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# Concentration estimates for SPDEs driven by fractional Brownian motion

Nils Berglund, Alexandra Blessing (Neamțu)

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## Abstract

The main goal of this work is to provide sample-path estimates for the solution of slowly time-dependent SPDEs perturbed by a cylindrical fractional Brownian motion. Our strategy is similar to the approach by Berglund and Nader for space-time white noise. However, the setting of fractional Brownian motion does not allow us to use any martingale methods. Using instead optimal estimates for the probability that the supremum of a Gaussian process exceeds a certain level, we derive concentration estimates for the solution of the SPDE, provided that the Hurst index  $H$  of the fractional Brownian motion satisfies  $H > \frac{1}{4}$ . As a by-product, we also obtain concentration estimates for one-dimensional fractional SDEs valid for any  $H \in (0, 1)$ .

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## 1 Introduction

Fractional Brownian motion (fBm) is a famous example of stochastic process used in order to model memory effects or long-range dependencies. An fBm is a centered, stationary Gaussian process parameterized by a so-called Hurst index/parameter  $H \in (0, 1)$ . For  $H = \frac{1}{2}$ , one recovers the classical Brownian motion. However, for  $H \in (\frac{1}{2}, 1)$  and  $H \in (0, \frac{1}{2})$ , fBm exhibits a different behaviour than Brownian motion. Its increments are no longer independent, but positively correlated for  $H > \frac{1}{2}$ , and negatively correlated for  $H < \frac{1}{2}$ . Fractional Brownian motion has been used to model a wide range of phenomena, extending from mathematical finance [19] to fluid dynamics [22].

Since fractional Brownian motion is neither a Markov process, nor a semi-martingale, it is a challenging task to construct solutions of SDEs/SPDEs driven by such a process and to analyze their dynamical properties [12, 13]. Here we contribute to this topic by deriving concentration estimates for slowly time-dependent SDEs perturbed by an additive fBm for all ranges of the Hurst index  $H \in (0, 1)$ , and for slowly time-dependent semilinear SPDEs perturbed by a cylindrical fBm, provided  $H \in (\frac{1}{4}, 1)$ . In this case, the well-posedness of the SPDEs is well-known [8, 9, 10, 21]. We mention that concentration estimates for SDEs driven by fBm with  $H \in (\frac{1}{2}, 1)$  were previously obtained in [11]. One of the main novelties of this work is to extend this finite-dimensional result to  $H \in (0, \frac{1}{2})$ . The tools used in [11], based on results in [6], break down in this case, since they require the covariance of the fBm to be increasing, and rely on Lyapunov-type equations for the variance of a non-autonomous fractional Ornstein–Uhlenbeck process.

The SDEs we consider in this work have the form

$$dx_t = f(\varepsilon t, x_t) dt + \sigma dW_t^H,$$

where the drift term  $f$  admits a so-called stable uniformly hyperbolic critical manifold, that is, a smooth curve on which  $f$  vanishes and has a uniformly negative  $x$ -derivative. In the deterministic case  $\sigma = 0$ , it is well known that the equation admits a so-called slow solution, staying  $\varepsilon$ -close to such a curve, and attracting nearby solutions exponentially fast. In the Brownian case  $H = \frac{1}{2}$ , it was shown in [1] that for  $\sigma > 0$ , sample paths are concentrated in a neighbourhood of size of order  $\sigma$  of such a slow solution. Our first main result, Theorem 3.6, provides similar concentration estimates for any  $H \in (0, 1)$ , with explicit bounds on the probability of leaving such a neighbourhood.

The SPDEs we consider have the form

$$d\phi(t, x) = [\Delta\phi(t, x) + f(\varepsilon t, \phi(t, x))] dt + \sigma dW^H(t, x),$$

where  $x$  belongs to the one-dimensional torus  $\mathbb{T}$ , and  $W^H(t, x)$  denotes a cylindrical fractional Brownian motion on  $\mathbb{T}$ . In the Brownian case  $H = \frac{1}{2}$ , concentration estimates near stable uniformly hyperbolic critical manifolds have been obtained in [2] for the one-dimensional torus, and in [3] for the two-dimensional torus  $\mathbb{T}^2$ , provided the equation is suitably renormalised. Our second main result, Theorem 4.6, extends the concentration results on  $\mathbb{T}$  to all  $H \in (\frac{1}{4}, 1)$ , for all fractional Sobolev norms of index  $s \in (0, 2H - \frac{1}{2})$ .

Numerous extensions and applications of these results are imaginable. For instance, one could investigate bifurcations in SDEs/SPDEs with fractional noise. For example [4] analyzes pitchfork bifurcations using finite-time Lyapunov-exponents and approximations with amplitude equations derived in [5], whereas [15] computes early-warning signs for fast-slow systems perturbed by additive fractional noise. In the Brownian case  $H = \frac{1}{2}$ , the works [2, 3] have obtained similar results for pitchfork and avoided transcritical bifurcations in slowly time-dependent SPDEs. Another exciting direction is given by concentration estimates for SDEs/SPDEs perturbed by multiplicative fractional noise, using tools from rough path theory and slow-fast systems [12].

This manuscript is structured as follows. Section 2 gives a precise definition of fBm, and provides a criterion allowing to estimate the probability that the supremum of a mean-square Hölder continuous Gaussian process exceeds a certain level. Section 3 deals with sample-path estimates for one-dimensional SDEs. The key result in this setting is a suitable upper bound on the variance of a non-autonomous fractional Ornstein–Uhlenbeck process. The concentration inequality is extended to semilinear SPDEs in Section 4, by analyzing the Fourier components of its solution and using Schauder-type estimates.

## 2 Fractional Brownian motion

In this section we collect basic results on fractional Brownian motion and Gaussian processes which will be required later on.

**Definition 2.1.** *A fractional Brownian motion (fBm)  $(W_t^H)_{t \geq 0}$  with Hurst index  $H \in (0, 1)$  is a centered Gaussian process with covariance*

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (2.1)$$

For  $H = \frac{1}{2}$  we recover the standard Brownian motion, whereas for  $H \neq \frac{1}{2}$  we obtain a process which is neither Markov nor a semi-martingale.

A useful estimate for the probability that the supremum of a Gaussian process exceeds a certain threshold is given by the following theorem [18, Theorem D.4]. It is obtained by comparing the probability of exceeding a certain level with the one of a suitable stationary process, using Slepian's lemma [20]. The mean-square Hölder continuity enables one to define such a process.

**Theorem 2.2.** *Let  $(X_t)_t$  be a continuous Gaussian process with zero mean on  $[0, T]$  for  $T > 0$ . Assume that  $(X_t)_t$  is mean-square Hölder continuous, i.e., there are constants  $G$  and  $\gamma$  such that*

$$\mathbb{E}[(X_t - X_s)^2] \leq G|t - s|^\gamma \quad \text{for all } t, s \in [0, T].$$

*Then there exists a constant  $K := K(G, \gamma)$  such that for  $c > 0$  and  $A \subset [0, T]$ , one has*

$$\mathbb{P}\left\{\sup_{t \in A} X_t > c\right\} \leq K T c^{2/\gamma} \exp\left\{-\frac{c^2}{2\sigma^2(A)}\right\},$$

where  $\sigma^2(A) := \sup_{t \in A} \text{Var}(X_t)$ .

**Remark 2.3.** By a simple scaling argument, one can infer that

$$K(G, \gamma) = G^{-1/\gamma} K(1, \gamma) =: G^{-1/\gamma} K_0(\gamma).$$

Indeed, the process  $\tilde{X}_t = G^{-1/2} X_t$  has Hölder constant 1, maximal variance  $G^{-1} \sigma^2(A)$ , and satisfies  $\sup_{t \in A} \tilde{X}_t = G^{-1/2} \sup_{t \in A} X_t$ .  $\diamond$

In our case, we use this result for a non-autonomous fractional Ornstein-Uhlenbeck process of Hurst index  $H \in (0, 1)$ , which is known to be mean-square Hölder continuous with exponent  $\gamma = 2H$ .

### 3 The one-dimensional SDE case

#### 3.1 Linear case

We start by considering linear fractional SDEs, driven by an fBm  $(W_t^H)_{t \geq 0}$  with Hurst parameter  $H \in (0, 1)$ , given by

$$dx_t = a(\varepsilon t)x_t dt + \sigma dW_t^H, \quad (3.1)$$

where  $\varepsilon, \sigma > 0$  are small parameters, and  $a : [0, T] \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^1$ . It will be convenient to scale time by a factor  $\varepsilon$ , which turns the SDE (3.1) into

$$dx_t = \frac{1}{\varepsilon} a(t)x_t dt + \frac{\sigma}{\varepsilon^H} dW_t^H. \quad (3.2)$$

We will assume that the function  $a$  satisfies

$$a(t) \leq -a_0, \quad |a'(t)| \leq a_1 \quad \forall t \in [0, T] \quad (3.3)$$

for some constants  $a_0, a_1 > 0$ , and write

$$\alpha(t) = \int_0^t a(s) ds, \quad \alpha(t, u) = \int_u^t a(s) ds.$$

In order to apply Theorem 2.2, we will need to control the variance of  $x_t$ . The following result is an adaptation of [14, Theorem 1.4.3] to the non-autonomous case.

**Lemma 3.1.** *Assume the initial condition  $x_0$  in (3.2) is deterministic. For any  $H \in (0, 1)$ , the variance of  $x_t$  satisfies the upper bound*

$$\text{Var}(x_t) \leq \frac{2H\sigma^2}{\varepsilon^{2H}} \int_0^t \left[ e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} - e^{\alpha(t)/\varepsilon} (1 - e^{\alpha(t,s)/\varepsilon}) s^{2H-1} \right] ds. \quad (3.4)$$

PROOF: The proof is based on the representation

$$x_t = x_0 e^{\alpha(t)/\varepsilon} + \frac{\sigma}{\varepsilon^H} \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} W^H(s) ds + \frac{\sigma}{\varepsilon^H} W^H(t)$$

obtained by integration by parts. Since  $x_0 e^{\alpha(t)/\varepsilon}$  is deterministic, we obtain

$$\begin{aligned} \text{Var}(x_t) = & \frac{\sigma^2}{\varepsilon^{2H}} \left[ \mathbb{E}[(W^H(t))^2] + 2 \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} \mathbb{E}[W^H(t)W^H(s)] ds \right. \\ & \left. + \int_0^t \int_0^t \frac{a(u)}{\varepsilon} \frac{a(v)}{\varepsilon} e^{\alpha(t,u)/\varepsilon} e^{\alpha(t,v)/\varepsilon} \mathbb{E}[W^H(u)W^H(v)] du dv \right]. \end{aligned} \quad (3.5)$$

By the expression (2.1) of the covariance function of the fBm, the first term in square brackets gives  $\mathbb{E}[(W^H(t))^2] = t^{2H}$ , while the second term becomes

$$2 \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} \mathbb{E}[W^H(t)W^H(s)] ds = \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} [t^{2H} + s^{2H} - (t-s)^{2H}] ds.$$

We split this into three integrals that we compute separately. The first one gives

$$I_1 := t^{2H} \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} ds = t^{2H} [e^{\alpha(t)/\varepsilon} - 1],$$

while the other two integrals can be evaluated using integration by parts, yielding

$$\begin{aligned} I_2 &:= \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} s^{2H} ds = -t^{2H} + 2H \int_0^t e^{\alpha(t,s)/\varepsilon} s^{2H-1} ds, \\ I_3 &:= - \int_0^t \frac{a(s)}{\varepsilon} e^{\alpha(t,s)/\varepsilon} (t-s)^{2H} ds = -e^{\alpha(t)/\varepsilon} t^{2H} + 2H \int_0^t e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} ds. \end{aligned}$$

We further split the last term in the expression (3.5) for the variance of  $x_t$  into three integrals. By symmetry, the first two are equal and add up to

$$\begin{aligned} I_4 &:= \int_0^t \int_0^t \frac{a(u)}{\varepsilon} \frac{a(v)}{\varepsilon} e^{\alpha(t,u)/\varepsilon} e^{\alpha(t,v)/\varepsilon} u^{2H} du dv \\ &= t^{2H} - t^{2H} e^{\alpha(t)/\varepsilon} + 2H(e^{\alpha(t)/\varepsilon} - 1) \int_0^t e^{\alpha(t,s)/\varepsilon} s^{2H-1} ds. \end{aligned}$$

The last integral, given by

$$- \int_0^t \int_0^t \frac{a(u)}{\varepsilon} \frac{a(v)}{\varepsilon} e^{\alpha(t,u)/\varepsilon} e^{\alpha(t,v)/\varepsilon} |u-v|^{2H} du dv,$$

can be dropped since  $a$  is negative. The result follows by exploiting cancellations, and writing the factor  $t^{2H}$  occurring in one remaining term  $-t^{2H} e^{\alpha(t)/\varepsilon}$  as the integral of  $2Hs^{2H-1}$ .  $\square$

Laplace asymptotics and our assumptions (3.3) on  $a$  allow us to obtain the following simplified expression for the variance for small  $\varepsilon$ .

**Lemma 3.2.** *There exists a constant  $r_1$ , depending only on  $a_0$ ,  $a_1$  and  $H$ , such that for any  $H \in (0, 1)$  and any  $t \geq 0$ , one has*

$$\text{Var}(x_t) \leq \frac{\sigma^2 2H\Gamma(2H)}{|a(t)|^{2H}} (1 + r_1 \varepsilon). \quad (3.6)$$

PROOF: It is sufficient to bound the first term in the integrand in (3.4), since the second term is negative. The assumptions (3.3) on  $a$  imply that whenever  $0 \leq s \leq t \leq T$ , one has

$$\alpha(t, s) \leq -a_0(t-s) \quad \text{and} \quad \alpha(t, s) \leq a(t)(t-s) + \frac{1}{2} a_1(t-s)^2.$$

Using the substitution  $y = a(t)(t-s)$ , we obtain

$$\int_0^t e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} ds = \frac{1}{|a(t)|^{2H}} \int_0^{|a(t)|t} e^{\alpha(t, t-\frac{y}{|a(t)|})/\varepsilon} y^{2H-1} dy.$$

If  $|a(t)|t > \frac{a_0}{a_1}$ , we split the integral at  $y = \frac{a_0}{a_1}$ . To bound the integral over the interval  $[0, \frac{a_0}{a_1}]$ , we use the fact that

$$\alpha\left(t, t - \frac{y}{|a(t)|}\right) \leq -y + \frac{a_1}{2a(t)^2} y^2 \leq -y + \frac{a_1}{2a_0} y^2,$$

to obtain

$$\begin{aligned} \int_0^{a_0/a_1} e^{\alpha(t,s)/\varepsilon} (t-s)^{2H-1} ds &\leq \frac{1}{|a(t)|^{2H}} \int_0^{a_0/a_1} y^{2H-1} \exp\left\{-\frac{1}{\varepsilon} \left[y - \frac{a_1}{2a_0} y^2\right]\right\} dy \\ &= \frac{\varepsilon^{2H}}{|a(t)|^{2H}} \int_0^{a_0/\varepsilon a_1} z^{2H-1} \exp\left\{-\left[z - \varepsilon \frac{a_1}{2a_0} z^2\right]\right\} dz. \end{aligned}$$

The integral can easily be shown to be bounded above by  $\Gamma(2H)[1 + \mathcal{O}(\varepsilon)]$ , see for instance [16, Theorem 8.1].

Using the fact that  $\alpha(t, t - \frac{y}{|a(t)|}) \leq -y$ , one finds that the integral over the remaining interval  $[\frac{a_0}{a_1}, |a(t)|t]$  is exponentially small in  $\varepsilon$ , and therefore negligible with respect to the error of order  $\varepsilon$ . Finally, if  $|a(t)|t < \frac{a_0}{a_1}$ , we can use the integral over  $[0, \frac{a_0}{a_1}]$  as an upper bound.  $\square$

**Remark 3.3.** For  $H = \frac{1}{2}$ , this result is consistent with the case of Brownian motion investigated by Berglund and Gentz in [1]. In particular, the  $H$ -dependent constant in (3.6) is given by  $2H\Gamma(2H) = 1$  for  $H = \frac{1}{2}$ .  $\diamond$

As a first consequence of the bound (3.6), we obtain a concentration result for the solutions of the linear SDE (3.2). This is based on Theorem 2.2, taking into account the scaling argument in Remark 2.3.

**Proposition 3.4** (Concentration estimate for the linear SDE). *Assume  $x_0 = 0$ . Then there exists a constant  $r_2$ , depending only on  $a_0$ ,  $a_1$  and  $H$ , such that*

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} |x_t| |a(t)|^H \geq h\right\} \leq C\left(T; \frac{h}{\sigma}, a_0\right) \exp\left\{-\kappa(\varepsilon) \frac{h^2}{2\sigma^2}\right\}, \quad (3.7)$$

where the prefactor and exponent are given by

$$C\left(T; \frac{h}{\sigma}, a_0\right) = \frac{2K_0(2H)T^2}{a_0} \left(\frac{h}{\sigma}\right)^{1/H}, \quad \kappa(\varepsilon) = \frac{1 - r_2\varepsilon}{2H\Gamma(2H)}. \quad (3.8)$$

PROOF: We introduce a partition  $0 = t_0 < t_1 < \dots < t_N = T$  of  $[0, T]$  given by  $t_k = k\varepsilon$  for  $0 \leq k \leq N-1 = \lfloor T/\varepsilon \rfloor$ , and write  $I_k = [t_k, t_{k+1}]$  for the  $k$ th interval in the partition. Then the probability on the left-hand side of (3.7) is bounded by

$$\begin{aligned} & \sum_{k=0}^{N-1} \mathbb{P} \left\{ \sup_{t \in I_k} |x_t| |a(t)|^H \geq h \right\} \\ & \leq \sum_{k=0}^{N-1} \mathbb{P} \left\{ \sup_{t \in I_k} |x_t| \geq h \inf_{t \in I_k} \frac{1}{|a(t)|^H} \right\} \\ & \leq 2K_0(2H)\varepsilon T \left( \frac{h}{\sigma} \right)^{1/H} \sum_{k=0}^{N-1} \inf_{t \in I_k} \frac{1}{|a(t)|} \exp \left\{ -\frac{h^2}{2} \left( \inf_{t \in I_k} \frac{1}{|a(t)|^{2H}} \right) \left( \sup_{t \in I_k} \text{Var}(x_t) \right)^{-1} \right\}. \end{aligned}$$

To obtain the last line, we have applied Theorem 2.2 to  $x_t$  with  $\gamma = 2H$  and  $G = \sigma^2/\varepsilon^{2H}$ , which is justified for  $H > \frac{1}{2}$  by [11, Theorem 3.7]. For  $H < \frac{1}{2}$  a computation similar to the one in Lemma 3.1 entails the mean-square Hölder continuity of the non-autonomous fractional Ornstein-Uhlenbeck process with the same coefficients  $\gamma = 2H$  and  $G = \sigma^2/\varepsilon^{2H}$ . Now we observe that, setting

$$\hat{\nu}(t) = \frac{2H\Gamma(2H)}{|a(t)|^{2H}},$$

(3.6) implies  $\text{Var}(x_t) \leq \sigma^2 \hat{\nu}(t)(1 + \mathcal{O}(\varepsilon))$  for all  $t \in [0, T]$ . The result thus follows from the regularity properties (3.3) of  $a$  and the fact that the length of  $I_k$  is bounded by  $\varepsilon$ .  $\square$

The proposition shows that as soon as  $h \gg \sigma$ , the sample paths  $(x_t)_{t \in [0, T]}$  are unlikely to leave a strip of width  $h|a(t)|^{-H}$  before time  $T$ . In other words, we obtain a “confidence strip” for these sample paths.

**Remark 3.5.** Instead of using a partition of spacing  $\varepsilon$ , one could choose a partition given by  $t_k = k\delta$  for an arbitrary  $\delta \in (0, T]$ . This yields an extra factor  $\varepsilon/\delta$  in the prefactor  $C$ , and an additional error term of order  $\delta$  in the exponent. Taking  $\delta < \varepsilon$  is not of interest, since it increases the prefactor while it does not improve the exponent. Otherwise, the optimal value of  $\delta$  has order  $\sigma^2/h^2$ , and yields a prefactor of order  $\varepsilon T^2(h/\sigma)^{2+1/H}$ . Because of the condition  $\delta \geq \varepsilon$ , this is only of interest if  $h^2 < \sigma^2/\varepsilon$ .  $\diamond$

### 3.2 Nonlinear case

This result can now easily be extended to concentration estimates for sample paths near stable slow manifolds of non-linear slowly time-dependent fractional SDEs. Consider the equation

$$dx_t = \frac{1}{\varepsilon} f(t, x_t) dt + \frac{\sigma}{\varepsilon^H} dW_t^H, \quad (3.9)$$

where  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^2$ . Assume that  $x^* : [0, T] \rightarrow \mathbb{R}$  is a stable uniformly hyperbolic slow manifold (or stable equilibrium branch), meaning that

- $f(t, x^*(t)) = 0$  for all  $t \in [0, T]$ ,
- and there exists  $a_0 > 0$  such that  $a^*(t) = \partial_x f(t, x^*(t)) \leq -a_0$  for all  $t \in [0, T]$ .

Note that by the implicit function theorem, such a function  $x^*$  is of class  $\mathcal{C}^2$  as well. Classical results by Tihonov and Fenichel ensure that for sufficiently small  $\varepsilon$ , the deterministic

equation  $\varepsilon \dot{x} = f(t, x)$  admits a particular solution  $\bar{x}(t)$  satisfying  $\bar{x}(t) = x^*(t) + \mathcal{O}(\varepsilon)$  uniformly in  $t \in [0, T]$ . We set

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t))$$

and observe that it is bounded above by  $-\bar{a}_0 = -a_0 + \mathcal{O}(\varepsilon)$ , which is still negative for  $\varepsilon$  small enough. We define the set

$$\mathcal{B}(h) = \{(t, x) : t \in [0, T], |x - \bar{x}(t)| |\bar{a}(t)|^H \leq h\},$$

which is a strip of width  $h |\bar{a}(t)|^{-H}$  around the graph of  $\bar{x}$ , and write  $\tau_{\mathcal{B}(h)}$  for the first-exit time of  $(x_t)_t$  from  $\mathcal{B}(h)$ . Then we have the following concentration result, which is the main result of this section.

**Theorem 3.6** (Concentration estimate for the nonlinear SDE). *There exist  $\varepsilon_0, h_0 > 0$  such that for  $\varepsilon \leq \varepsilon_0$  and  $h \leq h_0$ , the solution of (3.9) with initial condition  $x_0 = \bar{x}(0)$  satisfies*

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} \leq C\left(T; \frac{h}{\sigma}, \bar{a}_0\right) \exp\left\{-\kappa(\varepsilon) \frac{h^2}{2\sigma^2} [1 - \mathcal{O}(h)]\right\},$$

where the constants  $C(T; \frac{h}{\sigma}, \bar{a}_0)$  and  $\kappa(\varepsilon)$  are the same as in Proposition 3.4.

PROOF: The difference  $y_t = x_t - \bar{x}(t)$  satisfies the SDE

$$dy_t = \frac{1}{\varepsilon} [\bar{a}(t) + b(t, y_t)] dt + \frac{\sigma}{\varepsilon^H} dW_t^H,$$

where there exist constants  $M, d > 0$  such that  $|b(t, y)| \leq My^2$  whenever  $t \in [0, T]$  and  $|y| \leq d$ . Its solution can be represented as  $y_t = y_t^0 + y_t^1$ , where

$$y_t^0 = \frac{\sigma}{\varepsilon^H} \int_0^t e^{\alpha(t,s)/\varepsilon} dW_s^H, \quad y_t^1 = \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,s)/\varepsilon} b(s, y_s) ds.$$

For any decomposition  $h = h^0 + h^1$  with  $h^0, h^1 > 0$ , continuity of sample paths allows us to write

$$\begin{aligned} \mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} &= \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} |y_t| |\bar{a}(t)|^H \geq h\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} |y_t^0| |\bar{a}(t)|^H \geq h^0\right\} + \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} |y_t^1| |\bar{a}(t)|^H \geq h^1\right\}. \end{aligned} \quad (3.10)$$

Since

$$|y_{t \wedge \tau_{\mathcal{B}(h)}}^1| \leq \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,s)/\varepsilon} \frac{Mh^2}{|\bar{a}(s)|^H} ds \leq \frac{Mh^2}{\bar{a}_0^{1+H}},$$

the second term on the right-hand side of (3.10) vanishes for an  $h^1$  of order  $h^2$ . The result then follows from Proposition 3.4, taking  $h^0 = h - h^1 = h[1 - \mathcal{O}(h)]$ .  $\square$

## 4 The SPDE case

### 4.1 Linear case

We now turn to the analysis of linear SPDEs on the one-dimensional torus  $\mathbb{T}$ , of the form

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta + a(t)] \phi(t, x) dt + \frac{\sigma}{\varepsilon^H} dW^H(t, x), \quad (4.1)$$

where  $a : [0, T] \rightarrow \mathbb{R}$  satisfies again (3.3). The SPDE is driven by a cylindrical fractional Brownian motion  $(W^H(t))_{t \geq 0}$  with Hurst parameter  $H \in (\frac{1}{4}, 1)$ . This means that the noise is fractional-in-time and white-in-space. The existence of mild solutions for such linear SPDEs was established in [8, Example 3.1] for  $H \in (\frac{1}{4}, 1)$ .

The  $k$ th Fourier component of  $\phi$  satisfies

$$d\phi_k(t) = -\frac{1}{\varepsilon} \lambda_k(t) \phi_k(t) dt + \frac{\sigma}{\varepsilon^H} dW_k^H(t),$$

where  $\lambda_k(t)$  is the  $k$ th eigenvalue of  $-\Delta - a(t)$  given by  $\lambda_k(t) = (2\pi)^2 k^2 - a(t)$ . It has the order  $\langle k \rangle^2$ , where  $\langle k \rangle = \sqrt{1 + k^2}$ , and satisfies  $|\lambda_k(t)| \geq (2\pi)^2 k^2 + a_0$ .

**Remark 4.1.** Rescaling time as  $t = \mu_k \tilde{t}$ , where  $\mu_k = (a_0 + ck^2)^{-1}$  for a constant  $c > 0$ , we obtain

$$d\phi_k = -\frac{1}{\varepsilon} \tilde{\lambda}_k(\tilde{t}) \phi_k(\tilde{t}) d\tilde{t} + \frac{\sigma \mu_k^H}{\varepsilon^H} dW_k^H(\tilde{t}) \quad (4.2)$$

with  $\tilde{t} \in [0, T/\mu_k]$ , where  $\tilde{\lambda}_k(\tilde{t}) = \mu_k \lambda_k(\tilde{t}/\mu_k)$ , and consequently  $\tilde{\lambda}_k(\tilde{t}) \geq 1$  and  $|\tilde{\lambda}'_k(\tilde{t})| \leq a_1$  for all  $\tilde{t} \in [0, T/\mu_k]$ . This allows us to use Proposition 3.4 with values of  $a_0$  and  $a_1$  that do not depend on  $k$ . Therefore, we need not worry about a possible  $k$ -dependence of the error term  $r_2 \varepsilon$  in (3.8).  $\diamond$

We recall that the (fractional) Sobolev norm on  $H^s(\mathbb{T})$  for  $s > 0$  is given by

$$\|\phi(t, \cdot)\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\phi_k(t)|^2. \quad (4.3)$$

While one can work with this norm, it turns out that we can obtain slightly sharper bounds using a time-dependent Sobolev norm defined by

$$\|\phi(t, \cdot)\|_{s,t}^2 := \sum_{k \in \mathbb{Z}} a_{k,s}(t) |\phi_k|^2,$$

where we will choose  $a_{k,s}(t) = \lambda_k^H(t) \langle k \rangle^{s-2H}$ , so that  $a_{k,s}(t) \asymp \langle k \rangle^s$ . Note that both norms are equivalent, and are not sensitive to Fourier modes with large  $|k|$ . However, the time-dependent norm will give a sharper control for Fourier modes with small  $|k|$ , and especially for  $k = 0$ .

**Proposition 4.2** (Concentration estimate for the linear SPDE). *Let  $H \in (0, \frac{1}{4})$  and  $s \in (0, 2H - \frac{1}{2})$ . Then there exist constants  $c_0, c_1, r_2 > 0$  such that the solution of (4.1) with initial condition  $\phi(0, \cdot) = 0$  satisfies the concentration inequality*

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} \leq C \left( T; \frac{h}{\sigma}, s \right) \exp \left\{ -\kappa(\varepsilon) Q(s) \frac{h^2}{2\sigma^2} \right\},$$

where  $Q(s) \geq c_0(2H - \frac{1}{2} - s)$ ,  $\kappa(\varepsilon)$  is the same as in (3.8), and

$$C \left( T; \frac{h}{\sigma}, s \right) = 2K_0(2H) T^2 a_0^2 \left( Q(s)^{1/2} \frac{h}{\sigma} \right)^{1/H} [1 + \mathcal{O}(e^{-c_1 h^2 / \sigma^2})].$$

PROOF: It is known that the fractional stochastic convolution has continuous trajectories in  $H^s$  for  $s \in (0, 2H - \frac{1}{2})$ , see [8, Corollary 3.1] for  $H > \frac{1}{2}$ , respectively [9, Lemma 11.10] for  $H \in (\frac{1}{4}, \frac{1}{2})$ . For any decomposition  $h^2 = \sum_{k \in \mathbb{Z}} h_k^2$  with  $h_k > 0$  for all  $k \in \mathbb{Z}$ , we have

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} &= \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sum_{k \in \mathbb{Z}} a_{k,s}(t) |\phi_k(t)|^2 \geq h^2 \right\} \\ &\leq \sum_{k \in \mathbb{Z}} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} |\phi_k(t)|^2 \lambda_k(t)^{2H} \geq \frac{h_k^2}{\langle k \rangle^{2s-4H}} \right\} \end{aligned} \quad (4.4)$$

by the choice of the time-dependent coefficients  $a_{k,s}(t) = \lambda_k^H(t) \langle k \rangle^{s-2H}$ . According to Proposition 3.4, we have for each component  $\phi_k$  solving (4.2) that

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} |\phi_k(t)| \lambda_k(t)^H \geq \frac{h_k}{\langle k \rangle^{s-2H}} \right\} &= \mathbb{P} \left\{ \sup_{0 \leq \tilde{t} \leq T/\mu_k} |\phi_k(\tilde{t})| \frac{\tilde{\lambda}_k(\tilde{t})^H}{\mu_k^H} \geq \frac{h_k}{\langle k \rangle^{s-2H}} \right\} \\ &\leq C_k \exp \left\{ -\kappa(\varepsilon) \frac{h_k^2 \mu_k^{2H}}{2\tilde{\sigma}_k^2 \langle k \rangle^{2s-4H}} \right\}, \end{aligned}$$

where  $\tilde{\sigma}_k = \mu_k^H \sigma$  due to the scaling in (4.2), and

$$C_k = C \left( \frac{T}{\mu_k}, \frac{h_k \mu_k^H}{\tilde{\sigma}_k \langle k \rangle^{s-2H}}, 1 \right) = \frac{2K_0(2H)T^2}{\mu_k^2} \left( \frac{h_k}{\sigma \langle k \rangle^{s-2H}} \right)^{1/H}.$$

Plugging this in (4.4) and simplifying the factors  $\mu_k^H$  entails

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} \leq \sum_{k \in \mathbb{Z}} C_k \exp \left\{ -\kappa(\varepsilon) \frac{h_k^2}{2\sigma^2 \langle k \rangle^{2s-4H}} \right\}.$$

We pick  $\eta > 0$  and choose  $h_k^2 = Q(s) h^2 \langle k \rangle^{-(4H-2s-\eta)}$ , where the condition  $h^2 = \sum_{k \in \mathbb{Z}} h_k^2$  imposes

$$Q(s)^{-1} = \sum_{k \in \mathbb{Z}} \frac{1}{\langle k \rangle^{4H-2s-\eta}} < \infty.$$

This means that we need to have  $4H-2s-\eta > 1$ , and since both  $s$  and  $\eta$  must be positive we obtain the restriction  $H > \frac{1}{4}$ . The claimed lower bound on  $Q(s)$  follows from the behaviour of Riemann's zeta function  $\zeta(u)$  as  $u \rightarrow 1$ , choosing for instance  $\eta = 2H - s - \frac{1}{2}$ . Based on this choice of the  $h_k$ , we further obtain

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\phi(t, \cdot)\|_{s,t} \geq h \right\} &\leq \sum_{k \in \mathbb{Z}} C_k \exp \left\{ -\kappa(\varepsilon) Q(s) \frac{h^2 \langle k \rangle^\eta}{2\sigma^2} \right\} \\ &\leq 2K_0(2H)T^2 \left( Q(s)^{1/2} \frac{h}{\sigma} \right)^{1/H} \sum_{k \in \mathbb{Z}} \frac{\langle k \rangle^{\eta/(2H)}}{\mu_k^2} \exp \{ -\beta \langle k \rangle^\eta \}, \end{aligned}$$

where  $\beta := \kappa(\varepsilon) Q(s) \frac{h^2}{2\sigma^2}$ . We claim that the sum over  $k$  is dominated by the term  $k = 0$ . In fact, by an argument similar to the one in [2, Theorem 2.4], we can bound this sum by

$$f(0) + 2f(1) + \int_1^\infty f(x) dx, \quad \text{where} \quad f(x) := (a_0 + cx^2)(1+x^2)^\gamma e^{-\beta(1+x^2)^{\eta/2}},$$

with an exponent  $\gamma = \eta/(4H)$ . The integral can be shown to be of order  $e^{-c\beta}$  with  $c > 1$ , which yields the result.  $\square$

**Remark 4.3.** The condition  $H > \frac{1}{4}$  is consistent with the solution theory for such SPDEs driven by cylindrical fractional noise [8, 21].  $\diamond$

## 4.2 Nonlinear case

We finally consider the nonlinear SPDE given by

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + f(t, \phi(t, x))] dt + \frac{\sigma}{\varepsilon^H} dW^H(t, x).$$

We assume  $f(t, \phi) = -\partial_\phi U(t, \phi)$ , where the potential  $U$  can be decomposed as  $U(t, \phi) = P(t, \phi) + g(t, \phi)$ . Here  $P$  is polynomial of even degree  $2p$  with smooth bounded coefficients such that the leading order coefficient  $a_{2p}(t) > 0$  for all  $t \in [0, T]$ , and the function  $g \in \mathcal{C}^2([0, T] \times \mathbb{R}; \mathbb{R})$  satisfies the boundedness assumptions

$$|g(t, \phi)\phi^{-1}|, |\partial_\phi g(t, \phi)|, |\partial_{\phi\phi} g(t, \phi)|, |\partial_t g(t, \phi)| \leq \widetilde{M}$$

for all  $(t, \phi) \in [0, T] \times \mathbb{R}$ , for a constant  $\widetilde{M} > 0$ .

The well-posedness of semilinear SPDEs with drift given by  $f(t, \phi) = -\partial_\phi P(t, \phi)$  was established in [10, Section 4]. This remains valid for  $U(t, \phi) = P(t, \phi) + g(t, \phi)$ , due to the smoothness and boundedness assumptions on  $g$ . As before, we assume that we are in a stable situation, meaning that there exists a map  $\phi^* : [0, T] \rightarrow \mathbb{R}$  such that

- $f(t, \phi^*(t)) = 0$  for all  $t \in [0, T]$ ,
- there exists  $a_0 > 0$  such that  $a^*(t) = \partial_\phi f(t, \phi^*(t)) \leq -a_0$  for all  $t \in [0, T]$ .

Similarly to the finite-dimensional case, the deterministic PDE admits, according to [2, Proposition 2.3], a solution  $\bar{\phi}$  such that  $\bar{\phi}(t) = \phi^*(t) + \mathcal{O}(\varepsilon)$  for all  $t \in [0, T]$ . We set as before  $\bar{a}(t) = \partial_x f(t, \bar{\phi}(t))$  and introduce for  $s \in (0, 2H - \frac{1}{2})$

$$\mathcal{B}(h) := \{(t, \phi) : t \in [0, T], \|\phi(t, \cdot) - \bar{\phi}(t, \cdot)\|_{s,t} \leq h\}.$$

The SPDE for the difference  $\psi(t, x) := \phi(t, x) - \bar{\phi}(t, x)$  reads as

$$d\psi(t, x) = \frac{1}{\varepsilon} [\Delta\psi(t, x) + \bar{a}(t)\psi(t, x) + b(t, \psi(t, x))] dt + \frac{\sigma}{\varepsilon^H} dW^H(t, x), \quad (4.5)$$

where there exists constants  $M, d > 0$  such that  $|b(t, \psi)| \leq M\psi^2$  and  $|\partial_\psi b(t, \psi)| \leq M|\psi|$  for all  $t \in [0, T]$  and  $\psi \in \mathbb{R}$  with  $|\psi| \leq d$ . By the variation of constants formula, its solution is given by

$$\begin{aligned} \psi(t, x) &= \frac{\sigma}{\varepsilon^H} \int_0^t e^{\alpha(t,s)/\varepsilon} e^{[(t-s)/\varepsilon]\Delta} dW^H(s) + \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,s)/\varepsilon} e^{[(t-s)/\varepsilon]\Delta} b(s, \psi(s, x)) ds \\ &:= \psi^0(t, x) + \psi^1(t, x). \end{aligned}$$

As before,  $\alpha(t, s) = \int_s^t \bar{a}(r) dr$ . In order to analyze the stochastic convolution we rely on Schauder-type estimates.

**Lemma 4.4** (Schauder-type estimates). *Let  $f \in H^r$  with  $r \in (0, 2H - \frac{1}{2})$ . Then for all  $q < r + 2$ , there exists a constant  $c(q, r) > 0$  such that*

$$\|e^{t\Delta} f\|_{H^q} \leq c(q, r) t^{-\frac{q-r}{2}} \|f\|_{H^r}.$$

PROOF: This follows from regularizing properties of analytic semigroups [17, Theorem 6.13], according to which

$$\|e^{t\Delta} f\|_{D((-\Delta)^{q/2})} \leq c(q, r) t^{-\frac{q-r}{2}} \|f\|_{D((-\Delta)^{r/2})}$$

for  $q - r/2 < 1$ , leading to the restriction  $q < r + 2$ . The statement follows, considering that  $H^q = D((-\Delta)^{q/2})$ . Here we use the equivalence of the time-dependent (fractional) Sobolev norm with (4.3).  $\square$

For  $\psi(t, \cdot) \in H^s$ , one can easily prove that  $\beta(t) := b(t, \psi(t, \cdot))$  belongs to  $H^s$ . A proof of this statement in  $H^s$  for  $s \in (0, \frac{1}{2})$  relying on Young's inequality is provided in [2, Lemma 3.4]. We now apply the Schauder estimate in order to obtain a bound on  $\psi^1(t, \cdot)$  similar to [2, Corollary 3.6].

**Lemma 4.5.** *Assume that there exists  $r \in (0, 2H - \frac{1}{2})$  such that  $\beta(t) \in H^r$  for all  $t \in [0, T]$ . Then for all  $q < r + 2$  there exists a constant  $c(q, r) > 0$  such that for all  $t \in [0, T]$  we have*

$$\|\psi^1(t, \cdot)\|_{H^q} \leq c(q, r) \varepsilon^{\frac{q-r}{2}-1} \sup_{0 \leq s \leq t} \|\beta(s)\|_{H^q}.$$

PROOF: We have

$$\begin{aligned} \|\psi^1(t, x)\|_{H^q} &\leq \frac{1}{\varepsilon} \int_0^t e^{-a_0(t-s)/\varepsilon} \|e^{[(t-s)/\varepsilon]\Delta} \beta(s)\|_{H^q} ds \\ &\leq c(q, r) \varepsilon^{\frac{q-r}{2}-1} \sup_{0 \leq s \leq t} \|\beta(s)\|_{H^r} \int_0^t (t-s)^{-\frac{q-r}{2}} ds < \infty, \end{aligned}$$

since  $q < r + 2$ . In the last step we used the uniform negative bound on  $\bar{a}$  and the Schauder estimate.  $\square$

**Theorem 4.6** (Concentration estimate for the nonlinear SPDE). *For every  $s \in (0, 2H - \frac{1}{2})$  and any  $\nu > 0$ , there exists positive constants  $\varepsilon_0, h_0$  such that for  $\varepsilon \leq \varepsilon_0$  and  $h \leq h_0 \varepsilon^\nu$ , the solution of (4.5) with initial condition  $\phi(0, \cdot) = \bar{\phi}(0, \cdot)$  satisfies*

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} \leq C\left(T; \frac{h}{\sigma}, s\right) \exp\left\{-\kappa(\varepsilon) Q(s) \frac{h^2}{2\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\},$$

with the same  $C\left(T; \frac{h}{\sigma}, s\right)$ ,  $Q(s)$  and  $\kappa(\varepsilon)$  as in Proposition 4.2.

PROOF: For any decomposition  $h = h_0 + h_1$  we have

$$\begin{aligned} \mathbb{P}\{\tau_{\mathcal{B}(h)} \leq T\} &= \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi(t, \cdot)\|_{s,t} \geq h\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi^0(t, \cdot)\|_{s,t} + \|\psi^1(t, \cdot)\|_{s,t} \geq h\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi^0(t, \cdot)\|_{s,t} \geq h_0\right\} + \mathbb{P}\left\{\sup_{0 \leq t \leq \tau_{\mathcal{B}(h)}} \|\psi^1(t, \cdot)\|_{s,t} \geq h_1\right\}. \end{aligned} \quad (4.6)$$

We bound the first term using Proposition 4.2. For the second one we have, similarly to the proof of Theorem 3.6, that for  $t \leq \tau_{\mathcal{B}(h)}$

$$\|\psi^1(t, \cdot)\|_{H^q} \leq c(q, r) \varepsilon^{\frac{q-r}{2}-1} M h^2,$$

since  $\|\beta(t)\|_{H^r} \leq M \|\psi^1(t, \cdot)\|_{H^q}^2 \leq M h^2$ . Therefore the second term in (4.6) vanishes provided that  $h_1 = c(q, r) \varepsilon^{\frac{q-r}{2}-1} M h^2$ . The statement follows, choosing

$$h_0 = h - h_1 = h - c(q, r) \varepsilon^{\frac{q-r}{2}-1} M h^2 = h(1 - \mathcal{O}(h/\varepsilon^\nu))$$

for  $\nu = 1 - \frac{q-r}{2}$ .  $\square$

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