SUPPLEMENTARY MATERIAL TO: PROJECTION ESTIMATION OF A QUADRATIC FUNCTIONAL FROM INDIRECT OBSERVATIONS

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ABSTRACT. We consider the convolution model: $Y = X + \varepsilon$, where X and ε are independent. We aim to estimate $\int_{\mathbb{R}} f^2(x)dx$, where f is the unknown density of the signal X from n observations of Y. We introduce a new projection estimator based on expanding f in the Hermite basis. Convergence rates for f within the Sobolev-Hermite ball are provided for various error types. We also present an adaptive procedure inspired by Goldenshluger and Lepski (2011) to select the appropriate space, and we demonstrate an oracle inequality for the adaptive estimator. Numerical experiments are conducted to illustrate the effectiveness of our methodology.

Key words: Deconvolution; Hermite basis; Model selection; Projection estimator; Quadratic functional; U-statistics.

APPENDIX A. PROOF OF AUXILIARY RESULTS

A.1. **Proof of Lemma 7.1.** Let $\widetilde{Y}_k^{(j)}$ be a real random variable defined as:

$$\widetilde{Y}_k^{(j)} = U^*_{\varphi_j}(Y_k) - \mathbb{E}[U^*_{\varphi_j}(Y_k)],$$

where U_t is defined in (10). The variables $(\widetilde{Y}_k^{(j)})_{k\in\{1,\dots,n\}}$ are i.i.d. with $\mathbb{E}[\widetilde{Y}_k^{(j)}] = 0$ and $\mathbb{E}[(\widetilde{Y}_k^{(j)})^2] = \mathrm{Var}(\widetilde{Y}_k^{(j)})$. By definition of $T_{1,j}$ given (24), we have

$$\begin{aligned}
\mathrm{Cov}(T_{1,j}, T_{1,j'}) = \mathbb{E}[T_{1,j}T_{1,j'}] &= \frac{1}{n^2(n-1)^2}\sum_{k\neq\ell=1,k'\neq l'=1}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_{k'}^{(j')}\widetilde{Y}_{\ell'}^{(j')}\right] \\
&= \frac{1}{n^2(n-1)^2}\Big(\sum_{k\neq\ell=1,k'\neq l'=1,k=k',k\neq\ell'}^{n}\mathbb{E}[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_{k'}^{(j')}\widetilde{Y}_{\ell'}^{(j')}] + \sum_{k\neq\ell=1,k'\neq l'=1,k\neq k',k=\ell'}^{n}\mathbb{E}[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_{k'}^{(j')}\widetilde{Y}_{\ell'}^{(j')}]\Big) \\
&= \frac{2}{n^2(n-1)^2}\sum_{k\neq\ell=1,k'\neq l'=1,k\neq\ell'}^{n}\mathbb{E}[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_k^{(j')}\widetilde{Y}_{\ell'}^{(j')}] \\
&= \frac{2}{n^2(n-1)^2}\sum_{k\neq\ell=1,k'\neq l'=1,k\neq\ell',\ell=\ell'}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_k^{(j')}\widetilde{Y}_{\ell'}^{(j')}\right] \\
&= \frac{2}{n^2(n-1)^2}\sum_{k\neq\ell=1}^{n}\mathbb{E}[\widetilde{Y}_k^{(j)}\widetilde{Y}_k^{(j')}]\,\mathbb{E}\left[\widetilde{Y}_\ell^{(j)}\widetilde{Y}_\ell^{(j')}\right] = \frac{2}{n^2(n-1)^2}\sum_{k\neq\ell=1}^{n}\mathbb{E}[\widetilde{Y}_1^{(j)}\widetilde{Y}_1^{(j')}]\,\mathbb{E}[\widetilde{Y}_1^{(j)}\widetilde{Y}_1^{(j')}] \\
&= \frac{2}{n(n-1)}\left(\mathbb{E}\left[\left(U^*_{\varphi_j}(Y_1) - \mathbb{E}[U^*_{\varphi_j}(Y_1)]\right)\left(U^*_{\varphi_{j'}}(Y_1) - \mathbb{E}[U^*_{\varphi_{j'}}(Y_1)]\right)\right]\right)^2 \\
&= \frac{2}{n(n-1)}\left(\mathrm{Cov}(U^*_{\varphi_j}(Y_1), U^*_{\varphi_{j'}}(Y_1))\right)^2.
\end{aligned}$$

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By recalling that $\mathbb{E}[\widetilde{Y}_k^{(j)}] = a_j(f)$ by construction, it holds

$$\mathbb{E}[T_{1,j}T_{2,j'}] = \frac{2}{n^2(n-1)}a_{j'}(f)\sum_{k\neq\ell=1,l'=1}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_{\ell'}^{(j')}\right]$$
$$= \frac{2}{n^2(n-1)}a_{j'}(f)\left(\sum_{k\neq\ell=1,\ell'=1,\ell=\ell'}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_{\ell'}^{(j')}\right] + \sum_{k\neq\ell=1,\ell'=1,\ell\neq\ell'}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j)}\widetilde{Y}_{\ell'}^{(j')}\right]\right) = 0.$$

We have

$$\mathbb{E}[T_{2,j}T_{2,j'}] = \frac{4}{n^2}a_j(f)a_{j'}(f)\sum_{k=1,\ell=1}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_\ell^{(j')}\right] = \frac{4}{n^2}a_j(f)a_{j'}(f)\sum_{k=1}^{n}\mathbb{E}\left[\widetilde{Y}_k^{(j)}\widetilde{Y}_k^{(j')}\right]$$
$$= \frac{4}{n}a_j(f)a_{j'}(f)\,\mathbb{E}\left[\left(U^*_{\varphi_j}(Y_1) - \mathbb{E}[U^*_{\varphi_j}(Y_1)]\right)\left(U^*_{\varphi_{j'}}(Y_1) - \mathbb{E}[U^*_{\varphi_{j'}}(Y_1)]\right)\right]$$
$$= \frac{4}{n}a_j(f)a_{j'}(f)\mathrm{Cov}(U^*_{\varphi_j}(Y_1), U^*_{\varphi_{j'}}(Y_1)).$$

Lemma 7.1 is therefore proved.

A.2. **Proof of Lemma 3.1.** Let us begin by Part (ii). By Plancherel-Parseval Equality, we have

$$\mathbb{E}[|U^*_{f_m}(Y_1)|^2] = \int |U^*_{f_m}(z)|^2 f_Y(z)dy \leqslant \|f_Y\|_\infty \frac{1}{2\pi}\int \frac{|f^*_m(-u)|^2}{|f^*_\varepsilon(u)|^2}du$$
$$= \|f_Y\|_\infty\left(\frac{1}{2\pi}\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f^*_m(-u)|^2}{|f^*_\varepsilon(u)|^2}du + \frac{1}{2\pi}\int_{|u|>\sqrt{\rho m}}\frac{|f^*_m(-u)|^2}{|f^*_\varepsilon(u)|^2}du\right). \tag{A1}$$

Using the Cauchy Schwarz inequality on the sum, (5) and as $\|f\|^2 \leqslant \|f\|_\infty$, we have for the second integral

$$\int_{|u|>\sqrt{\rho m}}\frac{|f^*_m(-u)|^2}{|f^*_\varepsilon(u)|^2}du \leqslant 2\pi\sum_{j=0}^{m-1}a_j^2(f)\sum_{j=0}^{m-1}\int_{|u|>\sqrt{\rho m}}\frac{|\varphi_j(u)|^2}{|f^*_\varepsilon(u)|^2}du \leqslant 2\pi\|f\|_\infty\Lambda(m). \tag{A2}$$

Parseval's equality and $\|f\|^2 \leqslant \|f\|_\infty$ imply that

$$\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f^*_m(-u)|^2}{|f^*_\varepsilon(u)|^2}du \leqslant 2\pi\Delta^{(1)}(m)\|f\|_\infty.$$

It follows that

$$\mathbb{E}[|U^*_{f_m}(Y_1)|^2] \leqslant \|f\|_\infty\|f_Y\|_\infty\left(\Delta^{(1)}(m) + \Lambda(m)\right) = C_0'\left(\Delta^{(1)}(m) + \Lambda(m)\right),$$

where $C_0' = \|f\|_\infty\|f_Y\|_\infty$. This gives Part (ii).
Now, we demonstrate Part (i). The difference with Part (i) is related to the study of the first integral given in (A1). We have

$$\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f^*_m(-u)|^2}{|f^*_\varepsilon(u)|^2}du \leqslant 2\int_{|u|\leqslant\sqrt{\rho m}}\frac{|(f_m-f)^*(-u)|^2}{|f^*_\varepsilon(u)|^2}du + 2\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f^*(-u)|^2}{|f^*_\varepsilon(u)|^2}du$$
$$\leqslant 4\pi\Delta^{(1)}(m)\|f-f_m\|^2 + 2\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f^*(-u)|^2}{|f^*_\varepsilon(u)|^2}du.$$

Injecting this and (A2) in (A1), we get Part (i) and then the announced result.

A.3. **Proof of Proposition 7.2.**

A.3.1. *Proof of Part (i).* We use the following Lemma, which is proved in the sequel.

Lemma A.1. *Let $U_n(I_m)$ be defined in (36). Assume that $\|f\|_\infty < \infty$, Assumption (H5) hold and f_ε is square integrable. There exists a constant $C_\star > 0$ depending on $\|f\|_\infty, \|f_Y\|_\infty$, and f_ε such that for any $x > 0$,*

$$\mathbb{P}\left\{|U_n(I_m)| > \frac{C_\star}{n-1}\left(\sqrt{\Delta^{(1)}(m)\Delta(m)x} + \left(\sqrt{\Delta^{(1)}(m)\Delta(m)} + \Delta^{(1)}(m)\right)x + \frac{\Delta(m)x^2}{n}\right)\right\} \leqslant 5.54\exp(-x), \tag{A3}$$

where $\Delta(m)$ is defined in (12) and $\Delta^{(1)}(m)$ in (13).

Choose $x = 4\log(n)$ and set

$$\Omega_1(m) := \left\{|U_n(I_m)| \leqslant \frac{C_\star}{n-1}\left(\begin{array}{c}\sqrt{\Delta^{(1)}(m)\Delta(m)4\log(n)} + \left(\sqrt{\Delta^{(1)}(m)\Delta(m)} + \Delta^{(1)}(m)\right)4\log(n) \\ + \frac{\Delta(m)16\log^2(n)}{n}\end{array}\right)\right\}.$$

Let us write that

$$\begin{aligned}\mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\left\{\left(U_n^2(I_m) - \frac{\kappa_1}{36}V(m)\right)_+\right\}\right] &\leqslant \mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\left\{\left(U_n^2(I_m) - \frac{\kappa_1}{36}V_1(m)\right)_+\right\}\mathbb{1}_{\Omega_1(m)}\right] = \mathbb{E}_1 \\ &\quad + \mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\left\{\left(U_n^2(I_m) - \frac{\kappa_1}{36}V_1(m)\right)_+\right\}\mathbb{1}_{\Omega_1^c(m)}\right] = \mathbb{E}_2\,.\end{aligned}$$

On $\Omega_1(m)$ and as $m \in \mathcal{M}_n$, we have for $C'_\star > 0$ a numerical constant $U_n^2(I_m) \leqslant C'_\star V_1(m)$, where $V_1(m)$ is defined in (18). Indeed, it is easy to check $\forall m \in \mathcal{M}_n$

$$\frac{\Delta(m)^2\log^4(n)}{n^2(n-1)^2} \leqslant 4\rho m\frac{\Delta^{(1)}(m)^2\log^4(n)}{n^2(n-1)^2} \lesssim \frac{\log^2(n)}{n(n-1)},$$

and

$$\begin{aligned}&\left(\sqrt{\Delta^{(1)}(m)\Delta(m)4\log(n)} + \left(\sqrt{\Delta^{(1)}(m)\Delta(m)} + \Delta^{(1)}(m)\right)4\log(n)\right)^2\frac{1}{(n-1)^2} \\ &\qquad \lesssim \Delta^{(1)}(m)\Delta(m)\frac{\log^2(n)}{n(n-1)}.\end{aligned}$$

Thus, for $\kappa_1 \geqslant 36C'_\star$, it yields $\left(U_n^2(I_m) - \frac{\kappa_1}{36}V_1(m)\right) \leqslant 0$ and $\mathbb{E}_1$ is equal to zero.
For the other term, it holds

$$\mathbb{E}_2 \leqslant \sum_{m\in\mathcal{M}_n}\mathbb{E}\left[U_n^2(I_m)\mathbb{1}_{\Omega_1^c(m)}\right]. \tag{A4}$$

Moreover, by definition of $U_n(I_m)$ given in (36) and the Cauchy-Schwarz inequality, we have

$$|U_n(I_m)| \leqslant \frac{1}{n(n-1)}\sum_{k\neq\ell=1}^{n}\sqrt{\sum_{j=0}^{m-1}|v_{\varphi_j}(Y_k) - \mathbb{E}[v_{\varphi_j}(Y_k)]|^2}\sqrt{\sum_{j=0}^{m-1}|v_{\varphi_j}(Y_\ell) - \mathbb{E}[v_{\varphi_j}(Y_\ell)]|^2},$$

where v_t is given in (34) for $t \in S_m$. It yields by Bessel's inequality

$$\sum_{j=0}^{m-1}\left|v_{\varphi_j}(Y_k)\right|^2 \leqslant \frac{\Delta(m)}{2\pi}, \tag{A5}$$

by (5) and as $\|f\|_\infty < \infty$: $\left|\mathbb{E}[v_{\varphi_j}(Y_k)]\right|^2 \leqslant \|f\|_\infty$. It comes that

$$|U_n(I_m)| \leqslant 2\left(\frac{\Delta(m)}{2\pi} + \|f\|_\infty\right) \leqslant (1 \vee 4\|f\|_\infty)\Delta(m).$$

Injecting this in (A4), using $\Delta(m) \leqslant 2\sqrt{\rho m}\Delta^{(1)}(m)$ and by taking $x = 4\log(n)$ in (A3), we deduce that

$$\mathbb{E}_2 \leqslant 4\rho(1 \vee 4\|f\|_\infty)^2 \frac{n^2}{\log^2(n)} \sum_{m\in\mathcal{M}_n} m\mathbb{P}\left(\Omega_1^c(m)\right) \leqslant 4\rho\frac{(1 \vee 4\|f\|_\infty)^2}{n},$$

since $m \leqslant \sqrt{n}$ by assumption on $\mathcal{M}_n$. This gives Part (i) of Proposition 7.2.

A.3.2. *Proof of Part (ii).* We start by the centered process $\nu_{n,1}(\varphi_j)$ defined in (32). By the Cauchy-Schwarz inequality, it holds

$$\begin{aligned}
\mathbb{E}\left[\sup_{m\in\mathcal{M}_n} \nu_{n,1}^2(\varphi_j)\right] &\leqslant \sum_{m\in\mathcal{M}_n} \mathrm{Var}(\nu_{n,1}(\varphi_j)) \\
&= \frac{4}{n^2(n-1)^2} \sum_{m\in\mathcal{M}_n} \sum_{k\neq\ell=1}^{n} \mathrm{Var}\left(\sum_{j=0}^{m-1} (v_{\varphi_j}(Y_k) - \mathbb{E}[v_{\varphi_j}(Y_k)])\Psi_{\varphi_j}(Y_\ell)\right) \\
&= \frac{1}{n^2(n-1)^2} \sum_{m\in\mathcal{M}_n} \sum_{k\neq\ell=1}^{n} \mathbb{E}\left[\left|\sum_{j=0}^{m-1} (v_{\varphi_j}(Y_k) - \mathbb{E}[v_{\varphi_j}(Y_k)])\Psi_{\varphi_j}(Y_\ell)\right|^2\right] \\
&\leqslant \frac{1}{n^2(n-1)^2} \sum_{m\in\mathcal{M}_n} \sum_{k\neq\ell=1}^{n} \sum_{j=0}^{m-1} \mathbb{E}\left[\left|v_{\varphi_j}(Y_k) - \mathbb{E}[v_{\varphi_j}(Y_k)]\right|^2\right] \sum_{j=0}^{m-1} \mathbb{E}\left[\left|\Psi_{\varphi_j}(Y_\ell)\right|^2\right].
\end{aligned}$$

Thanks to (A5), it yields

$$\sum_{j=0}^{m-1} \mathbb{E}\left[\left|v_{\varphi_j}(Y_k) - \mathbb{E}[v_{\varphi_j}(Y_k)]\right|^2\right] \leqslant \sum_{j=0}^{m-1} \mathbb{E}\left[|v_{\varphi_j}(Y_k)|^2\right] \leqslant \frac{\Delta(m)}{2\pi}.$$

As the integral $\int_{|u|>\sqrt{\rho m}} \frac{e^{-\xi u^2}}{|f_\varepsilon^*(u)|^2}du$ is convergent if f_ε is ordinary smooth or super smooth if $\delta < 2$ or $\delta = 2$ with $\mu \leqslant \xi$ and, by definition of $\Psi_{\varphi_j}(Y_\ell)$ given (35) and (4), it follows that

$$\begin{aligned}
\mathbb{E}\left[\left|\Psi_{\varphi_j}(Y_\ell)\right|^2\right] &\leqslant \mathbb{E}\left[\left|\int_{|u|>\sqrt{\rho m}} e^{iuY_\ell} \frac{\varphi_j(u)}{f_\varepsilon^*(u)}du\right|^2\right] \\
&\leqslant C_\infty\, \mathbb{E}\left[\int_{|u|>\sqrt{\rho m}} \frac{e^{-\xi u^2}}{|f_\varepsilon^*(u)|^2}du \int_{|u|>\sqrt{\rho m}} e^{-\xi u^2}du\right] \\
&\lesssim e^{-\xi m} \int_{|u|>\sqrt{\rho m}} \frac{e^{-\xi u^2}}{|f_\varepsilon^*(u)|^2}du \lesssim e^{-\xi m}. \qquad \text{(A6)}
\end{aligned}$$

Consequently, we get,

$$\mathbb{E}\left[\sup_{m\in\mathcal{M}_n} \nu_{n,1}^2(\varphi_j)\right] \lesssim \frac{1}{n(n-1)} \sum_{m\in\mathcal{M}_n} m^2\Delta(m)e^{-\xi m} \lesssim \frac{1}{n}\sum_{m=1}^{+\infty} e^{-\frac{\xi m}{2}} \lesssim \frac{1}{n},$$

because $\Delta(m) \leqslant 2\sqrt{\rho m}\Delta^{(1)}(m) \leqslant \frac{2\sqrt{\rho m}}{n}$ by definition of $\mathcal{M}_n$.

Let us now turn to the control of $\mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\nu_{n,2}^2(\varphi_j)\right] \leqslant \sum_{m\in\mathcal{M}_n}\mathrm{Var}(\nu_{n,2}(\varphi_j))$ with $\nu_{n,2}(\varphi_j)$ defined in (38). Similarly to the study of $\nu_{n,1}(\varphi_j)$, we have

$$\mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\nu_{n,1}^2(\varphi_j)\right] \leqslant \frac{1}{n^2(n-1)^2}\sum_{m\in\mathcal{M}_n}\sum_{k\neq\ell=1}^{n}\sum_{j=0}^{m-1}\mathbb{E}\left[\left|\Psi_{\varphi_j}(Y_k)\right|^2\right]\sum_{j=0}^{m-1}\mathbb{E}\left[\left|\Psi_{\varphi_j}(Y_\ell)\right|^2\right].$$

Equation (A6) imply that

$$\mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\nu_{n,1}^2(\varphi_j)\right] \leqslant \frac{1}{n^2(n-1)^2} \lesssim \frac{1}{n(n-1)}\sum_{m=1}^{+\infty}e^{-\xi m} \lesssim \frac{1}{n(n-1)},$$

which ends the proof of Part (ii).

A.4. **Proof of Lemma A.1.** In order to prove this result, we use an exponential inequality for U-statistic of order 2. Let $x>0$. Applying Theorem 3.4.8 given in Giné and Nickl (2016) (see Section B.2), we have for C an absolute constant

$$\mathbb{P}\left\{n(n-1)|U_n(I_m)| > C\left(A_1\sqrt{x}+A_2x+A_3x^{\frac{3}{2}}+A_4x^2\right)\right\} \leqslant 5.54e^{-x},$$

where the constants $(A_i)_{i\in1,\dots,4}$ are given by:

$$\begin{aligned}
A_1^2 &= \sum_{j=2}^{n}\sum_{i=1}^{j-1}\mathbb{E}\left[I_m^2(Y_i,Y_j)\right]\\
A_2 &= \sup\left\{\mathbb{E}\left[\sum_{j=2}^{n}\sum_{i=1}^{j-1}I_m(Y_i,Y_j)\alpha_i(Y_i)\beta_j(Y_j)\right], \mathbb{E}[\sum_{i=1}^{n-1}\alpha_i^2(Y_i)]\leqslant 1, \mathbb{E}[\sum_{j=2}^{n}\beta_j^2(Y_j)]\leqslant 1\right\},\\
A_3^2 &= \max\left(\max_j\sup_x\left\{\sum_{i=1}^{j-1}\mathbb{E}_{Y_i}[I_m^2(Y_i,x)]\right\}, \max_i\sup_x\left\{\sum_{j=i+1}^{n}\mathbb{E}_{Y_j}[I_m^2(x,Y_j)]\right\},\right)\\
A_4 &= \sup_{x,y}|I_m(x,y)|,
\end{aligned}$$

with I_m is given in (36). As the map $(x,y)\mapsto I_m(x,y)$ is symetric in its entries (that is $I_m(x,y) = I_m(y,x)$)) and the random variables $Y_1,\dots,Y_n$ are i.i.d, the first three constants becomes (see also Giné and Nickl (2016), Section 3.4.3, pp. 176):

$$\begin{aligned}
A_1^2 &= \frac{n(n-1)}{2}\mathbb{E}[|I_m(Y_1,Y_2)|^2],\\
A_2 &= \frac{n}{2}\sup\left\{\mathbb{E}\left[I_m(Y_1,Y_2)\alpha(Y_1)\beta(Y_2)\right], \mathbb{E}[\alpha^2(Y_1)]\leqslant 1, \mathbb{E}[\beta^2(Y_2)]\leqslant 1\right\}\\
A_3^2 &= (n-1)\sup_x\left\{\mathbb{E}_{Y_1}[I_m^2(Y_1,x)]\right\} = (n-1)\sup_x\left\{\mathbb{E}_{Y_2}[I_m^2(x,Y_2)]\right\}.
\end{aligned}$$

Let us now compute these terms.

- **Computing of A_1.** Notice that for any real-valued function t, the function $x\mapsto v_t(x)$ (defined in (34)) is also real-valued. We have

$$\begin{aligned}
\mathbb{E}\left[|I_m(Y_1,Y_2)|^2\right] &= \sum_{j,k=0}^{m-1}\left\{\mathbb{E}\left[\left(v_{\varphi_j}(Y_1)-\mathbb{E}[v_{\varphi_j}(Y_1)]\right)\left(\overline{v_{\varphi_k}(Y_1)}-\overline{\mathbb{E}[v_{\varphi_k}(Y_1)]}\right)\right]\right\}^2\\
&\leqslant 2\sum_{j,k=0}^{m-1}\left|\mathbb{E}\left[v_{\varphi_j}(Y_1)v_{\varphi_k}(Y_1)\right]\right|^2 + 2\sum_{j,k=0}^{m-1}\left|\mathbb{E}[v_{\varphi_j}(Y_1)]\,\mathbb{E}[v_{\varphi_k}(Y_1)]\right|^2
\end{aligned}$$

Using the definition of v_t given in (34), as $\varphi_j^* = \sqrt{2\pi}(i^j)\varphi_j$ (see (5)) and Bessel's inequality (applied twice), it yields

$$\sum_{j,k=0}^{m-1} \left|\mathbb{E}\left[v_{\varphi_j}(Y_1)v_{\varphi_k}(Y_1)\right]\right|^2 = \frac{1}{2\pi}\sum_{j,k=0}^{m-1}\left|\int\int \mathbb{E}[e^{iuY_1+ivY_1}]\frac{\varphi_j(u)}{f_\varepsilon^*(u)}\frac{\varphi_j(v)}{f_\varepsilon^*(v)}\mathbb{1}_{|u,v|\leqslant\sqrt{\rho m}}dudv\right|^2$$
$$\leqslant \frac{1}{2\pi}\int\int \frac{|f_Y^*(u+v)|^2}{|f_\varepsilon^*(u)f_\varepsilon^*(v)|^2}\mathbb{1}_{|u,v|\leqslant\sqrt{\rho m}}dudv.$$

Since $\int |f_Y^*(u)|du \leqslant 2\pi\|f\|\|f_\varepsilon\|$ and $|f_Y^*(u)| \leqslant 1$, we get

$$\sum_{j,k=0}^{m-1} \left|\mathbb{E}\left[v_{\varphi_j}(Y_1)v_{\varphi_k}(Y_1)\right]\right|^2 \leqslant \|f\|_\infty^{\frac{1}{2}}\|f_\varepsilon\|\Delta(m)\Delta^{(1)}(m).$$

Again, with the Bessel inequality and (5), we have

$$\sum_{j=0}^{m-1} |\,\mathbb{E}[v_{\varphi_j}(Y_1)]|^2 = \frac{1}{2\pi}\sum_{j=0}^{m-1}|\int_{|u|\leqslant\sqrt{\rho m}} f^*(-u)\varphi_j(u)du|^2 \leqslant \frac{\|f^*\|^2}{2\pi} \leqslant \|f\|_\infty.$$

Therefore, we deduce $\mathbb{E}\left[|I_m(Y_1,Y_2)|^2\right] \leqslant 2\|f\|_\infty^{\frac{1}{2}}\|f_\varepsilon\|\Delta(m)\Delta^{(1)}(m) + 2\|f\|_\infty^2$ and

$$A_1 \leqslant 2n(\|f\|_\infty \vee \|f\|_\infty^{\frac{1}{4}}\|f_\varepsilon\|^{\frac{1}{2}})\sqrt{\Delta(m)\Delta^{(1)}(m)}.$$

- **Computing of A_2.** We have by the Cauchy Schwarz inequatily, $\mathbb{E}[\alpha_i^2(Y_1)] \leqslant 1$ and $\mathbb{E}[\beta_j^2(Y_2)] \leqslant 1$

$$\mathbb{E}\left[I_m(Y_1,Y_2)\alpha_i(Y_1)\beta_j(Y_2)\right] = \iint I_m(u,v)\alpha_i(u)\beta_j(v)f_Y(u)f_Y(v)dudv$$
$$\leqslant \sqrt{\iint |I_m(u,v)|^2 f_Y(u)f_Y(v)dudv}\sqrt{\iint \alpha_i^2(u)\beta_j^2(v)f_Y(u)f_Y(v)dudv}$$
$$\leqslant \sqrt{\mathbb{E}\left[|I_m(Y_1,Y_2)|^2\right]}.$$

Following the computation of A_1, we derive

$$A_2 \leqslant n(\|f\|_\infty \vee \|f\|_\infty^{\frac{1}{4}}\|f_\varepsilon\|^{\frac{1}{2}})\sqrt{\Delta(m)\Delta^{(1)}(m)}.$$

- **Computing of A_3.** Let fixed x. Set $\widetilde{a}_j = \frac{1}{2\pi}\int_{|u|\leqslant\sqrt{\rho m}} f^*(u)\varphi_j^*(-u)du$. By remarking that v_t is real-value and $\widetilde{a}_j$ is a real number, we have

$$\mathbb{E}_{Y_2}[|I_m(x,Y_2)|^2] = \sum_{j,k=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)(\overline{v_{\varphi_k}(x)}-\overline{\widetilde{a}_k})\,\mathbb{E}_{Y_2}\left[(v_{\varphi_j}(Y_2)-\widetilde{a}_j)(\overline{v_{\varphi_k}(Y_2)}-\overline{\widetilde{a}_k})\right]$$
$$= \sum_{j,k=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)(v_{\varphi_k}(x)-\widetilde{a}_k)\left(\int v_{\varphi_j}v_{\varphi_k}f_Y - \widetilde{a}_k\int v_{\varphi_j}f_Y - \widetilde{a}_j\int v_{\varphi_k}f_Y + \widetilde{a}_j\widetilde{a}_k\right).$$

Let us recall that $\mathbb{E}[v_{\varphi_j}(Y_1)] = \widetilde{a}_j$ by definition of v_t given in (34). Thus, Plancherel Equality and the orthogonality of (φ_j) imply

$$\begin{aligned}\mathbb{E}_{Y_2}[|I_m(x,Y_2)|^2] &\leqslant \sum_{j,k=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)(v_{\varphi_k}(x)-\widetilde{a}_k)\int v_{\varphi_j}v_{\varphi_k}f_Y = \int\left|\sum_{j=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)v_{\varphi_j}(y)\right|^2 f_Y(y)dy\\ &\leqslant \|f_Y\|_\infty\int\left|\sum_{j=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)v_{\varphi_j}(y)\right|^2 dy\\ &= \|f_Y\|_\infty\int\left|\left(z\mapsto\sum_{j=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)\frac{1}{2\pi}\frac{\varphi_j^*(-z)}{f_\varepsilon^*(z)}\mathbb{1}_{|z|\leqslant\sqrt{\rho m}}\right)^*(y)\right|^2 dy\\ &= \|f_Y\|_\infty\int_{|z|\leqslant\sqrt{\rho m}}\left|\sum_{j=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)\frac{\varphi_j(z)}{f_\varepsilon^*(z)}\right|^2 dz = \|f_Y\|_\infty\Delta^{(1)}(m)\sum_{j=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)^2.\end{aligned}$$

It comes that

$$\mathbb{E}_{Y_2}[|I_m(x,Y_2)|^2] \leqslant \|f_Y\|_\infty\Delta^{(1)}(m)\sum_{j=0}^{m-1}(v_{\varphi_j}(x)-\widetilde{a}_j)^2.$$

The Bessel and Plancherel equalities give (see (A5))

$$\begin{aligned}\sum_{j=0}^{m-1}|v_{\varphi_j}(x)-\widetilde{a}_j|^2 &\leqslant 2\left(\sum_{j=0}^{m-1}|v_{\varphi_j}(x)|^2+\sum_{j=0}^{m-1}\widetilde{a}_j^2\right)\\ &\leqslant 4(\|f\|_\infty\vee 1)\Delta(m).\end{aligned}$$

We derive that

$$\mathbb{E}_{Y_2}[|I_m(x,Y_2)|^2] \leqslant 4(\|f\|_\infty\vee 1)\|f_Y\|_\infty\Delta(m)\Delta^{(1)}(m),$$

and

$$A_3 \leqslant 2\sqrt{(\|f\|_\infty\vee 1)\|f_Y\|_\infty}\sqrt{n\Delta(m)\Delta^{(1)}(m)}.$$

- **Computing of A_4.** Note that

$$|I_m(x,y)| \leqslant \sum_{j=0}^{m-1}|(v_{\varphi_j}(x)-\widetilde{a}_j)(v_{\varphi_j}(y)-\widetilde{a}_j)|.$$

We bound successively the terms. By the Cauchy Schwarz inequality and (A5), it holds

$$\sum_{j=0}^{m-1}|v_{\varphi_j}(x)v_{\varphi_j}(y)| \leqslant \sqrt{\sum_{j=0}^{m-1}|v_{\varphi_j}(x)|^2}\sqrt{\sum_{j=0}^{m-1}|v_{\varphi_j}(y)|^2} \leqslant \frac{\Delta(m)}{2\pi},$$

and

$$\sum_{j=0}^{m-1}|v_{\varphi_j}(x)\widetilde{a}_j| \leqslant \sqrt{\sum_{j=0}^{m-1}|v_{\varphi_j}(x)|^2}\sqrt{\sum_{j=0}^{m-1}\widetilde{a}_j^2} \leqslant \sqrt{\frac{\|f\|_\infty}{2}\Delta(m)},$$

since $\sum_{j=0}^{m-1}\widetilde{a}_j^2 \leqslant \|f\|_\infty$. This implies $|I_m(x,y)| \leqslant 4(\frac{1}{2\pi} \vee \sqrt{\frac{\|f\|_\infty}{2}} \vee \|f\|_\infty)\Delta(m)$ and

$$A_4 \leqslant 12\left(\frac{1}{2\pi} \vee \sqrt{\frac{\|f\|_\infty}{2}} \vee \|f\|_\infty\right)\Delta(m).$$

Collecting the above evaluations, it yields $\forall x>0$

$$\mathbb{P}\left\{|U_n(I_m)| > \frac{C_\star}{n-1}\left(\sqrt{\Delta^{(1)}(m)\Delta(m)x} + \sqrt{\Delta^{(1)}(m)\Delta(m)}x + \frac{\sqrt{\Delta(m)\Delta^{(1)}(m)}x^{\frac{3}{2}}}{\sqrt{n}} + \frac{\Delta(m)x^2}{n}\right)\right\} \leqslant 5.54\exp(-x),$$

where $C_\star>0$ depending on f_ε, $\|f\|_\infty$ and $\|f_Y\|_\infty$. Since $2\frac{\sqrt{\Delta(m)\Delta^{(1)}(m)}x^{\frac{3}{2}}}{\sqrt{n}} \leqslant x\Delta^{(1)}(m) + \frac{\Delta(m)x^2}{n}$, it follows that

$$\mathbb{P}\left\{|U_n(I_m)| > \frac{C_\star}{n-1}\left(\sqrt{\Delta^{(1)}(m)\Delta(m)x} + \left(\sqrt{\Delta^{(1)}(m)\Delta(m)} + \Delta^{(1)}(m)\right)x + \frac{\Delta(m)x^2}{n}\right)\right\} \leqslant 5.54\exp(-x).$$

Hence the announced result.

A.5. **Proof of Proposition 7.3.** We only proof Part (i). The proof of Part (ii) can be found in Sacko (2020) (see Proof of Lemma 7.3, p. 23). Let us recall that $P_{n,1}(t) = \frac{1}{n}\sum_{k=1}^n \left(v_t(Y_k) - \mathbb{E}\left[v_t(Y_k)\right]\right)$, with v_t is given (34). Notice that

$$\begin{aligned}\mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\left\{\left(P_{n,1}^2(2f_m) - \frac{\kappa_1}{24}V(m)\right)_+\right\}\right] &\leqslant \sum_{m\in\mathcal{M}_n}\int_{\mathbb{R}^+}\mathbb{P}\left\{\left(P_{n,1}^2(2f_m) - \frac{\kappa_1}{24}V(m)\right)_+ > x\right\}dx \\ &\leqslant \sum_{m\in\mathcal{M}_n}\int_{\mathbb{R}^+}\mathbb{P}\left\{|P_{n,1}(2f_m)| \geqslant \sqrt{\frac{\kappa_1}{24}V(m)+x}\right\}dx.\end{aligned}$$

Now, we apply Bernstein inequality given in Appendix B.1. We need to evaluate two quantities b and v. We distinguish two cases.

The ordinary smooth case. By the Cauchy-Schwarz inequality, it holds

$$\begin{aligned}|v_{2f_m}(Y_k)| &= \frac{1}{2\pi}\left|\int_{|u|\leqslant\sqrt{\rho m}} e^{iuY_k}\frac{(2f_m)^*(-u)}{f_\varepsilon^*(u)}du\right| \leqslant \frac{1}{\pi}\sqrt{\|f_m^*\|^2\int_{|u|\leqslant\sqrt{\rho m}}\frac{1}{|f_\varepsilon(u)|^2}du} \\ &\leqslant \sqrt{2\|f\|_\infty\Delta(m)} = b.\end{aligned}$$

By the Plancherel theorem and under (**H5**), we take (see (19))

$$\mathbb{E}\left[|v_{2f_m}(Y_k)|^2\right] \leqslant 4\|f_Y\|_\infty\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f_m(u)|^2}{|f_\varepsilon^*(u)|^2}du = 4\|f_Y\|_\infty V_2(m)\frac{n}{\log(n)} = v.$$

Applying Bernstein inequality leads to have for any $\alpha\in]0,1[$

$$\begin{aligned}&\mathbb{P}\left\{|P_{n,1}(2f_m)| \geqslant \sqrt{\frac{\kappa_1}{24}V(m)+x}\right\} \\ &\leqslant 2\left\{\exp\left(-\frac{n}{4v}\left(\frac{\kappa_1}{24}V(m)+x\right)\right) \vee \exp\left(-\frac{n}{4b}\sqrt{\frac{\kappa_1}{24}V(m)+x}\right)\right\} \\ &\leqslant 2\left\{\exp\left(-\frac{n}{4v}\left(\frac{\kappa_1}{24}V(m)+x\right)\right) \vee \exp\left(-\frac{n\alpha}{4b}\sqrt{\frac{\kappa_1}{24}V(m)}\right)\exp\left(-\frac{n(1-\alpha)}{4b}\sqrt{x}\right)\right\}.\end{aligned}$$

Moreover, for any $\kappa_1 \geqslant 384\|f_Y\|_\infty p$, we have

$$\frac{n}{4v}\frac{\kappa_1}{24}V(m) \geqslant \frac{V_2(m)}{384\|f_Y\|_\infty V_2(m)}\log(n) \geqslant p\log(n),$$

where p is an integer which shall be specified in the sequel. It also holds for any $m \in \mathcal{M}_n$ and $\kappa_1 \geqslant 384\|f\|_\infty p^2/(\alpha^2 c_1')$

$$\frac{n\alpha}{4b}\sqrt{\frac{\kappa_1}{24}V(m)} \geqslant \frac{n\alpha}{4b}\sqrt{\frac{\kappa_1}{24}V_1(m)}\frac{n\alpha}{4\sqrt{2\|f\|_\infty\Delta(m)}}\sqrt{\frac{\kappa_1}{24}\Delta(m)\Delta^{(1)}(m)\frac{\log^2(n)}{n^2}} \geqslant p\log(n),$$

since $V(m) = V_1(m) + V_2(m) \geqslant V_1(m) = \Delta^{(1)}(m)\Delta(m)\frac{\log^2(n)}{n^2}$ and $\Delta^{(1)}(m) \geqslant c_1'$ (see (7)). Therefore, we obtain for κ_1 large enough

$$\mathbb{P}\left\{|P_{n,1}(2f_m)| \geqslant \sqrt{\frac{\kappa_1}{24}V(m) + x}\right\} \leqslant 2n^{-p}\left\{\exp\left(-\frac{n}{4v}x\right) \vee \exp\left(-\frac{n(1-\alpha)}{4b}\sqrt{x}\right)\right\}. \tag{A7}$$

The super smooth case. We take an upper bound of $\int_{|u|\leqslant\sqrt{\rho m}}\frac{|f_m(u)|^2}{|f_\varepsilon^*(u)|^2}du$ in such a way as to set $v = 4\|f\|_\infty\|f_Y\|_\infty\Delta^{(1)}(m)$ and b does not change. Again with $V(m) = V_1(m) + V_2(m) \geqslant V_2(m) = \Delta^{(1)}(m)\log(n)/n$, it holds for $\kappa_1 \geqslant 384\|f\|_\infty\|f_Y\|_\infty$ that $\frac{n}{4v}\frac{\kappa_1}{24}V(m) \geqslant p\log(n)$. Thanks to $V(m) \geqslant \Delta^{(1)}(m)\Delta(m)\frac{\log^2(n)}{n^2}$ and $\kappa_1 \geqslant 384\|f\|_\infty p^2/(\alpha^2 c_1')$, we get $\frac{n\alpha}{4b}\sqrt{\frac{\kappa_1}{24}V(m)} \geqslant p\log n$. We derive that (A7) holds for the super smooth case.

By choosing $p = 3$ with some changes variables, we deduce that

$$\begin{aligned}\mathbb{E}\left[\sup_{m\in\mathcal{M}_n}\left\{\left(P_{n,1}^2(2f_m) - \frac{\kappa_1}{24}V(m)\right)_+\right\}\right] &\leqslant 2n^{-p}\sum_{m\in\mathcal{M}_n}\int_{\mathbb{R}^+}\left\{\exp\left(-\frac{n}{4v}x\right) \vee \exp\left(-\frac{n(1-\alpha)}{4b}\sqrt{x}\right)\right\}dx \\ &\leqslant 2n^{-p}\sum_{m\in\mathcal{M}_n}\left(\frac{4\|f_Y\|_\infty\|f\|_\infty\Delta^{(1)}(m)}{n} \vee 32\|f\|_\infty\frac{\Delta(m)}{n^2(1-\alpha)^2}\right) \leqslant \frac{C_1^*}{n},\end{aligned}$$

where C_1^* is a constant depending on $\|f\|_\infty$, $\|f_Y\|_\infty$ and f_ε via (7). This concludes the proof.

Appendix B. Concentration inequalities

B.1. **Bernstein inequality.** Let $Y_1, \ldots, Y_n$ be n independent real random variables and $S_n = \sum_{i=1}^n (Y_i - \mathbb{E}[Y_i])$. Assume there exist two constants s^2 and b, such that $\mathrm{Var}(Y_i) \leqslant s^2$ and $|Y_i| \leqslant b$. Then, for all $x > 0$, we have

$$\mathbb{P}\left(|S_n| \geqslant nx\right) \leqslant 2\max\left(\exp\left(-\frac{nx}{4v^2}\right), \exp\left(-\frac{nx^2}{4b}\right)\right).$$

Moreover, it holds

$$\mathbb{P}\left(|S_n| \geqslant \sqrt{2ns^2x} + \frac{bx}{3}\right) \leqslant 2e^{-x}.$$

A proof of Bernstein inequality can be found in Birgé and Massart (1998), p. 366.

B.2. **Exponential inequalities for U-statistics of order two.** Let $Y_1, \dots, Y_n$ be n independent random variables defined on a Polish space S and let us consider the degenerate (or canonical) U-statistic of order 2: $U_n(h) = \sum_{i \neq j=1}^{n} h(Y_i, Y_j)$, where the function $h : S^2 \mapsto \mathbb{R}$ is such that $\mathbb{E}[|h(Y_i, Y_j)|] < \infty$ for any $1 \leqslant i, j \leqslant n$. Let A_1, A_2, A_3 and A_4 be defined as follows:

$$
\begin{aligned}
A_1^2 &= \sum_{j=2}^{n} \sum_{i=1}^{j-1} \mathbb{E}\left[h^2(Y_i, Y_j)\right] \\
A_2 &= \sup \left\{ \mathbb{E}\left[\sum_{j=2}^{n} \sum_{i=1}^{j-1} h(Y_i, Y_j)\alpha_i(Y_i)\beta_j(Y_j) \right], \mathbb{E}[\sum_{i=1}^{n-1} \alpha_i^2(Y_i)] \leqslant 1, \mathbb{E}[\sum_{j=2}^{n} \beta_j^2(Y_j)] \leqslant 1 \right\}, \\
A_3^2 &= \max \left(\max_j \sup_x \left\{ \sum_{i=1}^{j-1} \mathbb{E}_{Y_i}[h^2(Y_i, x)] \right\}, \max_i \sup_x \left\{ \sum_{j=i+1}^{n} \mathbb{E}_{Y_j}[h^2(x, Y_j)] \right\}, \right) \\
A_4 &= \sup_{x,y} |h(x, y)|,
\end{aligned}
$$

where $\mathbb{E}_Y[X]$ denotes the conditional expectation of X given Y. Then, there exists an absolute constant $C > 0$ such that for any $x > 0$,

$$
\mathbb{P}\left\{ |U_n(h)| \geqslant C \left(A_1\sqrt{x} + A_2 x + A_3 x^{\frac{3}{2}} + A_4 x^2 \right) \right\} \leqslant 5.54 e^{-x}.
$$

The Bernstein-type inequality for canonical U-statistics of order 2 is proven in Houdré and Reynaud-Bouret (2003) for real variables. The above result is a simplified version of Theorem 3.4.8 given in Giné and Nickl (2016).

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