\sum_{k=0}^2 \left(c_k \epsilon^{2k} E_k(0; V_2) + \epsilon^{2k-1} \|\sqrt{x}\partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right) + \|W_0\|_{L^2(\mathbb{R})}^2 + \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L^2(\mathbb{R})}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L^2(\mathbb{R})}^2 \right) \leq \delta_0 \epsilon^3 \quad (16)$$

for some constants δ_0, c_0, c_1, c_2 depending only on s, γ, v_+ . Then there exists a unique solution (W_0, V) to (14) on $(0, T)$ such that

$$V \in C([0, T]; H^2(\mathbb{R})) \cap L^2(0, T; H^3(\mathbb{R})).$$

Moreover, there exists a constant $C = C(s, v_+, \gamma, \delta_0)$ such that

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\int_{\mathbb{R}} V^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 dx ds \right) \\ & + \sup_{t \in [0, T]} \left(\sum_{k=0}^1 \epsilon^{2k+2} \left[\int_{\mathbb{R}} \left| \partial_x^k \left(\frac{\partial_x V}{v_\epsilon - 1} \right) \right|^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left| \partial_x^{k+1} \left(\frac{\partial_x V}{v_\epsilon - 1} \right) \right|^2 dx ds \right] \right) \leq C \epsilon^3. \end{aligned} \quad (17)$$

Remark 1.4. It follows from Proposition 1.2 that the coefficient of diffusion $\phi_\epsilon(v_\epsilon)$ (see (6) above) is singular and tends to $+\infty$ as x tends to $-\infty$. As a consequence, we cannot close the estimates on V in the classical Sobolev spaces H^s . This is why we work with the weighted quantity $\eta := \partial_x V / (v_\epsilon - 1)$ instead (see (35)), which yields better estimates. The drawback of this method is that the time-independent quantity W_0 cannot be bounded uniformly in time in such weighted spaces. Hence we only obtain local in time well-posedness when $W_0 \neq 0$, with a time of existence T proportional to $1/\|\partial_x W_0\|_{H^1}^{2/\gamma}$.

Remark 1.5. The assumption $\sqrt{x}\partial_x^k W_0 \in L^2(\mathbb{R}_+)$ for $k = 0, 1, 2$ is classical for this kind of system and was already used in [13]. However, it is possible to remove this assumption and only assume that $W_0 \in H^2(\mathbb{R})$. One then obtains a shorter time of existence, proportional to $1/\|W_0\|_{H^2}^2$ (see Remark 3.11 below).

Remark 1.6. Using $v_\epsilon < v_+$ and the regularity of v_ϵ , the bound (17) implies that $V \in L^\infty(0, T; H^2(\mathbb{R})) \cap L^2(0, T; H^3(\mathbb{R}))$. Furthermore, by using the smallness assumption (16), we will show the lower bound $v > 1$ for every t, x . In other words, the perturbation does not reach the congested state (see Remark 3.1).

Under the same assumptions, we also prove a stability result.

Theorem 1.7 (Nonlinear stability of travelling wave solutions). *Let $T > 0$, $\gamma \geq 1$. Assume that the initial data (u_0, v_0) is such that*

$$u_0 - (u_\epsilon)_{t=0} \in W_0^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}), \quad \frac{\partial_x [u_0 - (u_\epsilon)_{t=0}]}{v_\epsilon - 1} \in L^2(\mathbb{R}), \quad v_0 - (v_\epsilon)_{t=0} \in W_0^{2,1}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad (18)$$

and the associated integrated initial data (W_0, V_0) satisfies (16). Then there exists a unique global solution (u, v) to (1) on $[0, T]$ which satisfies

$$\begin{aligned} u - u_\epsilon & \in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R})), \\ v - v_\epsilon & \in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})). \end{aligned}$$

In particular, there exists a constant $C_1 > 0$ dependent on γ, v_+, T such that

$$\|u - u_\epsilon\|_{L^\infty(0, T; H^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^\infty(0, T; H^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^2(0, T; H^2(\mathbb{R}))} \leq C_1.$$

There also exists a constant $C_2 > 0$ dependent on γ, v_+, T, ϵ such that

$$\|u - u_\epsilon\|_{L^\infty(0, T; L^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^\infty(0, T; L^1(\mathbb{R}))} \leq C_2.$$

As a consequence of the estimates derived during the course of proving the above theorem, we are also able to assert global existence with $T = +\infty$ and long-time stability if we remove the perturbation on the desired velocity.

Corollary 1.8. *If we additionally assume that*

$$(u - u_\epsilon)(0) = (\partial_x \phi_\epsilon(v) - \partial_x \phi_\epsilon(v_\epsilon))(0), \quad (19)$$

i.e. that $(w - w_\epsilon)(0) = 0$, then the solution (u, v) is defined on $\mathbb{R}_+ \times \mathbb{R}$, satisfies (17) with $T = +\infty$ and additionally

$$\sup_{x \in \mathbb{R}} (|v - v_\epsilon|(t, x) + |u - u_\epsilon|(t, x)) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

2 Properties of the travelling wave solutions

2.1 Existence and asymptotic behavior of the travelling waves

We give here the proof of Proposition 1.2.

Proof. We first prove the existence of travelling wave solutions to (5):

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u - \partial_x(\phi_\epsilon(v) \partial_x u) = 0, \end{cases}$$

We look for a travelling wave solution of this system, i.e. a pair (u_ϵ, v_ϵ) where u_ϵ, v_ϵ are functions of the variable $\xi = x - st$, s being the speed of propagation of the solution. We also suppose that $(u_\epsilon, v_\epsilon) \rightarrow (u_\pm, v_\pm)$ and that $(u'_\epsilon, v'_\epsilon) \rightarrow (0, 0)$ as ξ goes to $\pm\infty$. We first obtain

$$\begin{cases} -sv'_\epsilon - u'_\epsilon = 0, \\ -su'_\epsilon - (\phi_\epsilon(v_\epsilon)u'_\epsilon)' = 0. \end{cases}$$

Integrating this equation between ξ and $+\infty$, we get

$$\begin{cases} sv_\epsilon + u_\epsilon = sv_+ + u_+, \\ su_\epsilon + \phi_\epsilon(v_\epsilon)u'_\epsilon = su_+. \end{cases}$$

The first equation yields $u_\epsilon = sv_+ + u_+ - sv_\epsilon$. Substituting this into the second line, we deduce that v_ϵ solves the following ODE:

$$v'_\epsilon = \frac{s(v_+ - v_\epsilon)}{\phi_\epsilon(v_\epsilon)} = \frac{s(v_+ - v_\epsilon)v_\epsilon(v_\epsilon - 1)^{\gamma+1}}{\epsilon\gamma}. \quad (20)$$

Assuming $v_+ > 1$ and $s > 0$, we may deduce that $v_- = 1$, and $u_- = s(v_+ - 1) + u_+$, or equivalently $s = (u_- - u_+)/v_+$. The function v_ϵ is therefore an increasing function taking values in the interval $(1, v_+)$ (see Figure 1). The Cauchy-Lipschitz theorem thus yields that v_ϵ is the unique (up to a translation) global solution of (20), as stated in Proposition 1.2. Note that, as v_ϵ approaches 1, the diffusion coefficient $\phi_\epsilon(v_\epsilon)$ tends to $+\infty$.

Let us now make this statement more quantitative. From now on, suppose that $v_\epsilon(0) = (1 + v_+)/2$, i.e. v_ϵ is halfway between 1 and v_+ . Let $\xi < 0$. From the ODE (20) and the monotonicity of v_ϵ , we obtain the following bound:

$$\frac{s(v_+ - 1)(v_\epsilon(\xi) - 1)^{\gamma+1}}{2\epsilon\gamma} \leq v'_\epsilon(\xi) \leq \frac{s(v_+ - 1)(v_+ + 1)(v_\epsilon(\xi) - 1)^{\gamma+1}}{2\epsilon\gamma}.$$

Dividing by $(v_\epsilon(\xi) - 1)^{\gamma+1}$ and integrating between ξ and 0 yields

$$-\frac{s(v_+ - 1)}{2\epsilon\gamma} \xi \leq -\frac{2\gamma}{\gamma(v_+ - 1)^\gamma} + \frac{1}{\gamma(v_\epsilon(\xi) - 1)^\gamma} \leq -\frac{s(v_+ - 1)(v_+ + 1)}{2\epsilon\gamma} \xi,$$

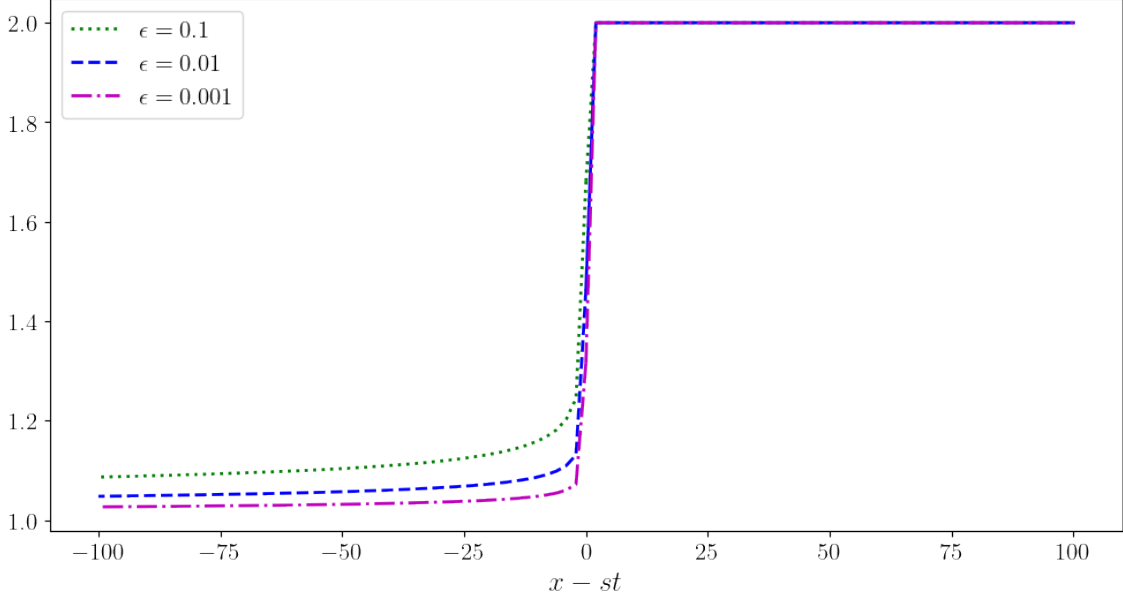


Figure 1: The profile of v_ϵ , where we fix $v_\epsilon(0) = (1 + v_+)/2$ and $v_+ = 2, \gamma = 5$.

i.e.

$$1 + \left(B - \frac{A_0}{\epsilon} \xi\right)^{-1/\gamma} \leq v_\epsilon(\xi) \leq 1 + \left(B - \frac{A_1}{\epsilon} \xi\right)^{-1/\gamma},$$

where

$$A_0 := \frac{s(v_+ - 1)(v_+ + 1)}{2}, \quad A_1 := \frac{s(v_+ - 1)}{2}, \quad \text{and } B := \left(\frac{2}{v_+ - 1}\right)^\gamma.$$

Estimate (11) follows. When $\xi > 0$, one has similarly

$$\frac{s(v_+ + 1)(v_+ - 1)^{\gamma+1}}{2^{\gamma+2}\epsilon\gamma} \leq \frac{v'_\epsilon(\xi)}{v_+ - v_\epsilon(\xi)} \leq \frac{sv_+(v_+ - 1)^{\gamma+1}}{\epsilon\gamma},$$

i.e.

$$\frac{A_2}{\epsilon} \leq \frac{v'_\epsilon(\xi)}{v_+ - v_\epsilon(\xi)} \leq \frac{A_3}{\epsilon},$$

with

$$A_2 := \frac{s(v_+ + 1)(v_+ - 1)^{\gamma+1}}{2^{\gamma+2}\gamma} \quad \text{and} \quad A_3 := \frac{sv_+(v_+ - 1)^{\gamma+1}}{\gamma}.$$

Integrating between 0 and ξ yields

$$\frac{A_2}{\epsilon} \xi \leq -\ln[v_+ - v_\epsilon(\xi)] + \ln\left[\frac{v_+ - 1}{2}\right] \leq \frac{A_3}{\epsilon} \xi,$$

i.e.

$$\frac{v_+ - 1}{2} \exp\left(-\frac{A_2}{\epsilon} \xi\right) \geq v_+ - v_\epsilon(\xi) \geq \frac{v_+ - 1}{2} \exp\left(-\frac{A_3}{\epsilon} \xi\right),$$

which is the desired estimate. □

2.2 Passage to the integrated system and reformulation of the equations

We now want to study the stability of the travelling wave solution obtained in the previous section. In order to do so, we first rewrite (5) in term of the unknowns v and $w = u - \phi_\epsilon(v)\partial_x v$:

$$\begin{cases} \partial_t w = 0, \\ \partial_t v - \partial_x w - \partial_x(\phi_\epsilon(v)\partial_x v) = 0. \end{cases} \quad (21)$$

Let (w_ϵ, v_ϵ) denote the travelling wave solution. It solves the same system:

$$\begin{cases} \partial_t w_\epsilon = 0, \\ \partial_t v_\epsilon - \partial_x w_\epsilon - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x v_\epsilon) = 0. \end{cases} \quad (22)$$

Taking the difference between these two systems yields

$$\begin{cases} \partial_t(w - w_\epsilon) = 0, \\ \partial_t(v - v_\epsilon) - \partial_x(w - w_\epsilon) - \partial_x(\phi_\epsilon(v)\partial_x v) + \partial_x(\phi_\epsilon(v_\epsilon)\partial_x v_\epsilon) = 0. \end{cases} \quad (23)$$

Following [12, 22], we introduce W, V such that

$$W(t, x) := \int_{-\infty}^x (w(t, x') - w_\epsilon(t, x')) dx' \quad \text{and} \quad V(t, x) := \int_{-\infty}^x (v(t, x') - v_\epsilon(t, x')) dx'. \quad (24)$$

Integrating (23) between $-\infty$ and x yields for (W, V) the following system:

$$\begin{cases} \partial_t W = 0, \\ \partial_t V - \partial_x W - \phi_\epsilon(v)\partial_x v + \phi_\epsilon(v_\epsilon)\partial_x v_\epsilon = 0. \end{cases}$$

Introducing ψ_ϵ such that $\psi'_\epsilon = \phi_\epsilon$, we obtain that $W = W_0$ is constant and

$$\partial_t V - \partial_x W_0 - \partial_x(\psi_\epsilon(v) - \psi_\epsilon(v_\epsilon)) = 0. \quad (25)$$

We first notice that we can replace v by $v_\epsilon + \partial_x V$. In order to write the system as a linearized part around v_ϵ plus a perturbation part, we also move the terms involving ψ_ϵ to the right-hand side and subtract $\partial_x(\phi_\epsilon(v_\epsilon)\partial_x V)$ from both sides:

$$\partial_t V - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x V) = \partial_x [\psi_\epsilon(v_\epsilon + \partial_x V) - \psi_\epsilon(v_\epsilon) - \psi'_\epsilon(v_\epsilon)\partial_x V].$$

This previous equation can be written in the following compact way:

$$\partial_t V - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x V) = \partial_x H(\partial_x V), \quad (26)$$

with

$$H(f) := \psi_\epsilon(v_\epsilon + f) - \psi_\epsilon(v_\epsilon) - \psi'_\epsilon(v_\epsilon)f. \quad (27)$$

2.3 Preliminary estimates

We give in this section some estimates on the functions $v_\epsilon, \psi_\epsilon, H$ and their derivatives.

Lemma 2.1. *(Estimates on v_ϵ) For any $k \geq 1$, there exists a constant $C = C(k, \gamma, v_+, s)$ such that*

$$\forall \epsilon > 0, \quad \left| \partial_x^k v_\epsilon \right| \leq \frac{C(v_\epsilon - 1)^{k\gamma+1}}{\epsilon^k}. \quad (28)$$

Proof. By using the ODE (20) satisfied by v_ϵ , one can show by induction that for every $k \geq 1$, there exists a function f_k independent of ϵ such that

- f_k is smooth on \mathbb{R}_+^* ,

- $f_k(1) \neq 0$,
- $\partial_x^k v_\epsilon = f_k(v_\epsilon)(v_\epsilon - 1)^{k\gamma+1}/\epsilon^k$.

Since f_k is continuous on $[1, v_+]$ and $1 < v_\epsilon < v_+$, the factor $f_k(v_\epsilon)$ is uniformly bounded. \square

Lemma 2.2. (*Estimates on ψ_ϵ*) Let $\bar{v} > 1$ be arbitrary. For any $k \geq 1$, there exists $C = C(k, \gamma, \bar{v})$ such that, for every $v \in (1, \bar{v})$, for every $\epsilon > 0$,

$$\left| \psi_\epsilon^{(k)}(v) \right| \leq \frac{C\epsilon}{(v-1)^{\gamma+k}}. \quad (29)$$

Proof. As in the previous lemma, one can show that there exists a smooth function g_k defined on \mathbb{R}_+^* , independent of ϵ , such that $\psi_\epsilon^{(k)}(v) = \epsilon g_k(v)(v-1)^{-\gamma-k}$ and $g_k(1) \neq 0$. Since g_k is smooth, we can bound it on $[1, \bar{v}]$. \square

As a consequence of Lemmas 2.1 and 2.2, we obtain the following estimates:

Lemma 2.3. The functions $\phi_\epsilon^{1/2} \partial_x v_\epsilon$ and $\phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1}$, $k \geq 2$, belong to $L^2(\mathbb{R})$, with a time-independent $L^2(\mathbb{R})$ norm.

Proof. We first see with Proposition 1.2 and the ODE (20) satisfied by v_ϵ that, for any $k \geq 1$, $\partial_x^k v_\epsilon \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Since ϕ_ϵ and $v_\epsilon - 1$ go to a positive and finite limit as ξ goes to $+\infty$, the integrability of the functions $\phi_\epsilon^{1/2} \partial_x v_\epsilon$ and $\phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1}$ at $+\infty$ is a consequence of the one of $\partial_x^k v_\epsilon$.

Concerning the integrability when $\xi \rightarrow -\infty$, we first see with Lemma 2.1 that there exists $C = C(\epsilon) > 0$ such that

$$|\phi_\epsilon^{1/2} \partial_x v_\epsilon| \leq C(v_\epsilon - 1)^{(\gamma+1)/2}, \quad \text{and} \quad \left| \phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1} \right| \leq C(v_\epsilon - 1)^{(2k-1)\gamma/2-1/2}.$$

By Proposition 1.2, we deduce that

$$|\phi_\epsilon^{1/2} \partial_x v_\epsilon| \leq C \left(B - \frac{A_1}{\epsilon} \xi \right)^{-1/2-1/(2\gamma)} \quad \text{and} \quad \left| \phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1} \right| \leq C \left(B - \frac{A_1}{\epsilon} \xi \right)^{1/2+1/(2\gamma)-k} \in L^2(\mathbb{R}_-)$$

since $k \geq 2$. Note that we consider functions that depend only on the variable $\xi = x - st$, and so the L^2 -norms are independent of time. \square

Lemma 2.4. (*Bounds on H*) Let f such that there exists $\delta < 1$ with $\| \frac{f}{v_\epsilon - 1} \|_\infty \leq \delta$. Then one has the following bounds on $H(f)$:

$$|H(f)| \leq \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} f^2, \quad (30)$$

$$|\partial_x H(f)| \leq \frac{C}{(v_\epsilon - 1)^2} f^2 + \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |f| |\partial_x f|, \quad (31)$$

$$|\partial_x^2 H(f)| \leq \frac{C(v_\epsilon - 1)^{\gamma-2}}{\epsilon} f^2 + \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |f| |\partial_x^2 f| + \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x f|^2, \quad (32)$$

for some $C = C(s, \gamma, \delta, v_+)$.

Proof. We first prove (30). Recall that

$$H(f) = \psi_\epsilon(v_\epsilon + f) - \psi_\epsilon(v_\epsilon) - \psi'_\epsilon(v_\epsilon) f.$$

By Taylor's theorem,

$$|H(f)| \leq \frac{f^2}{2} \sup_{|v-v_\epsilon| \leq |f|} |\psi_\epsilon^{(2)}(v)|.$$

By the hypothesis on f , for any v such that $|v - v_\epsilon| \leq |f|$, it holds $0 < (1 - \delta)(v_\epsilon - 1) < v - 1 < 2v^+$. Lemma 2.2 then implies that $|\psi_\epsilon^{(2)}(v)| \leq C\epsilon(v - 1)^{-(\gamma+2)}$. Hence

$$|H(f)| \leq \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} f^2,$$

for some constant C depending on γ, v_+, δ . This is the first inequality. The other inequalities are proved in the same way. We differentiate H with respect to x and obtain

$$\partial_x H(f) = \partial_x v_\epsilon \left(\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon) - \psi_\epsilon^{(2)}(v_\epsilon)f \right) + \partial_x f (\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon)).$$

which yields (31) by the same arguments and Lemmas 2.1, 2.2. We now differentiate a second time with respect to x :

$$\begin{aligned} \partial_{xx} H(f) &= \partial_{xx} v_\epsilon \left(\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon) - \psi_\epsilon^{(2)}(v_\epsilon)f \right) + (\partial_x f)^2 \psi_\epsilon^{(2)}(v_\epsilon + f) \\ &\quad + (\partial_x v_\epsilon)^2 \left(\psi_\epsilon^{(2)}(v_\epsilon + f) - \psi_\epsilon^{(2)}(v_\epsilon) - \psi_\epsilon^{(3)}(v_\epsilon)f \right) \\ &\quad + 2\partial_x f \partial_x v_\epsilon \left(\psi_\epsilon^{(2)}(v_\epsilon + f) - \psi_\epsilon^{(2)}(v_\epsilon) \right) + \partial_{xx} f (\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon)). \end{aligned}$$

which yields (32) by similar computations. \square

3 Well-posedness for the integrated system

3.1 Basic energy estimate

The goal of this section is to prove the existence of strong solutions to the equation

$$\partial_t V - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x V) = \partial_x H(\partial_x V). \quad (33)$$

Since W_0 is constant in time, we can control it for positive times through suitable assumptions on the initial data. Thus without loss of generality we treat W_0 as a perturbation of the system and prove the existence of a strong solution V to (33). Our strategy is to employ a fixed-point argument, which requires us to first derive appropriate energy estimates. The first (zero-th order) estimate is obtained by multiplying (33) by V :

$$\frac{1}{2} \int_{\mathbb{R}} V^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 dx d\tau = \frac{1}{2} \int_{\mathbb{R}} V^2(0) dx + \int_0^t \int_{\mathbb{R}} V(\partial_x H(\partial_x V) + \partial_x W_0) dx d\tau. \quad (34)$$

From the left-hand side of the equation, we see that the natural energy space is $V \in L^\infty([0, T], L^2(\mathbb{R}))$ and $\sqrt{\phi_\epsilon(v_\epsilon)}\partial_x V \in L^2((0, T) \times \mathbb{R})$. Let us now move to the right-hand side of (34) and try to close the estimate. For the term containing W_0 , an integration by parts and Young's inequality yield

$$\left| \int_0^t \int_{\mathbb{R}} V \partial_x W_0 \right| = \left| - \int_0^t \int_{\mathbb{R}} W_0 \partial_x V \right| \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{W_0^2}{\phi_\epsilon(v_\epsilon)}.$$

Hence this term can be bounded through suitable assumptions on W_0 . For the term containing $H(\partial_x V)$, we also perform an integration by parts and use the estimate (30) on H that we computed before:

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} V \partial_x H(\partial_x V) \right| &= \left| - \int_0^t \int_{\mathbb{R}} H(\partial_x V) \partial_x V \right| \leq C\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V|^3}{(v_\epsilon - 1)^{\gamma+2}} \\ &\leq C\epsilon \left\| \frac{1}{\phi_\epsilon(v_\epsilon)(v_\epsilon - 1)^{\gamma+1}} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 \frac{|\partial_x V|}{v_\epsilon - 1}. \end{aligned}$$

By recalling that $\phi_\epsilon(v_\epsilon) = \frac{\epsilon\gamma}{v_\epsilon(v_\epsilon - 1)^{\gamma+1}}$ (see (6)), we obtain that

$$\left| \int_0^t \int_{\mathbb{R}} V \partial_x H(\partial_x V) \right| \leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 \frac{|\partial_x V|}{v_\epsilon - 1} \leq C \left\| \frac{\partial_x V}{v_\epsilon - 1} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2,$$

where $C = C(s, \gamma, \delta)$. In order to close the estimate, we need that the norm of the quantity $\partial_x V/(v_\epsilon - 1)$ in $L^\infty([0, T] \times \mathbb{R})$ be small enough. The ideas of the proof of the well-posedness of (33) are thus the following:

- Derive a-priori energy estimates for the quantities $\partial_x V/(v_\epsilon - 1)$ and $\partial_x[\partial_x V/(v_\epsilon - 1)]$.
- Use the fact that $\partial_x V/(v_\epsilon - 1) \in L^\infty([0, T], H^1(\mathbb{R}))$ and the injection $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ to complete the estimates and bound these quantities in the same function spaces as chosen for V .
- Use a fixed-point argument to obtain the local-in-time existence and uniqueness of strong solutions to (33).

Remark 3.1. Note that the assumption that $\|\partial_x V/(v_\epsilon - 1)\|_{L_{t,x}^\infty}$ is small, which enables to close the previous estimate, is also the one needed to bound the quantity $H(\partial_x V)$ and its derivatives (see Lemma 2.4). This assumption has a simple interpretation; by recalling that $\partial_x V = v - v_\epsilon$, we see that it prevents the perturbation v from reaching 1. In other words, this assumption ensures that $\rho < 1$, i.e. that the perturbation stays in the uncongested state.

Remark 3.2. As explained in the previous remark, it is crucial to bound the quantity $\partial_x V/(v_\epsilon - 1)$. Note that adding the factor $1/(v_\epsilon - 1)$ is not free, and the price to pay is that a lot of commutators appear and the equations become more complicated (see the next subsection). However, even if the weight $1/(v_\epsilon - 1)$ is singular, the commutators are not difficult to bound. For instance,

$$\partial_x \left(\frac{\partial_x V}{v_\epsilon - 1} \right) = \frac{\partial_x^2 V}{v_\epsilon - 1} - \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \partial_x V.$$

From the estimate on v'_ϵ that we derived in Lemma (2.1), it follows that the coefficient $v'_\epsilon/(v_\epsilon - 1)^2$ is bounded by $C(v_\epsilon - 1)^{\gamma-1}/\epsilon$. Hence this coefficient is uniformly bounded in t, x as soon as $\gamma \geq 1$, which is our hypothesis throughout the paper. The computations of the next subsections show that $\gamma \geq 1$ is the only hypothesis needed in order to obtain good bounds for every quantity which appears due to commutators. Furthermore, the factor $1/\epsilon$ in the coefficient $v'_\epsilon/(v_\epsilon - 1)^2$ of the commutator suggests that the operator ∂_x scales as $1/\epsilon$ for solutions of the equations, i.e. that $\|\partial_x f\| \cong \|f\|/\epsilon$. This observation is reflected in the definition of the norm to be seen in Section 3.3.

3.2 Formulation of the system for higher order estimates

In this subsection we describe our approach for deriving higher order energy estimates. Our estimates will heavily involve the quantity

$$\eta \equiv \eta(\partial_x V) := \frac{\partial_x V}{v_\epsilon - 1}. \quad (35)$$

The evolution equation for η reads as

$$\begin{aligned} \partial_t \eta - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \eta) &= \frac{\partial_x^2 W_0}{v_\epsilon - 1} + \frac{\partial_x^2 H}{v_\epsilon - 1} + \frac{s \eta v'_\epsilon}{v_\epsilon - 1} \\ &+ \frac{\phi_\epsilon(v_\epsilon) v'_\epsilon \partial_x^2 V}{(v_\epsilon - 1)^2} + \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \right) + \frac{1}{v_\epsilon - 1} \partial_x(\phi'_\epsilon(v_\epsilon) v'_\epsilon \partial_x V), \end{aligned} \quad (36)$$

which may be expressed as

$$\partial_t \eta - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \eta) = \mathcal{L}_\epsilon(\eta) + S_\epsilon(\partial_x V), \quad (37)$$

where

$$\begin{aligned}\mathcal{L}_\epsilon(\eta) &:= \frac{s\eta v'_\epsilon}{v_\epsilon - 1} + \frac{\phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \left(\partial_x \eta + \frac{\eta v'_\epsilon}{v_\epsilon - 1} \right) + \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \right) + \frac{1}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon)v'_\epsilon(v_\epsilon - 1)\eta), \\ S_\epsilon(\partial_x V) &:= \frac{\partial_x^2 [W_0 + H(\partial_x V)]}{v_\epsilon - 1}.\end{aligned}$$

Differentiating (37), we obtain

$$\partial_t(\partial_x \eta) - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x^2 \eta) = \partial_x S_\epsilon(\partial_x V) + \mathcal{L}_\epsilon(\partial_x \eta) + \mathcal{C}_\epsilon(\eta), \quad (38)$$

where $\mathcal{C}_\epsilon(\eta) := [\partial_x, \partial_x(\phi_\epsilon \partial_x \cdot)]\eta + [\partial_x, \mathcal{L}_\epsilon]\eta = \partial_x(\partial_x \phi_\epsilon(v_\epsilon)\partial_x \eta) + [\partial_x, \mathcal{L}_\epsilon]\eta$ appears due to commutators and

$$\begin{aligned}[\partial_x, \mathcal{L}_\epsilon]\eta &:= s\eta \partial_x \left(\frac{v'_\epsilon}{v_\epsilon - 1} \right) + \partial_x \left(\frac{\phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \right) \partial_x \eta \\ &\quad + \partial_x \left(\phi_\epsilon(v_\epsilon)(v'_\epsilon)^2 \frac{1}{(v_\epsilon - 1)^2} \right) \eta + \partial_x \left(\eta \partial_x \left(\frac{\phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \right) \right) \\ &\quad - \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \partial_x (\phi'_\epsilon v'_\epsilon (v_\epsilon - 1)\eta) + \frac{1}{v_\epsilon - 1} \partial_x (\partial_x (\phi'_\epsilon(v_\epsilon)v'_\epsilon(v_\epsilon - 1))\eta).\end{aligned}$$

The equations (37) and (38) will be used to derive key estimates that allow us to eventually conclude the fixed-point argument.

Note that the operators \mathcal{L}_ϵ and \mathcal{C}_ϵ , that appear in equations (37) and (38), are linear. In order to improve readability, we give bounds for these two operators in the following lemma.

Lemma 3.3. *There exists $C = C(\gamma, s, v_+) > 0$ such that, for every $\alpha \in (0, 1)$, for every $\epsilon > 0$, for any η sufficiently regular, there holds*

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta)\eta \right| \leq \alpha \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon)(\partial_x \eta)^2 + \frac{C}{\alpha \epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon)(\partial_x V)^2 \quad (39)$$

and

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{C}_\epsilon(\eta)\partial_x \eta \right| \leq \alpha \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon)(\partial_x^2 \eta)^2 + \frac{C}{\alpha \epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left((\partial_x \eta)^2 + \frac{(\partial_x V^2)}{\epsilon^2} \right). \quad (40)$$

We defer the proof of this lemma to Appendix B.

3.3 Construction of global strong solutions

In this subsection we lay out the framework for our fixed point argument before finally stating the result of this section, i.e. the well posedness of (33) in an appropriate space. Firstly, we fix $T > 0$ arbitrary. For $t \in [0, T]$, we define the energies

$$E_0(t; V) := \int_{\mathbb{R}} V^2(t) dx, \quad D_0(t; V) := \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V(t)|^2 dx, \quad (41)$$

as well as

$$E_k(t; V) := \int_{\mathbb{R}} |\partial_x^{k-1} \eta|^2(t) dx, \quad D_k(t; V) := \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x^k \eta|^2(t) dx, \quad \text{for } k = 1, 2. \quad (42)$$

Then given V_1 , we introduce the system

$$\begin{cases} \partial_t V_2 - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x V_2) = \partial_x H(\partial_x V_1), \\ V_2|_{t=0} = V_0, \end{cases} \quad (43)$$

and the application $\mathcal{A}^\epsilon : V_1 \mapsto V_2$, where $V_1, V_2 \in \mathcal{X}$ and

$$\mathcal{X} = \{V : E_k(t, V) \in L^\infty(0, T) \text{ and } D_k(t; V) \in L^1(0, T) \text{ for } k = 0, 1, 2.\},$$

$$\|V\|_{\mathcal{X}}^2 := \sup_{t \in [0, T]} \left(\sum_{k=0}^2 c_k \epsilon^{2k} \left[E_k(t; V(t)) + \int_0^t D_k(\tau; V(\tau)) d\tau \right] \right),$$

where $c_k = c_k(s, \gamma, v_+)$, for $k = 0, 1, 2$, are constants independent of ϵ which are to be determined. For a proof that the map \mathcal{A}^ϵ is well defined, see Appendix A. For $\delta > 0$ we define the ball

$$B_\delta := \{V \in \mathcal{X} : \|V\|_{\mathcal{X}} < \delta \epsilon^{3/2}\}. \quad (44)$$

The remainder of this section aims to prove the following result.

Proposition 3.4. *Assume that for some $\delta_0 = \delta_0(s, v_+, \gamma) > 0$ it holds*

$$\sum_{k=0}^2 \left(c_k \epsilon^{2k} E_k(0; V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right) + \|W_0\|_{L_x^2}^2 + \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) \leq \delta_0 \epsilon^3, \quad (45)$$

where $c_k = c_k(s, \gamma, v_+)$ for $k = 0, 1, 2$ are positive constants. Then there exists $\delta = \delta(s, \gamma, v_+)$ such that

1. the ball B_δ is stable by \mathcal{A}^ϵ ,
2. the map \mathcal{A}^ϵ is a contraction on B_δ .

Consequently, \mathcal{A}^ϵ has a unique fixed point in B_δ .

3.4 Stability of the ball B_δ

We let $\delta > 0$, $V_1 \in B_\delta$ and $V_2 := \mathcal{A}^\epsilon(V_1)$. The aim of this subsection is to show that we can find $\delta > 0$ such that $V_2 \in B_\delta$. First, let us clarify the meaning of some notation we will use in this section.

- We denote by $C = C(s, \gamma, v_+)$ an arbitrary positive constant independent of ϵ . This constant may change, even within the same line. We also denote $C' = C'(s, \gamma, v_+, c_0, c_1, c_2)$ another positive constant that may additionally depend on the constants c_0, c_1, c_2 appearing in the definition of the norm $\|\cdot\|_{\mathcal{X}}$.
- To ease notation, we will often shorten $\phi_\epsilon(v_\epsilon), \psi_\epsilon(v_\epsilon)$ to $\phi_\epsilon, \psi_\epsilon$ respectively. For $i = 1, 2$ we also adopt the notation $\eta_i := \frac{\partial_x V_i}{v_\epsilon - 1}$.
- We adopt the notation $X_t Y_x := X(0, T; Y(\mathbb{R}))$ for appropriate function spaces X and Y .

Next, we make note of some useful estimates which will be repeatedly used in the remainder of the paper. We have up to a constant independent of ϵ ,

$$\begin{aligned} \phi_\epsilon(v_\epsilon) &= \frac{\epsilon \gamma}{v_\epsilon (v_\epsilon - 1)^{\gamma+1}}, \\ \phi_\epsilon^{(k)}(v_\epsilon) &\cong \frac{\phi_\epsilon(v_\epsilon)}{(v_\epsilon - 1)^k}, \end{aligned} \quad (46)$$

$$\partial_x v_\epsilon = v'_\epsilon \cong \frac{1}{\epsilon} (v_\epsilon - 1)^{\gamma+1}. \quad (47)$$

As a consequence of (46) and (47), we have $(v_\epsilon - 1)^2 \partial_x v_\epsilon \leq C/\epsilon$ and $(v_\epsilon - 1)^2 / \phi_\epsilon \leq C/\epsilon$, provided $\gamma \geq 1$. Let us now note a Gagliardo-Nirenberg-Sobolev interpolation inequality which we will use. For $f \in H^1(\mathbb{R})$,

$$\|f\|_{L_x^\infty} \leq C \|\partial_x f\|_{L_x^2}^{\frac{1}{2}} \|f\|_{L_x^2}^{\frac{1}{2}}, \quad (48)$$

Finally, we mention some guidelines which our estimates will follow.

- We will often artificially place a factor of ϕ_ϵ into the integral in order to obtain an expression which is a function of the energies (41)-(42). This results in the multiplication of a factor of $1/\epsilon$. For example, to estimate the term $A := \int_0^t \int_{\mathbb{R}} \partial_x V \partial_x \eta \, dx d\tau$, we have

$$A \leq \left\| \frac{1}{\phi_\epsilon} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x V| |\sqrt{\phi_\epsilon} \partial_x \eta| \, dx d\tau \leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2}. \quad (49)$$

- To estimate terms involving the expression $\partial_x^2 V / (v_\epsilon - 1)$, we will make use of the identity

$$\frac{\partial_x^2 V}{v_\epsilon - 1} = \partial_x \eta + \frac{v'_\epsilon \partial_x V}{(v_\epsilon - 1)^2}. \quad (50)$$

3.4.1 Estimates for $k = 0$

Fix $t \in [0, T]$. We multiply (43) by V_2 and integrate on $\mathbb{R} \times (0, t)$ to obtain as before:

$$\frac{1}{2} \int_{\mathbb{R}} V_2^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x V_2|^2 dx d\tau = \frac{1}{2} \int_{\mathbb{R}} V_2^2(0) dx + \int_0^t \int_{\mathbb{R}} V_2 (\partial_x H(\partial_x V_1) + \partial_x W_0) dx d\tau. \quad (51)$$

Integrating by parts, we have

$$\int_0^t \int_{\mathbb{R}} V_2 \partial_x (H(\partial_x V_1) + W_0) = - \int_0^t \int_{\mathbb{R}} H(\partial_x V_1) \partial_x V_2 - \int_0^t \int_{\mathbb{R}} W_0 \partial_x V_2 =: G_1 + G_2. \quad (52)$$

Using the estimate on H and (48), and introducing $\eta_2 = \frac{\partial_x V_2}{v_\epsilon - 1}$,

$$\begin{aligned} |G_1| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \frac{|\partial_x V_1|^2}{v_\epsilon - 1} |\partial_x V_2| = C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x V_1|^2 |\eta_2| \\ &\leq C \|\eta_2\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2}^2 \leq C \|\eta_2\|_{L_t^\infty L_x^2}^{1/2} \|\partial_x \eta_2\|_{L_t^\infty L_x^2}^{1/2} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2}^2 \\ &\leq \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned} \quad (53)$$

Remark 3.5. Note the following estimate, which can be obtained from (48) and will also be used later on:

$$\|\eta\|_{L_{t,x}^\infty} \leq \frac{C'}{\epsilon^{3/2}} \|V\|_{\mathcal{X}}. \quad (54)$$

We now estimate G_2 . Young's inequality gives that

$$\left| \int_0^t \int_{\mathbb{R}} W_0 \partial_x V_2 \right| \leq \frac{1}{4} \int_0^t D_0(\tau; V_2) \, d\tau + \int_0^t \int_{\mathbb{R}} \frac{W_0^2}{\phi_\epsilon}.$$

Using Fubini and the fact that W_0 is independant of time, one then has

$$\int_0^t \int_{\mathbb{R}} \frac{W_0^2}{\phi_\epsilon} = \int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} = \int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi < 0} \, d\tau dx + \int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi > 0} \, d\tau dx,$$

where $\xi := x - s\tau$. By using Equation (11), one then obtain for the first integral the estimate

$$\int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi < 0} \, d\tau dx \leq \frac{C}{\epsilon} \int_{\mathbb{R}} W_0^2 \int_0^{+\infty} \left(B - \frac{A_1}{\epsilon} \xi \right)^{-1-1/\gamma} \mathbf{1}_{\xi < 0} \, d\tau dx \leq C \int_{\mathbb{R}} W_0^2 \, dx,$$

and for the second integral

$$\int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi > 0} \, d\tau dx \leq \frac{C}{\epsilon} \int_{\mathbb{R}} W_0^2 \int_0^{x/s} \mathbf{1}_{\xi > 0} \, d\tau dx = \frac{C}{\epsilon} \int_{\mathbb{R}} W_0^2 x \mathbf{1}_{x \geq 0} \, dx.$$

Returning to (51), we get

$$\begin{aligned} & \int_{\mathbb{R}} V_2^2(t) dx + \frac{3}{2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V_2|^2 dx d\tau \\ & \leq \int_{\mathbb{R}} V_2^2(0) dx + C \|W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} W_0\|_{L^2(\mathbb{R}_+)} + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}, \end{aligned}$$

i.e.

$$E_0(t; V_2) + \frac{3}{2} \int_0^t D_0(\tau; V_2) d\tau \leq E_0(0; V_2) + C \|W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \quad (55)$$

Remark 3.6. Note that (55) does not depend on $T > 0$. This is due to the assumption that $\sqrt{x} W_0 \in L^2(\mathbb{R}_+)$. If this assumption is removed, one can still obtain a bound, but it is not global in time anymore. Indeed, we can simply apply Holder and Young's inequality to get

$$\left| \int_0^t \int_{\mathbb{R}} W_0 \partial_x V_2 \right| \leq \|\phi_\epsilon^{-1/2}\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_2\|_{L_{t,x}^2} \|W_0\|_{L_{t,x}^2} \leq \frac{Ct}{\epsilon} \|W_0\|_{L_x^2}^2 + \frac{1}{4} \int_0^t D_0(\tau; V_2) d\tau.$$

Thus instead of (55) we obtain

$$E_0(t; V_2) + \frac{3}{2} \int_0^t D_0(\tau; V_2) d\tau \leq E_0(0; V_2) + \frac{Ct}{\epsilon} \|W_0\|_{L_x^2}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \quad (56)$$

3.4.2 Estimates for $k = 1$

From (37), we deduce that η_2 solves

$$\partial_t \eta_2 - \partial_x (\phi_\epsilon(v_\epsilon) \partial_x \eta_2) = \mathcal{L}_\epsilon(\eta_2) + S_\epsilon(\partial_x V_1).$$

Multiplying by η_2 and integrating in space and time, we get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} |\eta_2(t)|^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x \eta_2|^2 dx d\tau - \frac{1}{2} \int_{\mathbb{R}} |\eta_2(0)|^2 dx \\ & = \int_0^t \int_{\mathbb{R}} \eta_2 \frac{\partial_x^2 H(\partial_x V_1)}{v_\epsilon - 1} dx d\tau + \int_0^t \int_{\mathbb{R}} \eta_2 \frac{\partial_x^2 W_0}{v_\epsilon - 1} dx d\tau + \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta_2) \eta_2 dx d\tau \\ & =: \sum_{n=1}^3 I_n. \end{aligned} \quad (57)$$

We now estimate each of I_1, I_2, I_3 . For I_1 , we first integrate by parts to get

$$I_1 = - \int_0^t \int_{\mathbb{R}} \partial_x \eta_2 \frac{\partial_x H(\partial_x V_1)}{v_\epsilon - 1} + \int_0^t \int_{\mathbb{R}} \eta_2 \frac{v'_\epsilon \partial_x H(\partial_x V_1)}{(v_\epsilon - 1)^2} =: (1a) + (1b).$$

Using (31),

$$|(1a)| \leq C \int_0^t \int_{\mathbb{R}} |\partial_x \eta_2| \left| \frac{1}{v_\epsilon - 1} \right| \left(\left| \frac{\partial_x V_1}{v_\epsilon - 1} \right|^2 + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x V_1| |\partial_x^2 V_1| \right).$$

The first term can be estimated as

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} |\partial_x \eta_2| \left| \frac{1}{v_\epsilon - 1} \right| \left| \frac{\partial_x V_1}{v_\epsilon - 1} \right|^2 \leq \left\| \frac{1}{\phi_\epsilon(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta_2\|_{L_{t,x}^2} \\ & \leq \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}, \end{aligned}$$

where we have used (48). Note that the identity (50) implies that

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \leq \|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L_{t,x}^2} + \|v'_\epsilon (v_\epsilon - 1)^{-2} \sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon} \|V_1\|_{\mathcal{X}}. \quad (58)$$

Therefore, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+1}} |\partial_x \eta_2| \left| \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right| \left| \frac{\partial_x V_1}{v_\epsilon - 1} \right| &\leq \left\| \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+1} \phi_\epsilon} \right\|_{L_{t,x}^\infty} \|\eta_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x \eta_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \\ &\leq \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned}$$

Next note that using (31),

$$|(1b)| \leq \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon}{(v_\epsilon - 1)^3} |\partial_x V_2| |\partial_x H(\partial_x V_1)| \leq \int_0^t \int_{\mathbb{R}} |\partial_x V_2| \frac{v'_\epsilon}{(v_\epsilon - 1)^3} \left(\left| \frac{\partial_x V_1}{v_\epsilon - 1} \right|^2 + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x V_1| |\partial_x^2 V_1| \right).$$

Then estimating in the same way as (1a), we also find that (1b) $\leq (C'/\epsilon^{7/2}) \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}$ and so

$$|I_1| \leq \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \quad (59)$$

Integrating by parts,

$$I_2 = - \int_0^t \int_{\mathbb{R}} \frac{\partial_x W_0}{v_\epsilon - 1} \partial_x \eta_2 + \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon \partial_x W_0}{(v_\epsilon - 1)^2} \frac{\partial_x V_2}{v_\epsilon - 1}.$$

We first compute using Young's inequality that

$$\left| \int_0^t \int_{\mathbb{R}} \frac{\partial_x W_0}{v_\epsilon - 1} \partial_x \eta_2 \right| \leq \frac{1}{8} \int_0^t D_1(\tau; V_2) d\tau + C \int_0^t \int_{\mathbb{R}} \frac{1}{\phi_\epsilon} \left(\frac{\partial_x W_0}{v_\epsilon - 1} \right)^2.$$

The last integral can be estimated by

$$\int_0^t \int_{\mathbb{R}} \frac{1}{\phi_\epsilon} \left(\frac{\partial_x W_0}{v_\epsilon - 1} \right)^2 \leq \int_{\mathbb{R}} (\partial_x W_0)^2 \int_0^t \frac{1}{\phi_\epsilon (v_\epsilon - 1)^2} (\mathbf{1}_{\xi < 0} + \mathbf{1}_{\xi > 0}) d\tau dx,$$

with $\xi := x - s\tau$. The bound (11) then yields for the part $\{\xi < 0\}$ the estimate

$$\begin{aligned} \int_0^t \frac{1}{\phi_\epsilon (v_\epsilon - 1)^2} \mathbf{1}_{\xi < 0} &\leq \frac{C}{\epsilon} \int_0^t \left(B - \frac{A_1}{\epsilon} \xi \right)^{-1+1/\gamma} \mathbf{1}_{\xi < 0} \\ &\leq C \left[\left(B - \frac{A_1}{\epsilon} (x - st) \right)^{1/\gamma} - \left(B - \frac{A_1}{\epsilon} (x - \max(x, 0)) \right)^{1/\gamma} \right] \mathbf{1}_{x-st \leq 0}. \end{aligned}$$

Note that the bound obtained in the end is a positive nondecreasing function that goes to $+\infty$ as t goes to $+\infty$, for every x . The inequality (9) shows that this bound is optimal, i.e. that a similar lower bounds holds for this integral. Hence it seems complicated to bound the contribution of the integral I_2 uniformly in time. However, it is always possible to write a bound of the form

$$\int_0^t \frac{1}{\phi_\epsilon (v_\epsilon - 1)^2} \mathbf{1}_{\xi < 0} \leq C \left(\frac{t}{\epsilon} \right)^{1/\gamma},$$

uniformly in x . For the part $\{\xi > 0\}$, one has as before

$$\int_{\mathbb{R}} (\partial_x W_0)^2 \int_0^t \frac{1}{\phi_\epsilon (v_\epsilon - 1)^2} \mathbf{1}_{\xi > 0} d\tau dx \leq \frac{C}{\epsilon} \int_{\mathbb{R}} (\partial_x W_0)^2 \int_0^{x/s} \mathbf{1}_{\xi > 0} d\tau dx = \frac{C}{\epsilon} \int_{\mathbb{R}} (\partial_x W_0)^2 x \mathbf{1}_{x \geq 0} dx.$$

For the second integral appearing in I_2 , we use the same splitting between $\xi < 0$ and $\xi > 0$, together with (11) to obtain again

$$\left| \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon \partial_x W_0}{(v_\epsilon - 1)^2} \frac{\partial_x V_2}{v_\epsilon - 1} \right| \leq \frac{1}{2\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2.$$

This leads to

$$|I_2| \leq \frac{1}{8} \int_0^t D_1(\tau; V_2) d\tau + \frac{1}{2\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2. \quad (60)$$

By comparison with (55), we see that multiplying by the singular weight $1/(v_\epsilon - 1)$ prevents the estimates from being uniform in time when $W_0 \neq 0$.

Remark 3.7. *If we do not suppose that $\sqrt{x} \partial_x W_0 \in L^2(\mathbb{R}_+)$, but only that $W_0 \in H^1(\mathbb{R})$, it is still possible to obtain a weaker estimate:*

$$\begin{aligned} |I_2| &\leq \left\| \frac{1}{(v_\epsilon - 1)\sqrt{\phi_\epsilon}} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \eta_2\|_{L_{t,x}^2} \|\partial_x W_0\|_{L_{t,x}^2} + \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^3 \sqrt{\phi_\epsilon}} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_2\|_{L_{t,x}^2} \|\partial_x W_0\|_{L_{t,x}^2} \\ &\leq \frac{Ct}{\epsilon} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau + \frac{1}{8} \int_0^t D_1(\tau; V_2) d\tau. \end{aligned}$$

Finally, we use Equation (39) of Lemma 3.3 with $\alpha = 1/8$ in order to bound I_3 :

$$|I_3| = \left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta_2) \eta_2 \right| \leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x \eta_2)^2 + \frac{C}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x V_2)^2$$

Collecting our estimates for $I_1 - I_3$ and returning to (57), we find

$$\begin{aligned} E_1(t; V_2) + \frac{3}{2} \int_0^t D_1(\tau; V_2) d\tau &\leq E_1(0; V_2) + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau, \end{aligned} \quad (61)$$

which yields, after multiplication by ϵ^2 ,

$$\begin{aligned} \epsilon^2 E_1(t; V_2) + \frac{3}{2} \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau &\leq \epsilon^2 E_1(0; V_2) + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + C\epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + C \int_0^t D_0(\tau; V_2) d\tau. \end{aligned} \quad (62)$$

Remark 3.8. *If we do not suppose that $\sqrt{x} \partial_x W_0$ belong to $L^2(\mathbb{R}_+)$, we still have*

$$\epsilon^2 E_1(t; V_2) + \frac{3}{2} \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau \leq \epsilon^2 E_1(0; V_2) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + Ct\epsilon \|\partial_x W_0\|_{L_{t,x}^2}^2.$$

3.4.3 Estimates for $k = 2$

Equation (38) gives that $\partial_x \eta_2$ solves

$$\partial_t(\partial_x \eta_2) - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x^2 \eta_2) = \partial_x S_\epsilon(\partial_x V_1) + \mathcal{L}_\epsilon(\partial_x \eta_2) + \mathcal{C}_\epsilon(\eta_2).$$

Multiplying by $\partial_x \eta_2$ and integrating in space and time leads to

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} |\partial_x \eta_2(t)|^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x^2 \eta_2|^2 dx d\tau - \frac{1}{2} \int_{\mathbb{R}} |\partial_x \eta_2(0)|^2 dx \\
&= \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 H(\partial_x V_1)}{v_\epsilon - 1} \partial_x^2 \eta_2 + \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 W_0}{v_\epsilon - 1} \partial_x^2 \eta_2 + \int_0^t \mathcal{L}_\epsilon(\partial_x \eta_2) \partial_x \eta_2 + \int_0^t \int_{\mathbb{R}} \partial_x \eta_2 \mathcal{C}_\epsilon(\eta_2) \quad (63) \\
&=: \sum_{n=1}^4 J_n.
\end{aligned}$$

We now estimate J_1, J_2, J_3, J_4 . Using (32),

$$\begin{aligned}
|J_1| &\leq C \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \eta_2|}{v_\epsilon - 1} \left(\frac{(v_\epsilon - 1)^{\gamma-2}}{\epsilon} |\partial_x V_1|^2 + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x V_1| |\partial_x^3 V_1| + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^2 V_1|^2 \right) \\
&\leq \frac{C}{\epsilon^2} \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} + C \epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V_1|}{v_\epsilon - 1} \frac{|\partial_x^2 \eta_2|}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^3 V_1| \quad (64) \\
&+ C \epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \eta_2|}{v_\epsilon - 1} \frac{|\partial_x^2 V_1|^2}{(v_\epsilon - 1)^{\gamma+2}}.
\end{aligned}$$

We have that

$$\begin{aligned}
\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V_1|}{v_\epsilon - 1} \frac{|\partial_x^2 \eta_2|}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^3 V_1| &\leq C \|\eta_1\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^3 V_1|}{v_\epsilon - 1} \phi_\epsilon |\partial_x^2 \eta_2| \\
&\leq C \|\eta_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x^2 \eta_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2}.
\end{aligned}$$

We compute that

$$\frac{\partial_x^3 V_1}{v_\epsilon - 1} = \partial_x^2 \eta_1 + \frac{2(v'_\epsilon)}{v_\epsilon - 1} \partial_x \eta_1 + \frac{\partial_x V_1}{v_\epsilon - 1} \left[\frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^2} + \partial_x \left(\frac{v'_\epsilon}{v_\epsilon - 1} \right) \right],$$

hence

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon^2} \|V_1\|_{\mathcal{X}}, \quad (65)$$

which yields with the previous computations that

$$\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V_1|}{v_\epsilon - 1} \frac{|\partial_x^2 \eta_2|}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^3 V_1| \leq \frac{C'}{\epsilon^{11/2}} \|V_1\|_{\mathcal{X}} \|V_2\|_{\mathcal{X}}^2.$$

To estimate the final integral appearing in (64), let us first note that since

$$\partial_x(\sqrt{\phi_\epsilon} \partial_x \eta_1) = \frac{v'_\epsilon \phi'_\epsilon}{2\sqrt{\phi_\epsilon}} \partial_x \eta_1 + \sqrt{\phi_\epsilon} \partial_x^2 \eta_1,$$

we have that

$$\|\partial_x(\sqrt{\phi_\epsilon} \partial_x \eta_1)\|_{L_{t,x}^2} \leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L_{t,x}^2} + \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_1\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon^2} \|V_1\|_{\mathcal{X}}. \quad (66)$$

Therefore,

$$\|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L_t^2 L_x^\infty} \leq \|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L_{t,x}^2}^{1/2} \|\partial_x(\sqrt{\phi_\epsilon} \partial_x \eta_1)\|_{L_{t,x}^2}^{1/2} \leq \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}} \quad (67)$$

Using this estimate and (50), we have that

$$\begin{aligned}
\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \eta_2|}{v_\epsilon - 1} \frac{|\partial_x^2 V_1|^2}{(v_\epsilon - 1)^{\gamma+2}} &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x^2 \eta_2| \frac{|\partial_x^2 V_1|^2}{(v_\epsilon - 1)^2} \leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x^2 \eta_2| \left(|\partial_x \eta_1|^2 + \frac{(v'_\epsilon)^2 |\partial_x V_1|^2}{(v_\epsilon - 1)^4} \right) \\
&\leq C \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_2\|_{L^2_{t,x}} \left(\|\partial_x \eta_1\|_{L^\infty_t L^2_x} \|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L^2_t L^\infty_x} + \frac{C}{\epsilon^2} \|\eta_1\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L^2_{t,x}} \right) \\
&\leq \frac{C'}{\epsilon^{11/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}.
\end{aligned}$$

In summary, we find $J_1 \leq (C'/\epsilon^{11/2}) \|V_2\|_{\mathcal{X}} \|V_1\|_{\mathcal{X}}^2$.

Remark 3.9. From the estimate for J_1 we can deduce that

$$\|\sqrt{\phi_\epsilon} |\partial_x \eta|^2\|_{L^2_{t,x}} \leq \frac{C'}{\epsilon^{5/2}} \|V\|_{\mathcal{X}}, \quad (68)$$

$$\|\sqrt{\phi_\epsilon} |\partial_x V|^2\|_{L^2_{t,x}} \leq \frac{C'}{\epsilon} \|V\|_{\mathcal{X}}, \quad (69)$$

$$\|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_t L^\infty_x} \leq \frac{C'}{\sqrt{\epsilon}} \|V\|_{\mathcal{X}}, \quad (70)$$

which will be useful for later estimates.

For J_2 , we proceed as in the $k = 1$ case and obtain an estimate similar to (60):

$$|J_2| = \left| \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 W_0}{v_\epsilon - 1} \partial_x^2 \eta_2 \right| \leq \frac{1}{6} \int_0^t D_2(\tau; V_2) d\tau + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x^2 W_0\|_{L^2_x}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2.$$

Remark 3.10. If $\sqrt{x} \partial_x^2 W_0 \notin L^2(\mathbb{R}_+)$, we still have using the Holder and Young inequalities that

$$|J_2| \leq \frac{C}{\sqrt{\epsilon}} \|\partial_x^2 W_0\|_{L^2_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_2\|_{L^2_{t,x}} \leq \frac{Ct}{\epsilon} \|\partial_x^2 W_0\|_{L^2_x}^2 + \frac{1}{6} \int_0^t D_2(\tau; V_2) d\tau.$$

We now use Equation (39) of Lemma 3.3 with $\alpha = 1/6$ to obtain that

$$|J_3| = \left| \int_0^t \mathcal{L}_\epsilon(\partial_x \eta_2) \partial_x \eta_2 \right| \leq \frac{1}{6} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x^2 \eta_2)^2 + \frac{C}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x \eta_2)^2.$$

It now remains to estimate the term involving \mathcal{C}_ϵ , i.e. J_4 . We use this time Equation (40) of Lemma 3.3 with $\alpha = 1/6$ to get

$$|J_4| = \left| \int_0^t \int_{\mathbb{R}} \partial_x \eta_2 \mathcal{C}_\epsilon(\eta_2) \right| \leq \frac{1}{6} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x^2 \eta)^2 + \frac{C}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon \left((\partial_x \eta_2)^2 + \frac{(\partial_x V_2^2)}{\epsilon^2} \right).$$

Returning to (63) with our estimates for $K_1 - K_7$ and $J_1 - J_3$, we have

$$\begin{aligned}
E_2(t; V_2) + \int_0^t D_2(\tau; V_2) d\tau &\leq E_2(0; V_2) + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x^2 W_0\|_{L^2_x}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{C'}{\epsilon^{11/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\
&\quad + \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V_2) d\tau + \frac{C}{\epsilon^4} \int_0^t D_0(\tau; V_2) d\tau.
\end{aligned}$$

We may assume that each $C > 0$ appearing on the right hand-side is bounded by $B_0 = B_0(\gamma, v_+, s) > 0$. Thus letting $k_0 := 1/(2B_0)$ and multiplying by $k_0 \epsilon^4$ results in

$$\begin{aligned}
k_0 \epsilon^4 E_2(t; V_2) + k_0 \epsilon^4 \int_0^t D_2(\tau; V_2) d\tau &\leq k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon} \right)^{1/\gamma} \epsilon^4 \|\partial_x^2 W_0\|_{L^2_x}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\
&\quad + \frac{\epsilon^2}{2} \int_0^t D_1(\tau; V_2) d\tau + \frac{1}{2} \int_0^t D_0(\tau; V_2) d\tau + \epsilon^3 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2.
\end{aligned} \quad (71)$$

Adding to what we found for $k = 1$, i.e. (62), we find after simplifying the terms containing D_1 that

$$\begin{aligned} & \epsilon^2 E_1(t; V_2) + \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau + k_0 \epsilon^4 E_2(t; V_2) + k_0 \epsilon^4 \int_0^t D_2(\tau; V_2) d\tau \\ & \leq \epsilon^2 E_1(0; V_2) + k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(C \epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + C \epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon^3 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + \left(C + \frac{1}{2} \right) \int_0^t D_0(\tau; V_2) d\tau. \end{aligned}$$

Again, we may assume that each $C + 1/2, C > 0$ that appear on the right-hand side is bounded by $B_1 = B_1(\gamma, v_+, s) > 0$. Without loss of generality, we may assume that $B_1 > 1$. Thus letting $k_1 := 1/(2B_1)$ and multiplying by k_1 yields

$$\begin{aligned} & k_1 \epsilon^2 E_1(t; V_2) + k_1 \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau + k_1 k_0 \epsilon^4 E_2(t; V_2) + k_1 k_0 \epsilon^4 \int_0^t D_2(\tau; V_2) d\tau \\ & \leq k_1 \epsilon^2 E_1(0; V_2) + k_1 k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + \epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon^3 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{1}{2} \int_0^t D_0(\tau; V_2) d\tau. \end{aligned}$$

Adding this inequality to the one that we obtained for $k = 0$, i.e. (55), and simplifying the terms depending on D_0 , we get

$$\begin{aligned} & E_0(t; V_2) + \int_0^t D_0(\tau; V_2) d\tau + k_1 \epsilon^2 \left(E_1(t; V_2) + \int_0^t D_1(\tau; V_2) d\tau \right) + k_1 k_0 \epsilon^4 \left(E_2(t; V_2) + \int_0^t D_2(\tau; V_2) d\tau \right) \\ & \leq E_0(0; V_2) + k_1 \epsilon^2 E_1(0; V_2) + k_1 k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + \epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon^4 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + C \|W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} W_0\|_{L^2(\mathbb{R}_+)}^2. \end{aligned}$$

Again, we may assume that each $C > 0$ in the right-hand side is bounded by $B_2 = B_2(s, \gamma, v_+) > 1$. We set $k_2 := 1/B_2$ and multiply by k_2 . In order to simplify computations, we also define

$$c_0 := k_2, \quad c_1 := k_2 k_1, \quad c_2 := k_2 k_1 k_0. \quad (72)$$

We then obtain

$$\begin{aligned} \sum_{k=0}^2 c_k \epsilon^{2k} \left(E_k(t; V_2) + \int_0^t D_k(\tau; V_2) d\tau \right) & \leq \sum_{k=0}^2 \left(c_k \epsilon^{2k} E_k(0; V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right) + \|W_0\|_{L_x^2}^2 \\ & \quad + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned}$$

We use the inequality $t^{1/\gamma} \leq T^{1/\gamma}$ in the right-hand side, and then take the supremum in time in the left-hand side to obtain

$$\begin{aligned} \sup_{t \in [0, T]} \left(\sum_{k=0}^2 c_k \epsilon^{2k} \left[E_k(t; V_2) + \int_0^t D_k(\tau; V_2) d\tau \right] \right) & \leq \sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0, V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right] \\ & \quad + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + \|W_0\|_{L_x^2}^2 + \epsilon^2 \left(\frac{T}{\epsilon}\right)^{1/\gamma} \left(\|\partial_x W_0\|_{L_{t,x}^2}^2 + \epsilon^2 \|\partial_x^2 W_0\|_{L_x^2}^2 \right). \end{aligned} \quad (73)$$

Note that, when $W_0 = 0$, one can take $T = +\infty$ and obtain a global bound. Using the constants c_0, c_1, c_2 in

the definition of the norm $\|\cdot\|_{\mathcal{X}}$, (73) can be recast as

$$\begin{aligned} \|V_2\|_{\mathcal{X}}^2 &\leq \sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0, V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right] + \|W_0\|_{L_x^2}^2 \\ &\quad + \epsilon^2 \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\|\partial_x W_0\|_{L_{t,x}^2}^2 + \epsilon^2 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ &\leq C' \delta^2 \epsilon^{3/2} \|V_2\|_{\mathcal{X}} + \delta_0^2 \epsilon^3, \end{aligned}$$

where we have used our assumption on the initial data (45). Taking $\delta < 1$ to be such that $\delta < 1/(\sqrt{2}C')$, defining $\delta_0 := \delta/2$ and using Young's inequality gives us

$$\|V_2\|_{\mathcal{X}}^2 \leq \delta^2 \epsilon^3, \quad (74)$$

and so $V_2 \in B_\delta$ as required. This completes the first part of Proposition 3.4.

Remark 3.11. *When we do not suppose that $\sqrt{x} \partial_x^k W_0 \in L^2(\mathbb{R}_+)$ for $k = 0, 1, 2$, we still obtain that*

$$\|V_2\|_{\mathcal{X}}^2 \leq \sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0; V_2) + T \epsilon^{2k-1} \|\partial_x^k W_0\|_{L_x^2}^2 \right] + C' \delta^2 \epsilon^{3/2} \|V_2\|_{\mathcal{X}}.$$

Hence the proof of the proposition still works after suitable modification of the assumption on the initial data. In this case, one has to impose the condition

$$\sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0; V_2) + T \epsilon^{2k-1} \|\partial_x^k W_0\|_{L_x^2}^2 \right] \leq \delta_0^2 \epsilon^3.$$

3.5 The map \mathcal{A}^ϵ is a contraction

In this subsection we aim to show that the map \mathcal{A}^ϵ is a contraction. We consider two elements $V_1, V_1' \in B_\delta$ and define $V_2 := \mathcal{A}^\epsilon(V_1)$ and $V_2' := \mathcal{A}^\epsilon(V_1')$. Our goal is to show that there exists $\delta < 1$ with $\|V_2 - V_2'\|_{\mathcal{X}} \leq \delta \|V_1 - V_1'\|_{\mathcal{X}}$. Defining $\tilde{V}_1 := V_1 - V_1'$ and $\tilde{V}_2 := V_2 - V_2'$, the following equations are satisfied:

$$\begin{cases} \partial_t \tilde{V}_2 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \tilde{V}_2) = \partial_x(H(\partial_x V_1) - H(\partial_x V_1')), \\ \partial_t \tilde{\eta}_2 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \tilde{\eta}_2) = \frac{\partial_x^2(H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} + \mathcal{L}_\epsilon(\tilde{\eta}_2), \\ \partial_t \partial_x \tilde{\eta}_2 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x^2 \tilde{\eta}_2) = \partial_x \left(\frac{\partial_x^2(H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} \right) + \mathcal{L}_\epsilon(\partial_x \tilde{\eta}_2) + \mathcal{C}_\epsilon(\tilde{\eta}_2). \end{cases} \quad (75)$$

We see that \tilde{V}_2 , $\tilde{\eta}_2$ and $\partial_x \tilde{\eta}_2$ solve the same equations than $V_2, \eta_2, \partial_x \eta_2$ respectively, up to two differences:

- The constant term W_0 is equal to zero here, as well as the initial data,
- The nonlinear terms (depending on H) are different.

With this in mind, multiplying the equations by $\tilde{V}_2, \tilde{\eta}_2, \partial_x \tilde{\eta}_2$ respectively and using again Lemma 3.3 (taking advantage of the linearity of \mathcal{L}_ϵ and \mathcal{C}_ϵ), we see that the computations of the previous section can be repeated. We get a similar estimate after suitable modification of the nonlinear terms. Since here W_0 and the initial data are equal to zero, we are left with

$$\begin{aligned} \|\tilde{V}_2\|_{\mathcal{X}}^2 &\leq C \left| \int_0^t \int_{\mathbb{R}} \tilde{V}_2 \partial_x(H(\partial_x V_1) - H(\partial_x V_1')) \right| + C \epsilon^2 \left| \int_0^t \int_{\mathbb{R}} \tilde{\eta}_2 \left(\frac{\partial_x^2(H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} \right) \right| \\ &\quad + C \epsilon^4 \left| \int_0^t \int_{\mathbb{R}} \partial_x \tilde{\eta}_2 \partial_x \left(\frac{\partial_x^2(H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} \right) \right| =: C \sum_{k=1}^3 \epsilon^{2k-2} |L_k|. \end{aligned} \quad (76)$$

To estimate each L_k we will need the following result.

Lemma 3.12. *If $f_1, f_2 \in \mathcal{X}$ are such that $\| \frac{f_1}{v_\epsilon - 1} \|_\infty + \| \frac{f_2}{v_\epsilon - 1} \|_\infty \leq \delta$ for some $\delta < 1$, then there exists a constant $C > 0$ independent of ϵ such that*

$$|H(f_1) - H(f_2)| \leq \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} |f_1 - f_2| (|f_1| + |f_2|), \quad (77)$$

$$|\partial_x H(f_1) - \partial_x H(f_2)| \leq \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} \left(\frac{v'_\epsilon}{v_\epsilon - 1} |f_1 - f_2| (|f_1| + |f_2|) + |\partial_x(f_1 - f_2)| |f_1| + |\partial_x f_2| |f_1 - f_2| \right), \quad (78)$$

$$\begin{aligned} |\partial_x^2 H(f_1) - \partial_x^2 H(f_2)| &\leq \frac{C(v_\epsilon - 1)^{\gamma-2}}{\epsilon} |f_1 - f_2| (|f_1| + |f_2|) \\ &+ \frac{C}{(v_\epsilon - 1)^2} (|\partial_x(f_1 - f_2)| |f_2| + |f_1 - f_2| |\partial_x f_1|) \\ &+ \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} |\partial_x(f_1 - f_2)| (|\partial_x f_1| + |\partial_x f_2|) \\ &+ \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} \left(|f_1 - f_2| |\partial_x^2 f_1| + |\partial_x^2(f_1 - f_2)| |f_2| + \frac{(\partial_x f_2)^2}{v_\epsilon - 1} |f_1 - f_2| \right). \end{aligned} \quad (79)$$

The proof follows the same approach as that of Lemma 2.4, and a very similar result can be seen in Lemma 3.4 of [12]. Thus we omit the proof. Integrating by parts and using (77),

$$\begin{aligned} |L_1| &= \left| \int_0^t \int_{\mathbb{R}} \partial_x \tilde{V}_2 (H(\partial_x V_1) - H(\partial_x V'_1)) \right| \leq \int_0^t \int_{\mathbb{R}} |\partial_x \tilde{V}_2| \left(\frac{\phi_\epsilon}{v_\epsilon - 1} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) \right) \\ &\leq C \int_0^t \int_{\mathbb{R}} \sqrt{\phi_\epsilon} |\partial_x \tilde{V}_2| \frac{|\partial_x \tilde{V}_1|}{v_\epsilon - 1} \left(\sqrt{\phi_\epsilon} |\partial_x V_1| + \sqrt{\phi_\epsilon} |\partial_x V'_1| \right) \\ &\leq \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \end{aligned}$$

Using the definition of the norm,

$$|L_1| \leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \quad (80)$$

To deal with L_2 , we first integrate by parts to get

$$\begin{aligned} L_2 &= - \int_0^t \int_{\mathbb{R}} \tilde{\eta}_2 \frac{v'_\epsilon (\partial_x H(\partial_x V_1) - \partial_x H(\partial_x V'_1))}{(v_\epsilon - 1)^2} - \int_0^t \int_{\mathbb{R}} \frac{\partial_x \tilde{\eta}_2}{v_\epsilon - 1} (\partial_x H(\partial_x V_1) - \partial_x H(\partial_x V'_1)) \\ &=: (2a) + (2b). \end{aligned} \quad (81)$$

Using (78),

$$\begin{aligned} |(2a)| &\leq C \int_0^t \int_{\mathbb{R}} |\tilde{\eta}_2| \frac{(v'_\epsilon)^2 \phi_\epsilon}{(v_\epsilon - 1)^4} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) + C \int_0^t \int_{\mathbb{R}} |\tilde{\eta}_2| \frac{v'_\epsilon \phi_\epsilon}{(v_\epsilon - 1)^3} |\partial_x^2 \tilde{V}_1| |\partial_x V_1| \\ &+ C \int_0^t \int_{\mathbb{R}} |\tilde{\eta}_2| \frac{v'_\epsilon \phi_\epsilon}{(v_\epsilon - 1)^3} |\partial_x^2 V'_1| |\partial_x \tilde{V}_1| \\ &\leq C \left\| \frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^4} \right\|_{L_{t,x}^\infty} \|\tilde{\eta}_2\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} (\|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} + \|\sqrt{\phi_\epsilon} \partial_x V'_1\|_{L_{t,x}^2}) \\ &+ C \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\tilde{\eta}_2\|_{L_{t,x}^\infty} \left(\|\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1}\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} + \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 V'_1}{v_\epsilon - 1}\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} \right). \end{aligned}$$

Using (48) and (58), we therefore find

$$|(2a)| \leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}).$$

For (2b) we use (78) to get

$$|(2b)| \leq \int_0^t \int_{\mathbb{R}} |\partial_x \tilde{\eta}_2| \frac{\phi_\epsilon}{(v_\epsilon - 1)^2} \left(\frac{v'_\epsilon}{v_\epsilon - 1} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) + |\partial_x^2 \tilde{V}_1| |\partial_x V_1| + |\partial_x^2 V'_1| |\partial_x \tilde{V}_1| \right) =: b_1 + b_2 + b_3.$$

We then have that

$$\begin{aligned} b_1 &\leq C \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2| |\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1| (|\eta_1| + |\eta'_1|) \\ &\leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} \left(\|\eta_1\|_{L_{t,x}^\infty} + \|\eta'_1\|_{L_{t,x}^\infty} \right) \\ &\leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \end{aligned}$$

Next,

$$\begin{aligned} b_2 &\leq \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2| |\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1}| |\eta_1| \leq C \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1}\|_{L_{t,x}^2} \\ &\leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V_1\|_{\mathcal{X}}. \end{aligned}$$

We find similarly that

$$b_3 \leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V'_1\|_{\mathcal{X}}, \quad (82)$$

which yields with the previous estimates that

$$\epsilon^2 |L_2| \leq \epsilon^2 \frac{C}{\epsilon^{7/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}) = \frac{C}{\epsilon^{3/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}).$$

We now estimate the term L_3 . Using (79) and (46),

$$\begin{aligned} |L_3| &\leq \frac{C}{\epsilon} \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| (v_\epsilon - 1)^{\gamma-3} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) + C \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \tilde{\eta}_2|}{(v_\epsilon - 1)^3} (|\partial_x^2 \tilde{V}_1| |\partial_x V_1| + |\partial_x \tilde{V}_1| |\partial_x^2 V_1|) \\ &\quad + C \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| \frac{\phi_\epsilon}{(v_\epsilon - 1)^2} |\partial_x^2 \tilde{V}_1| (|\partial_x^2 V_1| + |\partial_x^2 V'_1|) + C \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| \frac{\phi_\epsilon}{(v_\epsilon - 1)^2} (|\partial_x \tilde{V}_1| |\partial_x^3 V_1| + |\partial_x^3 \tilde{V}_1| |\partial_x V'_1|) \\ &\quad + C \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| \frac{\phi_\epsilon}{(v_\epsilon - 1)^3} |\partial_x^2 V'_1|^2 |\partial_x \tilde{V}_1| =: \sum_{n=1}^5 M_n. \end{aligned} \quad (83)$$

Using (58),

$$\begin{aligned} M_1 &\leq \frac{C}{\epsilon} \left\| \frac{(v_\epsilon - 1)^{\gamma-2}}{\phi_\epsilon} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} \left(\|\eta_1\|_{L_{t,x}^\infty} + \|\eta'_1\|_{L_{t,x}^\infty} \right) \\ &\leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \end{aligned}$$

Similarly,

$$\begin{aligned} M_2 &\leq C \left\| \frac{1}{\phi_\epsilon (v_\epsilon - 1)} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2\|_{L_{t,x}^2} \left(\|\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1}\|_{L_{t,x}^2} \|\eta_1\|_{L_{t,x}^\infty} + \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1}{v_\epsilon - 1}\|_{L_{t,x}^2} \right) \\ &\leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \end{aligned}$$

Next,

$$\begin{aligned} M_3 &\leq C \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2| \left| \sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1} \right| \left(\left| \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right| + \left| \frac{\partial_x^2 V_1'}{v_\epsilon - 1} \right| \right) \\ &\leq C \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \left(\left\| \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right\|_{L_t^\infty L_x^2} + \left\| \frac{\partial_x^2 V_1'}{v_\epsilon - 1} \right\|_{L_t^\infty L_x^2} \right). \end{aligned}$$

Using (50), (67) and (70),

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \leq \left\| \sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_1 \right\|_{L_t^2 L_x^\infty} + \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x \tilde{V}_1 \right\|_{L_t^2 L_x^\infty} \leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_1\|_{\mathcal{X}}. \quad (84)$$

On the other hand (50) also gives us

$$\left\| \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right\|_{L_t^\infty L_x^2} \leq C \|\partial_x \eta_1\|_{L_t^\infty L_x^2} + \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\eta_1\|_{L_t^\infty L_x^2} \leq \frac{C}{\epsilon^2} \|V_1\|_{\mathcal{X}}, \quad (85)$$

and so

$$M_3 \leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}).$$

For M_4 , we use again (65), thus,

$$\begin{aligned} M_4 &\leq C \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} + C \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \|\eta_1'\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 \tilde{V}_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \\ &\leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}). \end{aligned}$$

Lastly, using (58) and (85),

$$\begin{aligned} M_5 &\leq C \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1'}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \left\| \frac{\partial_x^2 V_1'}{v_\epsilon - 1} \right\|_{L_t^\infty L_x^2} \\ &\leq \frac{C}{\epsilon^7} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V_1'\|_{\mathcal{X}}^2 = \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V_1'\|_{\mathcal{X}} \left(\frac{\|V_1'\|_{\mathcal{X}}}{\epsilon^{3/2}} \right). \end{aligned}$$

Finally,

$$\begin{aligned} \epsilon^4 |L_3| &\leq \epsilon^4 \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}) \left(1 + \frac{\|V_1'\|_{\mathcal{X}}}{\epsilon^{3/2}} \right) \\ &\leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}) \left(1 + \frac{\|V_1'\|_{\mathcal{X}}}{\epsilon^{3/2}} \right) \end{aligned}$$

Gathering our estimates for L_1, L_2, L_3 and returning to (76), using the fact that $V_1, V_1' \in B_\delta$, we get

$$\begin{aligned} \|\tilde{V}_2\|_{\mathcal{X}}^2 &\leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}) \left(1 + \frac{\|V_1'\|_{\mathcal{X}}}{\epsilon^{3/2}} \right) \\ &\leq \delta C \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}}. \end{aligned} \quad (86)$$

Therefore taking $\delta < 1/C$ we obtain that \mathcal{A}^ϵ is a contraction, as required. This concludes the proof of Proposition 3.4.

4 Asymptotic stability of (u_ϵ, v_ϵ)

So far, we have proved that if (u, v) is a solution to the original system such that $v - v_\epsilon, u - u_\epsilon \in L_0^1(\mathbb{R})$ for all positive times then the system can be re-expressed in terms of the integrated quantities (V, W_0) , as (25). Furthermore, provided the initial energy is small enough (i.e. satisfying (45)) then there exists a unique strong solution to (25). In order to prove the existence and uniqueness of strong solutions to the original system (1), we will need the following result:

Proposition 4.1. *Provided that*

$$(u - u_\epsilon)(0) \in L_0^1(\mathbb{R}) \cap H^1(\mathbb{R}), \quad \frac{\partial_x(u - u_\epsilon)(0)}{v_\epsilon - 1} \in L^2(\mathbb{R}), \quad (87)$$

$$(v - v_\epsilon)(0) \in L_0^1(\mathbb{R}), \quad (88)$$

we have

1. $\|u - u_\epsilon\|_{L^\infty(0,t;L^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^\infty(0,t;L^1(\mathbb{R}))} \leq C(t, \epsilon)$,
2. $(u - u_\epsilon)(t), (v - v_\epsilon)(t) \in L_0^1(\mathbb{R})$ for each $t > 0$.

Our strategy of proof is as follows.

- **Step 1:** Establish the stability of the profile u_ϵ . This involves bounding the quantity $u - u_\epsilon$ in $L_t^\infty H_x^1$, which will in particular be needed for the next steps.
- **Step 2:** Use the previous estimates to show that $u - u_\epsilon$ and $v - v_\epsilon$ remain in L^1 for positive times.
- **Step 3:** Complete the remaining part of Proposition 4.1 using the estimates from the previous steps.

For the remainder of this section we define $v := v_\epsilon + \partial_x V$, $w := w_\epsilon + \partial_x W_0$ and $u := w + \phi_\epsilon(v)\partial_x v$. Recall that $\eta = \partial_x V/(v_\epsilon - 1)$ is bounded, with $\|\eta\|_{L_{t,x}^\infty} \leq C\epsilon^{-3/2}\|V\|_{\mathcal{X}} \leq C\delta$ since $V \in B_\delta$, hence $v \geq 1 + (1 - C\delta)(v_\epsilon - 1) > 1$ (up to reducing δ) and $\phi_\epsilon(v)$ is well defined. As a consequence, the pair (w, v) satisfies the system

$$\begin{cases} \partial_t w = 0, \\ \partial_t v - \partial_x w - \partial_x(\phi_\epsilon(v)\partial_x v) = 0. \end{cases}$$

4.1 Stability of the profile $u - u_\epsilon$

From the system solved by (w, v) , we deduce that u is a solution of

$$\partial_t(u - u_\epsilon) - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x(u - u_\epsilon)) = \partial_x[(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon))\partial_x u] =: G. \quad (89)$$

We see that the quantity G appearing on the right-hand side depends on the difference $\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)$. In order to control G , we will make use of the following lemma, which is analogous to Lemma 2.4.

Lemma 4.2. *Let $\Delta\phi_\epsilon(f) := \phi_\epsilon(v_\epsilon + f) - \phi_\epsilon(v_\epsilon)$. Then we have*

$$|\Delta\phi_\epsilon(f)| \leq C\phi_\epsilon(v_\epsilon)\frac{|f|}{v_\epsilon - 1}, \quad (90)$$

$$|\partial_x\Delta\phi_\epsilon(f)| \leq C\phi_\epsilon(v_\epsilon)\left(\frac{\partial_x v_\epsilon|f|}{(v_\epsilon - 1)^2} + \frac{|\partial_x f|}{v_\epsilon - 1}\right), \quad (91)$$

$$|\partial_x^2\Delta\phi_\epsilon(f)| \leq C\phi_\epsilon(v_\epsilon)\left(\frac{(\partial_x v_\epsilon)^2|f|}{(v_\epsilon - 1)^3} + \frac{\partial_x v_\epsilon|\partial_x f|}{(v_\epsilon - 1)^2} + \left|\frac{\partial_x f}{v_\epsilon - 1}\right|^2 + \frac{|\partial_x^2 f|}{v_\epsilon - 1}\right). \quad (92)$$

As a consequence, we also have that

$$|\phi_\epsilon(v)| = |\phi_\epsilon(v_\epsilon + \partial_x V)| \leq |\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)| + \phi_\epsilon(v_\epsilon) \leq (|\eta| + 1)\phi_\epsilon(v_\epsilon). \quad (93)$$

We omit the proof since it follows the same argument as that of Lemma 2.4. We now wish to prove the following existence result.

Lemma 4.3. *Suppose that $(U_0, V_0) \in H^2(\mathbb{R}) \times H^3(\mathbb{R})$ is such that (45) is satisfied by (W_0, V_0) , and that (87) is also satisfied. Consider the solution $(W_0, V) \in B_\delta \subset \mathcal{X}$ of (25) obtained in Proposition 3.4. Then there exists a unique solution $u - u_\epsilon$ to (89) such that*

$$u - u_\epsilon \in C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})). \quad (94)$$

Moreover, $u - u_\epsilon$ satisfies

$$\sup_{t \in [0, T]} \left(\sum_{k=0}^1 \left[E_k(t; u - u_\epsilon) + \int_0^t D_k(\tau; u - u_\epsilon) d\tau \right] \right) \leq C, \quad (95)$$

where $C = C(\epsilon, \delta, v_+, \gamma, s)$.

Proof. We start from the equation satisfied by $u - u_\epsilon$:

$$\partial_t(u - u_\epsilon) - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x(u - u_\epsilon)) = \partial_x[(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon))\partial_x u] = G.$$

Multiplying by $u - u_\epsilon$ and integrating by parts where appropriate, we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} |u - u_\epsilon|^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x(u - u_\epsilon)|^2 dx d\tau - \frac{1}{2} \int_{\mathbb{R}} |u - u_\epsilon|^2(0) dx \\ = - \int_0^t \int_{\mathbb{R}} \partial_x(u - u_\epsilon)(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u dx d\tau. \end{aligned}$$

With (90) and the triangle inequality, the right-hand side can be bounded by

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} \partial_x(u - u_\epsilon)(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u \right| &\leq C \|\eta\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x u\|_{L_{t,x}^2} \\ &\leq C \|\eta\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2}^2 + C \|\eta\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x u_\epsilon\|_{L_{t,x}^2}^2 \\ &\leq C \delta \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2}^2 + C(\delta, v_+, s, \epsilon), \end{aligned}$$

by using $\partial_x u_\epsilon = -s \partial_x v_\epsilon$ and $\|\eta\|_\infty \leq \delta$. Hence we obtain the following estimate (up to reducing δ):

$$\int_{\mathbb{R}} |u - u_\epsilon|^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x(u - u_\epsilon)|^2 dx \leq \int_{\mathbb{R}} |u - u_\epsilon|^2(0) dx + C(\delta, v_+, s, \epsilon).$$

We now wish to find an estimate for $\partial_x(u - u_\epsilon)$. Notice that (89) has the same structure as the equation for V (i.e. (33)), up to the source term. As we did for V , we define

$$\mu := \mu(\partial_x u) = \frac{\partial_x(u - u_\epsilon)}{v_\epsilon - 1}. \quad (96)$$

The computations that we did for V imply that μ is solution of

$$\partial_t \mu - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \mu) + \mathcal{L}_\epsilon(\mu) = \frac{\partial_x G}{v_\epsilon - 1}, \quad (97)$$

where \mathcal{L}_ϵ is the same as in Equation (37). Multiplying by μ and integrating yields

$$\frac{1}{2} \int_{\mathbb{R}} \mu^2(t) + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x \mu)^2 = \frac{1}{2} \int_{\mathbb{R}} \mu^2|_{t=0} - \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\mu) \mu + \int_0^t \int_{\mathbb{R}} \mu \frac{\partial_x G}{v_\epsilon - 1}. \quad (98)$$

By (39), for $\alpha = 1/2$, there exists $C > 0$ such that

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\mu) \mu \right| \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x \mu)^2 + C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) [\partial_x(u - u_\epsilon)]^2.$$

The first integral can be absorbed by the left-hand side of (98). The second integral can be bounded by using the energy estimate satisfied by $u - u_\epsilon$. Hence it is enough to bound the last integral in the right-hand side of (98). We compute that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \mu \frac{\partial_x G}{v_\epsilon - 1} &= - \int_0^t \int_{\mathbb{R}} \partial_x \mu \frac{G}{v_\epsilon - 1} + \int_0^t \int_{\mathbb{R}} \mu \frac{\partial_x v_\epsilon G}{(v_\epsilon - 1)^2} \\ &= - \int_0^t \int_{\mathbb{R}} \partial_x \mu \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \mu - \int_0^t \int_{\mathbb{R}} \partial_x \mu \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x \mu (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x^2 (u - u_\epsilon)}{v_\epsilon - 1} - \int_0^t \int_{\mathbb{R}} \partial_x \mu (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x^2 u_\epsilon}{v_\epsilon - 1} \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{v_\epsilon - 1} \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \mu + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{v_\epsilon - 1} \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{(v_\epsilon - 1)^2} (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 (u - u_\epsilon) + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{(v_\epsilon - 1)^2} (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon \\ &=: \sum_{k=1}^8 I_k. \end{aligned}$$

By (91) and Young's inequality, for any $\alpha > 0$,

$$\begin{aligned} |I_1| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\mu \partial_x \mu| \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \\ &\leq C \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2} \left(\frac{1}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon)\|_{L_{t,x}^2} \|\eta\|_{L_{t,x}^\infty} + \|\mu\|_{L_t^\infty L_x^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \right) \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C\delta^2}{\alpha} \left(\frac{1}{\epsilon^2} \|\sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon)\|_{L_{t,x}^2}^2 + \|\mu\|_{L_t^\infty L_x^2}^2 \right), \end{aligned}$$

where the last line uses the fact that

$$\|\eta\|_{L^\infty} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \leq \frac{C}{\epsilon^{3/2}} \delta \epsilon^{3/2} \leq C\delta.$$

(see Equation (84)). Similarly,

$$\begin{aligned} |I_2| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x \mu| \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C}{\alpha} \left(\|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2}^2 \left\| \frac{\partial_x v_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty}^2 + \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1}\|_{L_{t,x}^2}^2 \right) \|\frac{\partial_x u_\epsilon}{v_\epsilon - 1}\|_{L_{t,x}^\infty}^2 \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + C(\alpha, \epsilon, \delta, s, v_+, \gamma). \end{aligned}$$

By (90),

$$\begin{aligned} |I_3| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x \mu| |\eta| \frac{|\partial_x^2 (u - u_\epsilon)|}{v_\epsilon - 1} \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C\delta^2}{\alpha} \left(\|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{1}{\epsilon^2} \|\sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon)\|_{L_{t,x}^2}^2 \right) \end{aligned}$$

by using

$$\frac{\partial_x^2(u - u_\epsilon)}{v_\epsilon - 1} = \partial_x \mu + \frac{\partial_x v_\epsilon \partial_x(u - u_\epsilon)}{(v_\epsilon - 1)^2}.$$

Then,

$$\begin{aligned} |I_4| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x \mu| |\eta| \frac{\partial_x^2 u_\epsilon}{v_\epsilon - 1} \leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C\delta^2}{\alpha} \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 u_\epsilon}{v_\epsilon - 1}\|_{L_{t,x}^2}^2 \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + C(\alpha, \epsilon, \delta, s, v_+, \gamma), \\ |I_5| &\leq C \int_0^t \int_{\mathbb{R}} \mu^2 \phi_\epsilon \frac{\partial_x v_\epsilon}{v_\epsilon - 1} \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \\ &\leq \frac{C\delta}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon [\partial_x(u - u_\epsilon)]^2 + \frac{1}{\epsilon} \|\mu\|_{L_t^\infty L_x^2} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \\ &\leq \alpha \|\mu\|_{L_t^\infty L_x^2}^2 + C(\alpha, \epsilon, \delta, s, v_+, \gamma), \\ |I_6| &\leq C \int_0^t \int_{\mathbb{R}} |\mu| \phi_\epsilon \frac{\partial_x v_\epsilon}{v_\epsilon - 1} \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\ &\leq \frac{C}{\epsilon^2} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \left(\frac{1}{\epsilon} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \right) \\ &\leq C(\epsilon, \delta, s, v_+, \gamma), \\ |I_7| &\leq C \int_0^t \int_{\mathbb{R}} |\mu| \phi_\epsilon |\eta| \frac{\partial_x v_\epsilon \partial_x^2(u - u_\epsilon)}{(v_\epsilon - 1)^2} \\ &\leq \frac{C}{\epsilon} \|\eta\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \left(\left\| \sqrt{\phi_\epsilon} \partial_x \mu \right\|_{L_{t,x}^2} + \frac{1}{\epsilon} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \right) \\ &\leq \alpha \left\| \sqrt{\phi_\epsilon} \partial_x \mu \right\|_{L_{t,x}^2}^2 + C(\alpha, \epsilon, \delta, v_+, s, \gamma), \end{aligned}$$

and

$$|I_8| \leq C \int_0^t \int_{\mathbb{R}} |\mu| \phi_\epsilon |\eta| \frac{\partial_x v_\epsilon |\partial_x^2 u_\epsilon|}{(v_\epsilon - 1)^2} \leq \frac{C}{\epsilon^3} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} \leq C(\epsilon, \delta, s, v_+, \gamma).$$

Hence, coming back to Equation (98), by taking $\alpha > 0$ sufficiently small, up to reducing δ , we obtain that there exists $\alpha_0 > 0$ such that

$$\alpha_0 \left(\|\mu\|_{L_t^\infty L_x^2}^2 + \left\| \sqrt{\phi_\epsilon} \partial_x \mu \right\|_{L_{t,x}^2}^2 \right) \leq \frac{1}{2} \|\mu_0\|_{L_x^2}^2 + C(\alpha_0, \epsilon, \delta, s, v_+, \gamma),$$

with $C(\alpha_0, \epsilon, \delta, s, v_+, \gamma) \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

With this estimate, the existence and uniqueness of $u - u_\epsilon$ is classical. It remains to verify that $u - u_\epsilon$ is continuous in time. Let $f \in H^1(\mathbb{R})$ with $\|f\|_{H^1(\mathbb{R})} = 1$. Then using $\partial_t(v - v_\epsilon) = \partial_x(u - u_\epsilon)$, we have

$$\begin{aligned} \langle \partial_t(v - v_\epsilon), f \rangle_{H^{-1} \times H^1} &= (\partial_x(u - u_\epsilon), f) = (u - u_\epsilon, \partial_x f) \\ &\leq \|(u - u_\epsilon)(t)\|_{L_x^2} \end{aligned}$$

and so $\|\partial_t(v - v_\epsilon)\|_{L_t^\infty H_x^{-1}} \leq \|u - u_\epsilon\|_{L_t^\infty L_x^2}$. Since $v - v_\epsilon = \partial_x V$, we have that $\partial_t \partial_x V \in L^\infty(0, T; H^{-1}(\mathbb{R})) \subset L^2(0, T; H^{-1}(\mathbb{R}))$. Since we also have $\partial_x V \in L^2(0, T; H^2(\mathbb{R}))$, it follows (e.g. from Theorem II.5.13. of [5]) that $\partial_x V \in C([0, T]; H^1(\mathbb{R}))$ and so $V \in C([0, T]; H^2(\mathbb{R}))$. Now since $w = u - \phi_\epsilon(v) \partial_x v$ and $w_\epsilon = u_\epsilon - \phi_\epsilon(v_\epsilon) \partial_x v_\epsilon$, we have that $\partial_x W_0 = u - u_\epsilon + \phi'_\epsilon(v) \partial_x v - \phi'_\epsilon(v_\epsilon) \partial_x v_\epsilon$. After rearranging, we find that

$$u - u_\epsilon = \partial_x W_0 - \phi'_\epsilon(v) \partial_x V + (\phi'_\epsilon(v) - \phi'_\epsilon(v_\epsilon)) \partial_x v_\epsilon.$$

The continuity in time of $W_0, v_\epsilon, \phi'_\epsilon(v)$ and V imply that $u - u_\epsilon \in C([0, T]; H^1(\mathbb{R}))$ as claimed. \square

4.2 Bounding $(v - v_\epsilon)$ in $L^1(\mathbb{R})$

We first obtain a L_x^1 bound for $u - u_\epsilon$. Let $\{j_n\}_{n \in \mathbb{N}} \subset C^2(\mathbb{R})$ be a convex approximation of $|\cdot|$ with $|j'_n| \leq C$, $j''_n > 0$. For example, we may take $j_n(z) := \sqrt{z^2 + n^{-1}} - \sqrt{n^{-1}}$. Multiplying (89) by $j'_n(u - u_\epsilon)$, we get

$$\begin{aligned} & \int_{\mathbb{R}} j_n(u - u_\epsilon)(t) dx - \int_{\mathbb{R}} j_n(u - u_\epsilon)(0) dx + \int_0^t \int_{\mathbb{R}} j''_n(u - u_\epsilon) \phi_\epsilon(v) |\partial_x(u - u_\epsilon)|^2 \\ &= \int_0^t \int_{\mathbb{R}} j'_n(u - u_\epsilon) ((\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon + (\partial_x \phi_\epsilon(v) - \partial_x \phi_\epsilon(v_\epsilon)) \partial_x u_\epsilon). \end{aligned} \quad (99)$$

Using $\partial_x u_\epsilon = -s \partial_x v_\epsilon$, (90), (91) and (58), the right-hand side can be bounded by

$$\begin{aligned} & C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \partial_x V \frac{|\partial_x^2 v_\epsilon|}{v_\epsilon - 1} + C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left(\frac{\partial_x v_\epsilon |\partial_x V|}{(v_\epsilon - 1)^2} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \partial_x v_\epsilon \\ & \leq C \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \\ & \quad + C \|\sqrt{\phi_\epsilon} \partial_x v_\epsilon\|_{L^2_{t,x}} \left(\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^2_{t,x}} + \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} \right). \end{aligned}$$

By Lemma 2.3,

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} + \|\sqrt{\phi_\epsilon} \partial_x v_\epsilon\|_{L^2_{t,x}} \leq C(s, \gamma, v_+, \epsilon) t^{1/2},$$

Hence we obtain a global bound of the form $C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}$ for the right-hand side. Note that the second term on the left-hand side of (99) has a positive sign and therefore can be discarded. Returning to (99) and taking $n \rightarrow \infty$, we get

$$\|(u - u_\epsilon)(t)\|_{L^1_x} \leq \|(u - u_\epsilon)(0)\|_{L^1_x} + C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}. \quad (100)$$

We now wish to find an estimate for $v - v_\epsilon$. From (5) we have that

$$\partial_t(v - v_\epsilon) = \partial_x(u - u_\epsilon).$$

Multiplying this equation by $j'_n(v - v_\epsilon)$ where j_n is defined as above and integrating in space, we have

$$\frac{d}{dt} \int_{\mathbb{R}} j_n(v - v_\epsilon) dx = \int_{\mathbb{R}} j'_n(v - v_\epsilon) \partial_x(u - u_\epsilon) dx \leq \int_{\mathbb{R}} |\partial_x(u - u_\epsilon)| dx. \quad (101)$$

Therefore, to obtain a $L^1(\mathbb{R})$ estimate for $v - v_\epsilon$ it is sufficient to bound $\partial_x(u - u_\epsilon)$ in $L^1(\mathbb{R})$. Now, the evolution equation of $\partial_x(u - u_\epsilon)$ can be expressed as

$$\begin{aligned} \partial_t \partial_x(u - u_\epsilon) - \partial_x(\phi_\epsilon(v) \partial_x^2(u - u_\epsilon)) &= \partial_x((\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon) + \partial_x[\partial_x(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u] \\ &\quad + \partial_x[\partial_x \phi_\epsilon(v_\epsilon) \partial_x(u - u_\epsilon)]. \end{aligned}$$

Multiplying by $j'_n(\partial_x(u - u_\epsilon))$ and integrating by parts where appropriate,

$$\begin{aligned} & \int_{\mathbb{R}} j_n(\partial_x(u - u_\epsilon))(t) dx - \int_{\mathbb{R}} j_n(\partial_x(u - u_\epsilon))(0) dx + \int_0^t \int_{\mathbb{R}} j''_n(\partial_x(u - u_\epsilon)) \phi_\epsilon(v) |\partial_x^2(u - u_\epsilon)|^2 \\ &= \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x((\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon) + \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x[\partial_x(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u] \\ &\quad + \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x[\partial_x \phi_\epsilon(v_\epsilon) \partial_x(u - u_\epsilon)] =: \sum_{n=1}^3 I_n. \end{aligned} \quad (102)$$

Firstly using (91),

$$\begin{aligned}
|I_1| &= \left| \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) ((\partial_x \phi_\epsilon(v) - \partial_x \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon + (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^3 u_\epsilon) \right| \\
&\leq C \int_0^t \int_{\mathbb{R}} |j'_n(\partial_x(u - u_\epsilon))| \phi_\epsilon(v_\epsilon) \left(|\partial_x^2 v_\epsilon| \left(\frac{\partial_x v_\epsilon |\partial_x V|}{(v_\epsilon - 1)^2} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) + \frac{|\partial_x^3 v_\epsilon|}{v_\epsilon - 1} |\partial_x V| \right) \\
&\leq \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 v_\epsilon}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \left(\frac{C}{\epsilon} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \partial_x^2 V \right\|_{L_{t,x}^2} \right) \\
&\leq C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}.
\end{aligned}$$

Then

$$I_2 = \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x^2 [\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)] \partial_x u + \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x [\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)] \partial_x^2 u =: K_1 + K_2.$$

Using (92),

$$\begin{aligned}
|K_1| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left(\frac{(\partial_x v_\epsilon)^2 |\partial_x V|}{(v_\epsilon - 1)^3} + \frac{\partial_x v_\epsilon |\partial_x^2 V|}{(v_\epsilon - 1)^2} + \left| \frac{\partial_x^2 V}{v_\epsilon - 1} \right|^2 + \frac{|\partial_x^3 V|}{v_\epsilon - 1} \right) |\partial_x u| \\
&\leq C \left\| \sqrt{\phi_\epsilon} \partial_x u \right\|_{L_{t,x}^2} \left(\left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \right. \\
&\quad \left. + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^2 L^\infty} \left\| \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^\infty L^2} \right),
\end{aligned}$$

and the triangle inequality

$$\left\| \sqrt{\phi_\epsilon} \partial_x u \right\|_{L_{t,x}^2} \leq \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \partial_x u_\epsilon \right\|_{L_{t,x}^2}$$

enables to bound K_1 . For K_2 , (91) yields

$$\begin{aligned}
|K_2| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left(\frac{\partial_x v_\epsilon |\partial_x V|}{(v_\epsilon - 1)^2} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) |\partial_x^2 u| \\
&\leq \left(\left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \right) \left(\left\| \sqrt{\phi_\epsilon} \partial_x^2(u - u_\epsilon) \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \partial_x^2 u_\epsilon \right\|_{L_{t,x}^2} \right).
\end{aligned}$$

For I_3 , we decompose

$$\begin{aligned}
|I_3| &= C \left| \int_0^t \int_{\mathbb{R}} j'_n [(\phi_\epsilon''(v_\epsilon)(\partial_x v_\epsilon)^2 + \phi_\epsilon'(v_\epsilon) \partial_x^2 v_\epsilon) \partial_x(u - u_\epsilon) + \phi_\epsilon'(v_\epsilon) \partial_x v_\epsilon \partial_x^2(u - u_\epsilon)] \right| \\
&\leq C \left[\left\| \frac{\phi_\epsilon''(v_\epsilon) v_\epsilon'}{\phi_\epsilon(v_\epsilon)} \right\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x v_\epsilon \right\|_{L_{t,x}^2} + \left\| \frac{\phi_\epsilon'(v_\epsilon)(v_\epsilon - 1)}{\phi_\epsilon(v_\epsilon)} \right\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \right] \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \\
&\quad + C \left\| \frac{\phi_\epsilon'(v_\epsilon)(v_\epsilon - 1)}{\phi_\epsilon(v_\epsilon)} \right\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x v_\epsilon \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2(u - u_\epsilon)}{v_\epsilon - 1} \right\|_{L_{t,x}^2}.
\end{aligned}$$

Passing to the limit $n \rightarrow \infty$ with the help of Fatou's lemma gives us

$$\left\| \partial_x(u - u_\epsilon)(t) \right\|_{L_x^1} \leq \left\| \partial_x(u - u_\epsilon)(0) \right\|_{L_x^1} + C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}. \quad (103)$$

We have now shown that $(u - u_\epsilon)(t)$, $(v - v_\epsilon)(t)$ belong to $L^1(\mathbb{R})$ for any $t > 0$. Note that since the equations for both quantities are conservative, we actually have that $(u - u_\epsilon)(t)$, $(v - v_\epsilon)(t) \in L_0^1(\mathbb{R})$. This marks the end of the proof of Proposition 4.1.

4.3 Concluding the proofs of Theorem 1.7 and Corollary 1.8

We are now in a position to justify the equivalence between the original and integrated systems, which will allow us to conclude the proof of Theorem 1.7. Firstly, consider the original system with some initial data (u_0, v_0) which satisfies the assumptions of Theorem 1.7, and let (V, W) be the corresponding solution of the integrated system (25). Since we defined $(u, v) := (u_\epsilon + \partial_x U, v_\epsilon + \partial_x V)$, it follows that (u, v) is a solution to the original system. Moreover, Lemma 4.3 tells us that $\partial_x U \in C([0, T]; H^1(\mathbb{R}))$ and Theorem 1.3 tells us that $\partial_x V \in C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))$. Thus, $(u, v) \in (u_\epsilon, v_\epsilon) + (C([0, T]; H^1(\mathbb{R})))^2$ solves the original system and using Proposition 4.1 we have that $(u, v)(t) \in L_0^1(\mathbb{R})$ for any $t \in (0, T]$. It remains to show that the solution is unique. To this end, suppose that (u, v) is another solution to the original system with initial data (u_0, v_0) satisfying the assumptions of Theorem 1.7. Then we may write $(u, v) = (u_\epsilon, v_\epsilon) + (f, g)$ for some $f, g \in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R}))$. Now, define the integrated quantities

$$U(t, x) := \int_{-\infty}^x (u(t, z) - u_\epsilon(t, z)) dz, \quad V(t, x) := \int_{-\infty}^x (v(t, z) - v_\epsilon(t, z)) dz,$$

and

$$W := U + \phi_\epsilon(v) - \phi_\epsilon(v_\epsilon).$$

Then (V, W) is a solution of the integrated system (25). Furthermore, thanks to the hypotheses on the initial data (u, v) we have that $\partial_x V \in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))$. Therefore $V \in \mathcal{X}$. Moreover, due to the assumptions on the initial data, we in fact have $V \in B_\delta$ and so since \mathcal{A}^ϵ is a contraction on B_δ we easily deduce that V is uniquely determined by the initial data. Since W is constant in time, we conclude that the pair (W, V) is unique and therefore so is (u, v) .

Finally, let us explain how Corollary 1.8 is obtained. With the additional assumption (19), the inequality (73) becomes

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\sum_{k=0}^2 c_k \epsilon^{2k} \left[E_k(t; V_2) + \int_0^t D_k(\tau; V_2) d\tau \right] \right) \\ & \leq C(E_0(0; V_2) + \epsilon^2 E_1(0; V_2) + \epsilon^4 E_2(0; V_2)) + \frac{C}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}, \end{aligned} \quad (104)$$

and so we may take $T = +\infty$ which gives us a solution defined on $\mathbb{R}_+ \times \mathbb{R}$. Then using (48) we have that there exists $C > 0$ independent of time such that

$$\|(v - v_\epsilon)(t)\|_{L_x^\infty} \leq C \|(v - v_\epsilon)(t)\|_{L_x^2}^{1/2} \|\partial_x(v - v_\epsilon)(t)\|_{L_x^2}^{1/2}, \quad (105)$$

where the right hand-side tends to 0 as $t \rightarrow \infty$ since $V \in L^2(\mathbb{R}_+; H^2(\mathbb{R}))$. This shows that $(v - v_\epsilon)(t) \rightarrow 0$ as $t \rightarrow \infty$. Similarly, the bounds obtained in Lemma 4.3 imply that

$$\|(u - u_\epsilon)(t)\|_{H_x^1} \rightarrow 0 \text{ as } t \rightarrow \infty$$

and so we also find that $(u - u_\epsilon)(t) \rightarrow 0$ as $t \rightarrow \infty$, which completes the proof.

A Proof that the map \mathcal{A}^ϵ is well defined

We give here a proof that the map \mathcal{A}^ϵ of Section 3.3 is well defined, i.e. we show that Equation (43):

$$\begin{cases} \partial_t V_2 - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x V_2) = \partial_x H(\partial_x V_1), \\ V_2|_{t=0} = V_0, \end{cases}$$

is well posed. Equation (43) rewrites as

$$\begin{cases} \partial_t V_2 = \partial_x(\phi_\epsilon \partial_x V_2) + f, \\ V_2|_{t=0} = V_0, \end{cases} \quad (106)$$

where $f = \partial_x H(\partial_x V_1) + \partial_x W_0$. Since the operator $\partial_x(\phi_\epsilon \partial_x \cdot)$ is uniformly elliptic, one can hope that it is the infinitesimal generator of an analytic evolution operator $(U(t, \tau))_{t, \tau \geq 0}$. The unique solution of (106) is then given by

$$V_2(t) = U(t, 0)V_0 + \int_0^t U(t, \tau)f(\tau)d\tau.$$

The proof that $\partial_x(\phi_\epsilon \partial_x \cdot)$ generates an analytic evolution operator is classical. Note that one has to be careful because the coefficient $\phi_\epsilon = \phi_\epsilon(t, x)$ depends on time and is unbounded. In order to simplify the situation, we take advantage of the fact that ϕ_ϵ is a function of the variable $\xi := x - st$. A change of variable $(t, x) \mapsto (t, \xi) \in (0, T) \times \mathbb{R}$ then yields that (106) can be written as

$$\begin{cases} \partial_t V_2 = \partial_\xi(\phi_\epsilon(\xi)\partial_\xi V_2) + s\partial_\xi V_2 + f, \\ V_2|_{t=0} = V_0, \end{cases}$$

Hence in these coordinates, the operator $\partial_\xi(\phi_\epsilon(\xi)\partial_\xi \cdot) + s\partial_\xi$ is independant of time. Consider the Hilbert space $H = L^2(\mathbb{R}, \mathbb{C})$, and define the unbounded operator

$$T_c V := \phi_\epsilon^{1/2} \partial_\xi V, \quad \text{with} \quad D(T_c) = C_c^\infty(\mathbb{R}).$$

Let T be the closure of T_c , i.e.

$$D(T) = \{V \in H, \phi_\epsilon^{1/2} \partial_\xi V \in H, \exists V_n \in C_c^\infty(\mathbb{R}), V_n \rightarrow V, \phi_\epsilon^{1/2} \partial_\xi V_n \rightarrow \phi_\epsilon^{1/2} \partial_\xi V\}.$$

The adjoint T^* of T is given by

$$D(T^*) = \{V \in H, \partial_\xi(\phi_\epsilon^{1/2} V) \in H\}, \quad \text{with} \quad T^* V = -\partial_\xi(\phi_\epsilon^{1/2} V).$$

Indeed, if $V \in D(T^*)$, then $T^* V = -\partial_\xi(\phi_\epsilon^{1/2} V)$ in the sense of distributions. Conversely, if V and $\partial_\xi(\phi_\epsilon^{1/2} V) \in H$, then $\langle -\partial_\xi(\phi_\epsilon^{1/2} V), \psi \rangle_H = \langle V, T\psi \rangle_H$ for every $\psi \in C_c^\infty(\mathbb{R})$, and the same is true for every $\psi \in D(T)$ by density.

Since T is closed and densely defined, T^*T defined by

$$D(T^*T) = \{V \in D(T), TV \in D(T^*)\}, \quad T^*TV = T^*(TV) = -\partial_\xi(\phi_\epsilon(\xi)\partial_x V)$$

is self-adjoint and positive (see Theorem 13.13 in [20], Chapter 13). It follows that $-T^*T$ is the infinitesimal generator of an analytic semigroup. Now define the closed operator $B := s\partial_\xi$, with $D(B) = H^1(\mathbb{R})$. Since there exists $\alpha > 0$ such that $\phi_\epsilon \geq \alpha$, we deduce that $D(T^*T) \subset D(T) \subset D(B)$. Furthermore, for every $\epsilon > 0$, for every $V \in D(T^*T)$, one has that

$$\begin{aligned} \|BV\| &\leq \frac{s}{\alpha} \|TV\| = \frac{s}{\alpha} |\langle V, T^*TV \rangle_H|^{1/2} \leq \frac{s}{\alpha} \|V\|^{1/2} \|T^*TV\|^{1/2} \\ &\leq \epsilon \|T^*TV\| + \frac{s^2}{4\alpha^2\epsilon} \|V\|, \end{aligned}$$

by Young's inequality. By Theorem 12.37 in [19], it follows that $-T^*T + B$ is the infinitesimal generator of an analytic semigroup.

Note that in fact it is possible to prove that

$$D(T) = \{V \in H, \phi_\epsilon^{1/2} \partial_x V \in H\}.$$

Indeed, let θ a smooth function such that $0 \leq \theta \leq 1$, $\theta|_{[-\frac{1}{3}, \frac{1}{3}]} \equiv 1$, $\text{supp}(\theta) \subset [-1, 1]$ and $\int_{\mathbb{R}} \theta = 1$. For $V \in H$ with $\phi_\epsilon^{1/2} \partial_\xi V \in H$, define

$$V_n(\xi) := \theta\left(\frac{\xi}{n}\right) (\rho_n * V)(\xi),$$

where $\rho_n := n\theta(n\cdot) \in C_c^\infty(\mathbb{R})$ is an approximation of unity. Then $V_n \in C_c^\infty(\mathbb{R})$ and $V_n \rightarrow V$ in L^2 . Furthermore,

$$\phi_\epsilon^{1/2} \partial_\xi V_n = \frac{\phi_\epsilon^{1/2}}{n} \theta'\left(\frac{\xi}{n}\right) (\rho_n * V)(\xi) + \phi_\epsilon^{1/2} \theta\left(\frac{\xi}{n}\right) (\rho_n * \partial_\xi V)(\xi).$$

Since $\phi_\epsilon^{1/2}(\xi) \leq C(1 + |\xi|^{1/2+1/(2\gamma)})$ and θ is compactly supported,

$$\left| \frac{\phi_\epsilon^{1/2}}{n} \theta' \left(\frac{\xi}{n} \right) \right| \leq C n^{1/(2\gamma)-1/2} \mathbf{1}_{n/3 \leq |\xi| \leq n}.$$

Hence

$$\frac{\phi_\epsilon^{1/2}}{n} \theta' \left(\frac{\xi}{n} \right) (\rho_n * V)(\xi) \rightarrow 0 \quad \text{and} \quad \phi_\epsilon^{1/2} \theta \left(\frac{\xi}{n} \right) (\rho_n * \partial_\xi V)(\xi) \rightarrow \phi_\epsilon^{1/2} \partial_\xi V.$$

Hence $V \in D(T)$.

B Proof of Lemma 3.3

We give here the proof of Lemma 3.3.

Proof. We first do the computations of the estimate for \mathcal{L}_ϵ . We compute that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta) \eta &= \int_0^t \int_{\mathbb{R}} \frac{s \eta^2 v'_\epsilon}{v_\epsilon - 1} dx d\tau + \int_0^t \int_{\mathbb{R}} \frac{\phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \eta \left(\partial_x \eta + \frac{\eta v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} \eta \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau + \int_0^t \int_{\mathbb{R}} \frac{\eta}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1) \eta) dx d\tau =: \sum_{k=1}^4 I_k. \end{aligned}$$

Let us fix $\alpha \in (0, 1)$. For I_1 , we exploit the smallness of v'_ϵ (i.e. (47)) to get

$$|I_1| = \left| \int_0^t \int_{\mathbb{R}} \frac{s \eta^2 v'_\epsilon}{v_\epsilon - 1} dx d\tau \right| \leq \left\| \frac{s v'_\epsilon}{\phi_\epsilon(v_\epsilon - 1)^3} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x V|^2 \leq \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V) d\tau.$$

Next,

$$\begin{aligned} |I_2| &= \left| \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \eta \left(\partial_x \eta + \frac{\eta v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau \right| \\ &\leq \int_0^t \int_{\mathbb{R}} \left| \frac{v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \right| |\eta| |\partial_x \eta| + \int_0^t \int_{\mathbb{R}} \left| \frac{v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \frac{v'_\epsilon \partial_x V}{(v_\epsilon - 1)^2} \frac{\partial_x V}{v_\epsilon - 1} \right| \\ &\leq \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2} + \left\| \frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^4} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2}^2 \\ &\leq \frac{C}{\epsilon} \left(\int_0^t D_0(\tau; V) d\tau \right)^{\frac{1}{2}} \left(\int_0^t D_1(\tau; V) d\tau \right)^{\frac{1}{2}} + \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V) d\tau \\ &\leq \frac{\alpha}{3} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau, \end{aligned}$$

where we used Young's inequality for the last line. Similarly,

$$\begin{aligned} |I_3| &= \left| \int_0^t \int_{\mathbb{R}} \eta \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau \right| \\ &\leq \int_0^t \int_{\mathbb{R}} |\partial_x \eta| \left| \frac{\eta v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \right| \leq \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2} \\ &\leq \frac{\alpha}{3} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau. \end{aligned}$$

To deal with the last term, we first integrate by parts to get

$$I_4 = \int_0^t \int_{\mathbb{R}} \frac{\eta}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1) \eta) dx d\tau = - \int_0^t \int_{\mathbb{R}} \frac{\partial_x \eta}{v_\epsilon - 1} v'_\epsilon \phi'_\epsilon \partial_x V + \int_0^t \int_{\mathbb{R}} \frac{(v'_\epsilon)^2 \eta}{(v_\epsilon - 1)^2} \phi'_\epsilon \partial_x V.$$

Then in the same spirit as the previous estimates and with the estimate (46) on ϕ'_ϵ ,

$$\begin{aligned} |I_4| &\leq \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L^2_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} + \left\| \frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^4} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}}^2 \\ &\leq \frac{\alpha}{3} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau. \end{aligned}$$

We finally put all terms together to obtain that

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta) \eta \right| \leq \sum_{k=1}^4 |I_k| \leq \alpha \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau.$$

We now move on to the proof of the second estimate. Again, we fix $\alpha \in (0, 1)$ and compute that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \partial_x \eta C_\epsilon(\eta) &= - \int_0^t \int_{\mathbb{R}} \partial_x \phi_\epsilon \partial_x^2 \eta \partial_x \eta + \int_0^t \int_{\mathbb{R}} \eta \partial_x \eta \partial_x \left(\frac{sv'_\epsilon}{v_\epsilon - 1} \right) \\ &\quad + \int_0^t \int_{\mathbb{R}} (\partial_x \eta)^2 \partial_x \left(\phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \right) + \int_0^t \int_{\mathbb{R}} \eta \partial_x \eta \partial_x \left(\frac{(v'_\epsilon)^2 \phi_\epsilon(v_\epsilon)}{(v_\epsilon - 1)^2} \right) \\ &\quad + \int_0^t \int_{\mathbb{R}} \partial_x \eta \partial_x \left(\eta \partial_x \left(\frac{v'_\epsilon \phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} \right) \right) - \int_0^t \int_{\mathbb{R}} \partial_x \eta \frac{v'_\epsilon \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1) \eta)}{(v_\epsilon - 1)^2} \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\partial_x \eta}{v_\epsilon - 1} \partial_x (\partial_x (v'_\epsilon \phi'_\epsilon(v_\epsilon) (v_\epsilon - 1) \eta)) =: \sum_{n=1}^7 K_n. \end{aligned} \tag{107}$$

We can estimate each of these terms using the same strategy as our estimates for $I_1 - I_4$. This leads to

$$\begin{aligned} |K_1| &\leq \left\| \frac{\partial_x \phi_\epsilon}{\phi_\epsilon} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta\|_{L^2_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L^2_{t,x}} \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_1(\tau; V) d\tau, \\ |K_2| &= \left| \int_0^t \int_{\mathbb{R}} \frac{sv'_\epsilon}{v_\epsilon - 1} ((\partial_x \eta)^2 + \eta \partial_x^2 \eta) \right| \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^4} \int_0^t D_0(\tau; V) d\tau, \\ |K_3| &= 2 \left| \int_0^t \int_{\mathbb{R}} \phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \partial_x^2 \eta \partial_x \eta \right| \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_1(\tau; V) d\tau, \\ |K_4| &= \left| \int_0^t \int_{\mathbb{R}} \frac{(v'_\epsilon)^2 \phi_\epsilon}{(v_\epsilon - 1)^2} (\eta \partial_x^2 \eta + (\partial_x \eta)^2) \right| \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\epsilon^4} \int_0^t D_0(\tau; V) d\tau, \\ |K_5| &= \left| \int_0^t \int_{\mathbb{R}} \partial_x \left(\phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \right) \eta \partial_x^2 \eta \right|, \end{aligned}$$

and we compute with the estimate (46) and Lemma 2.1 that

$$\left| \partial_x \left(\phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \right) \right| \leq \frac{C}{\epsilon^2} \phi_\epsilon (v_\epsilon - 1)^{2\gamma} \leq \frac{C}{\epsilon^2} \phi_\epsilon,$$

hence

$$|K_5| \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\alpha \epsilon^4} \int_0^t D_0(\tau; V) d\tau.$$

Next,

$$K_6 = - \int_0^t \int_{\mathbb{R}} (\partial_x \eta)^2 \frac{v'_\epsilon \phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1)}{(v_\epsilon - 1)^2} - \int_0^t \int_{\mathbb{R}} \eta \partial_x \eta \frac{v'_\epsilon \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1))}{(v_\epsilon - 1)^2}.$$

With the estimate (46) and Lemma 2.1, we see that

$$|\partial_x[\phi'_\epsilon v'_\epsilon(v_\epsilon - 1)]| \leq \frac{C}{\epsilon}(v_\epsilon - 1)^\gamma. \quad (108)$$

Hence

$$|K_6| \leq \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\epsilon^4} \int_0^t D_0(\tau; V) d\tau.$$

To estimate K_7 , we first integrate by parts to obtain

$$K_7 = - \int_0^t \int_{\mathbb{R}} \frac{\eta \partial_x^2 \eta}{v_\epsilon - 1} \partial_x[\phi'_\epsilon v'_\epsilon(v_\epsilon - 1)] + \int_0^t \int_{\mathbb{R}} \frac{\eta_2 \partial_x \eta}{(v_\epsilon - 1)^2} v'_\epsilon \partial_x[\phi'_\epsilon v'_\epsilon(v_\epsilon - 1)].$$

Then using again (108) and the Holder and Young inequalities,

$$\begin{aligned} |K_7| &\leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} \left(\left\| \frac{(v_\epsilon - 1)^{\gamma-2}}{\phi_\epsilon} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta\|_{L^2_{t,x}} + \left\| \frac{v'_\epsilon (v_\epsilon - 1)^{\gamma-3}}{\phi_\epsilon} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L^2_{t,x}} \right) \\ &\leq \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^4} \int_0^t D_0(\tau; V) d\tau + \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau. \end{aligned} \quad (109)$$

Summing $|K_1|, \dots, |K_7|$ finally yields the desired estimate. \square

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