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A holographic global uniqueness in passive imaging

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Abstract

We consider a radiation solution ψ for the Helmholtz equation in an exterior region in \mathbb{R}^3 . We show that the restriction of ψ to any ray L in the exterior region is uniquely determined by its imaginary part $\text{Im } \psi$ on an interval of this ray. As a corollary, the restriction of ψ to any plane X in the exterior region is uniquely determined by $\text{Im } \psi$ on an open domain in this plane. These results have holographic prototypes in the recent work [Novikov, arXiv:2310.11127]. In particular, these and known results imply a holographic type global uniqueness in passive imaging and for the Gelfand-Krein-Levitan inverse problem in the monochromatic case. Some other surfaces for measurements instead of the planes X are also considered.

Key words: Helmholtz equation, Schrödinger equation, radiation solutions, Gelfand-Krein-Levitan problem, passive imaging, holographic global uniqueness

AMS subject classification: 35J05, 35J08, 35P25, 35R30

1 Introduction

We consider the Helmholtz equation

$$-\Delta\psi(x) = \kappa^2\psi(x), \quad x \in \mathcal{U}, \quad \kappa > 0, \quad (1)$$

where Δ is the Laplacian in x , and \mathcal{U} is an exterior region in \mathbb{R}^3 that is \mathcal{U} is an open connected set in \mathbb{R}^3 consisting of all points outside a closed bounded regular surface S (as in [3], [24]).

For equation (1) we consider the radiation solutions ψ that is the solutions satisfying the Sommerfeld's radiation condition

$$|x| \left(\frac{\partial}{\partial |x|} - i\kappa \right) \psi(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, \quad (2)$$

uniformly in $x/|x|$. We assume that $\psi \in C^2$ in the closure of \mathcal{U} .

Let

$$L = L_{x_0, \theta} = \{x \in \mathbb{R}^3 : x = x_0 + s\theta, 0 < s < +\infty\}, \quad x_0 \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^2, \quad (3)$$

where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 .

In the present work we show that, for any radiation solution ψ and any ray $L \subset \mathcal{U}$, the restriction of ψ to L is uniquely determined by its imaginary part $\text{Im } \psi$ on an arbitrary interval of L ; see Theorem 1 in Section 2.

As a corollary, we also obtain that, for any radiation solution ψ and any plane $X \subset \mathcal{U}$, the restriction of ψ to X is uniquely determined by $\text{Im } \psi$ on an arbitrary open domain of X ; see Theorem 2 in Section 2.

Our studies are motivated by the problem of Gelfand-Krein-Levitan and by passive imaging; see, e.g., [1], [2], [4], [6], [10], [11], [14], [23].

The Gelfand-Krein-Levitan problem (in its fixed energy version in dimension $d = 3$) consists in determining the potential v in the Schrödinger equation

$$-\Delta\psi(x) + v(x)\psi(x) = \kappa^2\psi(x) + \delta(x - y), \quad x, y \in \mathbb{R}^3, \quad \kappa > 0, \quad (4)$$

from the imaginary part of the outgoing Green function $R_v^+(x, y, \kappa)$ for one κ and all x, y on some part of the boundary of a domain containing the support of v . Here, δ is the Dirac delta function. In this problem $\text{Im } R_v^+$ is related to the spectral measure of the Schrödinger operator $H = -\Delta + v$. In addition, $R_v^+(x, y, \kappa)$, for fixed y , can be defined as the solution ψ of equation (4) with the radiation condition (2). For more details, see [4], [1] and references therein.

Equation (4) at fixed κ can be also considered as the Helmholtz equation of acoustics or electrodynamics for monochromatic waves, where $v(x) = v(x, \kappa)$ is related to the perturbation of the refraction index. In particular, the aforementioned mathematical problem of recovering v from boundary values of $\text{Im } R_v^+$ arises in the framework of passive acoustic tomography (in ultrasonics, ocean acoustics, local helioseismology), where $\text{Im } R_v^+$ is related to cross correlations of wave fields generated by random sources. For more details, see [23], [6], [11], [1], [2] and references therein.

Suppose that

$$\text{supp } v \subset \mathbb{R}^3 \setminus \overline{\mathcal{U}}, \quad y \in \mathbb{R}^3 \setminus \overline{\mathcal{U}}, \quad (5)$$

where $\overline{\mathcal{U}}$ is the closure of \mathcal{U} (and \mathcal{U} is as in (1)). Then, in view of its definition, $\psi = R_v^+(x, y, \kappa)$ is a radiation solution of equation (1).

Therefore, the aforementioned results on recovering a radiation solution ψ from $\text{Im } \psi$ give a reduction of the Gelfand-Krein-Levitan problem (of inverse spectral theory and passive imaging in dimension $d = 3$) to the inverse scattering problem of finding v in (4) from boundary values of R_v^+ .

This reduction and known results imply, in particular, that v in (4) is uniquely determined by $\text{Im } R_v^+(x, y, \kappa)$ for one κ and all x, y on an arbitrary open domain \mathcal{D} of X , where $\text{supp } v \subset \Omega$, X is a plane in $\mathbb{R}^3 \setminus \overline{\Omega}$, Ω is an open bounded connected domain in \mathbb{R}^3 , $\overline{\Omega}$ is the closure of Ω ; see Theorem 3 in Section 2.

We also consider other surfaces for measurements instead of the planes X ; see Example 1 and Theorems 4 and 5 in Section 2.

The main results of this work are presented in more detail and proved in Sections 2–7.

2 Main results

In this work our key result is as follows.

Theorem 1. *Let ψ be a radiation solution of equation (1) as in (2). Let L be a ray as in (3) such that $L \subset \mathcal{U}$, where \mathcal{U} is the region in (1). Then ψ on L is uniquely determined by $\text{Im } \psi$ on Λ , where Λ is an arbitrary non-empty open interval of L .*

As a corollary, we also get, in particular, the following result.

Theorem 2. *Let ψ be a radiation solution of equation (1) as in (2). Let X be a two-dimensional plane in \mathbb{R}^3 such that $X \subset \mathcal{U}$. Then ψ on X is uniquely determined by $\text{Im } \psi$ on \mathcal{D} , where \mathcal{D} is an arbitrary non-empty open domain of X .*

Theorem 1 is proved in Section 4 using the Atkinson-Wilcox expansion of [3], [24] for the radiation solutions ψ of equation (1), and a modified version of holographic techniques of [17], [19]. In particular, in this proof we use Proposition 1 of Section 3, which yields a two-point approximation for ψ in terms of $\text{Im } \psi$. This two-point approximation is also of independent interest.

Note that ψ and $\text{Im } \psi$ are real-analytic on \mathcal{U} , and, therefore, on L in Theorem 1 and on X in Theorem 2. Because of this analyticity, Theorem 1 reduces to the case when $\Lambda = L$ and Theorem 2 reduces to the case when $\mathcal{D} = X$.

Using this reduction, Theorem 2 is proved as follows. We assume that $\mathcal{D} = X$. Then to determine ψ at an arbitrary $x \in X$, we consider a ray $L = L_{x_0, \theta} \subset X$ such that $x \in L$ and use Theorem 1 for this L .

Corollary 1. *Under the assumptions of Theorem 2, the imaginary part $\text{Im } \psi$ on \mathcal{D} , uniquely determines ψ in the entire region \mathcal{U} .*

Corollary 1 follows from Theorem 2, formula (15) recalled in Subsection 3.3, and analyticity of ψ in \mathcal{U} .

Remark 1. *In the one dimensional case, an analog of Theorem 1 follows from a very simple form of one dimensional radiation solutions. In particular, in this case an analog of the two-point approximation of Proposition 1 is exact. The two dimensional case will be considered elsewhere using results of [13] in place of the three dimensional Atkinson-Wilcox expansion (9) used in the present work.*

Theorems 1 and 2, Corollary 1, and Remark 1 have holographic prototypes in [19].

Using Theorem 2 and known results on direct and inverse scattering we obtain the following global uniqueness theorem for the Gelfand-Krein-Levitan inverse problem mentioned in Introduction.

Theorem 3. *Let $v \in L^\infty(\mathbb{R}^3)$, $\text{supp } v \subset \Omega$, and $X \subset \mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is an open bounded connected domain in \mathbb{R}^3 , $\overline{\Omega}$ is the closure of Ω , and X is a two-dimensional plane. Let $R_v^+(x, y, \kappa)$ be the outgoing Green function for equation (4). Then v is uniquely determined by $\text{Im } R_v^+(x, y, \kappa)$ for one κ and all x, y on \mathcal{D} , where \mathcal{D} is an arbitrary non-empty open domain of X .*

In Theorem 3 we do not assume that v is real-valued, but we assume that property (19) formulated in Subsection 3.4 holds.

Remark 2. *Under the conditions that v is real-valued, $v \in L^\infty(\mathbb{R}^d)$, $\text{supp } v \subseteq \overline{\Omega}$, where Ω is an open bounded domain in \mathbb{R}^d with $\partial\Omega \in C^{2,1}$, $\overline{\Omega} = \Omega \cup \partial\Omega$, $d \geq 2$, a local uniqueness for determining v from $\text{Im } R_v^+(\cdot, \cdot, \kappa)$ on $\partial\Omega \times \partial\Omega$ for one κ is proved in [1]. The assumption that v is real-valued is very essential in this proof.*

Remark 3. *In connection with applications to helioseismology, it is natural to assume that v is complex-valued, $v \in L^\infty(\mathbb{R}^3)$, $v(x) = \tilde{v}(|x|)$, $\tilde{v}(r) = \alpha/r$, $r \geq r_0$, for some constants $\alpha \in \mathbb{R}$ and $r_0 > 0$. Let $M_r = \mathbb{S}_r^2 \times \mathbb{S}_r^2$ and*

$r_2 > r_1 > r_0$, where \mathbb{S}_r^2 is defined by (6). Under these assumptions, a global uniqueness for determining v from $\text{Im } R_v^+(\cdot, \cdot, \kappa)$ on $M_{r_1} \cup M_{r_2}$, for fixed κ and nonsingular pair r_1, r_2 , is proved in [2].

One can see that Theorem 3 contains a principal progress on the Gelfand-Krein-Levitan inverse problem in comparison with the results of [1], [2] mentioned in Remarks 2 and 3. In comparison with the aforementioned result of [1], Theorem 3 is global and does not assume that v is real-valued. In comparison with the aforementioned result [2], Theorem 3 does not assume that v is spherically symmetric.

Theorem 3 is proved in Section 5. In particular, in this proof we use Proposition 2 of Section 5, which states that v is uniquely determined by $R_v^+(\cdot, \cdot, \kappa)$ on $X \times X$ for fixed κ . To our knowledge, Proposition 2 is slightly different from the results reported in the literature. Therefore, just in case, for completeness of presentation this proposition is also proved in Section 5.

In connection with other surfaces of measurements instead of the planes X our results are as follows.

The results of Theorems 1 and 2 don't hold for some other curves in place of L and surfaces in place of X . An example is as follows.

Let

$$\mathbb{S}_r^2 = \{x \in \mathbb{R}^3 : |x| = r\}, \quad r > 0. \quad (6)$$

Example 1. Let $\psi(x) = G^+(x, \kappa)$, where G^+ is defined by (16). Then ψ is a non-zero radiation solution of equation (1) if $\{0\} \in \mathbb{R}^3 \setminus \mathcal{U}$, but $\text{Im } \psi \equiv 0$ on the spheres \mathbb{S}_r^2 for $r = n\pi/\kappa$, $n \in \mathbb{N}$.

Nevertheless, the uniqueness results of Theorem 2, Corollary 1, and Theorem 3 remain valid for many interesting surfaces instead of the planes X . Suppose that

$$\begin{aligned} \mathcal{B} &\text{ is an open bounded domain in } \mathbb{R}^3, \\ Y &= \partial\mathcal{B} \text{ is real-analytic and connected.} \end{aligned} \quad (7)$$

In particular, we have the following uniqueness theorems.

Theorem 4. Let ψ be a radiation solution of equation (1) as in (2). Let Y be a surface as in (7), where κ is not a Dirichlet eigenvalue for \mathcal{B} , and $\overline{\mathcal{B}} = \mathcal{B} \cup Y \subset \mathcal{U}$. Then ψ on \mathcal{U} is uniquely determined by $\text{Im } \psi$ on \mathcal{D} , where \mathcal{D} is an arbitrary non-empty open domain of Y .

Theorem 5. *Let $v \in L^\infty(\mathbb{R}^3)$, $\text{supp } v \subset \mathbb{R}^3 \setminus \overline{\mathcal{U}}$, where \mathcal{U} is as in (1), and property (19) hold for fixed $\kappa > 0$. Let Y be a surface as in (7), where κ is not a Dirichlet eigenvalue for \mathcal{B} , and $\overline{\mathcal{B}} = \mathcal{B} \cup Y \subset \mathcal{U}$. Then v is uniquely determined by $\text{Im } R_v^+(\cdot, \cdot, \kappa)$ on $\mathcal{D} \times \mathcal{D}$, where \mathcal{D} is an arbitrary non-empty open domain of Y .*

Theorems 4 and 5 are proved in Sections 6 and 7.

Note that the determinations in Theorems 1, 2, 3 and especially in Theorems 4 and 5 include analytic continuations. Studies on more stable reconstructions appropriate for numerical implementation will be continued elsewhere. In this respect one can use, in particular, results of Subsection 3.2.

3 Preliminaries

3.1 The Atkinson-Wilcox expansion

Let

$$B_r = \{x \in \mathbb{R}^3 : |x| < r\}, \quad r > 0. \quad (8)$$

If ψ is a radiation solution of equation (1) and $\mathbb{R}^3 \setminus B_r \subset \mathcal{U}$, then the following Atkinson-Wilcox expansion holds:

$$\psi(x) = \frac{e^{i\kappa|x|}}{|x|} \sum_{j=1}^{\infty} \frac{f_j(\theta)}{|x|^{j-1}} \quad \text{for } x \in \mathbb{R}^3 \setminus B_r, \quad \theta = \frac{x}{|x|}, \quad (9)$$

where the series converges absolutely and uniformly; see [3], [24].

3.2 A two-point approximation for ψ

Let

$$I(x) = |x| \text{Im } \psi(x), \quad x \in \mathcal{U}, \quad (10)$$

where ψ is a radiation solution of equation (1).

We have that

$$2iI(x) = e^{i\kappa|x|} f_1(\theta) - e^{-i\kappa|x|} \overline{f_1(\theta)} + O(|x|^{-1}) \quad \text{as } |x| \rightarrow +\infty, \quad (11)$$

uniformly in $\theta = x/|x|$, where f_1 is the leading coefficient in (9).

Proposition 1. *Let ψ be a radiation solution of equation (1). Then*

$$f_1(\theta) = \frac{1}{\sin(\kappa\tau)} \left(-e^{-i\kappa|y|} I(x) + e^{-i\kappa|x|} I(y) + O(|x|^{-1}) \right), \quad (12)$$

$$x, y \in L_{x_0, \theta}, \quad x_0 = 0, \quad y = x + \tau\theta, \quad \theta \in \mathbb{S}^2, \quad \tau > 0,$$

uniformly in θ , where f_1 is the coefficient in (11), I is defined by (10), $L_{x_0, \theta}$ is the ray defined by (3), and $\sin(\kappa\tau) \neq 0$ for fixed τ .

Formula (12) is a two-point approximation for f_1 and together with (9) also gives a two-point approximation for ψ in terms of I . For phaseless inverse scattering and holography formulas of such a type go back to [17] (see also [21], [20], [19]).

Remark 4. *If an arbitrary function I on $L_{0, \theta}$ satisfies (11), then formula (12) holds, for fixed $\theta \in \mathbb{S}^2$.*

We obtain (12) from the system of equations for f_1 and $\overline{f_1}$:

$$e^{i\kappa|x|} f_1(\theta) - e^{-i\kappa|x|} \overline{f_1(\theta)} = 2iI(x) + O(|x|^{-1}), \quad (13)$$

$$e^{i\kappa|y|} f_1(\theta) - e^{-i\kappa|y|} \overline{f_1(\theta)} = 2iI(y) + O(|y|^{-1}),$$

where x, y are as in (12). In particular, we use that $|y| = |x| + \tau$ and that

$$D = 2i\sin(\kappa\tau), \quad (14)$$

where D is the determinant of system (13).

In turn, (13) follows from (11).

3.3 A Green type formula

The following formula holds:

$$\psi(x) = 2 \int_X \frac{\partial G^+(x-y, \kappa)}{\partial \nu_y} \psi(y) dy, \quad x \in V_X, \quad (15)$$

$$G^+(x, \kappa) = -\frac{e^{i\kappa|x|}}{4\pi|x|}, \quad x \in \mathbb{R}^3, \quad (16)$$

where ψ is a radiation solution of equation (1), X and V_X are plane and open half-space in \mathcal{U} , where X is the boundary of V_X , ν is the outward normal to X relative to V_X ; see, for example, formula (5.84) in [5].

Recall that $R_0^+(x, y, \kappa) = -G^+(x-y, \kappa)$, where R_0^+ is the outgoing Green function for equation (4) with $v \equiv 0$.

3.4 Some facts of direct scattering

We consider equation (4) assuming for simplicity that

$$v \in L^\infty(\Omega), \quad v \equiv 0 \text{ on } \mathbb{R}^3 \setminus \overline{\Omega} \quad (17)$$

Ω is an open bounded connected domain in \mathbb{R}^3 .

The outgoing Green function R_v^+ for equation (4) satisfies the integral equation

$$R_v^+(x, y, \kappa) = -G^+(x - y, \kappa) + \int_{\Omega} G^+(x - z, \kappa) v(z) R_v^+(z, y, \kappa) dz, \quad (18)$$

where $x, y \in \mathbb{R}^3$, G^+ is given by (16).

Actually, in addition to (17), we assume that, for fixed $\kappa > 0$,

$$\text{equation (18) is uniquely solvable for } R_v^+(\cdot, y, \kappa) \in L^2(\Omega). \quad (19)$$

In particular, it is known that if v satisfies (17) and is real-valued (or $\text{Im } v \geq 0$), then (19) is fulfilled automatically; see, for example, [7].

We also consider the scattering wave functions ψ^+ for the homogeneous equation (4) (i.e. without δ):

$$\psi^+ = \psi^+(x, k) = e^{ikx} + \psi_{sc}^+(x, k), \quad x, k \in \mathbb{R}^3, \quad |k| = \kappa, \quad (20)$$

where ψ_{sc}^+ satisfies the radiation condition (2) at fixed k .

The following formulas hold:

$$R_v^+(x, y, \kappa) = R_v^+(y, x, \kappa), \quad x, y \in \mathbb{R}^3; \quad (21)$$

$$R_v^+(x, y, \kappa) = -\frac{e^{i\kappa|x|}}{4\pi|x|} \psi^+(y, -\kappa \frac{x}{|x|}) + O\left(\frac{1}{|x|^2}\right) \text{ as } |x| \rightarrow +\infty \quad (22)$$

at fixed y ;

$$\psi_{sc}^+(x, k) = \frac{e^{i\kappa|x|}}{|x|} A(k, \kappa \frac{x}{|x|}) + O\left(\frac{1}{|x|^2}\right) \text{ as } |x| \rightarrow +\infty \quad (23)$$

at fixed k , where A arising in (23) is the scattering amplitude for the homogeneous equation (4).

In connection with aforementioned facts concerning R_v^+ and ψ^+ see, e.g., Section 1 of Chapter IV of [9].

The scattering amplitude $A = A(k, l)$ arises in (23) for $k, l \in \mathbb{R}^3$, $k^2 = l^2 = \kappa^2$ (where $l = \kappa \frac{x}{|x|}$), can be written as

$$A = A(\theta, \theta', \kappa), \quad \theta = \frac{k}{|k|}, \quad \theta' = \frac{l}{|l|}, \quad (24)$$

and is defined on $\mathbb{S}^2 \times \mathbb{S}^2$ at fixed κ .

Remark 5. *It is well known that, under assumptions (17), (19), the scattering amplitude $A = A(\cdot, \cdot, \kappa)$ is real analytic on $\mathbb{S}^2 \times \mathbb{S}^2$.*

4 Proof of Theorem 1

4.1 Case $L \subseteq L_{0,\theta}$

First, we give the proof for the case when $L \subseteq L_{0,\theta}$. In this case it is essentially sufficient to prove that $\text{Im } \psi$ on L uniquely determines $f_j(\theta)$ in (9) for all j . Such a determination is presented below in this subsection. The rest follows from the convergence of the series in (9) and analyticity of ψ and $\text{Im } \psi$ on L .

The determination of f_1 follows from (12).

Suppose that f_1, \dots, f_n are determined, then the determination of f_{n+1} is as follows.

Let

$$\psi_n(x) = \frac{e^{i\kappa|x|}}{|x|} \sum_{j=1}^n \frac{f_j(\theta)}{|x|^{j-1}}, \quad \text{where } \theta = \frac{x}{|x|}, \quad (25)$$

$$I_n(x) = |x| \text{Im } \psi_n, \quad (26)$$

$$J_n(x) = |x|^n \text{Im}(I(x) - I_n(x)), \quad (27)$$

where x is as in (9), $I(x)$ is defined by (10).

We have that

$$2iI(x) = 2iI_n(x) + \frac{e^{i\kappa|x|}}{|x|^n} f_{n+1}(\theta) - \frac{e^{-i\kappa|x|}}{|x|^n} \overline{f_{n+1}(\theta)} + O(|x|^{-n-1}), \quad (28)$$

$$2iJ_n(x) = e^{i\kappa|x|} f_{n+1}(\theta) - e^{-i\kappa|x|} \overline{f_{n+1}(\theta)} + O(|x|^{-1}), \quad (29)$$

as $|x| \rightarrow +\infty$, uniformly in $\theta = x/|x|$, where I is defined by (10).

Due to (29) and Remark 4, we get

$$f_{n+1}(\theta) = \frac{1}{\sin(\kappa\tau)} \left(-e^{-i\kappa|y|} J_n(x) + e^{-i\kappa|x|} J_n(y) + O(|x|^{-1}) \right), \quad (30)$$

$$x, y \in L_{x_0, \theta}, \quad x_0 = 0, \quad y = x + \tau\theta, \quad \theta \in \mathbb{S}^2, \quad \tau > 0,$$

uniformly in θ , assuming that $\sin(\kappa\tau) \neq 0$ for fixed τ (where the parameter τ can be always fixed in such a way for fixed $\kappa > 0$).

Formulas (10), (25)-(27) and (30) determine f_{n+1} , give the step of induction for finding all f_j , and complete the proof of Theorem 1 for the case $L \subseteq L_{0, \theta}$.

4.2 General case

The general case reduces to the case of Subsection 4.1 by the change of variables

$$x' = x - q \quad (31)$$

for some fixed $q \in \mathbb{R}^3$ such that $L \subseteq L_{q, \theta}$.

In the new variables $x' \in \mathcal{U}' = \mathcal{U} - q$, we have that:

$$\psi = \frac{e^{i\kappa|x'|}}{|x'|} \sum_{j=1}^{\infty} \frac{f'_j(\theta)}{|x'|^{j-1}} \quad \text{for } x' \in \mathbb{R}^3 \setminus B_{r'}, \quad \theta = \frac{x'}{|x'|}, \quad (32)$$

for some new f'_j , where r' is such that $\mathbb{R}^3 \setminus B_{r'} \subset \mathcal{U}'$;

$$L \subseteq L_{q, \theta} = L_{0, \theta}. \quad (33)$$

In addition, the series in (32) converges absolutely and uniformly.

In view of (32), (33), we complete the proof of Theorem 1 by repeating the proof of Subsection 4.1.

Remark 6. *Our proof of Theorem 1 has a holographic prototype in [19]. Additional formulas for finding f_j from $\text{Im } \psi$ on L can be obtained proceeding also from the approaches of [18], [21], [22].*

5 Proof of Theorem 3

Under our assumptions on v , Ω , X , and κ , we have, in particular, that

$$R_v^+(\cdot, y, \kappa) \text{ is real-analytic on } X \setminus \{y\} \text{ for fixed } y \in X, \quad (34)$$

$$R_v^+(x, \cdot, \kappa) \text{ is real-analytic on } X \setminus \{x\} \text{ for fixed } x \in X. \quad (35)$$

Here, we use that $R_v^+(x, y, \kappa)$ satisfies (4) and, therefore, is real-analytic outside of $\text{supp } v \cup \{y\}$, and that symmetry (21) holds.

In view of (34), (35), Theorem 3 reduces to the case when $\mathcal{D} = X$.

In turn, Theorem 3 with $\mathcal{D} = X$ follows from Lemma 2 and Proposition 2 given below.

Lemma 1. *Under the conditions of Theorem 3, $\text{Im } R_v^+(\cdot, y, \kappa)$ on L uniquely determines $R_v^+(\cdot, y, \kappa)$ on L if $L = L_{y,\theta} \subset X$, where $L_{y,\theta}$ is defined by (3).*

Lemma 1 follows from Theorem 1 with appropriate \mathcal{U} such that (5) holds and, for example, from (34).

Lemma 2. *Under the conditions of Theorem 3, $\text{Im } R_v^+(\cdot, \cdot, \kappa)$ on $X \times X$ uniquely determines $R_v^+(\cdot, \cdot, \kappa)$ on $X \times X$.*

Lemma 2 follows from Lemma 1.

Proposition 2. *Under the conditions of Theorem 3, $R_v^+(\cdot, \cdot, \kappa)$ on $X \times X$ uniquely determines v (for fixed κ).*

Proposition 2 is proved as follows (for example).

First,

$$\psi = R_v^+(\cdot, x', \kappa) \text{ on } X \text{ uniquely determines } \psi = R_v^+(\cdot, x', \kappa) \text{ on } V_X \quad (36)$$

via formula (15), for each fixed $x' \in X$.

Second,

$$R_v^+(\cdot, x', \kappa) \text{ on } V_X \cup X \text{ uniquely determines } \psi^+(x', \kappa\theta) \text{ for } \theta \in \Theta_X^+ \quad (37)$$

via formula (22), for each fixed $x' \in X$,

where

$$\Theta_X^\pm = \{\theta \in \mathbb{S}^2 : \pm\theta\nu \geq 0\}, \quad (38)$$

where ν is the outward normal to X relative to V_X considered as the interior of X .

Third,

$$\begin{aligned} \psi = \psi_{sc}^+(\cdot, \kappa\theta) \text{ on } X \text{ uniquely determines } \psi = \psi_{sc}^+(\cdot, \kappa\theta) \text{ on } V_X \quad (39) \\ \text{via formula (15), for each fixed } x' \in X, \end{aligned}$$

where ψ^+ and ψ_{sc}^+ are related by formula (20).

Fourth,

$$\begin{aligned} \psi_{sc}^+(\cdot, \kappa\theta) \text{ on } V_X \cup X \text{ uniquely determines } A(\theta, \theta', \kappa) \text{ for } \theta' \in \Theta_X^- \quad (40) \\ \text{via formula (23), for each fixed } \theta \in \mathbb{S}^2, \end{aligned}$$

where Θ_X^- is defined in (38).

Therefore, we get that

$$\begin{aligned} R_v^+(\cdot, \cdot, \kappa) \text{ on } X \times X \text{ uniquely determines } A(\cdot, \cdot, \kappa) \text{ on } \Theta_X^+ \times \Theta_X^- \quad (41) \\ \text{via (36), (37), (39), (40).} \end{aligned}$$

In addition,

$$\begin{aligned} A(\cdot, \cdot, \kappa) \text{ on } \Theta_X^+ \times \Theta_X^- \text{ uniquely determines } A(\cdot, \cdot, \kappa) \text{ on } \mathbb{S}^2 \times \mathbb{S}^2 \quad (42) \\ \text{by analyticity,} \end{aligned}$$

in view of Remark 5.

Finally, Proposition 2 follows from (41), (42) and the result of [15] (see also [16], [12], [8]) that, under assumptions (17), (19), the scattering amplitude A at fixed κ uniquely determines v . In this result the assumption that v is real-valued or $\text{Im } v \geq 0$ is not essential and can be replaced by assumption (19).

This completes the proofs of Proposition 2 and Theorem 3.

6 Proof of Theorem 4

Under the assumptions of Theorem 4, the functions ψ and $\text{Im } \psi$ are real-analytic in \mathcal{U} , in general, and on Y , in particular.

Therefore, $\text{Im } \psi$ on \mathcal{D} uniquely determines $\text{Im } \psi$ on Y by analytic continuation, taking also into account that Y is real-analytic and connected.

In turn, $\text{Im } \psi$ on Y uniquely determines $\text{Im } \psi$ in \mathcal{B} by solving the Dirichlet problem for the Helmholtz equation.

In turn, $\text{Im } \psi$ in \mathcal{B} uniquely determines $\text{Im } \psi$ in \mathcal{U} by analytic continuation.

Finally, $\text{Im } \psi$ in \mathcal{U} uniquely determines ψ in \mathcal{U} in view of Corollary 1 in Section 2.

This completes the proof of Theorem 4.

7 Proof of Theorem 5

Under the assumptions of Theorem 5, we have, in particular, that:

$$R_v^+(\cdot, y, \kappa) \text{ is real-analytic in } \mathcal{U} \setminus \{y\} \text{ for fixed } y, \quad (43)$$

$$\text{Im } R_v^+(\cdot, y, \kappa) \text{ satisfies (1) and is real-analytic in } \mathcal{U} \text{ for fixed } y; \quad (44)$$

$$\begin{aligned} \psi = R_v^+(\cdot, y, \kappa) \text{ is a radiation solution of equation (1)} & \quad (45) \\ \text{with } \mathcal{U}' \text{ in place of } \mathcal{U}, \end{aligned}$$

where $y \in \mathcal{U}$, \mathcal{U}' is an arbitrary exterior region in \mathbb{R}^3 in the sense of \mathcal{U} and such that $\mathcal{U}' \subset \mathcal{U}$, $y \notin \overline{\mathcal{U}'}$.

Therefore, $\text{Im } R_v^+(\cdot, y, \kappa)$ on \mathcal{D} uniquely determines $\text{Im } R_v^+(\cdot, y, \kappa)$ in \mathcal{U} as $\text{Im } \psi$ on \mathcal{D} uniquely determines $\text{Im } \psi$ in \mathcal{U} in the proof of Theorem 4. Here, we use (44).

In turn, $\text{Im } R_v^+(\cdot, y, \kappa)$ in \mathcal{U} uniquely determines $R_v^+(\cdot, y, \kappa)$ in \mathcal{U} . This follows from (45), the final part of the proof of Theorem 4 with \mathcal{U}' in place of \mathcal{U} , and (43).

Thus, $\text{Im } R_v^+(\cdot, y, \kappa)$ on \mathcal{D} uniquely determines $R_v^+(\cdot, y, \kappa)$ in \mathcal{U} .

Similarly, $\text{Im } R_v^+(x, \cdot, \kappa)$ on \mathcal{D} uniquely determines $R_v^+(x, \cdot, \kappa)$ in \mathcal{U} , in view of (21).

Therefore, $\text{Im } R_v^+(\cdot, \cdot, \kappa)$ on $\mathcal{D} \times \mathcal{D}$ uniquely determines $R_v^+(\cdot, \cdot, \kappa)$ on $\mathcal{U} \times \mathcal{U}$.

Finally, $R_v^+(\cdot, \cdot, \kappa)$ on $\mathcal{U} \times \mathcal{U}$ uniquely determines v by different ways.

For example: $R_v^+(\cdot, \cdot, \kappa)$ on $\mathcal{U} \times \mathcal{U}$ uniquely determines $A(\cdot, \cdot, \kappa)$ on $\mathbb{S}^2 \times \mathbb{S}^2$, in view of (22), (23); A at fixed κ uniquely determines v as recalled at the end of proof of Theorem 3.

This completes the proof of Theorem 5.

Remark 7. *Formulas relating R_v^+ on $\mathbb{S}_r^2 \times \mathbb{S}_r^2$ and A on $\mathbb{S}^2 \times \mathbb{S}^2$ at fixed κ , where $v(x) = 0$ for $|x| > r$, \mathbb{S}_r^2 is defined by (6), were given for the first time in [4].*

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