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Concentration estimates for SPDEs driven by fractional Brownian motion

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Abstract

The main goal of this work is to provide sample-path estimates for the solution of slowly time-dependent SPDEs perturbed by a cylindrical fractional Brownian motion. Our strategy is similar to the approach by Berglund and Nader for space-time white noise. However, the setting of fractional Brownian motion does not allow us to use any martingale methods. Using instead optimal estimates for the probability that the supremum of a Gaussian process exceeds a certain level, we derive concentration estimates for the solution of the SPDE, provided that the Hurst index H of the fractional Brownian motion satisfies $H > \frac{1}{4}$. As a by-product, we also obtain concentration estimates for one-dimensional fractional SDEs valid for any $H \in (0, 1)$.

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1 Introduction

Fractional Brownian motion (fBm) is a famous example of stochastic process used in order to model memory effects or long-range dependencies. An fBm is a centered, stationary Gaussian process parameterized by a so-called Hurst index/parameter $H \in (0, 1)$. For $H = \frac{1}{2}$, one recovers the classical Brownian motion. However, for $H \in (\frac{1}{2}, 1)$ and $H \in (0, \frac{1}{2})$, fBm exhibits a different behaviour than Brownian motion. Its increments are no longer independent, but positively correlated for $H > \frac{1}{2}$, and negatively correlated for $H < \frac{1}{2}$. Fractional Brownian motion has been used to model a wide range of phenomena, extending from mathematical finance [19] to fluid dynamics [22].

Since fractional Brownian motion is neither a Markov process, nor a semi-martingale, it is a challenging task to construct solutions of SDEs/SPDEs driven by such a process and to analyze their dynamical properties [12, 13]. Here we contribute to this topic by deriving concentration estimates for slowly time-dependent SDEs perturbed by an additive fBm for all ranges of the Hurst index $H \in (0, 1)$, and for slowly time-dependent semilinear SPDEs perturbed by a cylindrical fBm, provided $H \in (\frac{1}{4}, 1)$. In this case, the well-posedness of the SPDEs is well-known [8, 9, 10, 21]. We mention that concentration estimates for SDEs driven by fBm with $H \in (\frac{1}{2}, 1)$ were previously obtained in [11]. One of the main novelties of this work is to extend this finite-dimensional result to $H \in (0, \frac{1}{2})$. The tools used in [11], based on results in [6], break down in this case, since they require the covariance of the fBm to be increasing, and rely on Lyapunov-type equations for the variance of a non-autonomous fractional Ornstein–Uhlenbeck process.

The SDEs we consider in this work have the form

$$dx_t = f(\varepsilon t, x_t) dt + \sigma dW_t^H,$$

where the drift term f admits a so-called stable uniformly hyperbolic critical manifold, that is, a smooth curve on which f vanishes and has a uniformly negative x -derivative. In the deterministic case $\sigma = 0$, it is well known that the equation admits a so-called slow solution, staying ε -close to such a curve, and attracting nearby solutions exponentially fast. In the Brownian case $H = \frac{1}{2}$, it was shown in [1] that for $\sigma > 0$, sample paths are concentrated in a neighbourhood of size of order σ of such a slow solution. Our first main result, Theorem 3.6, provides similar concentration estimates for any $H \in (0, 1)$, with explicit bounds on the probability of leaving such a neighbourhood.

The SPDEs we consider have the form

$$d\phi(t, x) = [\Delta\phi(t, x) + f(\varepsilon t, \phi(t, x))] dt + \sigma dW^H(t, x),$$

where x belongs to the one-dimensional torus \mathbb{T} , and $W^H(t, x)$ denotes a cylindrical fractional Brownian motion on \mathbb{T} . In the Brownian case $H = \frac{1}{2}$, concentration estimates near stable uniformly hyperbolic critical manifolds have been obtained in [2] for the one-dimensional torus, and in [3] for the two-dimensional torus \mathbb{T}^2 , provided the equation is suitably renormalised. Our second main result, Theorem 4.6, extends the concentration results on \mathbb{T} to all $H \in (\frac{1}{4}, 1)$, for all fractional Sobolev norms of index $s \in (0, 2H - \frac{1}{2})$.

Numerous extensions and applications of these results are imaginable. For instance, one could investigate bifurcations in SDEs/SPDEs with fractional noise. For example [4] analyzes pitchfork bifurcations using finite-time Lyapunov-exponents and approximations with amplitude equations derived in [5], whereas [15] computes early-warning signs for fast-slow systems perturbed by additive fractional noise. In the Brownian case $H = \frac{1}{2}$, the works [2, 3] have obtained similar results for pitchfork and avoided transcritical bifurcations in slowly time-dependent SPDEs. Another exciting direction is given by concentration estimates for SDEs/SPDEs perturbed by multiplicative fractional noise, using tools from rough path theory and slow-fast systems [12].

This manuscript is structured as follows. Section 2 gives a precise definition of fBm, and provides a criterion allowing to estimate the probability that the supremum of a mean-square Hölder continuous Gaussian process exceeds a certain level. Section 3 deals with sample-path estimates for one-dimensional SDEs. The key result in this setting is a suitable upper bound on the variance of a non-autonomous fractional Ornstein–Uhlenbeck process. The concentration inequality is extended to semilinear SPDEs in Section 4, by analyzing the Fourier components of its solution and using Schauder-type estimates.

2 Fractional Brownian motion

In this section we collect basic results on fractional Brownian motion and Gaussian processes which will be required later on.

Definition 2.1. *A fractional Brownian motion (fBm) $(W_t^H)_{t \geq 0}$ with Hurst index $H \in (0, 1)$ is a centered Gaussian process with covariance*

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (2.1)$$

For $H = \frac{1}{2}$ we recover the standard Brownian motion, whereas for $H \neq \frac{1}{2}$ we obtain a process which is neither Markov nor a semi-martingale.

A useful estimate for the probability that the supremum of a Gaussian process exceeds a certain threshold is given by the following theorem [18, Theorem D.4]. It is obtained by comparing the probability of exceeding a certain level with the one of a suitable stationary process, using Slepian's lemma [20]. The mean-square Hölder continuity enables one to define such a process.

Theorem 2.2. *Let $(X_t)_t$ be a continuous Gaussian process with zero mean on $[0, T]$ for $T > 0$. Assume that $(X_t)_t$ is mean-square Hölder continuous, i.e., there are constants G and γ such that*

$$\mathbb{E}[(X_t - X_s)^2] \leq G|t - s|^\gamma \quad \text{for all } t, s \in [0, T].$$

Then there exists a constant $K := K(G, \gamma)$ such that for $c > 0$ and $A \subset [0, T]$, one has

$$\mathbb{P}\left\{\sup_{t \in A} X_t > c\right\} \leq K T c^{2/\gamma} \exp\left\{-\frac{c^2}{2\sigma^2(A)}\right\},$$

where $\sigma^2(A) := \sup_{t \in A} \text{Var}(X_t)$.

Remark 2.3. By a simple scaling argument, one can infer that

$$K(G, \gamma) = G^{-1/\gamma} K(1, \gamma) =: G^{-1/\gamma} K_0(\gamma).$$

Indeed, the process $\tilde{X}_t = G^{-1/2} X_t$ has Hölder constant 1, maximal variance $G^{-1} \sigma^2(A)$, and satisfies $\sup_{t \in A} \tilde{X}_t = G^{-1/2} \sup_{t \in A} X_t$. \diamond

In our case, we use this result for a non-autonomous fractional Ornstein-Uhlenbeck process of Hurst index $H \in (0, 1)$, which is known to be mean-square Hölder continuous with exponent $\gamma = 2H$.

3 The one-dimensional SDE case

3.1 Linear case

We start by considering linear fractional SDEs, driven by an fBm $(W_t^H)_{t \geq 0}$ with Hurst parameter $H \in (0, 1)$, given by

$$dx_t = a(\varepsilon t)x_t dt + \sigma dW_t^H, \quad (3.1)$$

where $\varepsilon, \sigma > 0$ are small parameters, and $a : [0, T] \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 . It will be convenient to scale time by a factor ε , which turns the SDE (3.1) into

$$dx_t = \frac{1}{\varepsilon} a(t)x_t dt + \frac{\sigma}{\varepsilon^H} dW_t^H. \quad (3.2)$$

We will assume that the function a satisfies

$$a(t) \leq -a_0, \quad |a'(t)| \leq a_1 \quad \forall t \in [0, T] \quad (3.3)$$

for some constants $a_0, a_1 > 0$, and write

$$\alpha(t) = \int_0^t a(s) ds, \quad \alpha(t, u) = \int_u^t a(s) ds.$$

In order to apply Theorem 2.2, we will need to control the variance of x_t . The following result is an adaptation of [14, Theorem 1.4.3] to the non-autonomous case.

Lemma 3.1. *Assume the initial condition x_0 in (3.2) is deterministic. For any $H \in (0, 1)$, the variance of x_t satisfies the upper bound*

$$\text{Var}(x_t) \leq \frac{2H\sigma^2}{\varepsilon^{2H}} \int_0^t \left[e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} - e^{\alpha(t)/\varepsilon} (1 - e^{\alpha(t,s)/\varepsilon}) s^{2H-1} \right] ds. \quad (3.4)$$

PROOF: The proof is based on the representation

$$x_t = x_0 e^{\alpha(t)/\varepsilon} + \frac{\sigma}{\varepsilon^H} \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} W^H(s) ds + \frac{\sigma}{\varepsilon^H} W^H(t)$$

obtained by integration by parts. Since $x_0 e^{\alpha(t)/\varepsilon}$ is deterministic, we obtain

$$\begin{aligned} \text{Var}(x_t) = & \frac{\sigma^2}{\varepsilon^{2H}} \left[\mathbb{E}[(W^H(t))^2] + 2 \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} \mathbb{E}[W^H(t)W^H(s)] ds \right. \\ & \left. + \int_0^t \int_0^t \frac{a(u)}{\varepsilon} \frac{a(v)}{\varepsilon} e^{\alpha(t,u)/\varepsilon} e^{\alpha(t,v)/\varepsilon} \mathbb{E}[W^H(u)W^H(v)] du dv \right]. \end{aligned} \quad (3.5)$$

By the expression (2.1) of the covariance function of the fBm, the first term in square brackets gives $\mathbb{E}[(W^H(t))^2] = t^{2H}$, while the second term becomes

$$2 \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} \mathbb{E}[W^H(t)W^H(s)] ds = \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} [t^{2H} + s^{2H} - (t-s)^{2H}] ds.$$

We split this into three integrals that we compute separately. The first one gives

$$I_1 := t^{2H} \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} ds = t^{2H} [e^{\alpha(t)/\varepsilon} - 1],$$

while the other two integrals can be evaluated using integration by parts, yielding

$$\begin{aligned} I_2 &:= \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} s^{2H} ds = -t^{2H} + 2H \int_0^t e^{\alpha(t,s)/\varepsilon} s^{2H-1} ds, \\ I_3 &:= - \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} (t-s)^{2H} ds = -e^{\alpha(t)/\varepsilon} t^{2H} + 2H \int_0^t e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} ds. \end{aligned}$$

We further split the last term in the expression (3.5) for the variance of x_t into three integrals. By symmetry, the first two are equal and add up to

$$\begin{aligned} I_4 &:= \int_0^t \int_0^t \frac{a(u)}{\varepsilon} \frac{a(v)}{\varepsilon} e^{\alpha(t,u)/\varepsilon} e^{\alpha(t,v)/\varepsilon} u^{2H} du dv \\ &= t^{2H} - t^{2H} e^{\alpha(t)/\varepsilon} + 2H(e^{\alpha(t)/\varepsilon} - 1) \int_0^t e^{\alpha(t,s)/\varepsilon} s^{2H-1} ds. \end{aligned}$$

The last integral, given by

$$- \int_0^t \int_0^t \frac{a(u)}{\varepsilon} \frac{a(v)}{\varepsilon} e^{\alpha(t,u)/\varepsilon} e^{\alpha(t,v)/\varepsilon} |u-v|^{2H} du dv,$$

can be dropped since a is negative. The result follows by exploiting cancellations, and writing the factor t^{2H} occurring in one remaining term $-t^{2H} e^{\alpha(t)/\varepsilon}$ as the integral of $2Hs^{2H-1}$. \square

Laplace asymptotics and our assumptions (3.3) on a allow us to obtain the following simplified expression for the variance for small ε .

Lemma 3.2. *There exists a constant r_1 , depending only on a_0 , a_1 and H , such that for any $H \in (0, 1)$ and any $t \geq 0$, one has*

$$\text{Var}(x_t) \leq \frac{\sigma^2 2H\Gamma(2H)}{|a(t)|^{2H}} (1 + r_1 \varepsilon). \quad (3.6)$$

PROOF: It is sufficient to bound the first term in the integrand in (3.4), since the second term is negative. The assumptions (3.3) on a imply that whenever $0 \leq s \leq t \leq T$, one has

$$\alpha(t, s) \leq -a_0(t-s) \quad \text{and} \quad \alpha(t, s) \leq a(t)(t-s) + \frac{1}{2} a_1(t-s)^2.$$

Using the substitution $y = a(t)(t-s)$, we obtain

$$\int_0^t e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} ds = \frac{1}{|a(t)|^{2H}} \int_0^{|a(t)|t} e^{\alpha(t, t-\frac{y}{|a(t)|})/\varepsilon} y^{2H-1} dy.$$

If $|a(t)|t > \frac{a_0}{a_1}$, we split the integral at $y = \frac{a_0}{a_1}$. To bound the integral over the interval $[0, \frac{a_0}{a_1}]$, we use the fact that

$$\alpha\left(t, t - \frac{y}{|a(t)|}\right) \leq -y + \frac{a_1}{2a(t)^2} y^2 \leq -y + \frac{a_1}{2a_0} y^2,$$

to obtain

$$\begin{aligned} \int_0^{a_0/a_1} e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} ds &\leq \frac{1}{|a(t)|^{2H}} \int_0^{a_0/a_1} y^{2H-1} \exp\left\{-\frac{1}{\varepsilon} \left[y - \frac{a_1}{2a_0} y^2\right]\right\} dy \\ &= \frac{\varepsilon^{2H}}{|a(t)|^{2H}} \int_0^{a_0/\varepsilon a_1} z^{2H-1} \exp\left\{-\left[z - \varepsilon \frac{a_1}{2a_0} z^2\right]\right\} dz. \end{aligned}$$

The integral can easily be shown to be bounded above by $\Gamma(2H)[1 + \mathcal{O}(\varepsilon)]$, see for instance [16, Theorem 8.1].

Using the fact that $\alpha(t, t - \frac{y}{|a(t)|}) \leq -y$, one finds that the integral over the remaining interval $[\frac{a_0}{a_1}, |a(t)|t]$ is exponentially small in ε , and therefore negligible with respect to the error of order ε . Finally, if $|a(t)|t < \frac{a_0}{a_1}$, we can use the integral over $[0, \frac{a_0}{a_1}]$ as an upper bound. \square

Remark 3.3. For $H = \frac{1}{2}$, this result is consistent with the case of Brownian motion investigated by Berglund and Gentz in [1]. In particular, the H -dependent constant in (3.6) is given by $2H\Gamma(2H) = 1$ for $H = \frac{1}{2}$. \diamond

As a first consequence of the bound (3.6), we obtain a concentration result for the solutions of the linear SDE (3.2). This is based on Theorem 2.2, taking into account the scaling argument in Remark 2.3.

Proposition 3.4 (Concentration estimate for the linear SDE). *Assume $x_0 = 0$. Then there exists a constant r_2 , depending only on a_0 , a_1 and H , such that*

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} |x_t| |a(t)|^H \geq h\right\} \leq C\left(T; \frac{h}{\sigma}, a_0\right) \exp\left\{-\kappa(\varepsilon) \frac{h^2}{2\sigma^2}\right\}, \quad (3.7)$$

where the prefactor and exponent are given by

$$C\left(T; \frac{h}{\sigma}, a_0\right) = \frac{2K_0(2H)T^2}{a_0} \left(\frac{h}{\sigma}\right)^{1/H}, \quad \kappa(\varepsilon) = \frac{1 - r_2\varepsilon}{2H\Gamma(2H)}. \quad (3.8)$$

PROOF: We introduce a partition $0 = t_0 < t_1 < \dots < t_N = T$ of $[0, T]$ given by $t_k = k\varepsilon$ for $0 \leq k \leq N-1 = \lfloor T/\varepsilon \rfloor$, and write $I_k = [t_k, t_{k+1}]$ for the k th interval in the partition. Then the probability on the left-hand side of (3.7) is bounded by

$$\begin{aligned} & \sum_{k=0}^{N-1} \mathbb{P} \left\{ \sup_{t \in I_k} |x_t| |a(t)|^H \geq h \right\} \\ & \leq \sum_{k=0}^{N-1} \mathbb{P} \left\{ \sup_{t \in I_k} |x_t| \geq h \inf_{t \in I_k} \frac{1}{|a(t)|^H} \right\} \\ & \leq 2K_0(2H)\varepsilon T \left(\frac{h}{\sigma} \right)^{1/H} \sum_{k=0}^{N-1} \inf_{t \in I_k} \frac{1}{|a(t)|} \exp \left\{ -\frac{h^2}{2} \left(\inf_{t \in I_k} \frac{1}{|a(t)|^{2H}} \right) \left(\sup_{t \in I_k} \text{Var}(x_t) \right)^{-1} \right\}. \end{aligned}$$

To obtain the last line, we have applied Theorem 2.2 to x_t with $\gamma = 2H$ and $G = \sigma^2/\varepsilon^{2H}$, which is justified for $H > \frac{1}{2}$ by [11, Theorem 3.7]. For $H < \frac{1}{2}$ a computation similar to the one in Lemma 3.1 entails the mean-square Hölder continuity of the non-autonomous fractional Ornstein-Uhlenbeck process with the same coefficients $\gamma = 2H$ and $G = \sigma^2/\varepsilon^{2H}$. Now we observe that, setting

$$\hat{\nu}(t) = \frac{2H\Gamma(2H)}{|a(t)|^{2H}},$$

(3.6) implies $\text{Var}(x_t) \leq \sigma^2 \hat{\nu}(t)(1 + \mathcal{O}(\varepsilon))$ for all $t \in [0, T]$. The result thus follows from the regularity properties (3.3) of a and the fact that the length of I_k is bounded by ε . \square

The proposition shows that as soon as $h \gg \sigma$, the sample paths $(x_t)_{t \in [0, T]}$ are unlikely to leave a strip of width $h|a(t)|^{-H}$ before time T . In other words, we obtain a ‘‘confidence strip’’ for these sample paths.

Remark 3.5. Instead of using a partition of spacing ε , one could choose a partition given by $t_k = k\delta$ for an arbitrary $\delta \in (0, T]$. This yields an extra factor ε/δ in the prefactor C , and an additional error term of order δ in the exponent. Taking $\delta < \varepsilon$ is not of interest, since it increases the prefactor while it does not improve the exponent. Otherwise, the optimal value of δ has order σ^2/h^2 , and yields a prefactor of order $\varepsilon T^2(h/\sigma)^{2+1/H}$. Because of the condition $\delta \geq \varepsilon$, this is only of interest if $h^2 < \sigma^2/\varepsilon$. \diamond

3.2 Nonlinear case

This result can now easily be extended to concentration estimates for sample paths near stable slow manifolds of non-linear slowly time-dependent fractional SDEs. Consider the equation

$$dx_t = \frac{1}{\varepsilon} f(t, x_t) dt + \frac{\sigma}{\varepsilon^H} dW_t^H, \quad (3.9)$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is of class \mathcal{C}^2 . Assume that $x^* : [0, T] \rightarrow \mathbb{R}$ is a stable uniformly hyperbolic slow manifold (or stable equilibrium branch), meaning that

- $f(t, x^*(t)) = 0$ for all $t \in [0, T]$,
- and there exists $a_0 > 0$ such that $a^*(t) = \partial_x f(t, x^*(t)) \leq -a_0$ for all $t \in [0, T]$.

Note that by the implicit function theorem, such a function x^* is of class \mathcal{C}^2 as well. Classical results by Tihonov and Fenichel ensure that for sufficiently small ε , the deterministic

equation $\varepsilon \dot{x} = f(t, x)$ admits a particular solution $\bar{x}(t)$ satisfying $\bar{x}(t) = x^*(t) + \mathcal{O}(\varepsilon)$ uniformly in $t \in [0, T]$. We set

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t))$$

and observe that it is bounded above by $-\bar{a}_0 = -a_0 + \mathcal{O}(\varepsilon)$, which is still negative for ε small enough. We define the set

$$\mathcal{B}(h) = \{(t, x) : t \in [0, T], |x - \bar{x}(t)| |\bar{a}(t)|^H \leq h\},$$

which is a strip of width $h|\bar{a}(t)|^{-H}$ around the graph of \bar{x} , and write $\tau_{\mathcal{B}(h)}$ for the first-exit time of $(x_t)_t$ from $\mathcal{B}(h)$. Then we have the following concentration result, which is the main result of this section.

Theorem 3.6 (Concentration estimate for the nonlinear SDE). *There exist $\varepsilon_0, h_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ and $h \leq h_0$, the solution of (3.9) with initial condition $x_0 = \bar{x}(0)$ satisfies*

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} \leq C\left(T; \frac{h}{\sigma}, \bar{a}_0\right) \exp\left\{-\kappa(\varepsilon) \frac{h^2}{2\sigma^2} [1 - \mathcal{O}(h)]\right\},$$

where the constants $C(T; \frac{h}{\sigma}, \bar{a}_0)$ and $\kappa(\varepsilon)$ are the same as in Proposition 3.4.

PROOF: The difference $y_t = x_t - \bar{x}(t)$ satisfies the SDE

$$dy_t = \frac{1}{\varepsilon} [\bar{a}(t) + b(t, y_t)] dt + \frac{\sigma}{\varepsilon^H} dW_t^H,$$

where there exist constants $M, d > 0$ such that $|b(t, y)| \leq My^2$ whenever $t \in [0, T]$ and $|y| \leq d$. Its solution can be represented as $y_t = y_t^0 + y_t^1$, where

$$y_t^0 = \frac{\sigma}{\varepsilon^H} \int_0^t e^{\alpha(t,s)/\varepsilon} dW_s^H, \quad y_t^1 = \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,s)/\varepsilon} b(s, y_s) ds.$$

For any decomposition $h = h^0 + h^1$ with $h^0, h^1 > 0$, continuity of sample paths allows us to write

$$\begin{aligned} \mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} &= \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} |y_t| |\bar{a}(t)|^H \geq h\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} |y_t^0| |\bar{a}(t)|^H \geq h^0\right\} + \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} |y_t^1| |\bar{a}(t)|^H \geq h^1\right\}. \end{aligned} \quad (3.10)$$

Since

$$|y_{t \wedge \tau_{\mathcal{B}(h)}}^1| \leq \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,s)/\varepsilon} \frac{Mh^2}{|\bar{a}(s)|^H} ds \leq \frac{Mh^2}{\bar{a}_0^{1+H}},$$

the second term on the right-hand side of (3.10) vanishes for an h^1 of order h^2 . The result then follows from Proposition 3.4, taking $h^0 = h - h^1 = h[1 - \mathcal{O}(h)]$. \square

4 The SPDE case

4.1 Linear case

We now turn to the analysis of linear SPDEs on the one-dimensional torus \mathbb{T} , of the form

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta + a(t)] \phi(t, x) dt + \frac{\sigma}{\varepsilon^H} dW^H(t, x), \quad (4.1)$$

where $a : [0, T] \rightarrow \mathbb{R}$ satisfies again (3.3). The SPDE is driven by a cylindrical fractional Brownian motion $(W^H(t))_{t \geq 0}$ with Hurst parameter $H \in (\frac{1}{4}, 1)$. This means that the noise is fractional-in-time and white-in-space. The existence of mild solutions for such linear SPDEs was established in [8, Example 3.1] for $H \in (\frac{1}{4}, 1)$.

The k th Fourier component of ϕ satisfies

$$d\phi_k(t) = -\frac{1}{\varepsilon} \lambda_k(t) \phi_k(t) dt + \frac{\sigma}{\varepsilon^H} dW_k^H(t),$$

where $\lambda_k(t)$ is the k th eigenvalue of $-\Delta - a(t)$ given by $\lambda_k(t) = (2\pi)^2 k^2 - a(t)$. It has the order $\langle k \rangle^2$, where $\langle k \rangle = \sqrt{1 + k^2}$, and satisfies $|\lambda_k(t)| \geq (2\pi)^2 k^2 + a_0$.

Remark 4.1. Rescaling time as $t = \mu_k \tilde{t}$, where $\mu_k = (a_0 + ck^2)^{-1}$ for a constant $c > 0$, we obtain

$$d\phi_k = -\frac{1}{\varepsilon} \tilde{\lambda}_k(\tilde{t}) \phi_k(\tilde{t}) d\tilde{t} + \frac{\sigma \mu_k^H}{\varepsilon^H} dW_k^H(\tilde{t}) \quad (4.2)$$

with $\tilde{t} \in [0, T/\mu_k]$, where $\tilde{\lambda}_k(\tilde{t}) = \mu_k \lambda_k(\tilde{t}/\mu_k)$, and consequently $\tilde{\lambda}_k(\tilde{t}) \geq 1$ and $|\tilde{\lambda}'_k(\tilde{t})| \leq a_1$ for all $\tilde{t} \in [0, T/\mu_k]$. This allows us to use Proposition 3.4 with values of a_0 and a_1 that do not depend on k . Therefore, we need not worry about a possible k -dependence of the error term $r_2 \varepsilon$ in (3.8). \diamond

We recall that the (fractional) Sobolev norm on $H^s(\mathbb{T})$ for $s > 0$ is given by

$$\|\phi(t, \cdot)\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\phi_k(t)|^2. \quad (4.3)$$

While one can work with this norm, it turns out that we can obtain slightly sharper bounds using a time-dependent Sobolev norm defined by

$$\|\phi(t, \cdot)\|_{s,t}^2 := \sum_{k \in \mathbb{Z}} a_{k,s}(t) |\phi_k|^2,$$

where we will choose $a_{k,s}(t) = \lambda_k^H(t) \langle k \rangle^{s-2H}$, so that $a_{k,s}(t) \asymp \langle k \rangle^s$. Note that both norms are equivalent, and are not sensitive to Fourier modes with large $|k|$. However, the time-dependent norm will give a sharper control for Fourier modes with small $|k|$, and especially for $k = 0$.

Proposition 4.2 (Concentration estimate for the linear SPDE). *Let $H \in (0, \frac{1}{4})$ and $s \in (0, 2H - \frac{1}{2})$. Then there exist constants $c_0, c_1, r_2 > 0$ such that the solution of (4.1) with initial condition $\phi(0, \cdot) = 0$ satisfies the concentration inequality*

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} \leq C \left(T; \frac{h}{\sigma}, s \right) \exp \left\{ -\kappa(\varepsilon) Q(s) \frac{h^2}{2\sigma^2} \right\},$$

where $Q(s) \geq c_0(2H - \frac{1}{2} - s)$, $\kappa(\varepsilon)$ is the same as in (3.8), and

$$C \left(T; \frac{h}{\sigma}, s \right) = 2K_0(2H) T^2 a_0^2 \left(Q(s)^{1/2} \frac{h}{\sigma} \right)^{1/H} [1 + \mathcal{O}(e^{-c_1 h^2 / \sigma^2})].$$

PROOF: It is known that the fractional stochastic convolution has continuous trajectories in H^s for $s \in (0, 2H - \frac{1}{2})$, see [8, Corollary 3.1] for $H > \frac{1}{2}$, respectively [9, Lemma 11.10] for $H \in (\frac{1}{4}, \frac{1}{2})$. For any decomposition $h^2 = \sum_{k \in \mathbb{Z}} h_k^2$ with $h_k > 0$ for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} &= \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sum_{k \in \mathbb{Z}} a_{k,s}(t) |\phi_k(t)|^2 \geq h^2 \right\} \\ &\leq \sum_{k \in \mathbb{Z}} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} |\phi_k(t)|^2 \lambda_k(t)^{2H} \geq \frac{h_k^2}{\langle k \rangle^{2s-4H}} \right\} \end{aligned} \quad (4.4)$$

by the choice of the time-dependent coefficients $a_{k,s}(t) = \lambda_k^H(t) \langle k \rangle^{s-2H}$. According to Proposition 3.4, we have for each component ϕ_k solving (4.2) that

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} |\phi_k(t)| \lambda_k(t)^H \geq \frac{h_k}{\langle k \rangle^{s-2H}} \right\} &= \mathbb{P} \left\{ \sup_{0 \leq \tilde{t} \leq T/\mu_k} |\phi_k(\tilde{t})| \frac{\tilde{\lambda}_k(\tilde{t})^H}{\mu_k^H} \geq \frac{h_k}{\langle k \rangle^{s-2H}} \right\} \\ &\leq C_k \exp \left\{ -\kappa(\varepsilon) \frac{h_k^2 \mu_k^{2H}}{2\bar{\sigma}_k^2 \langle k \rangle^{2s-4H}} \right\}, \end{aligned}$$

where $\bar{\sigma}_k = \mu_k^H \sigma$ due to the scaling in (4.2), and

$$C_k = C \left(\frac{T}{\mu_k}, \frac{h_k \mu_k^H}{\bar{\sigma}_k \langle k \rangle^{s-2H}}, 1 \right) = \frac{2K_0(2H)T^2}{\mu_k^2} \left(\frac{h_k}{\sigma \langle k \rangle^{s-2H}} \right)^{1/H}.$$

Plugging this in (4.4) and simplifying the factors μ_k^H entails

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} \leq \sum_{k \in \mathbb{Z}} C_k \exp \left\{ -\kappa(\varepsilon) \frac{h_k^2}{2\sigma^2 \langle k \rangle^{2s-4H}} \right\}.$$

We pick $\eta > 0$ and choose $h_k^2 = Q(s)h^2 \langle k \rangle^{-(4H-2s-\eta)}$, where the condition $h^2 = \sum_{k \in \mathbb{Z}} h_k^2$ imposes

$$Q(s)^{-1} = \sum_{k \in \mathbb{Z}} \frac{1}{\langle k \rangle^{4H-2s-\eta}} < \infty.$$

This means that we need to have $4H-2s-\eta > 1$, and since both s and η must be positive we obtain the restriction $H > \frac{1}{4}$. The claimed lower bound on $Q(s)$ follows from the behaviour of Riemann's zeta function $\zeta(u)$ as $u \rightarrow 1$, choosing for instance $\eta = 2H - s - \frac{1}{2}$. Based on this choice of the h_k , we further obtain

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} &\leq \sum_{k \in \mathbb{Z}} C_k \exp \left\{ -\kappa(\varepsilon) Q(s) \frac{h^2 \langle k \rangle^\eta}{2\sigma^2} \right\} \\ &\leq 2K_0(2H)T^2 \left(Q(s)^{1/2} \frac{h}{\sigma} \right)^{1/H} \sum_{k \in \mathbb{Z}} \frac{\langle k \rangle^{\eta/(2H)}}{\mu_k^2} \exp \{ -\beta \langle k \rangle^\eta \}, \end{aligned}$$

where $\beta := \kappa(\varepsilon)Q(s)\frac{h^2}{2\sigma^2}$. We claim that the sum over k is dominated by the term $k = 0$. In fact, by an argument similar to the one in [2, Theorem 2.4], we can bound this sum by

$$f(0) + 2f(1) + \int_1^\infty f(x) dx, \quad \text{where} \quad f(x) := (a_0 + cx^2)(1+x^2)^\gamma e^{-\beta(1+x^2)^{\eta/2}},$$

with an exponent $\gamma = \eta/(4H)$. The integral can be shown to be of order $e^{-c\beta}$ with $c > 1$, which yields the result. \square

Remark 4.3. The condition $H > \frac{1}{4}$ is consistent with the solution theory for such SPDEs driven by cylindrical fractional noise [8, 21]. \diamond

4.2 Nonlinear case

We finally consider the nonlinear SPDE given by

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + f(t, \phi(t, x))] dt + \frac{\sigma}{\varepsilon^H} dW^H(t, x).$$

We assume $f(t, \phi) = -\partial_\phi U(t, \phi)$, where the potential U can be decomposed as $U(t, \phi) = P(t, \phi) + g(t, \phi)$. Here P is polynomial of even degree $2p$ with smooth bounded coefficients such that the leading order coefficient $a_{2p}(t) > 0$ for all $t \in [0, T]$, and the function $g \in \mathcal{C}^2([0, T] \times \mathbb{R}; \mathbb{R})$ satisfies the boundedness assumptions

$$|g(t, \phi)\phi^{-1}|, |\partial_\phi g(t, \phi)|, |\partial_{\phi\phi} g(t, \phi)|, |\partial_t g(t, \phi)| \leq \widetilde{M}$$

for all $(t, \phi) \in [0, T] \times \mathbb{R}$, for a constant $\widetilde{M} > 0$.

The well-posedness of semilinear SPDEs with drift given by $f(t, \phi) = -\partial_\phi P(t, \phi)$ was established in [10, Section 4]. This remains valid for $U(t, \phi) = P(t, \phi) + g(t, \phi)$, due to the smoothness and boundedness assumptions on g . As before, we assume that we are in a stable situation, meaning that there exists a map $\phi^* : [0, T] \rightarrow \mathbb{R}$ such that

- $f(t, \phi^*(t)) = 0$ for all $t \in [0, T]$,
- there exists $a_0 > 0$ such that $a^*(t) = \partial_\phi f(t, \phi^*(t)) \leq -a_0$ for all $t \in [0, T]$.

Similarly to the finite-dimensional case, the deterministic PDE admits, according to [2, Proposition 2.3], a solution $\bar{\phi}$ such that $\bar{\phi}(t) = \phi^*(t) + \mathcal{O}(\varepsilon)$ for all $t \in [0, T]$. We set as before $\bar{a}(t) = \partial_x f(t, \bar{\phi}(t))$ and introduce for $s \in (0, 2H - \frac{1}{2})$

$$\mathcal{B}(h) := \{(t, \phi) : t \in [0, T], \|\phi(t, \cdot) - \bar{\phi}(t, \cdot)\|_{s,t} \leq h\}.$$

The SPDE for the difference $\psi(t, x) := \phi(t, x) - \bar{\phi}(t, x)$ reads as

$$d\psi(t, x) = \frac{1}{\varepsilon} [\Delta\psi(t, x) + \bar{a}(t)\psi(t, x) + b(t, \psi(t, x))] dt + \frac{\sigma}{\varepsilon^H} dW^H(t, x), \quad (4.5)$$

where there exists constants $M, d > 0$ such that $|b(t, \psi)| \leq M\psi^2$ and $|\partial_\psi b(t, \psi)| \leq M|\psi|$ for all $t \in [0, T]$ and $\psi \in \mathbb{R}$ with $|\psi| \leq d$. By the variation of constants formula, its solution is given by

$$\begin{aligned} \psi(t, x) &= \frac{\sigma}{\varepsilon^H} \int_0^t e^{\alpha(t,s)/\varepsilon} e^{[(t-s)/\varepsilon]\Delta} dW^H(s) + \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,s)/\varepsilon} e^{[(t-s)/\varepsilon]\Delta} b(s, \psi(s, x)) ds \\ &:= \psi^0(t, x) + \psi^1(t, x). \end{aligned}$$

As before, $\alpha(t, s) = \int_s^t \bar{a}(r) dr$. In order to analyze the stochastic convolution we rely on Schauder-type estimates.

Lemma 4.4 (Schauder-type estimates). *Let $f \in H^r$ with $r \in (0, 2H - \frac{1}{2})$. Then for all $q < r + 2$, there exists a constant $c(q, r) > 0$ such that*

$$\|e^{t\Delta} f\|_{H^q} \leq c(q, r) t^{-\frac{q-r}{2}} \|f\|_{H^r}.$$

PROOF: This follows from regularizing properties of analytic semigroups [17, Theorem 6.13], according to which

$$\|e^{t\Delta} f\|_{D((-\Delta)^{q/2})} \leq c(q, r) t^{-\frac{q-r}{2}} \|f\|_{D((-\Delta)^{r/2})}$$

for $q - r/2 < 1$, leading to the restriction $q < r + 2$. The statement follows, considering that $H^q = D((-\Delta)^{q/2})$. Here we use the equivalence of the time-dependent (fractional) Sobolev norm with (4.3). \square

For $\psi(t, \cdot) \in H^s$, one can easily prove that $\beta(t) := b(t, \psi(t, \cdot))$ belongs to H^s . A proof of this statement in H^s for $s \in (0, \frac{1}{2})$ relying on Young's inequality is provided in [2, Lemma 3.4]. We now apply the Schauder estimate in order to obtain a bound on $\psi^1(t, \cdot)$ similar to [2, Corollary 3.6].

Lemma 4.5. *Assume that there exists $r \in (0, 2H - \frac{1}{2})$ such that $\beta(t) \in H^r$ for all $t \in [0, T]$. Then for all $q < r + 2$ there exists a constant $c(q, r) > 0$ such that for all $t \in [0, T]$ we have*

$$\|\psi^1(t, \cdot)\|_{H^q} \leq c(q, r) \varepsilon^{\frac{q-r}{2}-1} \sup_{0 \leq s \leq t} \|\beta(s)\|_{H^q}.$$

PROOF: We have

$$\begin{aligned} \|\psi^1(t, x)\|_{H^q} &\leq \frac{1}{\varepsilon} \int_0^t e^{-a_0(t-s)/\varepsilon} \|e^{[(t-s)/\varepsilon]\Delta} \beta(s)\|_{H^q} ds \\ &\leq c(q, r) \varepsilon^{\frac{q-r}{2}-1} \sup_{0 \leq s \leq t} \|\beta(s)\|_{H^r} \int_0^t (t-s)^{-\frac{q-r}{2}} ds < \infty, \end{aligned}$$

since $q < r + 2$. In the last step we used the uniform negative bound on \bar{a} and the Schauder estimate. \square

Theorem 4.6 (Concentration estimate for the nonlinear SPDE). *For every $s \in (0, 2H - \frac{1}{2})$ and any $\nu > 0$, there exists positive constants ε_0, h_0 such that for $\varepsilon \leq \varepsilon_0$ and $h \leq h_0 \varepsilon^\nu$, the solution of (4.5) with initial condition $\phi(0, \cdot) = \bar{\phi}(0, \cdot)$ satisfies*

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} \leq C\left(T; \frac{h}{\sigma}, s\right) \exp\left\{-\kappa(\varepsilon) Q(s) \frac{h^2}{2\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\},$$

with the same $C\left(T; \frac{h}{\sigma}, s\right)$, $Q(s)$ and $\kappa(\varepsilon)$ as in Proposition 4.2.

PROOF: For any decomposition $h = h_0 + h_1$ we have

$$\begin{aligned} \mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} &= \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi(t, \cdot)\|_{s,t} \geq h\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi^0(t, \cdot)\|_{s,t} + \|\psi^1(t, \cdot)\|_{s,t} \geq h\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi^0(t, \cdot)\|_{s,t} \geq h_0\right\} + \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi^1(t, \cdot)\|_{s,t} \geq h_1\right\}. \end{aligned} \quad (4.6)$$

We bound the first term using Proposition 4.2. For the second one we have, similarly to the proof of Theorem 3.6, that for $t \leq \tau_{\mathcal{B}(h)}$

$$\|\psi^1(t, \cdot)\|_{H^q} \leq c(q, r) \varepsilon^{\frac{q-r}{2}-1} M h^2,$$

since $\|\beta(t)\|_{H^r} \leq M \|\psi^1(t, \cdot)\|_{H^q}^2 \leq M h^2$. Therefore the second term in (4.6) vanishes provided that $h_1 = c(q, r) \varepsilon^{\frac{q-r}{2}-1} M h^2$. The statement follows, choosing

$$h_0 = h - h_1 = h - c(q, r) \varepsilon^{\frac{q-r}{2}-1} M h^2 = h(1 - \mathcal{O}(h/\varepsilon^\nu))$$

for $\nu = 1 - \frac{q-r}{2}$. \square

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