



HAL
open science

Some questions related to linguistic variables

Ronald Yager

► **To cite this version:**

Ronald Yager. Some questions related to linguistic variables. Bulletin pour les sous-ensembles flous et leurs applications, 1982, 10, pp.54–65. hal-04559263

HAL Id: hal-04559263

<https://hal.science/hal-04559263>

Submitted on 25 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some Questions Related to Linguistic Variables

Ronald R. Yager

Machine Intelligence Institute

Iona College

New Rochelle, NY 10801

Tech. Report #MII-201

Abstract

We distinguish between fuzzy subsets used to describe a concept and fuzzy subsets used to define a linguistic variable in terms of a concept. We introduce the idea of consistency of a fuzzy subset when used to define a variable. We show that this requires that the possibility that the variable lies in the realization set is one. We address the problem that arises when we use a fuzzy subset expressed over a subset of the realization set to define the value of a linguistic variable.

We introduce the idea of a multivalued linguistic variable.

Introduction

L. A. has developed the concept of a linguistic variable as a cornerstone of his theory of approximate reasoning [1]. With this idea he allows a variable to be defined to have a value equal to a fuzzy subset. From this fuzzy subset he then induces a possibility distribution associated with the variable. With the use of the rules of inference, from the theory of approximate reasoning, he then can use this possibility distribution to make inferences about this and other variables. In applying this theory certain anti-intuitive results arise if the fuzzy subset associated with the linguistic variable is subnormal. A fuzzy subset is subnormal if its maximal membership grade is less than one.

Example: Consider a variable V described over the base set

$X = \{1, 2, 3, 4\}$ Consider the proposition

$P = V$ is "not X ."

We then define the value of V as $V =$ "not X ." Since "not X " =

$\left\{ \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4} \right\}$ we then get that

$$V = \left\{ \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4} \right\}$$

If we apply the entailment principle* to our situation we can get the infer that

$$V = \left\{ \frac{1}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4} \right\} = \left\{ \frac{1}{1} \right\} = 1$$

This appears counter intuitive.

.....

*The entailment principle, which is a rule of inference in the theory of approximate reasoning states that if F and G are two fuzzy subsets of X such that $F \subset G$, then $V = F \Rightarrow V = G$.

We feel that Zadeh's theory of approximate reasoning is fundamentally correct and provides a very useful tool for building models to mimic human reasoning with imprecise information. However, we feel that it requires a slight modification to avoid the problems associated with sub-normal fuzzy sets. We shall here suggest such a modification which perhaps will help.

The essence of our modification resides in the fact that a careful distinction must be made between using a fuzzy subset to describe a concept and using a fuzzy subset to define the value of variable.

The crux of the issue lies in the selection of the base set used to express the fuzzy subset in question.

Take a concept such as young if I want to describe the concept young I can select any base set X and express the concept young as a fuzzy subset of X. I can do this, for example, if I select $X = \{5, 10, 15, 20, 25, 30, 35, 40, 45\}$ I can express my description of young as

$$\text{Young} = \left\{ \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{.8}{20}, \frac{.5}{25}, \frac{.1}{30}, \frac{0}{35}, \frac{0}{40}, \frac{0}{45} \right\}^*$$

However, I can also take as my set X the set $X = \{a, b, c, d\}$ where a is Allan's age, b is Barbara's age, c is Charlies' age and d is Debbie's age. I can then describe the concept young as a fuzzy subset of X as for example

$$\text{Young} = \left\{ \frac{.5}{a}, \frac{.6}{b}, \frac{.7}{c}, \frac{.2}{d} \right\}$$

In the above cases for any element $x \in X$, young (x) indicates the degree to which x has the property of being young.

.....

*The description of young in terms of the elements of is X a subjective reflection of my concept of young. The membership grade indicates the degree to which an element has the property young as I perceive it.

I can also describe the concept young over the set $X = \{ \text{young, "about 15", "about 30", old} \}$. As an extreme case I can describe the concept young over the set $X = \{ \text{red, blue, green} \}$.

The important idea is that in using a fuzzy subset to describe some concept we can select any set X as our base set and the fact that the fuzzy subset is sub-normal causes no problems. All that this means is that there is no element in X fully having the property being described.

Defining Values of Fuzzy Subsets

Let us now turn to the question of using a fuzzy subset to define or represent the value of a variable. The point that we would like to make is that in using a fuzzy subset to define the value of a variable we must in reality express the fuzzy subset over a base set which includes all the values which the variable can really assume. This set of values which the variable can really assume is called the realization set or domain of the variable. This idea of realization set is a fundamental idea in the sense that the variable takes a unique value as an element in this set and that if I know this element then I know the value of the variable as best as reasonably can be expected. For example, if we consider the variable John's age at his last birthday then the domain or realization set is the set of integers $\{0, 1, 2, \dots, 150\}$. This is the realization set in the sense that John's age at his last birthday must be an element of this set and if I give his age as an element of this set it is known what I mean in the sense the elements of this set are atomic.

The idea of this realization set is similar to the concept of the set of all outcomes in a probabilistic experiment, it is complete.

As a ramification of this observation we can make the following definition.

Assume V is a variable with R its realization set. Let Q be a fuzzy subset of any set X . The statement or proposition $V = Q$ is consistent with this realization set if I am able to infer that there exists at least one element in R which has possibility one of occurring. Formally this definition can be stated as for $V = Q$ to be a consistent definition of the value V then

$$\text{Poss} [V \in R \mid V \in Q] = 1.$$

In the special case where $R \subset X$ we can obtain the following operational assertion of this fact.

We note first, that there exists some set D such that

$$X = R \cup D, D \cap R = \emptyset$$

R then can be described as a fuzzy subset of X , where

$$R(y) = 1 \text{ for } y \in R \text{ and } R(y) = 0 \text{ for } y \in D.$$

Using Zadeh's definition for possibility [2], we get

$$\text{Poss} [V \in R \mid V \in Q] = \max_{y \in X} [R(y) \wedge Q(y)] = 1 \text{ as necessary}$$

for consistency.

Since $R(y) = 0$ for $y \notin R$ and $R(y) = 1$ for $y \in R$

$$\text{Poss} [V \in R \mid V \in Q] = \max_{y \in R} [Q(y)] = 1.$$

Thus there exists some element $x^* \in R$ such that $Q(x^*) = 1$. We can conclude that any consistent value for V defined as a fuzzy subset Q of X , where $X \supset R$ must have at least one element with membership grade one in R .

Fundamentally what we indicated in the preceding is that in order to define a variable value equal to a fuzzy subset Q the fuzzy subset must be consistent with the realization set. What we have shown in the previous paragraph is that if Q is expressed as a fuzzy subset of some set containing the realization set that this amounts to the requirement that there exist some element in R having membership grade 1. Thus all consistent linguistic

values expressed over the realization set have at least one element equal one, are normal. If Q is not consistent we can't use it as the value for the variable.

A problem with subnormal fuzzy sets can arise due to the fact that in using a fuzzy subset to describe a concept we need not use a base set which includes the realization set.

Let us assume that V is a variable which has realization set R . Let \mathcal{A} indicate some concept. Assume our knowledge about V is \mathcal{A} . Let A be a fuzzy subset which describes the concept \mathcal{A} in terms of the elements of the set X . A is a fuzzy subset of X , where X need not include R and A may be subnormal. A is a description of our knowledge of \mathcal{A} . One cannot simply say that the value for $V = A$ for one must assure oneself that A is a consistent value for V . When $R \subseteq X$ we have a test to accomplish this, if there exists an $x \in R$ such that $A(x) = 1$ then A is consistent and we can use A to define the value of V . If there exists no such x then A is not a consistent definition of V and we can't say $V = A$.

When R is not contained in X we can't use this consistency test. Thus when A is expressed as a fuzzy subset of X and R is not contained in X we are uncertain as to the consistency of A as a value for V . However, if we want to use the information contained in the description A of \mathcal{A} in the theory of approximate reasoning to deduce possibility distributions we must extend this description of \mathcal{A} to the set R in a consistent manner.

Let A_E be the extension of A to be a fuzzy subset of R . What I can say is that for each element $a \in R - X$, $A_E(a) = \mathcal{U}$ = "unknown." However, for A_E to be a consistent extension of A then I require

$$\text{Poss} [V \in R \mid V \in A_E] = 1$$

If A is normal over X then my consistency condition is satisfied. However, if A is subnormal over X then to insure consistency of A_E there must exist some element $u \in R - X$ which allows for the possibility that $\text{Poss} [V \in R \mid V \in A_E] = 1$.

We must now show that the inclusion of the membership grade U^* for elements in $R - X$ allows this possibility.

Def: A fuzzy subset of R is said to have membership grade of type II if the membership values of any of the elements in R is a fuzzy subset of the unit interval, U .

Note: The membership grade U is itself a fuzzy subset of U and hence A_E has membership grade of type II.

Note: A crisp membership grade can be expressed as a type II membership grade. For example, the membership grade a can be expressed as $\left\{\frac{1}{a}\right\}$.

The inclusion of type II membership grades for the elements in R may result in the condition that $\text{Poss}[V \in R \mid V \in A_E]$ may be a fuzzy subset and thus not enabling us to determine if this is 1. To handle this situation we generalize our definition of the consistency of fuzzy subset over the realization set R .

Def. A fuzzy subset B defined over the set R is a consistent value for the variable V if the possibility that the $\text{Poss}[V \in R \mid V \in B] = 1$. Operationally this becomes the following:

Let $G = \text{Poss}[V \in R \mid V \in B] = \text{Max}_{y \in R} [R(y) \wedge B(y)]$ then what we require is that $C = \text{Poss}[One \mid G]$, where One is the fuzzy subset of I s.t. $one(1)=1$, $One(i)=0 \quad i \neq 1$, $one = \left\{\frac{1}{1}\right\}$, so such that $C = 1$, that is $\text{Max}_{i \in I} [G(i) \wedge One(i)] = 1$.

Theorem: If B is a type I fuzzy subset of R , then

$C = 1$, iff B is normal.

Proof: $G = \text{Max}_{y \in R} [R(y) \wedge B(y)] = \text{Max}_{y \in R} [B(y)] = b_{\max} = \left\{\frac{1}{b_{\max}}\right\}$

$C = \text{Poss}[one \mid G] = \text{Max}_{i \in I} [One(i) \wedge G(i)]$

This equals 1 iff $b_{\max} = 1$ else it equals zero.

.....
 * U is the fuzzy subset of the unit interval $[0, 1]$ defined by $U(x) = 1$ for all $x \in [0, 1]$. It is in essence the whole unit interval.

Thus this definition is an acceptable extension of our original definition.

Lemma: If B is a type II fuzzy subset of R then

$$\max_{y \in R} [R(y) \wedge B(y)] = \max_{y \in R} [B(y)]$$

Proof: We note that $R(y) = 1 = \left\{ \frac{1}{1} \right\}$. If B is a fuzzy subset of type

II then for any y, B(y) is a fuzzy subset of I, is $B(y) = \left\{ \frac{a_i}{i} \right\}$, $i \in I$, then $\left\{ \frac{1}{1} \right\} \wedge \left\{ \frac{a_i}{i} \right\} = \left\{ \frac{a_i \wedge 1}{1 \wedge i} \right\} = \left\{ \frac{a_i}{i} \right\} = B(y)$.

Lemma: Assume B_j are a family of normal fuzzy subsets of I, then

$$G = \max_j B_j,$$

G is a fuzzy subset of I such that

$$G(1) = 1$$

iff there exists at least one B_j such that $B_j(1) = 1$.

Proof: $G = B_1 \vee B_2 \vee B_3 \dots \vee B_n$

1) Sufficiency

We shall use induction. If B_1 is such that

$B_1(1) = 1$ then

$$(B_1 \vee B_2)(1) = \max_{j \vee k = 1} [B_1(j) \wedge B_2(k)], \text{ for } j=1, B_1(1)=1 \text{ and } j \vee k=1$$

for any k.

Choose k to be such that $B_2(k) = 1$ then theorem follows.

ii) necessity

$$(B_1 \vee B_2)(1) = \max_{j \vee k = 1} [B_1(j) \wedge B_2(k)] \cdot j \vee k = 1 \text{ iff either } j=1, \text{ or } k=1,$$

however if $B_1(1) \neq 1, B_2(1) = 1$, then $B_1(1) \wedge B_2(1) \neq 1$.

Lemma: Assume G is a fuzzy subset of I then

$$\text{Poss} [One | G] = 1 \text{ iff } G(1) = 1.$$

Proof: $\text{One}(1)=1$

$$\text{One}(i) = 0 \quad \forall i \neq 1$$

$$\text{Poss} [\text{One} | G] = \max_{i \in I} [\text{One}(i) \wedge G(i)] = G(1)$$

Theorem: Assume B is a fuzzy subset of R of type II then the consistency of B is 1 iff there exists at least one element $y \in R$ such that

$$(B(y))(1) = 1.$$

Corollary: If $B(y) = I$ then the consistency of B is one.

Thus if we include the elements in $R-X$ with membership grade I we get a consistent definition.

Let us summarize our approach.

Assume V is a variable ^{with} realization set R. Let X be a subset of R, $X \subseteq R$. Assume V is defined by the concept A which is described by the fuzzy subset A of X.

If $X = R$ then if A is such that there exists some $x \in R$ such that the membership grade $\mu_A(x) = 1$, i.e. $\mu_A(x) = 1$, then A is a consistent definition for V and we can use this value for V to generate a possibility distribution for purposes of approximate reasoning. If A doesn't satisfy this condition then A is inconsistent and we can't use it.*

If X is strictly contained in R and if A satisfies the above condition, some elements has membership grades one, we can proceed to generate a possibility distribution. If A doesn't satisfy the above condition when X is strictly contained in R we must extend A to R by defining $\mu_A(u) = I$ for all $u \in R-X$ this automatically assures consistency.

.....

*In the situation when A is inconsistent over the realization set another alternative approach is to say that we may wrong in what we think the realization set R and we may used $R = R \cup \{\text{else}\}$, else is some element we don't know, whose membership grade is I.

In the framework of using fuzzy subsets to define the value of variables great care must be taken to completely specify the realization set. Consider the variable John's friends at school. Let $X = \{x_1, \dots, x_n\}$ be the students at school. If I have the information that John has no friends at school then $Q = \{\frac{0}{x_1}, \dots, \frac{0}{x_n}\}$ represents this information. However, this is inconsistent with the set X . But we must realize that the realization set must include the possibility of nobody thus for this problem the realization set is $R = X \cup \{\text{nobody}\}$ Then $Q = \{\frac{1}{\text{nobody}}\}$.

Applications to Multiple Valued Variables

A problem which requires special attention concerns the situation in which a variable can assume multiple values. We shall make some preliminary comments on this topic. Zadeh [4] discusses similar problems related to the distinction between disjunctive and conjunctive fuzzy sets.

Consider a set $X = \{a, b, c, d, e\}$ where the letters stand for the initials of people. Let the fuzzy subset FJ of X describe the concept of a friend of John.

$$FJ = \left\{ \frac{1}{a}, \frac{1}{b}, \frac{.6}{c}, \frac{.3}{d}, \frac{0}{e}, \frac{0}{f} \right\}$$

Thus the membership grade, $FJ(c)$ indicates the degree to which c is a friend of John.

Consider the variable V , John's Friends. FJ is not the value of the variable V for the realization set R is not X . The values which the variable V can assume R , consists of the set of all subsets of X , 2^X .

Thus while FJ is not the value of V it can be used to obtain the value of V . To obtain V from the above we suggest the following:

For any subset $K \subset X$

$$V(K) = \left\{ \frac{K(a)}{FJ(a)}, \frac{K(b)}{FJ(b)}, \frac{K(c)}{FJ(c)}, \frac{K(d)}{FJ(d)}, \frac{K(e)}{FJ(e)}, \frac{K(f)}{FJ(f)} \right\}$$

For example, if $K = \{a, b\}$

$$V(\{a, b\}) = \left\{ \frac{1}{1}, \frac{1}{1}, \frac{0}{.6}, \frac{0}{.3}, \frac{0}{0}, \frac{0}{0} \right\} = 1$$

if $K = \{a, c, f\}$

$$V(\{a, c, f\}) = \left\{ \frac{1}{1}, \frac{0}{1}, \frac{1}{.6}, \frac{0}{.3}, \frac{0}{0}, \frac{1}{0} \right\} = \left\{ \frac{1}{1}, \frac{1}{.6}, \frac{1}{0} \right\}$$

Note for any subset consisting of a singleton the membership grade is its membership grade in FJ.

More generally if A is some concept described over some set X and V is some variable whose realization set R is a subset of X then we can express the value of V as a fuzzy subset of R obtained in the preceding manner.

hmf/01-14 /#12

References

- 1 Zadeh, L.A., "A theory of approximate reasoning," in Machine Intelligence Vol. 9, Hayes, J., Michie, D. and Mikulich, L. (eds.) Wiley, N.Y., 149-194, 1979.
2. Zadeh, L.A., "Fuzzy sets and information granularity," in Advances in Fuzzy Set Theory and Applications, Gupta, M.M. Ragade, R. K. and Yager, R.R. eds. North Holland, Amsterdam, 3-18, 1979.
- 3 Mizumoto, M. & Tanaka, K., "Some properties of fuzzy sets of type 2," Inf. & Control 31, 312-340, 1976.
- 4 Zadeh, L.A. "Test-score semantics for natural languages and meaning representation via PRUF," Technical note 241, SRI International, Mento Park, Ca., 1981.

hmf/15/#12