Two general and curious conjectures like Collatz's

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Abstract

In this paper, we will introduce two main conjectures. The first one may be seen as a general conjecture that encompasses the Collatz one and may be stated as following. For any positive integer $\boldsymbol{p \geq 0}$, let $\boldsymbol{d_p = 2^p + 1}$, $\boldsymbol{\alpha_p = 2^p + 2}$ and $\boldsymbol{\beta_p = 2^p}$. Starting with any positive integer $\boldsymbol{n \geq 1}$: If $\boldsymbol{n}$ is divided by $\boldsymbol{d_p}$ then divide it by $\boldsymbol{d_p}$, else multiply it by $\boldsymbol{\alpha_p}$ and add $\boldsymbol{\beta_p}$ times the remainder of $\boldsymbol{n}$ by $\boldsymbol{d_p}$. Repeating the process iteratively, it always reaches $\boldsymbol{2^p}$ after finite number of iterations. Except for $\boldsymbol{p = 1, 3, 4}$, the process for $\boldsymbol{p = 1}$ may reaches $\boldsymbol{2^1}$ or also $\boldsymbol{14}$, for $\boldsymbol{p = 3}$, the process may reaches $\boldsymbol{2^3}$ or also $\boldsymbol{280}$ and for $\boldsymbol{p = 4}$, the process may reaches $\boldsymbol{2^4}$ or also $\boldsymbol{1264}$. The classical Collatz conjecture may be seen as a special case of our conjecture for $\boldsymbol{p = 0}$ corresponding to $\boldsymbol{d_0 = 2}$, $\boldsymbol{\alpha_0 = 3}$ and $\boldsymbol{\beta_0 = 1}$. The second one, may be stated in its special case as following. Starting with a positive integer: If it is a multiple of 10 then remove all the zeros on the right, otherwise, multiply it by 6, add 4 times its last digit and divide the result by 5. Repeat the process infinitely. Regardless the starting number, the process eventually reaches 4 after a finite number of iterations. We will discuss the two conjectures more precisely, we will specify the trivial cycles and we will also give a general formulation of the second conjecture. We will discuss verification of both conjectures and give some graphs by specifying the corresponding backward mapping.

Keywords: Collatz conjecture, Syracuse Conjecture, $\boldsymbol{3n+1}$-problem, discrete mathematics

1 Introduction

The Collatz conjecture is one of the famous unsolved problem in mathematics. This conjecture is also well known as the $3n+1$ problem and is formulated as following: Starting with any initial integer ≥ 1, if the number is even divide it by 2 else multiply it by 3 and add 1. The Collatz conjecture asserts that repeating the process iteratively, it always reaches 1. More precisely, consider the operator $\widetilde{C} : \mathbb{N} :\longrightarrow \mathbb{N}$ given by

$$\widetilde{C}(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \not\equiv 0 \pmod 2. \end{cases} \tag{1}$$

The Collatz conjecture may be reformulated as following: For any $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, there is a positive integer $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $\widetilde{C}^{(k)}(n) = 1$, where the notation $\widetilde{C}^{(k)}$ stands for the k-th iterate of the mapping $\widetilde{C}$. It is also conjectured that the trivial cycle $(1 \to 4 \to 2 \to 1)$ is the unique cycle of the Collatz operator $\widetilde{C}$. A survey and extensive of literature on this subject has been given in [1, 2]. We may speed the

convergence to the cycle by dividing in the odd case by 2 which leads to the modified Collatz operator $C : \mathbb{N} :\longrightarrow \mathbb{N}$ given by

$$C(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ (3n+1)/2 & \text{if } n \not\equiv 0 \pmod 2. \end{cases} \tag{2}$$

The trivial unique cycle is now $(1 \to 2 \to 1)$. The Collatz conjecture has been verified experimentally with computer by many authors. For instance, in [3], the author claims that he verified in 2009 the classical Collatz conjecture up to $2^{62.3} \simeq 5.67 \times 10^{18}$ and in [4], the author claims that he verified in 2020 the same conjecture up to $2^{68} \simeq 2.95 \times 10^{20}$.

We have discussed, in [5], a general extension to the classical Collatz conjecture by considering the following operator $T : \mathbb{N} :\longrightarrow \mathbb{N}$ given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod d \\ (\alpha n + \beta[\kappa_0 n]_d)/d & \text{if } n \not\equiv 0 \pmod d, \end{cases} \tag{3}$$

where $\alpha > d \geq 2$ are two positive integers, β is another integer which may be negative and $\kappa_0 = \pm 1$. The integers d, α and β are such that: $\alpha \not\equiv 0 \pmod d$ and $\beta \not\equiv 0 \pmod d$. The notation $[n]_d$ denotes the remainder of n in the Euclidean division of n by d with the standard condition $0 \leq [n]_d < d$. The triplet (d, α, β) associated to T will be denoted by $(d, \alpha, \beta)_+$ and $(d, \alpha, \beta)_-$ for $\kappa_0 = +1$ and for $\kappa_0 = -1$, respectively and it will be called an admissible triplet if it satisfies the following condition

$$\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta > d \quad \text{and} \quad \beta = \kappa_0(\lambda_0 d^{\nu_0} - \alpha), \tag{4}$$

where the integers $d \geq 2$, $\lambda_0 \geq 1$, $\lambda_1 \geq 1$, $\nu_0 \geq 1$, $\nu_1 \geq 1$, $1 \leq \delta$ and $\kappa_0, \kappa_1 = \pm 1$ such that $\lambda_0 \not\equiv 0 \pmod d$ and $\lambda_1 \not\equiv 0 \pmod d$. The condition (4) is a necessary and sufficient condition that the operator $T : \mathbb{N} :\longrightarrow \mathbb{N}$ is well defined from $\mathbb{N}$ into $\mathbb{N}$.

Given $n \in \mathbb{N}$, the trajectory of n is the set $\Gamma(n) = \{n, T(n), T^{(2)}(n), \ldots\}$ of iterates starting by n. A cycle (if it exists), having k elements (or vertices), associated to the mapping T is a finite set Ω for which $T^{(k)}(x) = x$ for all $x \in \Omega$. We will denote by $\Omega(\omega) = \{\omega, T(\omega), T^{(2)}(\omega), \ldots, T^{(k-1)}(\omega)\}$ the cycle (if it exists) associated to T (or to the triplet $(d, \alpha, \beta)_\pm$) of length k where ω is the smallest element in the cycle. We will also use this abusive following notation

$$(\omega \to T(\omega) \to T^{(2)}(\omega) \to \ldots \to T^{(k-1)}(\omega) \to \omega),$$

to denote the cycle $\Omega(\omega)$. The order of an admissible triplet (d, α, β) is the number of its cycles. It may be zero, finite integer or infinite. We have the following definition.

Definition 1. *Let $(d, \alpha, \beta)_\pm$ be an admissible triplet with its associated mapping T given by (3). We will say that:*

- *The admissible triplet $(d, \alpha, \beta)_\pm$ is a weak admissible triplet if and only if it has at least one cycle and at most a finite number of cycles.*
- *The admissible triplet $(d, \alpha, \beta)_\pm$ is a strong admissible triplet if and only if $(d, \alpha, \beta)_\pm$ is a weak admissible triplet and the set $G_\infty(d, \alpha, \beta)$ is empty. Namely, the triplet has a finite number of cycles without any divergent trajectory.*

The classical Collatz conjecture may be now reformulated as following:

Conjecture 1 (of Collatz)**.** *The triplet $(2, 3, 1)_+$ is a finite strong triplet of order one. Its unique cycle is the trivial one $\Omega(1) = (1 \to 2 \to 1)$ of length 2.*

Many theoretical results and interesting properties on the classical Collatz sequence were given by many others, among them we cite [2, 6–10], the list is not exhaustive.

The remainder of this paper is organized as following: In Section 2, we will formulate a general conjecture that encompasses the classical Collatz one. In Section 3, we introduce a new general conjecture that behaves as the Collatz one. Written the Collatz conjecture in the binary base, the new proposed conjecture may be seen as the equivalent of the Collatz one in the decimal base. The short Section 4 is devoted to the verification of the conjectures. Section 5, is devoted to the algorithm given the

backward operator and which allows us the construction of graphs illustrating the conjecture.

2 The $(2^p+2)n+2^p$ conjecture modulo 2^p+1 as a general Collatz conjecture

In this section, we are interested to the specific case, where $d \geq 2$, $\lambda_1 = \nu_0 = \nu_1 = \delta = 1$, $\lambda_0 = 2$, $\kappa_0 = 1$ and $\kappa_1 = -1$ which leads, according to (4), to the triplet $(d, d+1, d-1)_+$. We will formulate and discuss a general conjecture related to the triplet $(d, d+1, d-1)_+$ that encompasses the classical Collatz conjecture. For more details on this extension, we refer to [5]. The operator $\widetilde{T}$ corresponding to the admissible triplet $(d, d+1, d-1)_+$ is given by

$$\widetilde{T}(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ (d+1)n + (d-1)[n]_d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \tag{5}$$

We have the following result

Theorem 1. *For $d \geq 2$, the triplet $(d, \alpha, \beta)_+ := (d, d+1, d-1)_+$ associated to the operator $\widetilde{T}$ given by (16) is at least a weak triplet. It has at least the following trivial cycle of length $2d-1$:*

$$\begin{aligned} \Omega(\beta) = \big(&\beta \to 2\beta d \to 2\beta \to 3\beta d \to 3\beta \to 4\beta d \to 4\beta \to \dots \\ &\dots \to k\beta \to (k+1)\beta d \to (k+1)\beta \to \dots \to (d-1)\beta \to d\beta d \to d\beta \to \beta\big). \end{aligned} \tag{6}$$

Proof. We have obviously, for $k \in \{1, 2, \dots, d-1\}$, $\widetilde{T}(k\beta) = (d+1)k\beta + (d-1)[k\beta]_d$. But $[k\beta]_d = [-k]_d = d-k$, it follows that $\widetilde{T}(k\beta) = (d+1)k\beta + (d-1)(d-k) = (k+1)\beta d$. So, for $k \in \{1, 2, \dots, d-1\}$, we have $\widetilde{T}^{(2k-1)}(\beta) = (k+1)\beta d$, $\widetilde{T}^{(2k)}(\beta) = (k+1)\beta$ and $\widetilde{T}^{(2d-1)}(\beta) = \beta$, which shows that the $\Omega(\beta)$ given by (8) is a cycle of length $2d-1$. □

To speed up the convergence to the cycle, we consider the following modified operator $T : \mathbb{N} :\longrightarrow \mathbb{N}$ given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ \dfrac{(d+1)n + (d-1)[n]_d}{d} & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \tag{7}$$

Then, we have the corollary

Corollary 1. *For $d \geq 2$, the triplet $(d, \alpha, \beta)_+ = (d, d+1, d-1)_+$ associated to the operator T given by (7) is at least a weak triplet. It has at least the following trivial cycle of length d:*

$$\Omega(\beta) = \big(\beta \to 2\beta \to 3\beta \to \dots \to k\beta \to (k+1)\beta \to \dots \to (d-1)\beta \to d\beta \to \beta\big). \tag{8}$$

In the rest of this paper, we are interested to the particular case where $d = 2^p + 1$ with $p \geq 0$ is a positive integer. Then,

$$d = d_p := 2^p + 1, \quad \alpha = \alpha_p := d_p + 1 = 2^p + 2, \quad \beta = \beta_p := d_p - 1 = 2^p. \tag{9}$$

The corresponding mapping, for $p \geq 0$, is given by

$$T_p(n) = \begin{cases} n/(2^p+1) & \text{if } n \equiv 0 \pmod{(2^p+1)} \\ \dfrac{(2^p+2)n + 2^p[n]_p}{2^p+1} & \text{if } n \not\equiv 0 \pmod{(2^p+1)}, \end{cases} \tag{10}$$

here $[n]_p := [n]_{d_p}$ stands for the rest in the Euclidean division of n by $d_p = 2^p + 1$. We observe that the case $p = 0$ corresponds to the classical Collatz case (2). We now ready to formulate our main conjecture which may be seen as a generalization of the classical Collatz conjecture .

Conjecture 2. *For all integer $p \geq 0$ the triplet $(2^p+1, 2^p+2, 2^p)_+$ is a strong admissible triplet.*

1. *For all integer $p \geq 0$ with $p \notin \{1,3,4\}$, the strong admissible triplet $(2^p+1, 2^p+2, 2^p)_+$ is of order one and its unique cycle is the trivial cycle of length 2^p+1:*

$$\Omega(2^p) = \Big(1 \cdot 2^p \to 2 \cdot 2^p \to 3 \cdot 2^p \to \ldots \to (2^p+1) \cdot 2^p \to 2^p\Big).$$

   For all integer $n \geq 1$, there exists an integer $k \geq 0$ such that $T_p^{(k)}(n) = 2^p$.
2. *For $p \in \{1,3,4\}$, the strong admissible triplet $(2^p+1, 2^p+2, 2^p)_+$ is of order two. Its first trivial cycle of length 2^p+1 is:*

$$\Omega(2^p) = \Big(1 \cdot 2^p \to 2 \cdot 2^p \to 3 \cdot 2^p \to \ldots \to (2^p+1) \cdot 2^p \to 2^p\Big).$$

   Its second trivial cycle in each case, is the cycle $\Omega(\omega_p)$ of length $9, 21, 49$, respectively and starting with $\omega_1 = 14$, $\omega_3 = 280$ and $\omega_4 = 1264$, respectively. The second cycle is given in each case as:

$$\begin{aligned}
\Omega(\omega_1) = {} & \big(14 \to 20 \to 28 \to 38 \to 52 \to 70 \to 94 \to 126 \to 42 \to 14\big),\\
\Omega(\omega_3) = {} & \big(280 \to 312 \to 352 \to 392 \to 440 \to 496 \to 552 \to 616 \to 688 \to 768 \to\\
& 856 \to 952 \to 1064 \to 1184 \to 1320 \to 1472 \to 1640 \to 1824 \to 2032 \to\\
& 2264 \to 2520 \to 280\big).\\
\Omega(\omega_4) = {} & \big(1264 \to 1344 \to 1424 \to 1520 \to 1616 \to 1712 \to 1824 \to 1936 \to 2064 \to\\
& 2192 \to 2336 \to 2480 \to 2640 \to 2800 \to 2976 \to 3152 \to 3344 \to 3552 \to\\
& 3776 \to 4000 \to 4240 \to 4496 \to 4768 \to 5056 \to 5360 \to 5680 \to 6016 \to\\
& 6384 \to 6768 \to 7168 \to 7600 \to 8048 \to 8528 \to 9040 \to 9584 \to 10160 \to\\
& 10768 \to 11408 \to 12080 \to 12800 \to 13568 \to 14368 \to 15216 \to 16112 \to\\
& 17072 \to 18080 \to 19152 \to 20288 \to 21488 \to 1264\big).
\end{aligned}$$

   For $p \in \{1,3,4\}$ and for all integer $n \geq 1$, there exists an integer $k \geq 0$ such that $T_p^{(k)}(n) \in \{2^p, \omega_p\}$.

Example 1. *The classical Collatz conjecture is a special case of the conjecture 2 , corresponding to $p = 0$ with the triplet $(d_0, \alpha_0, \beta_0)_+ = (1,2,3)_+$. For all $n \in \mathbb{N}$, $\exists k \geq 0$ such that $T_0^{(k)}(n) = 2^0 = 1$. Its unique cycle is the trivial cycle $(1 \to 2 \to 1)$ of length $2^p + 1 = 2$.*

Example 2. *For $p = 2$, we get the triplet $(d_2, \alpha_2, \beta_2)_+ = (5,6,4)_+$. The corresponding mapping is $T_2 : \mathbb{N} \longrightarrow \mathbb{N}$ given by*

$$T_2(n) = \begin{cases} n/5 & \text{if } n \equiv 0 \pmod 5 \\ \Big(6n + 4[n]_d\Big)/5 & \text{if } n \not\equiv 0 \pmod 5. \end{cases} \tag{11}$$

If the previous conjecture is true, then $(5,6,4)_+$ is an admissible strong triplet. Its unique trivial cycle is $\Omega(4) = (4 \to 8 \to 12 \to 16 \to 20 \to 4)$, of length $d_2 = 2^2+1 = 5$. For all $n \in \mathbb{N}$, $\exists k \geq 0$ such that $T_2^{(k)}(n) = 2^2 = 4$.

Let us examine some trajectories for this triplet. Starting from the integer $n = 95$ the trajectory is

$$95 \to 19 \to 26 \to 32 \to 40 \to 8 \to 12 \to 16 \to 20 \to 4.$$

But, tarting from the integer $n = 83$ the trajectory is:

$83 \to 102 \to 124 \to 152 \to 184 \to 224 \to 272 \to 328 \to 396 \to 476 \to 572 \to 688 \to$
$828 \to 996 \to 1196 \to 1436 \to 1724 \to 2072 \to 2488 \to 2988 \to 3588 \to 4308 \to 5172 \to$
$6208 \to 7452 \to 8944 \to 10736 \to 12884 \to 15464 \to 18560 \to 3712 \to 4456 \to 5348 \to$
$6420 \to 1284 \to 1544 \to 1856 \to 2228 \to 2676 \to 3212 \to 3856 \to 4628 \to 5556 \to$
$6668 \to 8004 \to 9608 \to 11532 \to 13840 \to 2768 \to 3324 \to 3992 \to 4792 \to 5752 \to$
$6904 \to 8288 \to 9948 \to 11940 \to 2388 \to 2868 \to 3444 \to 4136 \to 4964 \to 5960 \to 1192 \to$
$1432 \to 1720 \to 344 \to 416 \to 500 \to 100 \to 20 \to 4.$

Figure1 presents the trajectories starting from 95 and from 83, respectively.

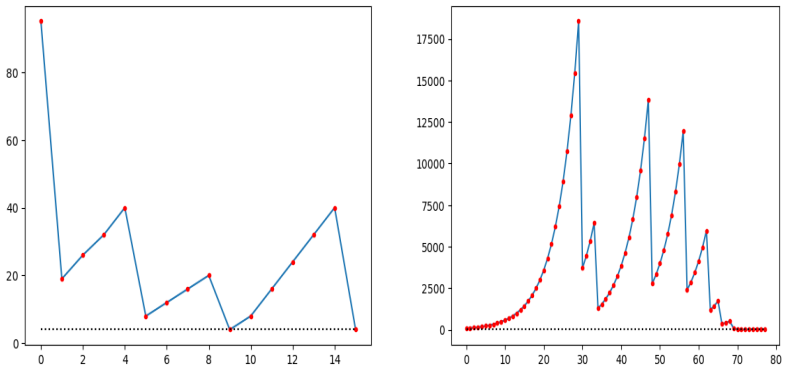

Fig. 1 Graphs of the trajectories stating from $n = 95$ (left) and from $n = 83$ (right).

Example 3. *For $p = 25$, we get the triplet $\big(33445533, 33445534, 33445532\big)_+$. The corresponding mapping $T_{25} : \mathbb{N} \longrightarrow \mathbb{N}$ is given by*

$$T_{25}(n) = \begin{cases} n/33554433 & \text{if } n \equiv 0 \pmod{33554433} \\ \Big(33554434n + 33554432[n]_d\Big)/33554433 & \text{if } n \not\equiv 0 \pmod{33554433}. \end{cases} \tag{12}$$

If the previous conjecture is true, then $\big(33445533, 33445534, 33445532\big)_+$ is an admissible strong triplet. Its unique cycle is the following trivial cycle of length 33554433:
$\Omega(33554432) = (33554432 \to 67108864 \to 100663296 \to 134217728 \to 167772160 \to$
$201326592 \to 234881024 \to 268435456 \to 301989888 \to 335544320 \to 369098752 \to$
$402653184 \to \ldots\ldots\ldots\ldots\ldots\ldots \to 1125899571298304 \to 1125899604852736 \to$
$1125899638407168 \to 1125899671961600 \to 1125899705516032 \to 1125899739070464$
$\to 1125899772624896 \to 1125899806179328 \to 1125899839733760 \to$
$1125899873288192 \to 1125899906842624 \to 1125899940397056 \to 33554432).$
and for all $n \in \mathbb{N}$ there exists $k \geq 0$ such that $T_{25}^{(k)}(n) = 2^{25} = 33554432$. We note that the maximum element of the trivial cycle $\Omega(33554432)$ is 1125899940397056.

3 Another nice conjecture and its generalization

In this section, we introduce a new conjecture, which behaves like the Collatz one. This conjecture may be stated as following.

Conjecture 3. *Starting with any positive integer $n \geq 1$:*

- *If n is a multiple of 10 then remove all the zeros on the right.*
- *Otherwise, multiply it by 12, add 8 times its last digit.*
- *Repeat the process infinitely.*

Then, regardless the starting number, the process eventually reaches 4 after a finite number of iterations.

It is clear that the last conjecture is related to the following operator $\widetilde{T} : \mathbb{N} :\longrightarrow \mathbb{N}$ given by

$$\widetilde{T}(n) = \begin{cases} n/10 & \text{if } n \equiv 0 \pmod{10} \\ 12n + 8[n]_{10} & \text{if } n \not\equiv 0 \pmod{10}. \end{cases} \tag{13}$$

The remainder $[n]_{10}$ in the Euclidian division of n by 10 is exactly equal to the last digit of the number n in the decimal base. We observe that the conjecture 3 looks like the classical Collatz conjecture written in the binary base which may be reformulated as following:

Conjecture 4. *In the binary base, starting with any positive integer $n \geq 1$:*

- *If n is a multiple of 10 then remove all the zeros on the right.*
- *Otherwise, multiply it by 11, add 1.*
- *Repeat the process infinitely.*

Then, regardless the starting number, the process eventually reaches 1 after a finite number of iterations.

To speed convergence we replace the operator given in (13) by the following one

$$T(n) = \begin{cases} n/10 & \text{if } n \equiv 0 \pmod{10} \\ \dfrac{12n + 8[n]_{10}}{10} & \text{if } n \not\equiv 0 \pmod{10}, \end{cases} \quad \text{or } T(n) = \begin{cases} n/10 & \text{if } n \equiv 0 \pmod{10} \\ \dfrac{6n + 4[n]_{10}}{5} & \text{if } n \not\equiv 0 \pmod{10}. \end{cases} \tag{14}$$

The conjecture 3 may be reformulated as following.

Conjecture 5. *The triplet $(10, 12, 8)_+$, associated to the operator T given by (14), is a strong admissible triplet with order one. Its unique trivial cycle is:*

$$\big(4 \to 8 \to 16 \to 24 \to 32 \to 40 \to 4\big), \tag{15}$$

and for all integer $n \geq 1$, there exists an integer $k \geq 0$ such that $T^{(k)}(n) = 4$.

For instance, starting with $n = 1$ gives the following trajectory $1 \to 2 \to 4$ and starting with $n = 20$ gives the following trajectory $20 \to 2 \to 4$. Another example by starting with $n = 75$, gives the trajectory

$$75 \to 94 \to 116 \to 144 \to 176 \to 216 \to 264 \to 320 \to 32 \to 40 \to 4.$$

Of course, the length of the trajectory is not always short. For example, for $n = 135$, we get the following trajectory

$$\begin{aligned} &135 \to 166 \to 204 \to 248 \to 304 \to 368 \to 448 \to 544 \to 656 \to 792 \to 952 \to 1144 \\ &\to 1376 \to 1656 \to 1992 \to 2392 \to 2872 \to 3448 \to 4144 \to 4976 \to 5976 \to 7176 \\ &\to 8616 \to 10344 \to 12416 \to 14904 \to 17888 \to 21472 \to 25768 \to 30928 \to 37120 \\ &\to 3712 \to 4456 \to 5352 \to 6424 \to 7712 \to 9256 \to 11112 \to 13336 \to 16008 \to 19216 \\ &\to 23064 \to 27680 \to 2768 \to 3328 \to 4000 \to 400 \to 40 \to 4. \end{aligned}$$

The maximun number reached in this trajectory is 37120. In Figure 2, the graphs represent the the trajectories starting from 75 and from 135, respectively.

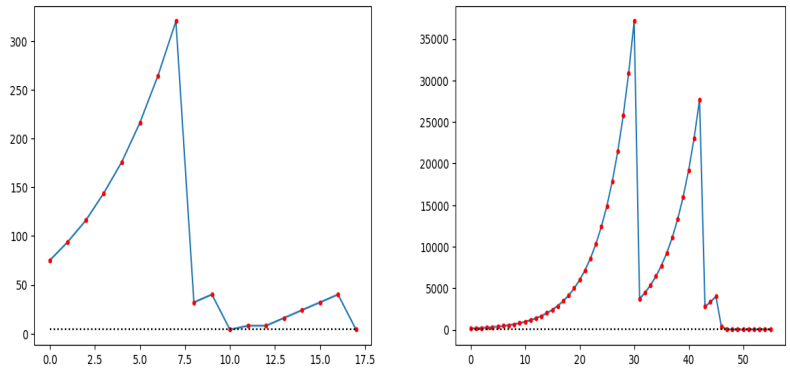

Fig. 2 Graphs of the trajectories stating from $n = 75$ (left) and from $n = 135$ (right).

We now give a general formulation of the conjecture 5 for other details, see [5]. Let $d = 2d' \geq 4$ be an even integer and choose $\lambda_1 = \nu_0 = \nu_1 = \kappa_0 = 1$, $\delta = \lambda_0 = 2$ and

$\kappa_1 = -1$ which leads, according to (4), to the triplet $(d, d+2, d-2)_+$. The mapping corresponding to this triplet is given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d}, \\ \dfrac{(d+2)n + (d-2)[n]_d}{d} & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \tag{16}$$

which may be simplified as

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ \dfrac{(d'+1)n + (d'-1)[n]_d}{d'} & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \tag{17}$$

Theorem 2. *For $d = 2d' \geq 4$ with d even integer, the admissible triplet $(d, \alpha, \beta)_+ = (d, d+2, d-2)_+$ has at least the following trivial cycle of length $\frac{1}{2}d + 1$:*

$$\Omega(\frac{1}{2}\beta) = \big(\frac{1}{2}\beta \to \beta \to 2\beta \to 3\beta \to \ldots \to d'\beta \to \frac{1}{2}\beta\big). \tag{18}$$

Proof. As $d = 2d'$, then $\alpha = 2(d'+1)$ and $\beta = 2(d'-1)$. We have obviously, $T(\frac{1}{2}\beta) = \frac{1}{2}\beta\ \dfrac{\alpha+\beta}{d} = \beta$. For $1 \leq k \leq d'-1$, we have $T(k\beta) = \dfrac{(d+2)k\beta + (d-2)[k\beta]_d}{d}$. But $k\beta = k(d-2)$, then $[k\beta]_d = [-2k]_d = d - 2k$. It follows that $T(k\beta) = \dfrac{(d+2)k\beta + (d-2)(d-2k)}{d} = (k+1)\beta$. Which gives $T^{(d'-1)}(\beta) = d'\beta$. But $d'\beta = 2d'(d'-1) = \frac{1}{2}\beta d$. It follows that $T^{(d'+1)}(\frac{1}{2}\beta) = \frac{1}{2}\beta$. which gives rise to the trivial cycle (18). □

Example 4. *For instance the admissible triplet $(12, 14, 10)_+$ has the trivial cycle $\Omega(5) = (5 \to 10 \to 20 \to 30 \to 40 \to 50 \to 60 \to 5)$ of length 7. But it has also two other cycles which are $\Omega(4) = 4 \to 8 \to 16 \to 22 \to 4)$ of length 6 and $\Omega(1305) = (1305 \to 1530 \to 1790 \to 2090 \to 2440 \to 2850 \to 3330 \to 3890 \to 4540 \to 5300 \to 6190 \to 7230 \to 8440 \to 9850 \to 11500 \to 13420 \to 15660 \to\to 1305)$ of length 17. It seems that the admissible triplet$(12, 14, 10)_+$ has only the three cycles.*

More interesting particular case is the one where $d = 2^p + 2$ with $p \geq 1$ is a positive integer. Then,

$$d = d_p := 2^p + 2, \quad \alpha = \alpha_p := d_p + 2 = 2^p + 4 \quad \beta = \beta_p := d_p - 2 = 2^p. \tag{19}$$

Thus the corresponding mapping $T := T_p$, for $p \geq 1$, is given by

$$T_p(n) = \begin{cases} n/(2^p+2) & \text{if } n \equiv 0 \pmod{(2^p+2)} \\ \dfrac{(2^{p-1}+2)n + 2^{p-1}[n]_p}{(2^{p-1}+1)} & \text{if } n \not\equiv 0 \pmod{(2^p+2)}, \end{cases} \tag{20}$$

here $[\ ]_p = [\ ]_{d_p}$ stands for the remainder in the Euclidean division by $d_p = 2^p + 2$. We observe that the case $p = 3$ corresponds to the case $d_3 = 10$, $\alpha_3 = 12$ and $\beta_p = 8$ which is exactly the case of our conjecture 5. We now formulate the following general main conjecture also obtained according to our computational experiences and tests supported by computer. We believe that the following general conjecture is true.

Conjecture 6. *1. For all integer $p \geq 1$ with $p \neq 2$, the triplet $(2^p + 2, 2^p + 4, 2^p)_+$ is an admissible strong triplet of order one, its unique cycle is the trivial cycle of length $2^{p-1} + 2$:*

$$\Omega(2^{p-1}) = \Big(2^{p-1} \to 2^p \to 2 \cdot 2^p \to 3 \cdot 2^p \to \ldots \to (2^{p-1}+1) \cdot 2^p \to 2^{p-1}\Big).$$

Then, for all integer $n \geq 1$, there exists an integer $k \geq 0$ such that $T_p^{(k)}(n) = 2^{p-1}$.

2. *If $p = 2$, the triplet $(6, 8, 4)_+$ is an admissible strong triplet of order two. Its first trivial cycle is $\Omega(2)$ of length 4 given as:*

$$\Omega(2) = \Big(2 \to 4 \to 8 \to 12 \to 2\Big),$$

Its second trivial cycle is the cycle $\Omega(74)$ of length 7 given as:

$$\Omega(74) = \big(74 \to 100 \to 136 \to 184 \to 248 \to 332 \to 444 \to 74\big).$$

Then, for all integer $n \geq 1$, there exists an integer $k \geq 0$ such that $T_2^{(k)}(n) \in \{2, 74\}$.

The last Conjecture 6 encompass our conjecture 5 obtained for $p = 3$ and associated to the operator given in (20). But, it does not encompass the classical Collatz conjecture. However, we can derive a modified version of the classical Collaz conjecture for $p = 1$, we get $(d, \alpha, \beta)_+ = (4, 6, 2)_+$, which gives the map T_1 as following:

$$T_1(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod 4 \\ \dfrac{3n + [n]_4}{2} & \text{if } n \not\equiv 0 \pmod 4, \end{cases} \quad \text{or} \quad T_1(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod 4 \\ \dfrac{3n+1}{2} & \text{if } n \equiv 1 \pmod 4, \\ \dfrac{3n+2}{2} & \text{if } n \equiv 2 \pmod 4, \\ \dfrac{3n+3}{2} & \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{21}$$

In this case where the previous conjecture is true, then the triplet $(4, 6, 2)_+$ is a strong admissible triplet of order one. Its unique trivial cycle is $(1 \to 2 \to 4 \to 1)$ and for all $n \geq 1$, there exists an integer $k \geq 0$ such that $T_1^{(k)}(n) = 2^{p-1} = 1$.

Let $p \geq 1$ be a positive integer and consider the following operator $T_p : \mathbb{N} :\longrightarrow \mathbb{N}$ given by

$$T_p(n) = \begin{cases} \dfrac{n}{2^p + 2} & \text{if } n \equiv 0 \pmod{d_p} \\ \dfrac{(2^p + 4)n + 2^p[n]_{d_p}}{2^p + 2} & \text{if } n \not\equiv 0 \pmod{d_p}. \end{cases} \tag{22}$$

The map map T_p may be written in the following form

$$T_p(n) = \begin{cases} \dfrac{n}{2^p + 2} & \text{if } n \equiv 0 \pmod{d_p} \\ \dfrac{(2^{p-1} + 2)n + 2^{p-1}[n]_{d_p}}{2^{p-1} + 1} & \text{if } n \not\equiv 0 \pmod{d_p}. \end{cases} \tag{23}$$

4 Verification and tests

In this section, we discuss verification concerning the main given conjecture 2. The verification of the conjecture 1 consists of performing an algorithm that looks like the algorithm 1. Clearly, if the conjecture 2 is not true for an integer $p \leq p_{max}$, then the while-loop in the Algorithm 1 never terminates for an integer n such that $1 \leq n \leq n_{max}$ where p_{max} and n_{max} are a great integers test. We have tested the Algorithm 1 with `Matlab`, `Python`, `SageMaths` and `Mathematica` by using 3 laptop computers. It was possible to work in parallel. With `Matlab`, we used Parallel Computing with 8 cores workers. Now, we may state that this conjecture is true from $p = 0$ up to $p = p_{max} = 25$ and starting by any integer n from $n = 1$ up to $n = n_{max} := 10^6$. For the fixed value $p = 2$, we took the tests further by changing the strategy. Instead of checking whether a trajectory reaches the terminal value 4, we checked whether for any starting integer n in an interval (N_1, N_2) there exists a number k of iterations such that $T_2^{(k)}(n) < n$ which led to Algorithm 2. This was faster than checking whether the trajectory reaches the cycle. In our test we took $N_1 = 10^6 + 1$ and we have reached $N_2 = 2^{40} = 1099511627776$. The speed was about 1.5×10^5 numbers per second and the elapsed time was about 7.33×10^6 seconds which is about 85 days. We may assert that our conjecture is true for $p = 2$ and for all n from 1 to $N_2 = 2^{40} = 1099511627776$.

Algorithm 1 Algorithm for testing conjecture 2.

Choose p_{max} and n_{max}
for $p = 0, \ldots, p_{max}$ **do**
  $d = 2^p + 1$; $\alpha = d + 1 = 2^p + 2$; $\beta = d - 1 = 2^p$;
  $S = \{\beta\}$
  if $p == 1$ **then**
    $S = \{\beta, 14\}$
  else
    if $p == 3$ **then**
      $S = \{\beta, 28\}$
    else
      if $p == 4$ **then**
        $S = \{\beta, 1264\}$
      end if
    end if
  end if
  for $n = 1, \ldots, n_{max}$ **do**
    $m = n$
    while $m \notin S$ **do**
      if $m \equiv 0 \pmod{d}$ **then**
        m =m/d
      else
        $m = (\alpha m + \beta[m]_d)/d$
      end if
    end while
  end for
end for

Algorithm 2 Algorithm for testing conjecture 2 for $p = 2$ we have $(d\alpha, \beta) = (5, 6, 4)$.

Choose N_1 and N_2, with $N_1 < N_2$
$d = 5$; $\alpha = d + 1 = 6$; $\beta = d - 1 = 4$;
for $n = N_1, \ldots, N_2$ **do**
  $m = n$
  while $m \geq n$ **do**
    if $m \equiv 0 \pmod{5}$ **then**
      m =m/5
    else
      $m = (6m + 4[m]_5)/5$
    end if
  end while
end for

5 Backward mapping and graphs

In this section we will give parts of the graphs representing the map T_p given by (10) for $p = 0, 1, 2$. For $p = 0$ the graph of the map T_0 is well known. To get the graph of the map T_p, it is crucial to give the backward mapping T_p^{-1}. The expression of T_p^{-1} for $p = 0, 1, 2$ may be obtained in each case as following:

$$T_0^{-1}(n) = \begin{cases} \{2n\} & \text{if } n \equiv 0 \text{ or } 1 \pmod 3, \\ \{2n, (2n-1)/3\} & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{24}$$

$$T_1^{-1}(n) = \begin{cases} \{3n, (3n-4)/4\} & \text{if } n \equiv 0 \pmod 4, \\ \{3n\} & \text{if } n \equiv 1 \text{ or } 3 \pmod 4, \\ \{3n, (3n-2)/4\} & \text{if } n \equiv 2 \pmod 4, \end{cases} \tag{25}$$

and

$$T_2^{-1}(n) = \begin{cases} \{5n, (5n-12)/6\} & \text{if } n \equiv 0 \pmod 6, \\ \{5n\} & \text{if } n \equiv 1 \text{ or } 3 \text{ or } 5 \pmod 6, \\ \{5n, (5n-4)/6, (5n-16)/6\} & \text{if } n \equiv 2 \pmod 6, \\ \{5n, (5n-8)/6\} & \text{if } n \equiv 4 \pmod 6. \end{cases} \tag{26}$$

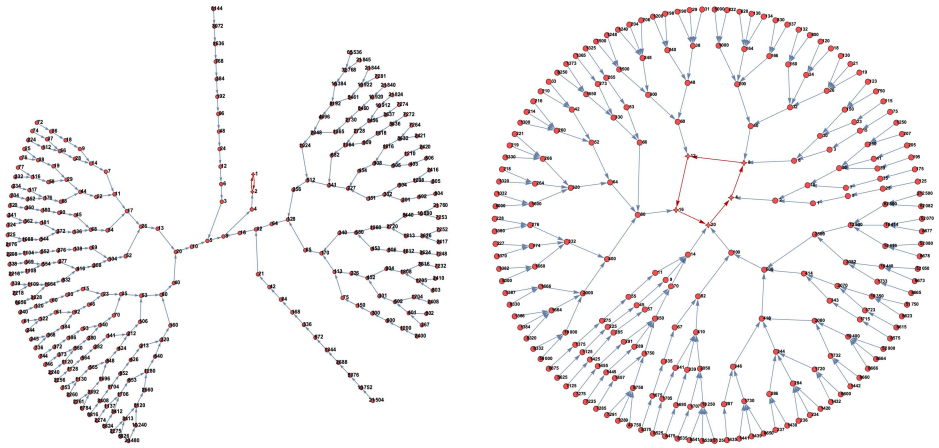

Fig. 3 Graph of a parts of the tree with its unique cycle corresponding to the triplet $(2, 3, 1)_+$ (left) and to triplet $(5, 6, 4)_+$.

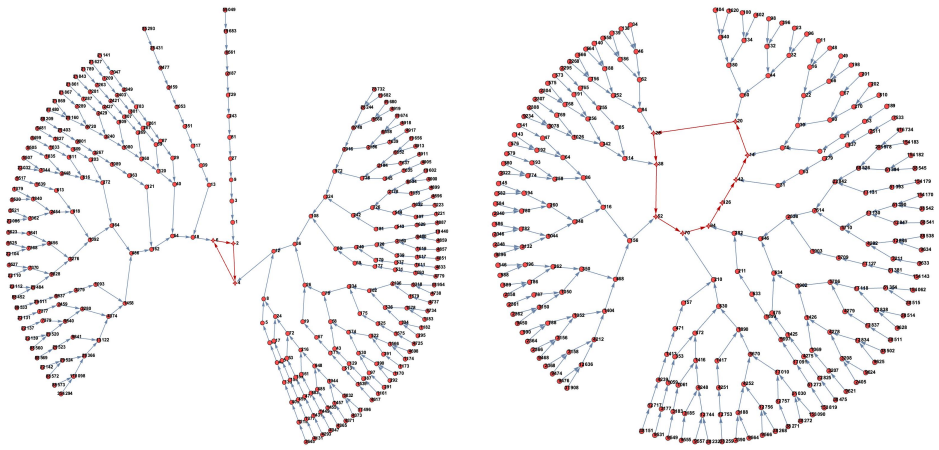

Fig. 4 Graph of parts of the tree corresponding to the triplet $(3, 4, 2)_+$, with its first cycle (right) and with its second cycle (right).

6 Conclusion

In this paper, we have introduced two general conjectures which behave like the classical Collatz conjecture. We have analyzed such conjectures and presented some examples. We have discussed the verification and we have presented some graphs. The main idea behind this paper is the introduction of such new and interesting conjectures.

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