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Generalized multi-view model: Adaptive density estimation under low-rank constraints

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Abstract

We study the problem of bivariate discrete or continuous probability density estimation under low-rank constraints. For discrete distributions, we assume that the two-dimensional array to estimate is a low-rank probability matrix. In the continuous case, we assume that the density with respect to the Lebesgue measure satisfies a generalized multi-view model, meaning that it is β -Hölder and can be decomposed as a sum of K components, each of which is a product of one-dimensional functions. In both settings, we propose estimators that achieve, up to logarithmic factors, the minimax optimal convergence rates under such low-rank constraints. In the discrete case, the proposed estimator is adaptive to the rank K . In the continuous case, our estimator converges with the L_1 rate $\min((K/n)^{\beta/(2\beta+1)}, n^{-\beta/(2\beta+2)})$ up to logarithmic factors, and it is adaptive to the unknown support as well as to the smoothness β and to the unknown number of separable components K . We present efficient algorithms for computing our estimators.

1 Introduction

Estimating discrete and continuous probability distributions is one of fundamental problems in statistics and machine learning. A classical density estimator for both discrete and continuous data is the histogram, while for continuous densities the most popular method is kernel density estimator (KDE) see, e.g., [Sil86, DG85, Sco92, Kle09, Tsy09, Gra17, WS19]. Asymptotically, these estimators can consistently recover any probability density on \mathbb{R}^m in the total variation (L_1) distance based on n independent identically distributed (iid) observations, if $nh^m \rightarrow \infty$, where h is the tuning parameter (bin width for the histogram in the continuous case and bandwidth for the KDE), and we assume that $h \rightarrow 0$, $n \rightarrow \infty$, see, e.g., [DG85]. On the other hand, smoothness assumptions on the underlying density are not enough to grant good accuracy of these estimators when m is large. Their rate of convergence drastically deteriorates with the dimension even if h is chosen optimally, and it remains true for any estimators under only smoothness assumptions. This gives rise to suggestions of density estimators that overcome the curse of dimensionality under more assumptions on the underlying density than only smoothness. An early suggestion is Projection Pursuit density estimation (PPDE) [FSS84], which is an iterative algorithm to estimate the density by finding a subspace spanned by a small number of significant components. One may consider PPDE as being adapted to the setting where there exists a linear map that

transforms the underlying random vector to a smaller dimensional random vector with independent components. This technique is rather popular but, to the best of our knowledge, theoretical guarantees on the performance of PPDE are not available, see a recent survey on PPDE in [WS19]. A related dimension reduction model for density estimation arises from independent component analysis (ICA), where one assumes the existence of a linear bijection of the underlying random vector to a random vector of the same dimension with independent components [ST04]. It is shown that under this model there is no curse of dimensionality in the sense that there exist estimators achieving one-dimensional rates [ST04, ST07, AAST10, LR20]. Finally, a recent line of work starting from [SADX14] and further developed in [AKS22a, AKS22b, KS19, SD13, VL21, Van23] deals with the multi-view model for density estimation, that is, a finite mixture model whose components are products of one-dimensional Lipschitz continuous probability densities. The fact that this model is free from the curse of dimensionality is demonstrated in [VL21] through a theoretical analysis of its sample complexity. However, [VL21] does not develop computationally tractable estimators.

In this paper, we focus on two-dimensional density estimation and consider a new model that generalizes the multi-view model in two aspects. We call it the *generalized multi-view model*. First, in contrast to the usual multi-view model, we do not assume the additive components to be products of densities but rather products of arbitrary functions. Second, we do not assume these functions to be Lipschitz continuous. We only need a β -Hölder continuity for some $\beta \in (0, 1]$ of the overall two-dimensional density, which is a sum of K such products. We propose a new estimator that is both computationally tractable and offers improved statistical guarantees by achieving the one-dimensional estimation rate to within a logarithmic factor. Our analysis deals with the L_1 risk of density estimators. As argued in [DG85], using the L_1 norm to characterize the error of density estimation has several advantages. Indeed, the L_1 distance between two densities is equivalent, up to a multiplicative constant factor, to the total variation distance between the corresponding probability measures. Moreover, the L_1 risk for density estimation is transformation invariant, which is not the case for the L_2 risk most often studied in the literature. From the technical point of view, it is more difficult to deal with the L_1 risk than with the L_2 risk.

Our approach to tackling generalized multi-view models is based on a reduction to the problem of estimating multivariate discrete distributions under a low-rank structure that we also consider in detail. This problem is of independent interest and it arises in many applications, in particular, if the aim is to explore correlations between categorical random variables, which is a particularly relevant subject [JBD17, DX09, DKK⁺19, TMM18]. Without structural assumptions, estimating a discrete distribution on a set of cardinality D in total variation distance is associated with the minimax risk of the order of $\sqrt{D/n}$, where n is the number of observations (see [KOPS15, HJW15] and Corollary 1 below). However, by imposing certain structure assumptions, it is possible to reduce the estimation risk. Several assumptions, including monotonicity, unimodality, t -modality, convexity, log-concavity or t -piecewise degree k -polynomial structure have been considered in the literature (see, for example, [CDGR18, DHS15, DHKR13]). In the present paper, we deal with a different setting where a multivariate discrete distribution is estimated under a low-rank structure. Specifically, for two integers $d_1, d_2 \geq 2$, we consider the problem of estimating a discrete distribution on a set of cardinality $D = d_1 d_2$ defined by a matrix of probabilities $P = (P_{ij})_{i \in [d_1], j \in [d_2]}$ with rank at most $K \geq 1$. This setting arises, for example, in the analysis of data represented as a table with n rows and 2 columns. Each row corresponds to one individual, while each column contains information about one feature of the individual, for instance, (1) eye color, and (2) hair color. We assume that the value of the cell in the table is a categorical random variable, for example, that hair color can only take 6 possible values: black, brown, red, blond, gray, white. Thus, each element of the ℓ -th column has a discrete distribution over $\{1, \dots, d_\ell\}$, for some $d_\ell \geq 2$, $\ell = 1, 2$. The individuals are assumed to be iid but the columns can be correlated. For instance, hair color can be correlated with eye color. In other words, each row can be viewed as a realization of a pair of correlated discrete random variables. If we are interested in possible associations between the two variables, we are lead to estimating their joint distribution. Assuming that P has low rank means that there exists a reduced representation of the correlation structure. A

basic unbiased estimator of P is a histogram Y/n where Y is a $d_1 \times d_2$ matrix such that its (i, j) -th entry Y_{ij} is the number of individuals whose row in the data table equals (i, j) . However, the histogram does not take advantage of low rank structure and it only attains the slow rate $\sqrt{d_1 d_2/n}$ in the total variation distance. We will suggest an estimator attaining a faster rate and show that it is minimax optimal up to logarithmic factors.

2 Summary of contributions and related work

The importance of low-rank structures in nonparametric density estimation has been discussed in several papers suggesting and analyzing various estimation methods [AKS22a, AKS22b, KS19, SADX14, SD13, VL21]. In [SADX14], the authors introduced the multi-view model and proposed a kernel method for learning it but did not study whether it can lead to an improvement in nonparametric density estimation. The paper [SD13] provided empirical evidence that hierarchical low-rank decomposition of kernel embeddings can lead to improved performance in density estimation. More recently, [AKS22a, AKS22b] used a low-rank characteristic function to improve nonparametric density estimation. Finally, [VL21] proved that there exists an estimator that converges at a rate $O(n^{-1/3})$ in the L_1 norm to any density satisfying the multi-view model with Lipschitz continuous marginals. This estimator is not constructed explicitly and is not computationally tractable in general. Furthermore, [VL21] shows that the standard histogram estimator can converge at a rate slower than $O(n^{-1/m})$ on the same class of densities, where m is the dimension of the data.

A related problem of estimating a low-rank probability matrix from discrete counts has been considered in [JO20], where the authors propose a polynomial time algorithm (called the curated SVD) for the case of square matrix, $d_1 = d_2 = d$, and requiring the exact knowledge of the rank K . They prove an upper bound on the total variation error of the curated SVD that scales as $\psi(K, d, n) := \sqrt{\frac{Kd}{n}} \wedge 1$ (to within a logarithmic factor in K, n) with probability at least $1 - d^{-2}$. They also state a lower bound with the rate $\psi(K, d, n)$ in *expectation* by referring to the lower bounds of [HJW15, KOPS15] for general discrete distributions on a set of cardinality $D = Kd$. Based on that, the authors in [JO20] claim minimax optimality of the rate $\psi(K, d, n)$, up to logarithmic factors. However, the lower bound obtained by this argument only holds under significant restrictions on K, d, n that are not specified in [JO20]. Indeed, the lower bounds of [HJW15, KOPS15] for general discrete distributions are only meaningful under some specific conditions on D, n . For example, the lower bound of [HJW15, Theorem 1] is vacuous for $D \asymp 1$ and the lower bound of [KOPS15, Lemma 8] is vacuous for $D \asymp \sqrt{n}$. These lower bounds are established only in expectation and it is not legitimate to compare them directly with upper bounds in probability derived in [JO20]. The upper bound in probability in [JO20, Theorem 2] can be transformed into a bound in expectation at the expense of adding a $O(d^{-2})$ term to the rate. This imposes one more restriction $d^{-2} \lesssim \psi(K, d, n)$ to match the lower bound up to a logarithmic factor. The algorithm suggested in [JO20] is based on normalization of the probability matrix by rescaling each row and column. Similar ideas have been developed in the literature on topic models, where topic matrices are estimated using an SVD-based technique on a rescaled corpus matrix [KW22]. Our approach uses completely different ideas. The algorithm that we suggest does not need to rescale the input matrix and relies on splitting the matrix into sub-matrices with entries of similar order of magnitude.

The contributions of the present work are as follows.

- We prove minimax lower bounds in the total variation norm for general discrete distributions on a set of cardinality D . We generalize [HJW15, KOPS15] in the sense that we derive lower bounds not only in expectation but also in probability and, in contrast to those works, we obtain the lower rate $\sqrt{\frac{D}{n}} \wedge 1$ for all $D, n \geq 1$ with no restriction. Next, under the low-rank matrix structure, we prove lower bounds of the order of $\psi(K, d, n)$ both in expectation and in probability, with no restriction on K, d, n , where $d = d_1 \vee d_2$. Moreover, we propose a computationally efficient algorithm to estimate a low-rank probability

matrix P and show that it attains the same rate $\psi(K, d, n)$ up to a logarithmic factor. Thus, we prove the minimax optimality of this rate and of our algorithm, up to a logarithmic factor. Unlike the curated SVD of [JO20], our algorithm applies to non-square matrices and it is adaptive to the unknown rank K .

- We propose a method of estimating β -Hölder densities for $\beta \in (0, 1]$ under the generalized multi-view model. Our algorithm achieves the rate of convergence $(K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)}$ up to a logarithmic factor on the class of densities that are (i) β -Hölder over an *unknown* sub-rectangle of $[0, 1]^2$ and (ii) represented as a sum of K separable components. In the two-dimensional case that we consider, we improve upon the prior work [VL21] in the following aspects:
 - Our estimator is computationally tractable.
 - The study in [VL21] was devoted to the case $\beta = 1$ and the standard multi-view model while we provide an extension to the generalized multi-view model described above and to any $\beta \in (0, 1]$.
 - We prove a lower bound showing that the above convergence rate is minimax optimal up to a logarithmic factor. We establish the explicit dependence of the minimax rate on K, n, β revealing, in particular, that it exhibits an elbow at $K \asymp n^{1/(2\beta+2)}$.
 - We propose an estimator, which is adaptive to the unknown number of separable components K , to the unknown smoothness β , and to the unknown support of the density.
- We provide a package for computation available at <https://github.com/hi-paris/Lowrankdensity>. We run a numerical experiment demonstrating the efficiency of our estimators both in discrete and continuous settings.

3 Notation

For two real numbers x, y , we define $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$. For $d \in \mathbb{N}$, we set $[d] = \{1, \dots, d\}$. For any probability vector or probability matrix P and for any $n \in \mathbb{N}^*$, we denote by $\mathcal{M}(P, n)$ the multinomial distribution with probability parameter P and sample size n .

For any matrix Λ , we denote by Λ_{ij} its (i, j) th entry and by $\text{rk}(\Lambda)$ its rank. We denote by $\Lambda_{IJ} = (\Lambda_{ij})_{i \in I, j \in J}$ an extraction of matrix $\Lambda \in \mathbb{R}^{d_1 \times d_2}$ corresponding to the sets of indices $I \subseteq [d_1]$ and $J \subseteq [d_2]$. We will use several matrix norms, namely, the operator norm denoted by $\|\Lambda\|$, the nuclear norm $\|\Lambda\|_*$, the Frobenius norm $\|\Lambda\|_F$, the entry-wise ℓ_1 -norm $\|\Lambda\|_1$, and the norms

$$\|\Lambda\|_{1,\infty} = \max_{j \in [d_2]} \sum_{i=1}^{d_1} |\Lambda_{ij}|,$$

$$\|\Lambda\|_{\square} = \|\Lambda\|_{1,\infty} \vee \|\Lambda^\top\|_{1,\infty}.$$

The notation $\|\cdot\|_{L_1}$ is used for the norms in $L_1([0, 1], \text{Leb})$ and in $L_1([0, 1]^2, \text{Leb})$, where Leb denotes the Lebesgue measure.

We denote by $\|\cdot\|$ the Euclidean norm in \mathbb{R}^2 , by $\text{Supp}(f) = f^{-1}(\mathbb{R} \setminus \{0\})$ the support of a real-valued function f , by $\mathbb{1}_A(\cdot)$ the indicator function of set A , and by $|A|$ the cardinality of finite set A . Throughout the paper, the absolute positive constants are denoted by C and may take different values on each appearance and we assume, unless otherwise stated, that $n, d_1, d_2 \geq 2$.

4 Discrete distributions

Consider first the setting with discrete distributions, which provides a base for studying continuous distributions in Section 5.

For integer $K \geq 1$, let \mathcal{T}_K be the class of all probability matrices of rank at most K :

$$\mathcal{T}_K = \left\{ P \in \mathbb{R}^{d_1 \times d_2} \mid \text{rk}(P) \leq K, \quad \sum_{(i,j) \in [d_1] \times [d_2]} P_{ij} = 1 \quad \text{and} \quad P_{ij} \geq 0, \quad \forall (i,j) \in [d_1] \times [d_2] \right\}. \quad (1)$$

Assume that for some unknown $P \in \mathcal{T}_K$ we are given iid observations X_1, \dots, X_n with distribution P , that is, $\mathbb{P}(X_k = (i,j)) = P_{ij}$ for all $k \in [n]$, $(i,j) \in [d_1] \times [d_2]$.

In this section, we consider the problem of minimax estimation of P on the class \mathcal{T}_K with respect to the norm $\|\cdot\|_1$. We derive the minimax optimal rate and propose a computationally efficient estimator achieving this rate, up to a logarithmic factor.

4.1 The estimator

We start by formally describing our algorithm. In the next subsection, we will provide an intuition behind its construction and sketch the ideas of proving the upper bounds on its performance.

Without loss of generality, assume that the total number of observations is even and equal to $2n$. We use sample splitting to define $H^{(1)} \in \mathbb{R}^{d_1 \times d_2}$ and $H^{(2)} \in \mathbb{R}^{d_1 \times d_2}$ as the matrices of empirical frequencies (the histograms) corresponding to the sub-samples (X_1, \dots, X_n) and (X_{n+1}, \dots, X_{2n}) respectively. In what follows, it will be useful to express the matrices P , $H^{(1)}$ and $H^{(2)}$ in terms of their columns and rows:

$$P = [C_1, \dots, C_{d_2}] = \begin{bmatrix} L_1 \\ \vdots \\ L_{d_1} \end{bmatrix} \quad \text{and} \quad H^{(\ell)} = [\widehat{C}_1^{(\ell)}, \dots, \widehat{C}_{d_2}^{(\ell)}] = \begin{bmatrix} \widehat{L}_1^{(\ell)} \\ \vdots \\ \widehat{L}_{d_1}^{(\ell)} \end{bmatrix}, \quad \text{for } \ell = 1, 2.$$

We set $T = \lfloor \log_2 d \rfloor - 1$, where $d \geq 2$ is a suitably chosen parameter. In this section, we take $d = d_1 \vee d_2$. Other choices of d will be used when applying Algorithm 1 as a building block for estimation of continuous densities. For any $t \in \{0, \dots, T\}$ we define

$$I_t = \left\{ i \in [d_1] : \|\widehat{L}_i^{(1)}\|_1 \in \left(\frac{1}{2^{t+1}}, \frac{1}{2^t} \right] \right\}, \quad J_t = \left\{ j \in [d_2] : \|\widehat{C}_j^{(1)}\|_1 \in \left(\frac{1}{2^{t+1}}, \frac{1}{2^t} \right] \right\} \quad (2)$$

and set

$$I_{T+1} = \left\{ i \in [d_1] : \|\widehat{L}_i^{(1)}\|_1 \leq \frac{1}{2^{T+1}} \right\}, \quad J_{T+1} = \left\{ j \in [d_2] : \|\widehat{C}_j^{(1)}\|_1 \leq \frac{1}{2^{T+1}} \right\}. \quad (3)$$

Next, for any $k = (t, t') \in \{0, \dots, T+1\}^2$ we define

$$U_k = I_t \times J_{t'}, \quad (4)$$

$$M^{(k)} = \left(H_{ij}^{(2)} \mathbf{1}_{\{(i,j) \in U_k\}} \right)_{i,j}, \quad P^{(k)} = \left(P_{ij} \mathbf{1}_{\{(i,j) \in U_k\}} \right)_{i,j}. \quad (5)$$

Algorithm 1: Estimation procedure

Input: $\alpha > 1, d \geq 2, N > 1$, integer $n \geq 1$, matrices $H^{(1)}, H^{(2)}$.

Output: Estimator \hat{P}^* of P .

If $n < 14\alpha d \log N$ **return** $\hat{P}^* = \frac{1}{2}(H^{(1)} + H^{(2)})$

Else: $T \leftarrow \lfloor \log_2(d) \rfloor - 1$

For $t, t' = 0, \dots, T + 1$:

1. $k \leftarrow (t, t'); \tau_k \leftarrow 12\sqrt{\alpha \frac{\log N}{n} 2^{-t \wedge t'}}$ **and define** $M^{(k)}$ **as in** (5)

2. $\hat{P}^{(k)} \leftarrow \operatorname{argmin}_{A \in \mathbb{R}^{d_1 \times d_2}} \left(\|M^{(k)} - A\|_F^2 + \tau_k \|A\|_* \right)$

3. $\hat{P} \leftarrow \sum_{k \in \{0, \dots, T+1\}^2} \hat{P}^{(k)}$

4. $\hat{P}_+ \leftarrow (\hat{P}_{ij} \vee 0)_{ij}$

5. **If** $\hat{P}_+ = 0_{\mathbb{R}^{d_1 \times d_2}}$ **return** $\hat{P}^* = \frac{1}{2}(H^{(1)} + H^{(2)})$

Return $\hat{P}^* = \frac{\hat{P}_+}{\|\hat{P}_+\|_1}$.

In what follows, $\text{Alg1}(\alpha, d, N, n, H^{(1)}, H^{(2)})$ stands for Algorithm 1 with input parameters $(\alpha, d, N, n, H^{(1)}, H^{(2)})$.

4.2 Intuition underlying Algorithm 1

Standard approaches to estimate a low-rank matrix from noisy data consist in using SVD-based methods. The main drawback of such methods is that they can be sub-optimal under non-isotropic noise. In particular, it is the case for the multinomial noise that we are dealing with in our setting since $nH^{(\ell)} \sim \mathcal{M}(P, n)$ for $\ell = 1, 2$. Any entry Y_{ij} of a multinomial matrix $Y \sim \mathcal{M}(P, n)$ has a binomial distribution, $Y_{ij} \sim \text{Bin}(n, P_{ij})$, with variance $nP_{ij}(1 - P_{ij})$ that varies across the indices (i, j) . To overcome this difficulty, Algorithm 1 splits the multinomial matrix into a logarithmic number of sub-matrices, on which the multinomial noise can be more carefully controlled. Then, each sub-matrix is de-noised separately using a nuclear norm penalized estimator. Recall that this estimator is based on soft thresholding of the singular values.

To appreciate why do we split the multinomial matrix as in equations (2) - (3), assume that $Y \sim \mathcal{M}(P, n)$ and for any two subsets $I \subset [d_1]$ and $J \subset [d_2]$, consider the extractions according to I and J :

$$Y_{IJ} = (Y_{ij})_{(i,j) \in I \times J}, \quad P_{IJ} = (P_{ij})_{(i,j) \in I \times J}, \quad \text{and} \quad W_{IJ} = \frac{Y_{IJ}}{n} - P_{IJ}.$$

By Lemma 8 the operator norm of W_{IJ} is controlled, with high probability and ignoring the logarithmic factors and smaller order terms, by a function of column-wise and row-wise sums of entries of P_{IJ} :

$$\begin{aligned} \|W_{IJ}\|^2 &\lesssim \frac{1}{n} \|P_{IJ}\|_{\square} = \frac{1}{n} \max \left(\max_{i \in I} \sum_{j \in J} P_{ij}, \max_{j \in J} \sum_{i \in I} P_{ij} \right) \\ &\leq \frac{1}{n} \max \left(\max_{i \in I} p_i, \max_{j \in J} q_j \right), \end{aligned} \tag{6}$$

where $p_i = \sum_{j=1}^{d_2} P_{ij}$ and $q_j = \sum_{i=1}^{d_1} P_{ij}$ are the marginal probabilities. This bound is accurate enough if we take

“balanced” subsets I, J such that

$$\forall i \in I : p_i \asymp \max_{i' \in I} p_{i'} \quad \text{and} \quad \forall j \in J : q_j \asymp \max_{j' \in J} q_{j'},$$

that is, we split the multinomial matrix according to similar values of p_i 's and q_j 's in order for the multinomial noise to be almost isotropic over the considered sub-matrices.

Since the values (p_1, \dots, p_{d_1}) and (q_1, \dots, q_{d_2}) are unknown we use sample splitting to obtain good enough estimators $(\hat{p}_1, \dots, \hat{p}_{d_1})$ and $(\hat{q}_1, \dots, \hat{q}_{d_2})$ from the first half of the data. To simplify the argument, here we do not discuss these pilot estimators and assume that an oracle gives us the exact values (p_1, \dots, p_{d_1}) and (q_1, \dots, q_{d_2}) .

Set $d = d_1 \vee d_2$, and for any $t \in \{0, \dots, \lfloor \log_2(d) - 1 \rfloor\}$, define the following index sets

$$\tilde{I}_t = \left\{ i : p_i \in \left(\frac{1}{2^{t+1}}, \frac{1}{2^t} \right] \right\}, \quad \tilde{J}_t = \left\{ j : q_j \in \left(\frac{1}{2^{t+1}}, \frac{1}{2^t} \right] \right\}.$$

For $T = \lfloor \log_2(d) \rfloor - 1$ define

$$\tilde{I}_T = [d_1] \setminus \bigcup_{t < T} \tilde{I}_t \quad \text{and} \quad \tilde{J}_T = [d_2] \setminus \bigcup_{t < T} \tilde{J}_t.$$

Fix $t, t' \in \{0, \dots, T\}$ and set $k = (t, t')$, $\tilde{M}_k = \frac{1}{n} Y_{\tilde{I}_t \tilde{J}_{t'}}$, $P_k = P_{\tilde{I}_t \tilde{J}_{t'}}$. Then $\text{rk}(P_k) \leq K$ and $\|P_k\|_{\square} \leq \frac{1}{2^{t \wedge t'}}$ by construction. The idea is now to take an estimator \hat{P}_k of P_k obtained from \tilde{M}_k by performing a soft-thresholding of its singular values, with a threshold based on Lemma 8, cf. (6). This strategy and standard guarantees for SVD soft-thresholding (nuclear norm penalized) estimators lead to the following bound on the error in the Frobenius norm, which holds with high probability:

$$\left\| \hat{P}_k - P_k \right\|_F^2 \lesssim \frac{K}{n} \|P_k\|_{\square} \lesssim \frac{K}{2^{t \wedge t'} n}.$$

Here and below, the sign \lesssim hides a logarithmic factor. This implies the following bound on the ℓ_1 -error over the cell k :

$$\left\| \hat{P}_k - P_k \right\|_1 \lesssim \sqrt{\frac{K}{2^{t \wedge t'} n} |\tilde{I}_t| |\tilde{J}_{t'}|}. \quad (7)$$

Denoting by \hat{P} the $d_1 \times d_2$ matrix obtained by concatenation of all the cells, and summing (7) over k we obtain

$$\|\hat{P} - P\|_1 \lesssim \sum_{t, t'=0}^T \sqrt{\frac{K}{2^{t \wedge t'} n} |\tilde{I}_t| |\tilde{J}_{t'}|}.$$

By the Cauchy-Schwarz inequality and the fact that $T = \lfloor \log_2(d) \rfloor - 1$ this bound can be simplified as follows:

$$\begin{aligned} \sum_{t, t'=0}^T \sqrt{\frac{K}{2^{t \wedge t'} n} |\tilde{I}_t| |\tilde{J}_{t'}|} &\leq \sqrt{\sum_{t, t'=0}^T \frac{K}{2^{t \wedge t'} n} |\tilde{I}_t| |\tilde{J}_{t'}| \sum_{t, t'=0}^T 1} \\ &\leq \log_2(d) \sqrt{\sum_{t, t'=0}^T \frac{K}{n} \left(\frac{1}{2^t} + \frac{1}{2^{t'}} \right) |\tilde{I}_t| |\tilde{J}_{t'}|} \\ &\lesssim \sqrt{\frac{K(d_1 + d_2)}{n}}, \end{aligned}$$

where the last inequality uses the relations

$$\sum_{t=0}^T |\tilde{I}_t| = d_1 \quad \text{and} \quad \sum_{t=0}^T 2^{-t} |\tilde{I}_t| = \sum_{t=0}^T \sum_{i \in \tilde{I}_t} 2^{-t} \leq \sum_{t=0}^T \sum_{i \in \tilde{I}_t} 2p_i \leq 2$$

(by the definition of \tilde{I}_t), and the analogous relations for the family $(\tilde{J}_t)_t$.

The above argument outlines a strategy for proving a bound of order $\sqrt{\frac{Kd}{n}}$ (up to a logarithmic factor) for the estimator defined by Algorithm 1. The exact statement of the result is given in Theorems 3 and 4 below.

4.3 Results for discrete distributions

We first provide minimax optimal rates for estimating a general discrete distribution on a finite set of size D . Without loss of generality, assume that this set is $\{1, \dots, D\}$. Let $\Delta_D = \left\{ p \in \mathbb{R}_+^D : \sum_{j=1}^D p_j = 1 \right\}$ be the set of all probability distributions on $\{1, \dots, D\}$. We denote by \mathbb{P}_p the probability measure of n iid observations drawn from p , by \mathbb{E}_p the expectation with respect to \mathbb{P}_p , and by $\inf_{\hat{p}}$ the infimum over all \mathbb{R}^D -valued estimators.

Theorem 1. *Let $D \geq 1, n \geq 1$. There exist two absolute positive constants c, c' such that*

$$\inf_{\hat{p}} \sup_{p \in \Delta_D} \mathbb{P}_p \left(\|\hat{p} - p\|_1 \geq c \left\{ \sqrt{\frac{D}{n}} \wedge 1 \right\} \right) \geq c', \quad (8)$$

and

$$\inf_{\hat{p}} \sup_{p \in \Delta_D} \mathbb{E}_p \|\hat{p} - p\|_1 \geq c \left\{ \sqrt{\frac{D}{n}} \wedge 1 \right\}. \quad (9)$$

The lower bound for the expectation (9) improves upon the bounds in [HJW15, KOPS15] that provide the same lower rate $\sqrt{\frac{D}{n}}$ under some additional conditions on D, n . The lower bound in probability (8) is new.

As a corollary of Theorem 1, we get the minimax optimal rate of convergence for the problem of estimating general discrete distributions. Given $\delta > 0$ and a class of discrete distributions $\mathcal{P} \subseteq \Delta_D$, we define the *minimax in probability rate* of estimation of $p \in \mathcal{P}$ based on an iid sample (X_1, \dots, X_n) from p as

$$\psi_\delta^*(n, \mathcal{P}) = \inf \left\{ \psi > 0 \mid \inf_{\hat{p}} \sup_{p \in \mathcal{P}} \mathbb{P}_p \left(\|\hat{p}(X_1, \dots, X_n) - p\|_1 > \psi \right) \leq \delta \right\}.$$

Corollary 1. *Let $D \geq 1, n \geq 1$. There exist two absolute positive constants c, c' such that, for all $\delta \in (0, c')$ we have*

$$c \left\{ \sqrt{\frac{D}{n}} \wedge 1 \right\} \leq \psi_\delta^*(n, \Delta_D) \leq c_\delta \left\{ \sqrt{\frac{D}{n}} \wedge 1 \right\}, \quad (10)$$

where $c_\delta > 0$ depends only on δ . Furthermore,

$$\inf_{\hat{p}} \sup_{p \in \Delta_D} \mathbb{E}_p \|\hat{p} - p\|_1 \asymp \sqrt{\frac{D}{n}} \wedge 1. \quad (11)$$

The proof of Corollary 1 follows immediately by combining Theorem 1 with the standard upper bound for the empirical frequency estimator (see, for example, Lemma 7 below).

Next, we obtain a lower bound for the class of discrete distributions \mathcal{T}_K defined by probability matrices of rank at most K , see (1).

Theorem 2 (Lower bounds for \mathcal{T}_K). *Let n, K, d_1, d_2 be positive integers. Set $d = d_1 \vee d_2$. There exist two absolute positive constants c, c' such that*

$$\inf_{\tilde{P}} \sup_{P \in \mathcal{T}_K} \mathbb{P}_P \left(\|\tilde{P} - P\|_1 \geq c \left\{ \sqrt{\frac{Kd}{n}} \wedge 1 \right\} \right) \geq c', \quad (12)$$

and

$$\inf_{\tilde{P}} \sup_{P \in \mathcal{T}_K} \mathbb{E}_P \|\tilde{P} - P\|_1 \geq c \left\{ \sqrt{\frac{Kd}{n}} \wedge 1 \right\}. \quad (13)$$

The next two theorems give upper bounds matching the lower rate of Theorem 2 up to a logarithmic factor.

Theorem 3 (Upper bound for \mathcal{T}_K in probability). *Let $\alpha > 1$, $d = d_1 \vee d_2$, $N \geq 2$, and let the estimator \hat{P}^* be obtained by $\text{Alg1}(\alpha, d, N, n, H^{(1)}, H^{(2)})$. Then there exist constants $C_0, C_1 > 0$ depending only on α such that*

$$\sup_{P \in \mathcal{T}_K} \mathbb{P}_P \left(\|\hat{P}^* - P\|_1 > C_1 \left\{ \sqrt{\frac{Kd}{n}} \log(d) \log^{1/2}(N) \wedge 1 \right\} \right) \leq C_0 (\log d)^2 d N^{-\alpha}.$$

Note that Theorem 3 can be used for several meaningful choices of N , such as $N = n$, $N = d$ or $N = d \vee n$.

Theorem 4 (Upper bound for \mathcal{T}_K in expectation). *Let $\alpha > 3/2$, $d = d_1 \vee d_2$, and let the estimator \hat{P}^* be obtained by $\text{Alg1}(\alpha, d, d \vee n, H^{(1)}, H^{(2)})$. Then there exists a constant $C > 0$ depending only on α such that*

$$\sup_{P \in \mathcal{T}_K} \mathbb{E}_P \|\hat{P}^* - P\|_1 \leq C \left\{ \sqrt{\frac{Kd}{n}} (\log n)^{3/2} \wedge 1 \right\}.$$

Theorem 2, Theorem 3 with $N = d$ and Theorem 4 lead to the following corollary, which provides the minimax rate for the class \mathcal{T}_K .

Corollary 2. *Let $\gamma > 0$ and $\delta = O(d^{-\gamma})$. There exist two constants $c, C > 0$ depending only on γ such that*

$$c \left\{ \sqrt{\frac{Kd}{n}} \wedge 1 \right\} \leq \psi_\delta^*(n, \mathcal{T}_K) \leq C \left\{ \sqrt{\frac{Kd}{n}} (\log d)^{3/2} \wedge 1 \right\} \quad (14)$$

and

$$c \left\{ \sqrt{\frac{Kd}{n}} \wedge 1 \right\} \leq \inf_{\tilde{P}} \sup_{P \in \mathcal{T}_K} \mathbb{E}_P \|\tilde{P} - P\|_1 \leq C \left\{ \sqrt{\frac{Kd}{n}} (\log n)^{3/2} \wedge 1 \right\}. \quad (15)$$

If K is substantially smaller than $d_1 \wedge d_2$ the rate of convergence $\sqrt{\frac{Kd}{n}}$ provided by Theorems 3 and 4 is much faster than the estimation rate $\sqrt{\frac{D}{n}}$ for general $D = d_1 d_2$ -dimensional discrete distributions. This characterizes the gain that is achieved due to the low-rank structure.

5 Continuous distributions

In this section, we use the ideas developed for discrete distributions in Section 4 to derive estimators of probability densities under the *generalized multi-view model*.

We start by defining the class of considered densities. Let $L > 0$ be a constant. For any $\beta \in (0, 1]$, we say that $f : [0, 1]^2 \rightarrow \mathbb{R}$ is a β -Hölder function if $|f(z) - f(z')| \leq L \|z - z'\|_\infty^\beta$ for all $z, z' \in \text{Supp}(f)$. We denote by

\mathcal{L}_β the set of all β -Hölder densities supported on rectangles contained in $[0, 1]^2$:

$$\mathcal{L}_\beta = \left\{ f : [0, 1]^2 \rightarrow \mathbb{R} \mid \exists r_1, r_2, R_1, R_2 \in [0, 1] \text{ s.t. } \begin{cases} \text{Supp}(f) = [r_1, R_1] \times [r_2, R_2], \\ f \text{ is } \beta\text{-Hölder on } \text{Supp}(f), \\ \int_{\text{Supp}(f)} f = 1 \text{ and } f \geq 0. \end{cases} \right\} \quad (16)$$

For integer $K \geq 1$, we define \mathcal{F}_K as the set of functions on $[0, 1]^2$ that are sums of K separable functions:

$$\mathcal{F}_K = \left\{ (x, y) \in [0, 1]^2 \mapsto \sum_{k=1}^K u_k(x)v_k(y) \in \mathbb{R} \mid u_k, v_k \in L_1[0, 1], \forall k \in [K] \right\}.$$

We consider the following set of β -Hölder probability densities

$$\mathcal{G}_{K,\beta} := \mathcal{L}_\beta \cap \mathcal{F}_K.$$

If $f \in \mathcal{G}_{K,\beta}$ we will say that f follows the *generalized multi-view model*. We emphasize that any function $f \in \mathcal{G}_{K,\beta}$ is only assumed to be β -Hölder on an unknown rectangle of the form $[r_1, R_1] \times [r_2, R_2]$ and not necessarily over the whole domain $[0, 1]^2$. In particular, $f \in \mathcal{G}_{K,\beta}$ can have jumps at the boundary of its support $[r_1, R_1] \times [r_2, R_2]$. Moreover, the functions u_k and v_k appearing in the decomposition $f(x, y) = \sum_{k=1}^K u_k(x)v_k(y)$ can take negative values, they need not be β -Hölder or continuous. Clearly, the set $\mathcal{G}_{K,\beta}$ contains all densities that are β -Hölder on their support and can be expressed as mixtures of product densities.

Since we consider density estimation under the L_1 risk it is not restrictive to assume that the support of the density is a compact set. Indeed, it has been highlighted in [IK84], [JLL04], [GL14], [CC21] that there exist no uniformly consistent estimators with respect to the L_1 norm on classes of Hölder continuous densities with unbounded support. On such classes, the minimax L_1 rate is of trivial order 1 regardless of the number of observations. Note also that in our setting the support of f is an *unknown* set. It allows us to handle densities that are not necessarily Hölder continuous on the whole domain $[0, 1]^2$. To the best of our knowledge, this setting was not explored in the prior work.

Assume that for some integer $K \geq 1$ and some unknown $f \in \mathcal{G}_{K,\beta}$, we are given n iid observations X_1, \dots, X_n distributed with probability density f . The goal is to estimate f based on X_1, \dots, X_n . The minimax rate for estimation of β -Hölder densities on $[0, 1]^2$ under the L_1 risk is known to be $n^{-\beta/(2\beta+2)}$ and this rate is attained by KDE and other basic density estimators [DG85, DL01]. We show that the minimax rate of convergence for the class $\mathcal{G}_{K,\beta}$ with rank K structure is faster than $n^{-\beta/(2\beta+2)}$ and we propose a computationally simple estimator that achieves this optimal rate up to a logarithmic factor.

Our estimator is defined in Algorithm 2. The main steps of Algorithm 2 can be summarized as follows.

- We divide the data in two subsamples of equal size. We estimate the support $[r_1, R_1] \times [r_2, R_2]$ by the smallest rectangle of the form $[\hat{r}_1, \hat{R}_1] \times [\hat{r}_2, \hat{R}_2]$ containing all of the data points in the first subsample.
- We partition the estimated support $[\hat{r}_1, \hat{R}_1] \times [\hat{r}_2, \hat{R}_2]$ into disjoint rectangular cells with edges of approximately the same length h^* in both dimensions. Dividing the second subsample in two equal parts, we construct two independent histogram matrices N and N' based on this partition.
- We apply Algorithm 1 with $H^{(1)} = N$ and $H^{(2)} = N'$, which outputs a matrix with entries corresponding to the cells of the partition. We define our density estimator as a function that takes a constant value in each cell, proportional to the output of Algorithm 1 in the cell.

Note also that we need to modify this scheme in some degenerate situations. Indeed, the partition of the rectangle $[\hat{r}_1, \hat{R}_1] \times [\hat{r}_2, \hat{R}_2]$ degenerates if either $\hat{R}_1 - \hat{r}_1$ or $\hat{R}_2 - \hat{r}_2$ is smaller than the size of the cell h^* .

If, for example, $\widehat{R}_1 - \widehat{r}_1$ is too small then, due to the β -Hölder property, f does not vary much in the first coordinate direction and we are lead to a one-dimensional density estimation problem over $[\widehat{r}_2, \widehat{R}_2]$, for which we use Algorithm 3.

In Algorithm 2 we assume without loss of generality that n is a multiple of 4. For $X \in \mathbb{R}^2$, we denote by $\Pi_1(X)$ and $\Pi_2(X)$ its first and second coordinates, respectively: $X = (\Pi_1(X), \Pi_2(X))$. We denote by $\text{Alg3}(Z_1, \dots, Z_n, K')$ the output of Algorithm 3 with input (Z_1, \dots, Z_n, K') .

Remark 1 (Choice of E_1 and E_2 in Algorithm 2). *In order to be able to apply the results of Section 4, we need the union of cells $C_{jj'}$ in Algorithm 2 to be such that*

$$\sum_{j \in E_1} \sum_{j' \in E_2} G_{jj'} = \sum_{j \in E_1} \sum_{j' \in E_2} G'_{jj'} = 1. \quad (17)$$

This condition cannot be guaranteed if we only take cells that form a partition of $[\widehat{r}_1, \widehat{R}_1] \times [\widehat{r}_2, \widehat{R}_2]$ since some data points from the second subsample $\{X_{n/2+1}, \dots, X_n\}$ may fall outside of the estimated support $[\widehat{r}_1, \widehat{R}_1] \times [\widehat{r}_2, \widehat{R}_2]$. Taking the sets of indices E_1 and E_2 as in Algorithm 2 ensures that the union of cells $(C_{jj'})_{j \in E_1, j' \in E_2}$ contains the whole domain $[0, 1]^2$ if $\widehat{R}_m - \widehat{r}_m > h^$ for $m \in \{1, 2\}$. In fact, under this condition, E_1 and E_2 are the sets of indices of smallest cardinality such that the union of $(C_{jj'})_{j \in E_1, j' \in E_2}$ contains $[0, 1]^2$. Of course, some of these cells $C_{jj'}$ may fall beyond $[0, 1]^2$. However, it does not affect the sums in (17) since $f = 0$ on such cells, so that the associated $G_{jj'}, G'_{jj'}$ vanish almost surely. Moreover, the order of magnitude of $|E_1|$ and $|E_2|$ remains controlled as needed. Indeed, we have $|E_1| \vee |E_2| \leq C/h^*$, which is sufficient for our purposes.*

In what follows we denote by \mathbb{P}_f the probability measure induced by (X_1, \dots, X_n) when X_i 's are iid distributed with density f , and by \mathbb{E}_f the corresponding expectation.

Theorem 5. *There exist constants $C'_0 > 0$, $C'_1 > 0$ depending only on α and L such that for the estimator \widehat{f} defined by Algorithm 2 with $\alpha > 1$ we have*

$$\sup_{f \in \mathcal{G}_{K, \beta}} \mathbb{P}_f \left(\|\widehat{f} - f\|_{L_1} > C'_1 \left\{ \left(\frac{K}{n} \right)^{\beta/(2\beta+1)} \log^{3/2} n \wedge n^{-\frac{\beta}{2\beta+2}} \right\} \right) \leq C'_0 (\log n)^2 n^{1/(2\beta+1) - \alpha}, \quad (18)$$

and for the estimator \widehat{f} defined by Algorithm 2 with $\alpha > 4/3$ we have

$$\sup_{f \in \mathcal{G}_{K, \beta}} \mathbb{E}_f \|\widehat{f} - f\|_{L_1} \leq C'_1 \left\{ \left(\frac{K}{n} \right)^{\beta/(2\beta+1)} \log^{3/2} n \wedge n^{-\frac{\beta}{2\beta+2}} \right\}. \quad (19)$$

Theorem 5 guarantees that the estimator \widehat{f} adapts to the best rate between $(K/n)^{\beta/(2\beta+1)}$, which is a “one-dimensional” rate as function of n but deteriorates as K grows, and $n^{-\beta/(2\beta+2)}$, which is the standard rate of estimating a β -Hölder two-dimensional density. This demonstrates a dimension reduction property. The elbow between the two rates occurs at $K \asymp n^{1/(2\beta+2)}$. Note that, in our setting with unknown support of f , the possibility to estimate the density even with the slow rate $n^{-\beta/(2\beta+2)}$ does not follow from the results on nonparametric density estimation developed in the prior work (see [GL14] and the references therein).

The following lower bound shows that the rate obtained in Theorem 5 cannot be improved up to a logarithmic factor. We derive even a stronger lower bound that holds for the subclass of $\mathcal{G}_{K, \beta}$ containing densities with support exactly $[0, 1]^2$. Define

$$\mathcal{L}_{[0, 1]^2} = \left\{ f : [0, 1]^2 \rightarrow \mathbb{R} \mid \text{Supp}(f) = [0, 1]^2 \right\}$$

and consider the class of densities $\mathcal{G}_{K, \beta}^\circ = \mathcal{G}_{K, \beta} \cap \mathcal{L}_{[0, 1]^2}$.

Algorithm 2: Two-dimensional density estimator

Input: $X_1, \dots, X_n \in \mathbb{R}^2$ with $n = 4k$ for an integer $k \geq 1$; $\alpha > 0$; $K \in \mathbb{N}$; $\beta \in (0, 1]$

For $m \in \{1, 2\}$, set $\hat{r}_m \leftarrow \min \{\Pi_m(X_i) : i \in \{1, \dots, \frac{n}{2}\}\}$ and $\hat{R}_m \leftarrow \max \{\Pi_m(X_i) : i \in \{1, \dots, \frac{n}{2}\}\}$.

If $(\frac{K}{n})^{\frac{\beta}{2\beta+1}} \log^{3/2}(n) \leq n^{-\frac{\beta}{2\beta+2}}$: set $K' = K$ **else:** set $K' = n^{\frac{1}{2\beta+2}}$.

$h^* \leftarrow (K'/n)^{1/(2\beta+1)}$.

If $\hat{R}_1 - \hat{r}_1 < h^*$:

$\hat{g} \leftarrow \text{Alg3}(\Pi_2(X_{n/2+1}), \dots, \Pi_2(X_n), K')$

$\hat{\phi}(x, y) := \frac{1}{\hat{R}_1 - \hat{r}_1} \mathbf{1}_{x \in [\hat{r}_1, \hat{R}_1]} \hat{g}(y)$.

Else If $\hat{R}_2 - \hat{r}_2 < h^*$:

$\hat{g} \leftarrow \text{Alg3}(\Pi_1(X_{n/2+1}), \dots, \Pi_1(X_n), K')$

$\hat{\phi}(x, y) := \frac{1}{\hat{R}_2 - \hat{r}_2} \mathbf{1}_{y \in [\hat{r}_2, \hat{R}_2]} \hat{g}(x)$.

Else:

For $m \in \{1, 2\}$, set $h_m = \ell_m^{-1}(\hat{R}_m - \hat{r}_m)$, where $\ell_m = \lfloor (\hat{R}_m - \hat{r}_m)/h^* \rfloor$

For $m \in \{1, 2\}$, set $E_m = \{-\lceil \hat{r}_m/h_m \rceil, \dots, \lceil (1 - \hat{r}_m)/h_m \rceil\}$

For $(j, j') \in E_1 \times E_2$:

$A_j \leftarrow [\hat{r}_1 + jh_1, \hat{r}_1 + (j+1)h_1)$.

$B_{j'} \leftarrow [\hat{r}_2 + j'h_2, \hat{r}_2 + (j'+1)h_2)$.

$C_{jj'} \leftarrow A_j \times B_{j'}$.

$G_{jj'} \leftarrow \frac{4}{n} \sum_{n/2+1 \leq i \leq 3n/4} \mathbf{1}_{X_i \in C_{jj'}}$; $G'_{jj'} \leftarrow \frac{4}{n} \sum_{3n/4+1 \leq i \leq n} \mathbf{1}_{X_i \in C_{jj'}}$;

If $(\frac{K}{n})^{\frac{\beta}{2\beta+1}} \log^{3/2}(n) > n^{-\frac{\beta}{2\beta+2}}$:

$\hat{P}^* \leftarrow (G + G')/2$, where $G = (G_{jj'})_{(j,j') \in E_1 \times E_2}$, $G' = (G'_{jj'})_{(j,j') \in E_1 \times E_2}$

Else:

$\hat{P}^* \leftarrow \text{Alg1}(\alpha, |E_1| \vee |E_2|, n, \frac{n}{4}, G, G')$

$\hat{\phi}(x, y) := \frac{1}{h_1 h_2} \sum_{j \in E_1} \sum_{j' \in E_2} \hat{P}^*_{jj'} \mathbf{1}_{(x,y) \in C_{jj'}}$

If $\hat{\phi} = 0$:

Return $\hat{f} = \mathbf{1}_{[0,1]^2}$

Return $\hat{f} = \frac{\hat{\phi}}{\|\hat{\phi}\|_{L_1}}$

Algorithm 3: One-dimensional density estimator

Input: Z_1, \dots, Z_n and $K' \geq 1$.

$\hat{r} \leftarrow \min \{Z_i : i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}\}; \hat{R} \leftarrow \max \{Z_i : i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}\}.$

$h^* \leftarrow (K'/n)^{1/(2\beta+1)}.$

If $\hat{R} - \hat{r} < h^*$, **then return** $\frac{1}{\hat{R} - \hat{r}} \mathbb{1}_{[\hat{r}, \hat{R}]}$.

Else:

$h = \ell^{-1}(\hat{R} - \hat{r})$, where $\ell = \lfloor (\hat{R} - \hat{r})/h^* \rfloor$

$A_j = [\hat{r} + (j-1)h, \hat{r} + jh)$, for $j = 1, \dots, \ell - 1$, and $A_\ell = [\hat{r} + (\ell-1)h, \hat{R}]$

For $j \in \{1, \dots, \ell\}$:

$N_j \leftarrow \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n \mathbb{1}_{Z_i \in A_j}$

Return $\hat{g} = \frac{1}{n - \lfloor \frac{n}{2} \rfloor} \sum_{j=1}^{\ell} \frac{1}{h} N_j \mathbb{1}_{A_j}$.

Theorem 6. Let $L > 0$, $\beta \in (0, 1]$. There exist two positive constants c, c' that can depend only on L and β such that

$$\inf_{\tilde{f}} \sup_{f \in \mathcal{G}_{K, \beta}^2} \mathbb{P}_f \left(\|\tilde{f} - f\|_{L_1} \geq c \left\{ (K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)} \right\} \right) \geq c', \quad (20)$$

and

$$\inf_{\tilde{f}} \sup_{f \in \mathcal{G}_{K, \beta}^2} \mathbb{E}_f \|\tilde{f} - f\|_{L_1} \geq c \left\{ (K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)} \right\}, \quad (21)$$

where $\inf_{\tilde{f}}$ denotes the infimum over all density estimators.

Consider now the minimax estimation rate on $\mathcal{G}_{K, \beta}$ with probability $\delta > 0$:

$$\bar{\psi}_\delta(n, \mathcal{G}_{K, \beta}) = \inf \left\{ t > 0 \mid \inf_{\tilde{f}} \sup_{f \in \mathcal{G}_{K, \beta}} \mathbb{P}_f \left(\|\tilde{f}(X_1, \dots, X_n) - f\|_{L_1} > t \right) \leq \delta \right\}. \quad (22)$$

Theorems 5 and 6 immediately imply the following corollary.

Corollary 3. For any $\gamma > 0$, $\beta \in (0, 1]$, $L > 0$, there exist two constants $c, C > 0$ depending only on γ, β and L such that

$$c \left\{ (K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)} \right\} \leq \bar{\psi}_{n^{-\gamma}}(n, \mathcal{G}_{K, \beta}) \leq C \left\{ (K/n)^{\beta/(2\beta+1)} (\log n)^{3/2} \wedge n^{-\beta/(2\beta+2)} \right\}.$$

6 Adaptive density estimation

The density estimators proposed in Section 5 require the knowledge of the number of separable components K and of smoothness β . In this section, we provide an estimator that is adaptive to both K and β .

Throughout this section, we assume that $n \geq 3$. We consider a grid $\{\beta_1, \dots, \beta_{\lceil \log(n) \log \log(n) \rceil}\}$ on the values of $\beta \in (0, 1]$, where

$$\beta_j = \left(1 + \frac{1}{\log n} \right)^{-j+1}.$$

Let $K_{\max} \geq 2$ be an integer. Set $m = K_{\max} \lceil \log(n) \log \log(n) \rceil$ and fix $\alpha > 4/3$. The adaptive estimator is obtained by the minimum distance choice from the family of m estimators $(\hat{f}_{(K,j)})_{K \in [K_{\max}], 1 \leq j \leq \lceil \log(n) \log \log(n) \rceil}$, where $\hat{f}_{(K,j)}$ is an output of Algorithm 2 with parameters $K, \beta = \beta_j, \alpha$ when the input sample is X_1, \dots, X_n . The minimum distance estimator (see, e.g., [DL01]) is defined as follows:

$$\hat{f}^* = \hat{f}_{(K^*,j^*)} \text{ where } (K^*, j^*) \in \underset{(K,j)}{\operatorname{argmin}} \max_{B \in \mathcal{B}} \left| \int_B \hat{f}_{(K,j)} - \mathbb{P}_n(B) \right|, \quad (23)$$

and $\mathcal{B} = \{B_{ii'}, i, i' = 1, \dots, m, i \neq i'\}$ with $B_{ii'} = \{x : \hat{f}_{i'}(x) > \hat{f}_i(x)\}$ for $i, i' \in \{(K, j) : K \in [K_{\max}], 1 \leq j \leq \lceil \log(n) \log \log(n) \rceil\}$. For a set $B \in \mathcal{B}$, the notation $\mathbb{P}_n(B)$ stands for the empirical probability measure of this set computed from the sample (X_1, \dots, X_n) .

Note that each $\hat{f}_{(K,j)}$ is a piece-wise constant function, so that the integrals in the definition of the adaptive estimator \hat{f}^* can be easily computed. To get \hat{f}^* we need $O(m^3)$ computations of such integrals [DL01], where m is logarithmic in n .

In what follows, we set $K_{\max} = \lceil \sqrt{n} \rceil$, which is essentially the smallest sufficient choice. Indeed, $\sqrt{n} = \sup_{\beta \in (0,1]} n^{\frac{1}{2\beta+2}}$, while (as discussed in Section 5) choosing K over the threshold $n^{\frac{1}{2\beta+2}}$ makes no sense since it does not bring any improvement compared to the standard two-dimensional rate $n^{-\frac{\beta}{2\beta+2}}$. The next theorem gives a bound on the L_1 -risk of the adaptive estimator \hat{f}^* .

Theorem 7. *There exists a constant $C > 0$ such that estimator \hat{f}^* defined in (23) with $K_{\max} = \lceil \sqrt{n} \rceil$ satisfies*

$$\sup_{f \in \mathcal{G}_{K,\beta}} \mathbb{E}_f \|\hat{f}^* - f\|_{L_1} \leq C \left\{ (K/n)^{\beta/(2\beta+1)} \log^{3/2} n \wedge n^{-\frac{\beta}{2\beta+2}} \right\}, \quad \forall K \geq 1, \beta \in (0, 1].$$

Note that the result of Theorem 7 holds for all $K \geq 1$ while the selection leading to \hat{f}^* is made only over the estimators $\hat{f}_{(K,j)}$ with $K \leq K_{\max} = \lceil \sqrt{n} \rceil$. This is because for $K > \lceil \sqrt{n} \rceil$ the estimators $\hat{f}_{(K,j)}$ do not depend on K , cf. the definition of Algorithm 2, and they achieve the same rate as $\hat{f}_{(\lceil \sqrt{n} \rceil, j)}$.

7 Numerical experiments

We present the results of numerical experiments on synthetic data. We have performed simulations with different values of parameters d, n and the number of components K . For the experiments, we use the Python implementation of our algorithm².

Figures 1 and 2 present numerical experiments with discrete distributions. We compare the total variation error of our estimator with that of the classical histogram estimator. In Figure 1, we fix $K = 1$ and $n = 10^5$, and apply the two estimators on square matrices of size ranging from $d = 10$ to $d = 1600$. To better appreciate the dependency on d , we also represent the same experiment on a logarithmic scale in Figure 2. We can see that the total variation error of the histogram estimator is approximately proportional to d , whereas the error of our estimator is approximately proportional to \sqrt{d} for the ranges of values represented in this figure.

Figure 3 presents the dependence of the total variation error on the rank K for fixed dimension $d = 100$ and fixed number of observations $n = 10^5$. We also provide in Figure 4 a representation of this error on a logarithmic scale, which shows that it grows as \sqrt{K} . These two figures are obtained for low-rank matrices close to the set used in the lower bound.

Finally, we provide simulations for the problem of density estimation (Figures 5 and 6). We compare the standard histogram density estimator with bin width $n^{-1/4}$ and our estimator defined by Algorithm 2 with $\beta = 1, K = 1$. We let n vary from 1000 to 10^6 . Again, we observe that our estimator performs better than the classical estimator and allows us to recover the one-dimensional estimation rate $n^{-1/3}$.

²The code of Lowrankdensity algorithm is available at <https://github.com/hi-paris/Lowrankdensity>

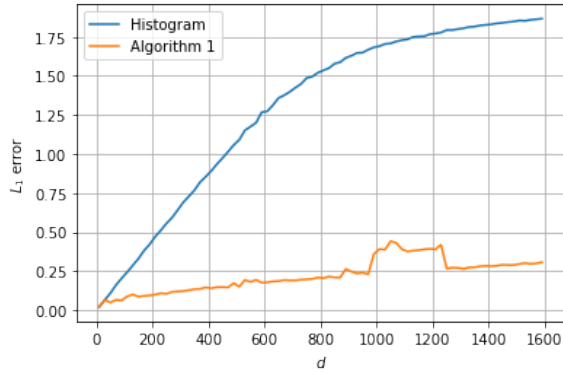


Figure 1: Total variation error of probability matrix estimators as a function of dimension d . Here, $n = 10^5$, $K = 1$.

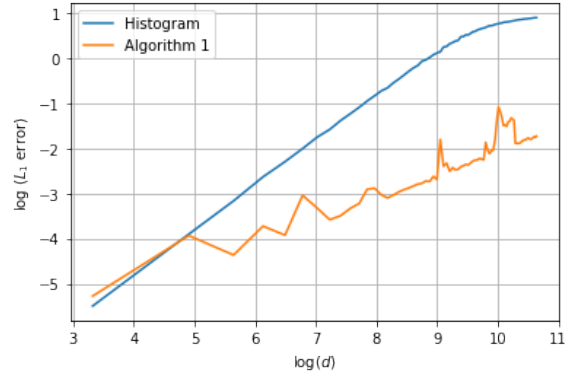


Figure 2: Total variation error of probability matrix estimators as a function of dimension d (on a log-log scale). Here, $n = 10^5$, $K = 1$.

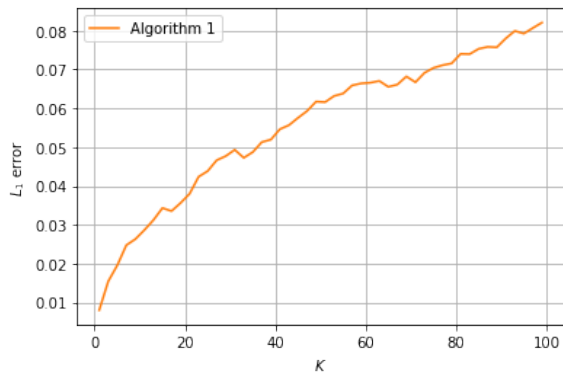


Figure 3: Total variation error of Algorithm 1 as a function of rank K . Here, $d = 100$ and $n = 10^5$.

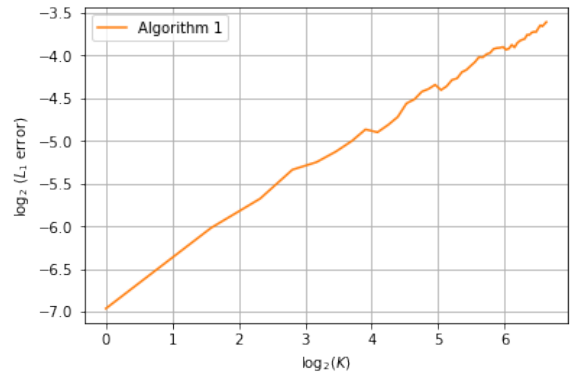


Figure 4: Total variation error of Algorithm 1 as a function of rank K (on a log-log scale). Here, $d = 100$, $n = 10^5$.

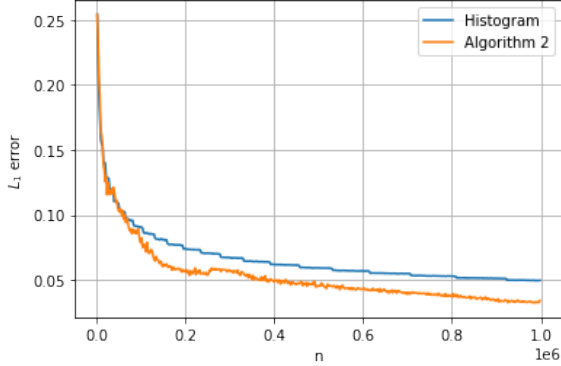


Figure 5: Total variation error of density estimators as a function of n for $K = 1$.

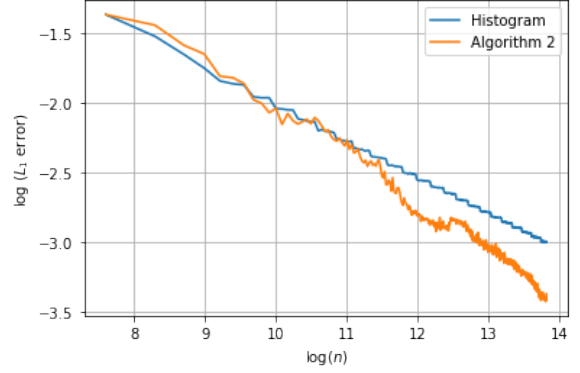


Figure 6: Total variation error of density estimators as a function of n for $K = 1$ (on a log-log scale).

8 Conclusion

In this paper, we obtained minimax near-optimal estimators for the problem of multinomial estimation under low-rank matrix constraints and for density estimation under the generalized multi-view model. In both cases, we demonstrated that the rank constraint can substantially reduce the worst-case estimation error. Our findings suggest that low-rank matrix model can be a powerful tool for dimension reduction in the context of density estimation. It would be interesting to extend our results to larger dimensions, in particular, to low-rank multinomial tensors, which presumably needs developing different tools.

9 Proofs

9.1 Upper bounds for discrete distributions

Proof of Theorem 3. Let \hat{P} be the matrix defined in Algorithm 1. The final estimator \hat{P}^* of Algorithm 1 is obtained from \hat{P} by the following transformation guaranteeing that \hat{P}^* is a probability matrix.

- (a) If $\hat{P}_{ij} \leq 0$ for all $(i, j) \in [d_1] \times [d_2]$ then $\hat{P}^* = \frac{1}{2}(H^{(1)} + H^{(2)})$.
- (b) In the opposite case, if $\hat{P}_{ij} > 0$ for some $(i, j) \in [d_1] \times [d_2]$ then $\hat{P}^* = \frac{\hat{P}_+}{\|\hat{P}_+\|_1}$, where $\hat{P}_+ = (\hat{P}_{ij} \vee 0)_{ij}$.

We first observe that it suffices to prove Theorem 3 for the estimator \hat{P} instead of \hat{P}^* since for any probability matrix P the following inequality holds:

$$\|P - \hat{P}^*\|_1 \leq 2\|P - \hat{P}\|_1. \quad (24)$$

Indeed, in case (a) this inequality is satisfied since $\|\hat{P}^* - P\|_1 \leq \frac{1}{2}(\|H^{(1)}\|_1 + \|H^{(2)}\|_1) + \|P\|_1 = 2$, while $\|\hat{P} - P\|_1 \geq \|P\|_1 = 1$. In case (b) we have

$$\begin{aligned} \|P - \hat{P}^*\|_1 &\leq \|P - \hat{P}_+\|_1 + \|\hat{P}_+ - \hat{P}^*\|_1 \leq \|P - \hat{P}\|_1 + \left\| \hat{P}_+ \left(1 - \frac{1}{\|\hat{P}_+\|_1}\right) \right\|_1 \\ &= \|P - \hat{P}\|_1 + \left| \|\hat{P}_+\|_1 - 1 \right| \leq \|P - \hat{P}\|_1 + \|P - \hat{P}_+\|_1 \end{aligned}$$

$$\leq 2\|P - \widehat{P}\|_1.$$

Also, in both cases (a) and (b) we have $\|\widehat{P}^* - P\|_1 \leq 2$. Therefore, to prove the theorem it suffices to show that $\|\widehat{P} - P\|_1 \leq C_1 \sqrt{\frac{Kd}{n}} \log(d) \log^{1/2}(N)$ with probability at least $1 - C_0(\log d)^2 dN^{-\alpha}$, where the constants $C_0, C_1 > 0$ depend only on α .

Consider first the case $n < 14\alpha d \log(N)$. Then Algorithm 1 outputs the estimator $\widehat{P}^* = \frac{1}{2}(H^{(1)} + H^{(2)})$ satisfying $\|\widehat{P}^* - P\|_1 \leq 2 = 2 \wedge C \sqrt{\frac{Kd}{n}} \log(d) \log^{1/2}(N)$ for a constant $C > 0$ depending only on α . Thus, the result of the theorem is granted if $n < 14\alpha d \log(N)$, and it remains to consider the case $n \geq 14\alpha d \log(N)$.

For the rest of this proof, we assume that $n \geq 14\alpha d \log(N)$. We recall the notation of Section 4.1 and define the random event

$$\mathcal{A} = \left\{ \begin{array}{l} \|\widehat{L}_i^{(1)}\|_1 \geq \frac{1}{4}\|L_i\|_1 \text{ for all } i \in [d_1] \text{ such that } \|L_i\|_1 \geq 14\alpha \frac{\log N}{n} \\ \text{and} \\ \|\widehat{C}_j^{(1)}\|_1 \geq \frac{1}{4}\|C_j\|_1 \text{ for all } j \in [d_2] \text{ such that } \|C_j\|_1 \geq 14\alpha \frac{\log N}{n} \end{array} \right\}.$$

Let us evaluate the probability $\mathbb{P}(\mathcal{A})$. We will use the following lemma proved in Section 10.

Lemma 1. *Let $Y \sim \mathcal{M}(P, n)$ be a multinomial $d_1 \times d_2$ random matrix. To any column $j \in [d_2]$, we associate a subset of row indices $V_j \subseteq [d_1]$ and a random variable $Z_j = \sum_{i \in V_j} Y_{ij}$. We define $\lambda_j = \sum_{i \in V_j} P_{ij} = \frac{1}{n} \mathbb{E}(Z_j)$. Let $\alpha > 0$, $N > 1$ be such that $n > 14\alpha \log(N)$. If $\lambda_j \in [\frac{14\alpha \log(N)}{n}, 1)$ for any $j \in J$, where $J \subseteq [d_2]$, then*

$$\mathbb{P}\left(\forall j \in J : \frac{Z_j}{n} \geq \frac{\lambda_j}{4}\right) \geq 1 - \frac{|J| + 1}{N^\alpha}.$$

By applying Lemma 1 with $V_j = [d_1]$ for all j , $Z_j = n\|\widehat{C}_j^{(1)}\|_1$, $\lambda_j = \|C_j\|_1 = \sum_{i=1}^{d_1} P_{ij}$, and $J = \{j \in [d_2] : \|C_j\|_1 \geq 14\alpha \frac{\log N}{n}\}$ we obtain

$$\mathbb{P}\left(\|\widehat{C}_j^{(1)}\|_1 \geq \frac{1}{4}\|C_j\|_1 \text{ for all } j \in [d_2] \text{ such that } \|C_j\|_1 \geq 14\alpha \frac{\log N}{n}\right) \geq 1 - \frac{d_2 + 1}{N}.$$

Quite analogously,

$$\mathbb{P}\left(\|\widehat{L}_i^{(1)}\|_1 \geq \frac{1}{4}\|L_i\|_1 \text{ for all } i \in [d_1] \text{ such that } \|L_i\|_1 \geq 14\alpha \frac{\log N}{n}\right) \geq 1 - \frac{d_1 + 1}{N}.$$

Therefore, since $d = d_1 \vee d_2 \geq 2$,

$$\mathbb{P}(\mathcal{A}) \geq 1 - \frac{d_1 + d_2 + 2}{N} \geq 1 - 3dN^{-\alpha}. \quad (25)$$

Recall that $T = \lfloor \log_2(d) \rfloor - 1$ and note that, for $n \geq 14\alpha d \log(N)$,

$$\forall t \in \{0, \dots, T+1\} : \frac{1}{2^{t-1}} \geq \frac{1}{d} \geq 14\alpha \frac{\log N}{n}.$$

Fix $k = (t, t') \in \{0, \dots, T+1\}^2$ and $(i, j) \in U_k$. Recalling the definition of U_k in (4) we get $\|\widehat{L}_i^{(1)}\|_1 \leq 2^{-t}$, so that on the event \mathcal{A} we have $\|L_i\|_1 \leq 2^{2-t} \vee 14\alpha \frac{\log N}{n} = 2^{2-t}$, and similarly, $\|C_j\|_1 \leq 2^{2-t'}$. By definition of U_k , we therefore have $\|P^{(k)}\|_{\square} \leq 2^{2-t \wedge t'}$ on the event \mathcal{A} . Moreover, $P^{(k)}$ has rank at most K .

For any fixed $H^{(1)} \in \mathcal{A}$, we apply Lemma 8 with $Q = P^{(k)}$ and $W_Q = W^{(k)} := M^{(k)} - P^{(k)}$. This yields that, with probability at least $1 - 2dN^{-\alpha}$, we have $\|W^{(k)}\|^2 \leq 9 \max \left\{ \|P^{(k)}\|_{\square} \frac{\alpha \log N}{n}, \left(\frac{\alpha \log N}{n} \right)^2 \right\} \leq 9\alpha \frac{\log N}{n} 2^{2-t\wedge t'}$ for any fixed $H^{(1)} \in \mathcal{A}$. By the definition of Algorithm 1,

$$\widehat{P}^{(k)} = \arg \min_{A \in \mathbb{R}^{d_1 \times d_2}} \|M^{(k)} - A\|_F^2 + \tau_k \|A\|_*,$$

where $\tau_k = 6\sqrt{\alpha \frac{\log N}{n} 2^{2-t\wedge t'}}$. Standard deterministic guarantees for nuclear norm penalized estimators (see, for example, [Gir21, Lemma 6.9]) imply that if $\|W^{(k)}\| \leq \tau_k/2$ and $P^{(k)}$ has rank at most K , then there exists an absolute constant $c^* > 0$ such that $\|\widehat{P}^{(k)} - P^{(k)}\|_F^2 \leq c^* K \tau_k^2$. Therefore, we have

$$\|\widehat{P}^{(k)} - P^{(k)}\|_F^2 \leq \frac{C K \log N}{n 2^{t\wedge t'-2}},$$

where the constant $C > 0$ depends only on α and this inequality holds with probability at least $1 - 2dN^{-\alpha}$ for any fixed $H^{(1)} \in \mathcal{A}$.

Now, note that since the entries of $H^{(1)}$ are non-negative and sum to 1 the definitions of I_t and $J_{t'}$ in (2) - (3) imply that $|I_t| \leq 2^{t+1} \wedge d$ and $|J_{t'}| \leq 2^{t'+1} \wedge d$. Thus, for U_k defined in (4) we have $|U_k| \leq (2^{t+1} \wedge d)(2^{t'+1} \wedge d)$. We use this bound to link the ℓ_1 -norm of $\widehat{P}^{(k)} - P^{(k)}$ to its Frobenius norm:

$$\begin{aligned} \|\widehat{P}^{(k)} - P^{(k)}\|_1 &\leq \sqrt{|U_k|} \|\widehat{P}^{(k)} - P^{(k)}\|_F \leq \sqrt{2(2^t \wedge d)(2^{t'} \wedge d)} \sqrt{\frac{C K \log N}{n 2^{t\wedge t'-2}}} \\ &\leq \sqrt{(2^{t\vee t'} \wedge d) \frac{8CK \log N}{n}}. \end{aligned}$$

Using the union bound we get that, for any fixed $H^{(1)} \in \mathcal{A}$, the estimator $\widehat{P} = \sum_{k \in \{0, \dots, T+1\}^2} \widehat{P}^{(k)}$ is such that, with probability at least $1 - 2(T+2)^2 dN^{-\alpha}$,

$$\|\widehat{P} - P\|_1 = \sum_{k \in \{0, \dots, T+1\}^2} \|\widehat{P}^{(k)} - P^{(k)}\|_1 \leq \sqrt{\frac{8CK \log N}{n}} \sum_{(t,t') \in \{0, \dots, T+1\}^2} 2^{(t\vee t')/2}.$$

In view of (25), this bound holds with probability over the joint distribution of $(H^{(1)}, H^{(2)})$, which is at least $1 - (2(T+2)^2 + 3)dN^{-\alpha} \geq 1 - C_0(\log d)^2 dN^{-\alpha}$ for an absolute constant $C_0 > 0$. Note that

$$\begin{aligned} \sum_{(t,t') \in \{0, \dots, T+1\}^2} 2^{(t\vee t')/2} &\leq 2 \sum_{(t,t') \in \{0, \dots, T+1\}^2: t \geq t'} 2^{t/2} = 2 \sum_{t=0}^{T+1} (t+1) 2^{t/2} \\ &\leq 2 \int_0^{T+1} (x+1) 2^{x/2} dx = 2 \left(\frac{2}{\log 2} \right)^2 [2^{(T+1)/2} \left(\frac{\log 2}{2} (T+1) - 1 \right) + 1] + 2 \frac{2}{\log 2} (2^{(T+1)/2} - 1) \\ &= 2^{(T+1)/2} \left(\frac{4}{\log 2} (T+1) + \frac{4}{\log 2} - \frac{8}{(\log 2)^2} \right) - \left(\frac{4}{\log 2} - \frac{8}{(\log 2)^2} \right) \leq \frac{4\sqrt{d} \log_2(d)}{\log 2}. \end{aligned}$$

It follows that $\|\widehat{P} - P\|_1 \leq C_1 \sqrt{\frac{Kd}{n}} \log(d) \log^{1/2}(N)$ with probability at least $1 - C_0(\log d)^2 dN^{-\alpha}$, where $C_0 > 0$ is an absolute constant and $C_1 > 0$ depends only on α . \square

Proof of Theorem 4. We follow the same argument as in the proof of Theorem 3 with the only difference that now we set $N = d \vee n$ in Lemmas 1 and 8. This leads to the bound

$$\|\widehat{P}^* - P\|_1 \leq C_1 \sqrt{\frac{Kd}{n}} \log^{3/2}(d \vee n),$$

which holds with probability at least $1 - C_0(\log(d \vee n))^2 d(d \vee n)^{-\alpha}$. Therefore, since $\|\widehat{P}^* - P\|_1 \leq 2$ and $\alpha > 3/2$ we have

$$\mathbb{E}_P \|\widehat{P}^* - P\|_1 \leq C_1 \sqrt{\frac{Kd}{n}} \log^{3/2}(d \vee n) + 2C_0(\log(d \vee n))^2 n^{1-\alpha} \leq C \sqrt{\frac{Kd}{n}} \log^{3/2}(d \vee n)$$

for some constant $C > 0$ depending only on α . Noticing that $\sqrt{\frac{Kd}{n}} \log^{3/2}(d \vee n) \wedge 1 = \sqrt{\frac{Kd}{n}} \log^{3/2}(n) \wedge 1$ for $d, n \geq 2$ completes the proof. \square

9.2 Lower bounds for discrete distributions

The aim of this subsection is to prove Theorems 1 and 2. Note that it suffices to prove Theorem 2. Indeed, Theorem 1 is obtained as a corollary of Theorem 2 by taking $K = d_1 = 1$ and $d_2 = D$.

Proof of Theorem 2. We first prove the lower bound in expectation (13) and then combine it with Lemma 6 (see Section 10) to deduce the bound in probability (12).

Proof of (13). Note first that the result is trivial if $d_1 = d_2 = 1$. Therefore, assume that $d_2 \geq 2$ and, without loss of generality, $d_2 \geq d_1$. Set $D_2 = \lfloor d_2/2 \rfloor$ and $D = 2KD_2$. Define $\gamma = \left(\frac{1}{4\sqrt{nD}} \wedge \frac{1}{2D} \right)$. For any $\epsilon = (\epsilon_{ij}) \in \{-1, 1\}^{K \times D_2}$, define a $d_1 \times d_2$ matrix P_ϵ with the following entries:

$$\forall (i, j) \in [d_1] \times [d_2] : P_\epsilon(i, j) = \begin{cases} \frac{1}{D} + \epsilon_{ij}\gamma & \text{if } i \leq K \text{ and } j \leq D_2, \\ \frac{1}{D} - \epsilon_{i(j-D_2)}\gamma & \text{if } i \leq K \text{ and } D_2 < j \leq 2D_2, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

Consider the set of $d_1 \times d_2$ matrices

$$\mathcal{P} := \left\{ P_\epsilon \mid \epsilon \in \{-1, 1\}^{K \times D_2} \right\}. \quad (27)$$

This set consists of 2^m matrices, where $m = KD_2$. Note also that $\mathcal{P} \subset \mathcal{T}_K$. Indeed, all matrices $P \in \mathcal{P}$ are of rank K and have non-negative entries summing up to 1. For any $\epsilon, \epsilon' \in \{-1, 1\}^{K \times D_2}$ we have

$$\|P_\epsilon - P_{\epsilon'}\|_1 = 2\gamma\rho(\epsilon, \epsilon'), \quad (28)$$

where $\rho(\epsilon, \epsilon') = \sum_{i=1}^K \sum_{j=1}^{D_2} \mathbb{1}_{\epsilon_{i,j} \neq \epsilon'_{i,j}}$ denotes the Hamming distance between ϵ and ϵ' .

We now apply Assouad's lemma (see Theorem 2.12(iv) in [Tsy09]). Let $\epsilon, \epsilon' \in \{-1, 1\}^{K \times D_2}$ be such that $\rho(\epsilon, \epsilon') = 1$. Denote by (i_0, j_0) , where $i_0 \in [K]$ and $j_0 \in [D_2]$, the unique pair of indices such that $\epsilon_{i_0, j_0} = -\epsilon'_{i_0, j_0}$. Then the χ^2 -divergence between P_ϵ and $P_{\epsilon'}$ satisfies

$$\begin{aligned} \chi^2(P_\epsilon, P_{\epsilon'}) &= \sum_{i=1}^K \sum_{j=1}^{2D_2} \frac{(P_\epsilon(i, j) - P_{\epsilon'}(i, j))^2}{P_{\epsilon'}(i, j)} \\ &= \frac{(P_\epsilon(i_0, j_0) - P_{\epsilon'}(i_0, j_0))^2}{P_{\epsilon'}(i_0, j_0)} + \frac{(P_\epsilon(i_0, j_0 + D_2) - P_{\epsilon'}(i_0, j_0 + D_2))^2}{P_{\epsilon'}(i_0, j_0 + D_2)} \\ &\leq \frac{8\gamma^2}{\frac{1}{D} - \gamma} \leq 16\gamma^2 D \quad \left(\text{since } \gamma \leq \frac{1}{2D} \right) \end{aligned}$$

$$= \frac{1}{n} \wedge \frac{4}{D} \leq \frac{1}{n}.$$

By Lemma 2.7 in [Tsy09], the χ^2 -divergence between the corresponding product measures satisfies

$$\chi^2(P_\epsilon^{\otimes n}, P_{\epsilon'}^{\otimes n}) = \left(1 + \chi^2(P_\epsilon, P_{\epsilon'})\right)^n - 1 \leq e - 1 \quad (29)$$

for all $\epsilon, \epsilon' \in \{-1, 1\}^{K \times D_2}$ such that $\rho(\epsilon, \epsilon') = 1$. Taking into account (28), (29), arguing as in [Tsy09, Example 2.2] we obtain and applying [Tsy09, Theorem 2.12(iv)] we obtain

$$\begin{aligned} \inf_{\tilde{P}} \max_{P \in \mathcal{P}} \mathbb{E}_P \|\tilde{P} - P\|_1 &\geq \frac{\gamma m}{4} \exp(1 - e) = \frac{D}{8} \exp(1 - e) \left(\frac{1}{4\sqrt{nD}} \wedge \frac{1}{2D} \right) \\ &\geq \frac{\exp(1 - e)}{32} \left\{ \sqrt{\frac{D}{n}} \wedge 1 \right\} \geq c \left\{ \sqrt{\frac{Kd_2}{n}} \wedge 1 \right\}, \end{aligned} \quad (30)$$

where $c > 0$ is an absolute constant. This proves (13). \square

Proof of (12). We apply Lemma 6, where we take \mathcal{P}_0 as the set of all $d_1 \times d_2$ matrices, \mathcal{P} as the set of matrices defined in (27), and we consider the metric $v(P, P') = \|P - P'\|_1$. Notice that, under these definitions, assumption (46) of Lemma 6 is satisfied with $U = \left[\frac{1}{D} \mathbb{1}_{i \leq K, j \leq 2D_2}\right]_{i,j}$ and $s = \gamma D = \left(\frac{1}{4} \sqrt{\frac{D}{n}} \wedge \frac{1}{2}\right)$, where D, D_2 are defined in the proof of (13). Moreover, due to (30) there exists an absolute constant $a > 0$ such that assumption (47) of Lemma 6 is satisfied with the same s . Thus, we can apply Lemma 6, which yields the desired lower bound in probability. \square

9.3 Upper bounds for continuous distributions

Proof of Theorem 5. Recall that in Algorithm 2 we assume that n is a multiple of 4. Note first that it suffices to consider the case $\beta > -\frac{2 \log((8L)^{-3/2} \wedge 1)}{\log(n)}$. Indeed, if the interval $(0, -\frac{2 \log((8L)^{-3/2} \wedge 1)}{\log(n)}]$ is non-empty and β belongs to this interval then the desired rate from equation (18) satisfies, for $n \geq 4$,

$$\begin{aligned} \left(\frac{K}{n}\right)^{\beta/(2\beta+1)} \log^{3/2} n \wedge n^{-\frac{\beta}{2\beta+2}} &\geq \left(\frac{1}{n}\right)^\beta \log^{3/2} n \wedge n^{-\frac{\beta}{2}} \\ &\geq ((8L)^{-3/2} \wedge 1)^2 \log^{3/2} 4 \wedge ((8L)^{-3/2} \wedge 1) =: a(L). \end{aligned}$$

On the other hand, since \hat{f} is a probability density we have the trivial bound $\|f - \hat{f}\|_{L_1} \leq 2$ for all probability densities f . Thus, we immediately get (18) with any $C'_1 > 2/a(L)$.

For the rest of this proof, we assume that $\beta > -\frac{2 \log((8L)^{-3/2} \wedge 1)}{\log(n)}$.

Fix a density $f \in \mathcal{G}_K$, and consider the marginal densities $g_1 : [0, 1] \rightarrow \mathbb{R}$ and $g_2 : [0, 1] \rightarrow \mathbb{R}$ defined by

$$g_1(x) = \int_0^1 f(x, y) dy \quad \text{and} \quad g_2(y) = \int_0^1 f(x, y) dx, \quad \forall x, y \in [0, 1].$$

The functions g_1 and g_2 are β -Hölder on $[r_1, R_1]$ and $[r_2, R_2]$, respectively. They are densities of random variables $\Pi_1(X_i)$ and $\Pi_2(X_i)$, respectively, where $\Pi_j(\cdot)$ denotes the projector onto the j -th coordinate. Let $q_-^{(j)}$ and $q_+^{(j)}$ be the quantiles of order $n^{-1/3}$ and $1 - n^{-1/3}$ of the probability measure induced by g_j :

$$\int_{-\infty}^{q_-^{(j)}} g_j = n^{-1/3} \quad \text{and} \quad \int_{q_+^{(j)}}^{+\infty} g_j = n^{-1/3}.$$

Since $f \in \mathcal{G}_K$ it follows from the definition in (16) that there exist real numbers $r_1, r_2, R_1, R_2 \in [0, 1]$ (depending on f) such that $\Delta_j := R_j - r_j > 0$, $j = 1, 2$, and

$$\begin{cases} \text{Supp}(f) = [r_1, R_1] \times [r_2, R_2], \\ f \text{ is } \beta\text{-H\"older over } \text{Supp}(f), \\ \int_{\text{Supp}(f)} f = 1 \text{ and } f \geq 0. \end{cases}$$

For $j = 1, 2$, let \hat{g}_j denote the output of Alg 3($\Pi_j(X_{n/2+1}), \dots, \Pi_j(X_n), K'$). Let \hat{r}_j and \hat{R}_j be the estimators of r_j and R_j defined in Algorithm 2. For the rest of this proof, we place ourselves on the event $\mathcal{E} \cap \mathcal{F}$, where

$$\begin{aligned} \mathcal{E} &= \left\{ \hat{r}_j < q_-^{(j)} < q_+^{(j)} < \hat{R}_j \text{ for } j = 1, 2 \right\}, \\ \mathcal{F} &= \left\{ \|g_j - \hat{g}_j\|_{L_1} \leq C(K'/n)^{\beta/(2\beta+1)} \text{ for } j = 1, 2 \right\}. \end{aligned}$$

Here, $C > 0$ is the constant from Lemma 2 and $K' = K$ if $(\frac{K}{n})^{\frac{\beta}{2\beta+1}} \log^{3/2}(n) \leq n^{-\frac{\beta}{2\beta+2}}$, otherwise $K' = n^{\frac{1}{2\beta+2}}$. By Lemma 2, if $n \geq 64$ and $K'/n \leq (8L)^{-3/2} \wedge 1$ then $\mathbb{P}(\mathcal{F}) \geq 1 - 10 \exp(-n^{1/3})$, and $\mathbb{P}(\mathcal{E}) \geq 1 - 4 \exp(-n^{2/3}/2)$, so that $\mathbb{P}(\mathcal{E} \cap \mathcal{F}) \geq 1 - 14 \exp(-n^{1/3})$. Here, the condition $K'/n \leq (8L)^{-3/2} \wedge 1$ is satisfied because, by the definition of K' , we have $K' \leq n^{\frac{1}{2\beta+2}} \leq n^{1/2} \leq n((8L)^{-3/2} \wedge 1)$, where the last inequality is due to the fact that $1 \geq \beta > -\frac{2 \log((8L)^{-3/2} \wedge 1)}{\log(n)}$.

Let $\hat{\Delta}_j = \hat{R}_j - \hat{r}_j$, $j = 1, 2$. We distinguish between the following two cases.

First case: $\hat{\Delta}_1 \wedge \hat{\Delta}_2 \leq h^*$. It suffices to assume that $\hat{\Delta}_1 \leq h^*$ since the case $\hat{\Delta}_2 \leq h^*$ is treated in the same way. If $\hat{\Delta}_1 \leq h^*$ the estimator $\hat{\phi}$ in Algorithm 2 has the form $\hat{\phi}(x, y) = \frac{1}{\hat{\Delta}_1} \mathbb{1}_{x \in [\hat{r}_1, \hat{R}_1]} \hat{g}_2(y)$ and we get

$$\begin{aligned} \|f - \hat{\phi}\|_{L_1} &\leq \int_{y=0}^1 \int_{x=r_1}^{R_1} \left| f(x, y) - \frac{g_2(y)}{\Delta_1} \right| dx dy + \int_{y=0}^1 \int_{x=r_1}^{R_1} \left| \frac{g_2(y)}{\Delta_1} - \hat{\phi}(x, y) \right| dx dy \\ &\leq \int_{y=0}^1 \int_{x=r_1}^{R_1} \left\{ \frac{1}{\Delta_1} \int_{r_1}^{R_1} \underbrace{|f(x, y) - f(x', y)|}_{\leq L\Delta_1^\beta} dx' \right\} dx dy + \int_{y=0}^1 \int_{x=r_1}^{R_1} \left| \frac{g_2(y)}{\Delta_1} - \frac{\hat{g}_2(y)}{\hat{\Delta}_1} \mathbb{1}_{x \in [\hat{r}_1, \hat{R}_1]} \right| dx dy \\ &\leq \int_{y=0}^1 \int_{x=r_1}^{R_1} L\Delta_1^\beta dx dy + \int_{y=0}^1 \left| g_2(y) \frac{\hat{\Delta}_1}{\Delta_1} - \hat{g}_2(y) \right| dy + \int_{y=0}^1 \int_{x \in [r_1, R_1] \setminus [\hat{r}_1, \hat{R}_1]} \frac{g_2(y)}{\Delta_1} dx dy \\ &\leq L\Delta_1^{\beta+1} + \|g_2 - \hat{g}_2\|_{L_1} + \|\hat{g}_2\|_{L_1} \left| 1 - \frac{\hat{\Delta}_1}{\Delta_1} \right| + \|g_2\|_{L_1} \frac{\Delta_1 - \hat{\Delta}_1}{\Delta_1} \\ &\leq L(h^*)^{\beta+1} \left(1 + 16n^{-1/(2\beta+1)} \right)^{\beta+1} + C \left(\frac{K'}{n} \right)^{\beta/(2\beta+1)} + 32n^{-1/(2\beta+1)} \\ &\leq C' \left(\frac{K'}{n} \right)^{\frac{\beta}{2\beta+1}} \\ &= C' \begin{cases} (K/n)^{\frac{\beta}{2\beta+1}} & \text{if } (K/n)^{\frac{\beta}{2\beta+1}} \log^{3/2}(n) \leq n^{-\frac{\beta}{2\beta+2}} \\ n^{-\frac{\beta}{2\beta+2}} & \text{otherwise} \end{cases} \\ &\leq C' \left(\frac{K}{n} \right)^{\beta/(2\beta+1)} \log^{3/2}(n) \wedge n^{-\frac{\beta}{2\beta+2}}. \end{aligned}$$

where the constant $C' > 0$ depends only on L . Here, we have used the facts that, by Lemma 2, if \mathcal{E} holds and $\widehat{\Delta}_1 \leq h^*$, then $\Delta_1 - \widehat{\Delta}_1 \leq 16\widehat{\Delta}_1 n^{-1/(2\beta+1)} \leq 16\Delta_1 n^{-1/(2\beta+1)}$ and on the event \mathcal{F} we have $\|g_2 - \widehat{g}_2\|_{L_1} \leq C(K'/n)^{\beta/(2\beta+1)}$. We conclude that, in the first case, the bound (18) of Theorem 5 is satisfied.

Second case: $\widehat{\Delta}_1 > h^*$ and $\widehat{\Delta}_2 > h^*$. In this case the estimator $\widehat{\phi}$ in Algorithm 2 has the form

$$\widehat{\phi}(x, y) = \frac{1}{h_1 h_2} \sum_{j \in E_1} \sum_{j' \in E_2} \widehat{P}_{jj'}^* \mathbf{1}_{(x, y) \in C_{jj'}}.$$

Recalling the notation of Algorithm 2 we have

$$b := |E_1| \vee |E_2| \leq 1 + 2 \lceil h_1^{-1} \rceil \vee 2 \lceil h_2^{-1} \rceil \leq C(n/K')^{1/(2\beta+1)}, \quad (31)$$

where $C > 0$ is an absolute constant. Since $f \in \mathcal{G}_{K, \beta}$ we have the representation $f(x, y) = \sum_{k=1}^K u_k(x) v_k(y)$ with some functions $u_k \in L_1[0, 1]$, $v_k \in L_1[0, 1]$ for $k \in [K]$. Introduce the matrix $P = (P_{ij})_{i \in E_1, j \in E_2}$ with entries

$$P_{ij} = \int_{C_{ij}} f(x, y) dx dy = \sum_{k=1}^K \int_{A_i} u_k(x) dx \int_{B_j} v_k(y) dy = \sum_{k=1}^K U_k(i) V_k(j), \quad (i, j) \in E_1 \times E_2,$$

where $U_k(i) = \int_{A_i} u_k(x) dx$ and $V_k(j) = \int_{B_j} v_k(y) dy$ for any $(i, j) \in E_1 \times E_2$ and $k \in [K]$. Set $U_k = (U_k(i))_{i \in E_1}$, $V_k = (V_k(j))_{j \in E_2}$. Then we can write

$$P = \sum_{k=1}^K U_k V_k^\top.$$

Matrix P has rank at most K . Consider now the histogram matrices $G = (G_{ij})_{i \in E_1, j \in E_2}$ and $G' = (G'_{ij})_{i \in E_1, j \in E_2}$ defined in Algorithm 2 with entries

$$G_{ij} = \frac{4}{n} \sum_{\ell=3n/4+1}^{3n/4} \mathbf{1}_{X_\ell \in C_{ij}} \quad \text{and} \quad G'_{ij} = \frac{4}{n} \sum_{\ell=3n/4+1}^n \mathbf{1}_{X_\ell \in C_{ij}}, \quad (i, j) \in E_1 \times E_2.$$

The matrices G and G' are mutually independent, and both $nG/4$ and $nG'/4$ follow the multinomial distribution $\mathcal{M}(P, n/4)$.

To alleviate the notation, we define the following two quantities

$$\psi_{\text{low-rank}} = (K/n)^{\frac{\beta}{2\beta+1}} \log^{3/2}(n) \quad \text{and} \quad \psi_{2D} = n^{-\frac{\beta}{2\beta+2}}.$$

By the definition of Algorithm 2, matrix \widehat{P}^* is the output of $\text{Alg1}(\alpha, b, n, \frac{n}{4}, G, G')$ with $b = |E_1| \vee |E_2|$, and $\alpha > 1$ if $\psi_{\text{low-rank}} \leq \psi_{2D}$ and $\widehat{P}^* = (G + G')/2$ if $\psi_{\text{low-rank}} > \psi_{2D}$. Therefore, if $\psi_{\text{low-rank}} \leq \psi_{2D}$, then Theorem 3 implies that, for some constants $C > 0$ depending only on α ,

$$\|\widehat{P}^* - P\|_1 \leq C \sqrt{\frac{Kb}{n}} \log(b) \log^{1/2}(n) \leq C \left(\frac{K}{n}\right)^{\beta/(2\beta+1)} (\log n)^{3/2}$$

with probability at least $1 - C_0(\log b)^2 b n^{-\alpha}$, where $C_0 > 0$ depends only on α .

If $\psi_{\text{low-rank}} > \psi_{2D}$, then (31) implies that $b \leq C n^{\frac{1}{2\beta+2}}$. Using this fact and Lemma 7 we obtain that, conditionally on $\mathcal{D}_1 = \{X_1, \dots, X_{3n/4}\}$,

$$\mathbb{P}\left(\|\widehat{P}^* - P\|_1 > C n^{\frac{\beta}{2\beta+2}} \mid \mathcal{D}_1\right) \leq \mathbb{P}\left(\|\widehat{P}^* - P\|_1 > \sqrt{\frac{b^2}{n/2}} + \sqrt{\frac{2\alpha \log(n)}{n/2}} \mid \mathcal{D}_1\right) \leq n^{-\alpha},$$

where the constant $C > 0$ depends only on α .

Combining the cases $\psi_{\text{low-rank}} \leq \psi_{2D}$ and $\psi_{\text{low-rank}} > \psi_{2D}$ and using the fact that $\|\widehat{P}^* - P\|_1 \leq 2$ we obtain the bound

$$\mathbb{P}\left(\|\widehat{P}^* - P\|_1 > C(\psi_{\text{low-rank}} \wedge \psi_{2D} \wedge 1)\right) \leq C_0(\log b)^2 bn^{-\alpha}, \quad (32)$$

where the constant $C > 0$ depends only on α . This bound will be used to control the stochastic component $\|\widehat{\phi} - \bar{f}\|_{L_1}$ of the L_1 -error of the estimator $\widehat{\phi}$, where \bar{f} is piecewise constant function defined as follows:

$$\bar{f}(x, y) = \frac{P_{ij}}{h_1 h_2} \quad \text{if } (x, y) \in C_{ij}.$$

We have $\int_{C_{ij}} \bar{f} = P_{ij}$. On the other hand, by the definition of Algorithm 2, $\widehat{\phi}(x, y) = \widehat{P}_{ij}^*/(h_1 h_2)$ for $(x, y) \in C_{i,j}$, and $\int_{C_{ij}} \widehat{\phi} = \widehat{P}_{ij}^*$.

The bias component of the error is $\|f - \bar{f}\|_{L_1}$. In order to control it, we need to distinguish between two cases. Indeed, f can be discontinuous at the boundaries of its rectangular support, which requires separately analyzing the behavior of \bar{f} on the cells C_{ij} that intersect the boundary of $\text{Supp}(f)$. Let $i_0, i_1 \in E_1$ be the indices such that $r_1 \in A_{i_0}$ and $R_1 \in A_{i_1}$ respectively. Similarly, let $j_0, j_1 \in E_2$ be the indices such that $r_2 \in B_{i_0}$ and $R_2 \in B_{i_1}$ respectively. We note that $i_0, j_0 \leq -1$ and that $i_1 \geq \ell_1$ and $j_1 \geq \ell_2$. We let \mathcal{B} denote the indices (i, j) of the cells C_{ij} intersecting the boundary of $\text{Supp}(f)$:

$$\begin{aligned} \mathcal{B} &= \{(i, j) \in E_1 \times E_2 : C_{i,j} \cap \partial \text{Supp}(f) \neq \emptyset\} \\ &= \left\{ (i, j) \in E_1 \times E_2 : \begin{cases} i \in \{i_0, i_1\} \text{ and } j \in [j_0, j_1] \\ \text{or} \\ j \in \{j_0, j_1\} \text{ and } i \in [i_0, i_1] \end{cases} \right\}. \end{aligned}$$

We define $\mathcal{C} = \bigcup_{(i,j) \in \mathcal{B}} C_{ij}$.

We first consider a cell C_{ij} that does not intersect the boundary of $\text{Supp}(f)$, which means that $C_{ij} \in \mathcal{C}^c$, that is, $(i, j) \in (E_1 \times E_2) \setminus \mathcal{B}$. For (x, y) in such cells C_{ij} we have

$$|f(x, y) - \bar{f}(x, y)| = \left| \frac{1}{h_1 h_2} \int_{C_{ij}} (f(x, y) - f(x', y')) dx' dy' \right| \leq CL(h^*)^\beta,$$

which yields that

$$\|f - \bar{f}\|_{L_1(\mathcal{C}^c)} = \sum_{(i,j) \in (E_1 \times E_2) \setminus \mathcal{B}} \int_{C_{ij}} |f(x, y) - \bar{f}(x, y)| dx dy \leq CL(h^*)^\beta. \quad (33)$$

We consider now the opposite case $(i, j) \in \mathcal{B}$, and we analyze the behavior of $\widehat{\phi}$ on \mathcal{C} . Note that, by construction, the sets C_{ij} with $(i, j) \in \mathcal{B}$ cannot belong to the rectangle $[\widehat{r}_1, \widehat{R}_1] \times [\widehat{r}_2, \widehat{R}_2]$, which is included in the interior of $\text{Supp}(f)$ and represents a union of sets C_{ij} with some (i, j) 's. Therefore, on the event \mathcal{E} , we have $\int_{\mathcal{C}} f \leq Cn^{-1/3} \leq n^{-\beta/(2\beta+1)}$ for an absolute constant $C > 0$. It follows that, on the event \mathcal{E} ,

$$\|f - \bar{f}\|_{L_1(\mathcal{C})} \leq \|f\|_{L_1(\mathcal{C})} + \|\bar{f}\|_{L_1(\mathcal{C})} = 2\|f\|_{L_1(\mathcal{C})} \leq Cn^{-\beta/(2\beta+1)}. \quad (34)$$

Combining (32)–(34) we obtain that

$$\|f - \widehat{\phi}\|_{L_1} \leq \|f - \bar{f}\|_{L_1} + \|\bar{f} - \widehat{\phi}\|_{L_1}$$

$$\begin{aligned}
&\leq \|f - \bar{f}\|_{L_1(C^c)} + \|f - \bar{f}\|_{L_1(C)} + \sum_{(i,j) \in E_1 \times E_2} \int_{C_{ij}} |\bar{f} - \hat{\phi}| \\
&\leq CL(h^*)^\beta + Cn^{-\beta}/(2\beta + 1) + \sum_{(i,j) \in E_1 \times E_2} |\hat{P}_{ij}^* - P_{ij}| \\
&\leq C \left(\left(\frac{K}{n} \right)^{\beta/(2\beta+1)} \log^{3/2} n \wedge n^{-\frac{\beta}{2\beta+2}} \right) \tag{35}
\end{aligned}$$

with probability at least $1 - C_0(\log b)^2 b n^{-\alpha} - 4 \exp(-n^{2/3}/2)$. Note also that $b \leq Cn^{1/(2\beta+1)}$. This implies that there exist constants $C'_0 > 0, C > 0$ depending only on α and L such that the bound (35) holds with probability at least $1 - C'_0(\log n)^2 n^{1/(2\beta+1)-\alpha}$. Next, since $\|\hat{\phi}\|_{L_1} \leq 1$ we have $\|f - \hat{\phi}\|_{L_1} \leq 2$. Thus, we can replace the bound in (35) by a stronger bound $C \left\{ \left(\frac{K}{n} \right)^{\beta/(2\beta+1)} \log^{3/2} n \wedge n^{-\frac{\beta}{2\beta+2}} \right\}$ that holds with the same probability. Furthermore, a bound of the same order is satisfied for $\|f - \hat{f}\|_{L_1}$ with the same probability. Indeed, an argument analogous to the proof of (24) yields that $\|f - \hat{f}\|_{L_1} \leq 2\|f - \hat{\phi}\|_{L_1}$. Thus, the bound (18) of Theorem 5 is proved. Next, if $\alpha > 4/3$ then the bound (19) for the expectation follows easily from (18) and the inequality $\|f - \hat{f}\|_{L_1} \leq 2$. Finally, note that the condition $n \geq 64$ used to apply Lemma 2 can be dropped since for $n < 64$ the result of the theorem follows from the trivial bound $\|f - \hat{f}\|_{L_1} \leq 2$. \square

Lemma 2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be a probability density. Assume that for some $r, R \in [0, 1]$, the function f is β -Hölder on $[r, R]$ and that $f = 0$ on $\mathbb{R} \setminus [r, R]$. Let Z_1, \dots, Z_n be iid random variables distributed with probability density f , where $n \geq 64$ is an even integer. Define $\hat{r} = \min \{Z_i : i \in \{1, \dots, \frac{n}{2}\}\}$ and $\hat{R} = \max \{Z_i : i \in \{1, \dots, \frac{n}{2}\}\}$. Let also $\Delta = R - r$ and $\hat{\Delta} = \hat{R} - \hat{r}$. Let q_- and q_+ be the quantiles of order $n^{-1/3}$ and $1 - n^{-1/3}$ of the probability measure induced by f :*

$$\int_{-\infty}^{q_-} f(x) dx = n^{-1/3}, \quad \int_{q_+}^{+\infty} f(x) dx = n^{-1/3}.$$

Define the event $\mathcal{E} = \{\hat{r} < q_- < q_+ < \hat{R}\}$ and set $h^* = (K'/n)^{1/(2\beta+1)}$, where $K' \geq 1$ is such that $K'/n \leq (8L)^{-3/2} \wedge 1$. Let \hat{g} be an output of $\text{Alg3}(Z_1, \dots, Z_n, K')$. Then the following holds.

1. $\mathbb{P}(\mathcal{E}) \geq 1 - 2e^{-n^{2/3}/2}$.
2. On the event \mathcal{E} , if $\hat{\Delta} \leq h^*$ then $\Delta < \hat{\Delta} \left[1 + 16n^{-1/(2\beta+1)} \right]$.
3. There exists a constant $C > 0$ depending only on L such that

$$\mathbb{P} \left(\|f - \hat{g}\|_{L_1} \leq C(K'/n)^{\beta/(2\beta+1)} \right) \geq 1 - 4 \exp(-n^{1/3}).$$

Proof of Lemma 2. The first assertion of the lemma follows from the inequalities

$$\mathbb{P}(\mathcal{E}^c) \leq \mathbb{P}(\hat{r} \geq q_-) + \mathbb{P}(\hat{R} \leq q_+) = \mathbb{P}(Z_1 \geq q_-)^{\frac{n}{2}} + \mathbb{P}(Z_1 \leq q_+)^{\frac{n}{2}} = 2 \left(1 - n^{-1/3} \right)^{\frac{n}{2}} \leq 2e^{-n^{2/3}/2}. \tag{36}$$

We now prove the second assertion of the lemma. On the event \mathcal{E} and if $\hat{\Delta} < h^*$, using the β -Hölder continuity of f we have

$$\frac{1}{2} \leq 1 - 2n^{-1/3} \leq \int_{\hat{r}}^{\hat{R}} f(x) dx \leq \hat{\Delta} \max_{[\hat{r}, \hat{R}]} f$$

$$\begin{aligned}
&\leq \widehat{\Delta} \min_{[\widehat{r}, \widehat{R}]} f + L\widehat{\Delta}^{\beta+1} \leq \widehat{\Delta} \min_{[\widehat{r}, \widehat{R}]} f + L(h^*)^{\beta+1} \\
&\leq \widehat{\Delta} \min_{[\widehat{r}, \widehat{R}]} f + \frac{1}{8},
\end{aligned}$$

where the last inequality follows from the fact that $L(h^*)^{\beta+1} = L(K'/n)^{(\beta+1)/(2\beta+1)} \leq \frac{1}{8}$ due to the assumption $K'/n \leq (8L)^{-3/2} \wedge 1$. Therefore, $f(\widehat{r}) \geq \min_{[\widehat{r}, \widehat{R}]} f \geq \frac{3}{8\widehat{\Delta}}$.

Set $\bar{r} := \widehat{r} - \frac{2}{f(\widehat{r})}n^{-1/(2\beta+1)}$ and let us prove that $r \geq \bar{r}$. Indeed, assume that, on the contrary, $r < \bar{r}$. Then, on the event \mathcal{E} and if $\widehat{\Delta} \leq h^*$, using the β -Hölder continuity of f we get

$$\begin{aligned}
n^{-1/3} &\geq \int_r^{\widehat{r}} f(x)dx \geq \int_{\bar{r}}^{\widehat{r}} f(x)dx \geq (\widehat{r} - \bar{r})f(\widehat{r}) - L(\widehat{r} - \bar{r})^{\beta+1}/(\beta+1) \\
&= 2n^{-1/(2\beta+1)} - \frac{L}{\beta+1} \left(\frac{2}{f(\widehat{r})} \right)^{\beta+1} n^{-(\beta+1)/(2\beta+1)} \\
&\geq 2n^{-1/3} - \frac{L}{\beta+1} \left(\frac{16\widehat{\Delta}}{3} \right)^{\beta+1} n^{-(\beta+1)/(2\beta+1)} \\
&\geq 2n^{-1/3} - \frac{L(h^*)^{\beta+1}}{\beta+1} \left(\frac{16}{3} \right)^{\beta+1} n^{-(\beta+1)/(2\beta+1)} \\
&\geq 2n^{-1/3} - \frac{1}{8(\beta+1)} \left(\frac{16}{3} \right)^{\beta+1} n^{-2/3} > n^{-1/3}
\end{aligned}$$

for all $n \geq 64$ and $\beta \in (0, 1]$, which is a contradiction. Thus, we have $r \geq \bar{r} \geq \widehat{r} - 8\widehat{\Delta}n^{-1/(2\beta+1)}$ and, similarly, $R \leq \widehat{R} + 8\widehat{\Delta}n^{-1/(2\beta+1)}$, which implies the desired inequality $\Delta \leq \widehat{\Delta}(1 + 16n^{-1/(2\beta+1)})$. This concludes the proof of the second assertion of the lemma.

Finally, we prove the third assertion of the lemma. We have, for any $t > 0$,

$$\begin{aligned}
\mathbb{P}(\|f - \widehat{g}\|_{L_1} > t) &\leq \mathbb{P}\left(\{\|f - \widehat{g}\|_{L_1} > t\} \cap \mathcal{E} \cap \{\widehat{\Delta} \leq h^*\}\right) \\
&\quad + \mathbb{P}\left(\{\|f - \widehat{g}\|_{L_1} > t\} \cap \mathcal{E} \cap \{\widehat{\Delta} > h^*\}\right) + \mathbb{P}(\mathcal{E}^c).
\end{aligned} \tag{37}$$

Consider the first probability on the right hand side of (37). Recall that if $\widehat{\Delta} < h^*$ the estimator of Algorithm 3 is $\widehat{g} = \frac{1}{\widehat{R} - \widehat{r}} \mathbf{1}_{[\widehat{r}, \widehat{R}]}$. By continuity of f over $[r, R]$, there exists $x_0 \in [r, R]$ such that $f(x_0)(R - r) = \int_r^R f = 1$. Thus, using the second assertion of the lemma we obtain that, on the event \mathcal{E} and if $\widehat{\Delta} \leq h^*$,

$$\begin{aligned}
\|f - \widehat{g}\|_{L_1} &= \int_r^R |f(x) - \widehat{g}(x)|dx = \int_r^{\widehat{r}} f(x)dx + \int_{\widehat{r}}^{\widehat{R}} |f(x) - \widehat{g}(x)|dx + \int_{\widehat{R}}^R f(x)dx \\
&\leq \int_{\widehat{r}}^{\widehat{R}} |f(x) - f(x_0)|dx + \int_{\widehat{r}}^{\widehat{R}} |f(x_0) - \widehat{g}(x)|dx + 2n^{-1/3} \\
&\leq L|R - r|^{\beta+1} + \int_{\widehat{r}}^{\widehat{R}} |f(x_0) - \widehat{g}(x)|dx + 2n^{-1/3} \\
&\leq L\left(\widehat{\Delta} + 16\widehat{\Delta}n^{-1/(2\beta+1)}\right)^{\beta+1} + (\widehat{R} - \widehat{r}) \left| \frac{1}{R - r} - \frac{1}{\widehat{R} - \widehat{r}} \right| + 2n^{-1/3} \\
&\leq 25L(h^*)^{\beta+1} + \frac{\Delta - \widehat{\Delta}}{\Delta} + 2n^{-1/3} \\
&\leq 25L(h^*)^{\beta+1} + \frac{\widehat{\Delta}16n^{-1/(2\beta+1)}}{\Delta} + 2n^{-1/3}
\end{aligned}$$

$$\begin{aligned} &\leq 25L(K'/n)^{(\beta+1)/(2\beta+1)} + 18n^{-1/3} \quad (\text{since } \widehat{\Delta}/\Delta \leq 1) \\ &\leq (25L + 18)(K'/n)^{\beta/(2\beta+1)} \end{aligned}$$

for all $n \geq 64$ and $\beta \in (0, 1]$. We conclude that the first probability on the right hand side of (37) vanishes for all $t > (25L + 18)(K'/n)^{\beta/(2\beta+1)}$.

Next, consider the second probability on the right hand side of (37). Fix the subsample $\mathcal{D}_1 = (Z_1, \dots, Z_{n/2})$ such that $\widehat{\Delta} > h^*$. Then the partition $(A_j)_{j \in E}$ defined in Algorithm 3 is also fixed, where $E = \{1, \dots, \ell\}$. By continuity of f over A_j , we define $x_j \in A_j$ such that $f(x_j)h = \int_{A_j} f$. For any $j \in E$, we define

$$p_j = \int_{A_j} f(x)dx, \quad \text{and} \quad \widehat{p}_j = \int_{A_j} \widehat{g}(x)dx = \frac{2N_j}{n},$$

where $N_j = \sum_{i=\lfloor n/2 \rfloor + 1}^n \mathbb{1}_{Z_i \in A_j}$, see the definition of Algorithm 3. Note that \widehat{g} is supported on $[\widehat{r}, \widehat{R}]$ and $(A_j)_{j \in E}$ is a partition of $[\widehat{r}, \widehat{R}]$. Then we have the following bound on the estimation error of \widehat{g} , which is valid for any fixed subsample \mathcal{D}_1 such that event \mathcal{E} holds and $\widehat{\Delta} > h^*$:

$$\begin{aligned} \|f - \widehat{g}\|_{L_1} &= \int_r^{\widehat{r}} f(x)dx + \int_{\widehat{R}}^R f(x)dx + \sum_{j \in E} \int_{A_j} |f(x) - \widehat{g}(x)|dx \\ &\leq 2n^{-1/3} + \sum_{j \in E} \int_{A_j} |f(x) - \widehat{g}(x)|dx \quad (\text{since } \mathcal{E} \text{ holds}) \\ &\leq 2n^{-1/3} + \sum_{j \in E} \int_{A_j} (|f(x) - f(x_j)| + |f(x_j) - \widehat{g}(x)|) dx \\ &\leq 2n^{-1/3} + \sum_{j \in E} \left\{ Lh^{\beta+1} + |p_j - \widehat{p}_j| \right\} \\ &\leq 2n^{-1/3} + Lh^\beta + \sum_{j=1}^{\ell} |p_j - \widehat{p}_j| \quad (\text{since } h = \widehat{\Delta}/\ell \leq 1/\ell) \\ &\leq 2n^{-1/3} + 2L(K'/n)^{\beta/(2\beta+1)} + \sum_{j=1}^{\ell} |p_j - \widehat{p}_j|, \end{aligned} \tag{38}$$

where the last inequality uses the fact that $h = \frac{\widehat{\Delta}}{\lfloor \widehat{\Delta}/h^* \rfloor} = h^* \frac{\widehat{\Delta}/h^*}{\lfloor \widehat{\Delta}/h^* \rfloor} \leq 2h^* = 2(K'/n)^{1/(2\beta+1)}$ for $\widehat{\Delta} > h^*$. Lemma 7 stated below yields that, for any $\delta \in (0, 1)$,

$$\mathbb{P} \left(\sum_{j=1}^{\ell} |p_j - \widehat{p}_j| > 2\sqrt{\frac{2(\ell + \log(1/\delta))}{n}} \mid \mathcal{D}_1 \right) \leq \delta.$$

Set $\delta = \exp(-n^{1/(2\beta+1)})$. Then combining this bound with (38) and the fact that $\ell = \lfloor \widehat{\Delta}/h^* \rfloor \leq 1/h^* \leq n^{1/(2\beta+1)}$ we obtain that there exists a constant $C_* > 0$ depending only on L such that

$$\mathbb{P} \left(\|f - \widehat{g}\|_{L_1} > C_*(K'/n)^{\beta/(2\beta+1)} \mid \mathcal{D}_1 \right) \leq \exp(-n^{1/(2\beta+1)}) \quad \text{if } \mathcal{D}_1 \text{ is such that } \mathcal{E} \text{ holds and } \widehat{\Delta} > h^*.$$

It follows that

$$\mathbb{P} \left(\{ \|f - \widehat{g}\|_{L_1} > C_*(K'/n)^{\beta/(2\beta+1)} \} \cap \mathcal{E} \cap \{ \widehat{\Delta} > h^* \} \right) \leq \exp(-n^{1/(2\beta+1)}).$$

Plugging this bound and (36) in (37) and recalling that the first probability on the right hand side of (37) vanishes for $t > (25L + 18)(K'/n)^{\beta/(2\beta+1)}$ we find that, for $C > \max(25L + 18, C_*)$,

$$\mathbb{P}(\|f - \hat{g}\|_{L_1} > C(K'/n)^{\beta/(2\beta+1)}) \leq \exp(-n^{1/(2\beta+1)}) + \mathbb{P}(\mathcal{E}^c) \leq 4\exp(-n^{1/3}),$$

where we have used the first assertion of the lemma and the fact that $\exp(-n^{2/3}/2) \leq \exp(-n^{1/3})$ for $n \geq 8$. \square

9.4 Lower bounds for continuous distributions

Proof of Theorem 6. Without loss of generality, we prove the lower bound over the class $\mathcal{G}_{K+1,\beta}$ rather than over $\mathcal{G}_{K,\beta}$. We define $f_0 = \mathbb{1}_{[0,1]^2}$ and set $K' = \lfloor n^{1/(2\beta+2)} \rfloor \wedge K$, $H = \lceil (n/K')^{1/(2\beta+1)} \rceil$, $h_x = 1/K'$, $h_y = 1/H$. Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be an infinitely many times differentiable function with support $(-1/2, 1/2)$ satisfying the β -Hölder condition with Hölder constant $1/2$ (see (2.34) in [Tsy09] for an example of such function). For $i, j \in \mathbb{Z}$, set $V_{i,h_x}(x) = \varphi\left(\frac{x-x_i^-}{h_x/2}\right) - \varphi\left(\frac{x-x_i^+}{h_x/2}\right)$, where $x_i^- = (i - \frac{3}{4})h_x$ and $x_i^+ = (i - \frac{1}{4})h_x$ and analogously $V_{j,h_y}(y) = \varphi\left(\frac{y-y_j^-}{h_y/2}\right) - \varphi\left(\frac{y-y_j^+}{h_y/2}\right)$, where $y_j^- = (j - \frac{3}{4})h_y$ and $y_j^+ = (j - \frac{1}{4})h_y$. The supports of functions $\varphi\left(\frac{\cdot-x_i^-}{h_x/2}\right)$ and $\varphi\left(\frac{\cdot-x_i^+}{h_x/2}\right)$ are disjoint and each of these functions is β -Hölder with Hölder constant $(1/2)(h_x/2)^{-\beta}$. Hence, $V_{i,h_x}(\cdot)$ is β -Hölder with Hölder constant $(h_x/2)^{-\beta}$. Note also that the functions V_{i,h_x} integrate to 0 and have disjoint supports in $[0, 1]$ for different i 's. The same is true for the functions V_{j,h_y} .

Set $c_* = 1/4 \wedge 1/(2L)$. For $\omega = (\omega_{ij})$, where $\omega_{ij} \in \{0, 1\}$ for all i, j , we define the functions f_ω as follows:

$$f_\omega(x, y) = f_0(x, y) + c_* L h_y^\beta \sum_{i=1}^{K'} \sum_{j=1}^H \omega_{ij} V_{i,h_x}(x) V_{j,h_y}(y).$$

First, we check that each f_ω belongs to $\mathcal{G}_{K'+1,\beta} \subseteq \mathcal{G}_{K+1,\beta}$.

- We have $\int_{[0,1]^2} f_\omega = 1$ by construction.
- We have $f_\omega(x, y) \geq 1/2$ for all $(x, y) \in [0, 1]^2$ since $c_* \leq (2L)^{-1}$, $h_y \leq 1$, and the functions $(x, y) \mapsto V_{i,h_x}(x)V_{j,h_y}(y)$ have disjoint supports and take values in $[-1, 1]$.
- We now check the Hölder condition. As the supports C_{ij} of functions $(x, y) \mapsto V_{i,h_x}(x)V_{j,h_y}(y)$ are disjoint and $f_\omega = 1$ at the boundary of each cell C_{ij} , it suffices to show that f_ω is β -Hölder with Hölder constant $L/2$ on each cell C_{ij} to obtain that f_ω is β -Hölder with Hölder constant L on $[0, 1]^2$. Fix any two points $z, z' \in [0, 1]^2$, belonging to the same cell C_{ij} . Then, writing $z = (x, y)$, $z' = (x', y')$ and recalling that V_{i,h_x}, V_{j,h_y} take values in $[-1, 1]$ and are β -Hölder with Hölder constants $(h_x/2)^{-\beta}$, $(h_y/2)^{-\beta}$ we obtain:

$$\begin{aligned} |f_\omega(z) - f_\omega(z')| &= c_* L h_y^\beta \left| V_{i,h_x}(x)V_{j,h_y}(y) - V_{i,h_x}(x')V_{j,h_y}(y') \right| \\ &\leq c_* L h_y^\beta \left| V_{i,h_x}(x) - V_{i,h_x}(x') \right| + c_* L h_y^\beta \left| V_{j,h_y}(y) - V_{j,h_y}(y') \right| \\ &\leq c_* L h_y^\beta \left[\left(\frac{|x-x'|}{h_x/2} \right)^\beta + \left(\frac{|y-y'|}{h_y/2} \right)^\beta \right] \\ &\leq \frac{L}{2} \|z - z'\|_\infty^\beta, \end{aligned}$$

where we have used the facts that $h_y \leq h_x$ and $c_* \leq 1/4$.

- Function f_ω belongs to $\mathcal{F}_{K'+1} \subseteq \mathcal{F}_{K+1}$ since it admits a separable representation

$$f_\omega(x, y) = f_0(x, y) + \sum_{i=1}^{K'} U_i(x) \left(\sum_{j=1}^H c_* L h_y^\beta \omega_{ij} V_{j,h_y}(y) \right), \quad U_i(x) := V_{i,h_x}(x).$$

Thus, $\mathcal{G}' := \{f_\omega : \omega \in \{0, 1\}^{K' \times H}\}$ is a subset of $\mathcal{G}_{K'+1, \beta}$, and it suffices to prove the lower bound with the required rate on this subset. We first prove the lower bound in expectation (21). We use the version of Assouad's lemma given in [Tsy09, Theorem 2.12(iv)]. For any $\omega, \omega' \in \{0, 1\}^{K' \times H}$ that only differ in one entry, that is, for exactly one (i_0, j_0) , we have $\omega_{i_0, j_0} \neq \omega'_{i_0, j_0}$ and $\omega_{i, j} = \omega'_{i, j}$ for all other i, j , the χ^2 -divergence between the densities f_ω and $f_{\omega'}$ is bounded as follows:

$$\chi^2(f_{\omega'}, f_\omega) = \int \frac{(c_* L h_y^\beta V_{i_0, h_x}(x) V_{j_0, h_y}(y))^2}{f_\omega(x, y)} \leq \int_{C_{i_0, j_0}} 2(c_* L h_y^\beta)^2 = 2(c_* L h_y^\beta)^2 h_x h_y \leq \frac{\tilde{c}}{n},$$

where $\tilde{c} > 0$ depends only on L, β . Using Lemma 2.7 in [Tsy09] we obtain that the χ^2 -divergence between the corresponding product densities satisfies

$$\chi^2(f_{\omega'}^{\otimes n}, f_\omega^{\otimes n}) = \left(1 + \chi^2(f_{\omega'}, f_\omega)\right)^n - 1 \leq e^{\tilde{c}} - 1.$$

Also, note that for all i, j we have

$$\|V_{i, h_x}(x) V_{j, h_y}(y)\|_{L_1} = 4 \int \varphi\left(\frac{x}{h_x/2}\right) dx \int \varphi\left(\frac{y}{h_y/2}\right) dy = 16 h_x h_y \left(\int \varphi(x) dx\right)^2.$$

Therefore, applying [Tsy09, Theorem 2.12(iv)] and arguing as in [Tsy09, Example 2.2] we obtain

$$\inf_{\hat{f}} \sup_{f_\omega \in \mathcal{G}'} \mathbb{E} \|\hat{f} - f_\omega\|_{L_1} \geq \frac{K' H}{8} (c_* L h_y^\beta) c'' h_x h_y = \frac{c_* c''}{8} L h_y^\beta \geq c \left((K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)} \right), \quad (39)$$

where $c'' = 16 \left(\int \varphi(x) dx\right)^2 \exp(1 - e^{\tilde{c}})$ and $c > 0$ is a constant depending only on L and β . This implies the lower bound in expectation (21).

To obtain the lower bound in probability (20) we apply Lemma 6 with $\mathcal{P} = \mathcal{P}_0 = \mathcal{G}'$, $v(f, g) = \|f - g\|_{L_1}$ for any $f, g \in \mathcal{G}'$, and $U = \mathbb{1}_{[0, 1]^2} \in \mathcal{G}'$. Note that there exists a constant $C > 0$ such that for any $f \in \mathcal{G}'$ we have $v(U, f) \leq s$, where $s = C \left((K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)} \right)$. Therefore, taking into account the bound in expectation (39) we can apply Lemma 6 with a small enough constant $a > 0$ to get (20). \square

9.5 Upper bound for the adaptive estimator

Lemma 3. *Let $0 < \delta < 1$. For any probability density f we have that, with \mathbb{P}_f -probability at least $1 - \delta$,*

$$\|\hat{f}^* - f\|_{L_1} \leq 3 \min_{(K, j)} \|\hat{f}_{(K, j)} - f\|_{L_1} + 2 \sqrt{\frac{2 \log(2m^2/\delta)}{n}}, \quad (40)$$

and

$$\mathbb{E}_f \|\hat{f}^* - f\|_{L_1} \leq 3 \min_{(K, j)} \mathbb{E}_f \|\hat{f}_{(K, j)} - f\|_{L_1} + 2 \sqrt{\frac{2 \log(2m^2)}{n}}. \quad (41)$$

Proof. This lemma is a simple corollary of Theorem 6.3 in [DL01], which states that

$$\|\hat{f}^* - f\|_{L_1} \leq 3 \min_{(K, j)} \|\hat{f}_{(K, j)} - f\|_{L_1} + 4\Delta,$$

where $\Delta = \max_{B \in \mathcal{B}} \left| \int_B f - \mathbb{P}_n(B) \right|$. Since $|\mathcal{B}| = m(m-1)$ it follows from Hoeffding's inequality and the union bound that, for any $t > 0$,

$$\mathbb{P}_f(\Delta > t) \leq 2m(m-1) \exp(-2nt^2).$$

This proves (40). Inequality (41) follows from the fact that $\mathbb{E}_f \max_{B \in \mathcal{B}} \left| \int_B f - \mathbb{P}_n(B) \right| \leq \sqrt{\frac{\log(2|\mathcal{B}|)}{2n}}$ (see [DL01, Lemma 2.2]). \square

Proof of Theorem 7. To establish the adaptivity of \widehat{f}^* , we will use the inclusions between the Hölder classes:

$$\forall 0 < \beta \leq \beta' \leq 1: \quad \mathcal{L}_{\beta'} \subseteq \mathcal{L}_{\beta}.$$

Indeed, for any $0 < \beta \leq \beta' \leq 1$, any $f \in \mathcal{L}_{\beta'}$ and any $z, z' \in \text{Supp}(f)$, we have $\|z - z'\|_{\infty} \leq 1$, and consequently

$$|f(z) - f(z')| \leq L \|z - z'\|_{\infty}^{\beta'} \leq L \|z - z'\|_{\infty}^{\beta}.$$

The embedding of the classes $(\mathcal{G}_{K,\beta})_{\beta \in (0,1]}$ immediately follows:

$$\forall 0 < \beta \leq \beta' \leq 1, \forall K \in \mathbb{N}: \quad \mathcal{G}_{K,\beta'} \subseteq \mathcal{G}_{K,\beta}.$$

Assume first that $\beta \in [\beta_j, \beta_{j-1}]$ for some $j \in \{2, \dots, \lceil \log(n) \log \log(n) \rceil\}$. Using the inclusion $\mathcal{G}_{K,\beta} \subseteq \mathcal{G}_{K,\beta_j}$ and (41) we can bound from above the risk of \widehat{f}^* over the class $\mathcal{G}_{K,\beta}$ as follows:

$$\begin{aligned} \sup_{f \in \mathcal{G}_{K,\beta}} \mathbb{E}_f \|\widehat{f}^* - f\|_{L_1} &\leq \sup_{f \in \mathcal{G}_{K,\beta}} \left\{ 3 \min_{(K,i)} \mathbb{E}_f \|\widehat{f}_{(K,i)} - f\|_{L_1} + C \sqrt{\frac{\log(m)}{n}} \right\} \\ &\leq \sup_{f \in \mathcal{G}_{K,\beta}} 3 \mathbb{E}_f \|\widehat{f}_{(K,j)} - f\|_{L_1} + C \sqrt{\frac{\log(K_{\max} \log(n) \log \log(n))}{n}} \\ &\leq \sup_{f \in \mathcal{G}_{K,\beta_j}} 3 \mathbb{E}_f \|\widehat{f}_{(K,j)} - f\|_{L_1} + C \sqrt{\frac{\log(n)}{n}} \quad \text{since } K_{\max} = \lceil \sqrt{n} \rceil \\ &\leq C \left\{ \left(\frac{K}{n} \right)^{\beta_j/(2\beta_j+1)} \log^{3/2} n \wedge n^{-\frac{\beta_j}{2\beta_j+2}} \right\} \quad \text{by Theorem 5.} \end{aligned} \quad (42)$$

Note that, for $c \in \{1, 2\}$,

$$\frac{\beta}{2\beta + c} - \frac{\beta_j}{2\beta_j + c} \leq \frac{c(\beta_{j-1} - \beta_j)}{(2\beta + c)(2\beta_j + c)} = \frac{c\beta_j}{(2\beta + c)(2\beta_j + c) \log(n)} \leq \frac{1}{\log(n)}.$$

Combining this with the bound (42) we obtain

$$\begin{aligned} \sup_{f \in \mathcal{G}_{K,\beta}} \mathbb{E} \|\widehat{f}^* - f\|_{L_1} &\leq C \left\{ (n/K)^{-\beta/(2\beta+1) + \frac{1}{\log(n)}} (\log n)^{3/2} \wedge n^{-\frac{\beta}{2\beta+2} + \frac{1}{\log(n)}} \right\} \\ &\leq C \left\{ (n/K)^{-\beta/(2\beta+1)} (\log n)^{3/2} \wedge n^{-\frac{\beta}{2\beta+2}} \right\}. \end{aligned}$$

This proves the theorem for $\beta \in [\beta_M, \beta_1] = [\beta_M, 1]$, where $M = \lceil \log(n) \log \log(n) \rceil$. Finally, consider the values $\beta \in (0, \beta_M)$. We have

$$\frac{\beta}{2\beta + 2} \leq \frac{\beta}{2\beta + 1} \leq \frac{\beta_M}{2\beta_M + 1} \leq \beta_M = \left(1 + \frac{1}{\log(n)} \right)^{-M} \leq \exp \left(-\frac{\log(n) \log \log(n)}{\log(n)} \right) = \frac{1}{\log(n)}.$$

Thus, for $\beta \in (0, \beta_M)$,

$$(K/n)^{\beta/(2\beta+1)} \wedge n^{-\beta/(2\beta+2)} \geq n^{-\frac{1}{\log(n)}} = 1/e.$$

The desired result follows immediately from this inequality and the fact that $\sup_{f \in \mathcal{G}_{K,\beta}} \mathbb{E}_f \|\widehat{f}^* - f\|_{L_1} \leq 2$. \square

10 Auxiliary results

First, we recall the definition of the multinomial distribution. Given a finite set Ω and a probability distribution $P = (P_\omega)_{\omega \in \Omega}$ on Ω , we say that a random vector $Y = (Y_\omega)_{\omega \in \Omega}$ follows the multinomial distribution $\mathcal{M}(P, n)$, $n \in \mathbb{N}^*$, if for all $(n_\omega)_{\omega \in \Omega} \in \mathbb{N}^\Omega$ satisfying $\sum_{\omega \in \Omega} n_\omega = n$, we have:

$$\mathbb{P}(Y_\omega = n_\omega, \forall \omega \in \Omega) = \frac{n!}{\prod_{\omega \in \Omega} n_\omega!} \prod_{\omega \in \Omega} P_\omega^{n_\omega}.$$

We denote by $\text{Poi}(\lambda)$ the Poisson distribution with mean λ .

Lemma 4 ([SW09], p. 486). *Let $\zeta \sim \text{Poi}(\lambda)$ be a Poisson random variable with mean λ . Then*

$$\mathbb{P}(\zeta \leq \lambda - x) \leq \exp\left(-x - (\lambda - x) \log\left(\frac{\lambda - x}{\lambda}\right)\right), \quad \forall x \in (0, \lambda), \quad (43)$$

$$\mathbb{P}(\zeta \geq \lambda + x) \leq \exp\left(x - (\lambda + x) \log\left(\frac{\lambda + x}{\lambda}\right)\right), \quad \forall x > 0. \quad (44)$$

Lemma 5 (Poissonization). *Let $d \geq 2$, $\lambda > 0$, and let $p = (p_1, \dots, p_d)$ be a probability distribution on $\Omega = \{1, \dots, d\}$. Let $\tilde{n} \sim \text{Poi}(\lambda)$ and let $\tilde{Y} = (\tilde{Y}_1, \dots, \tilde{Y}_d)$ be a random vector such that $\tilde{Y} | \tilde{n} \sim \mathcal{M}(p, \tilde{n})$. Then the entries \tilde{Y}_j are mutually independent and $\tilde{Y}_j \sim \text{Poi}(\lambda p_j)$ for all $j \in \{1, \dots, d\}$.*

Proof. For any $(n_1, \dots, n_d) \in \mathbb{N}^d$ we have

$$\begin{aligned} \mathbb{P}(\tilde{Y}_1 = n_1, \dots, \tilde{Y}_d = n_d) &= \sum_{k=0}^{\infty} \mathbb{P}(\tilde{n} = k) \mathbb{P}(\tilde{Y}_1 = n_1, \dots, \tilde{Y}_d = n_d | \tilde{n} = k) \\ &= \sum_{k=0}^{\infty} \left(\frac{k!}{n_1! \dots n_d!} \prod_{j=1}^d p_j^{n_j} \right) \frac{e^{-\lambda} \lambda^k}{k!} \mathbb{1}_{n_1 + \dots + n_d = k} \\ &= \left(\frac{\lambda^{n_1} \dots \lambda^{n_d}}{n_1! \dots n_d!} \prod_{j=1}^d p_j^{n_j} \right) e^{-\lambda} \sum_{k=0}^{\infty} \mathbb{1}_{n_1 + \dots + n_d = k} \\ &= \prod_{j=1}^d \frac{e^{-\lambda p_j} (\lambda p_j)^{n_j}}{n_j!}. \end{aligned}$$

□

Proof of Lemma 1. The elements of the sum $Z_j = \sum_{i \in V_j} Y_{ij}$ are not mutually independent. To overcome this difficulty, we use poissonization. Let $\tilde{n} \sim \text{Poi}(n/2)$ and let $\tilde{Y} | \tilde{n} \sim \mathcal{M}(P, \tilde{n})$. We define $\tilde{Z}_j = \sum_{i \in V_j} \tilde{Y}_{ij}$ for any $j \in J$. Lemma 5 implies that $\tilde{Y}_{ij} \sim \text{Poi}(nP_{ij}/2)$ for all i, j , and the random variables $(\tilde{Y}_{ij})_{i,j}$ are mutually independent. It follows that \tilde{Z}_j has a Poisson distribution: $\tilde{Z}_j \sim \text{Poi}(\sum_{i \in V_j} nP_{ij}/2)$, and the random variables $(\tilde{Z}_j)_j$ are mutually independent. At the same time, for any $k \geq 1$ the conditional distribution of \tilde{Z}_j given $\tilde{n} = k$ coincides with the distribution of Z_j under $Y \sim \mathcal{M}(P, k)$.

It follows from (43) with $x = \lambda/2$ and $\lambda = n\lambda_j/2$ that

$$\mathbb{P}\left(\tilde{Z}_j \leq \frac{n\lambda_j}{4}\right) \leq \exp\left(-\frac{n\lambda_j}{4}(1 - \log 2)\right) \leq \exp\left(-\frac{7}{2}(1 - \log 2)\alpha \log(N)\right) < N^{-\alpha}.$$

Hence,

$$\mathbb{P}\left(\forall j \in J : \tilde{Z}_j \geq \frac{n\lambda_j}{4}\right) \geq 1 - \frac{|J|}{N^\alpha}.$$

On the other hand,

$$\begin{aligned}
\mathbb{P}\left(\forall j \in J : \tilde{Z}_j \geq \frac{n\lambda_j}{4}\right) &\leq \mathbb{P}(\tilde{n} > n) + \mathbb{P}\left(\forall j \in J : \tilde{Z}_j \geq \frac{n\lambda_j}{4}, \text{ and } \tilde{n} \leq n\right) \\
&= \mathbb{P}(\tilde{n} > n) + \sum_{k \leq n} \mathbb{P}\left(\forall j \in J : \tilde{Z}_j \geq \frac{n\lambda_j}{4} \mid \tilde{n} = k\right) \mathbb{P}(\tilde{n} = k) \\
&= \mathbb{P}(\tilde{n} > n) + \sum_{k \leq n} \mathbb{P}_{Z \sim \mathcal{M}(P, k)}\left(\forall j \in J : Z_j \geq \frac{n\lambda_j}{4}\right) \mathbb{P}(\tilde{n} = k) \\
&\leq \mathbb{P}(\tilde{n} > n) + \mathbb{P}_{Y \sim \mathcal{M}(P, n)}\left(\forall j \in J : Z_j \geq \frac{n\lambda_j}{4}\right), \tag{45}
\end{aligned}$$

where we have used the inequality $\mathbb{P}_{Y \sim \mathcal{M}(P, k)}\left(\forall j \in J : Z_j \geq \frac{n\lambda_j}{4}\right) \leq \mathbb{P}_{Y \sim \mathcal{M}(P, n)}\left(\forall j \in J : Z_j \geq \frac{n\lambda_j}{4}\right)$ that holds for all $k \leq n$ due to stochastic dominance since, under $Y \sim \mathcal{M}(P, k)$, each Y_{ij} has a binomial distribution with parameters (P_{ij}, k) . It follows that

$$\mathbb{P}_{Y \sim \mathcal{M}(P, n)}\left(\forall j \in J : Z_j \geq \frac{n\lambda_j}{4}\right) \geq 1 - \frac{|J|}{N^\alpha} - \mathbb{P}(\tilde{n} > n).$$

Applying (44) with $\zeta = \tilde{n}$ and $\lambda = x = n/2$ we get

$$\mathbb{P}(\tilde{n} > n) \leq \exp(n(1/2 - \log 2)) \leq \exp(14(1/2 - \log 2)\alpha \log(N)) \leq \frac{1}{N^\alpha}.$$

Combining the last two displays yields the lemma. \square

The following lemma, that may be of independent interest, provides a tool for deducing lower bounds in probability from lower bounds in expectation.

Lemma 6 (Deducing lower bound in probability from lower bound in expectation). *Let \mathcal{P}_0 be a metric space with metric $v : \mathcal{P}_0 \times \mathcal{P}_0 \rightarrow \mathbb{R}_+$, and let \mathcal{P} be a subset of \mathcal{P}_0 with the property that there exists $U \in \mathcal{P}_0$ such that*

$$v(U, P) \leq s, \quad \forall P \in \mathcal{P}, \tag{46}$$

where $s > 0$. Let $\{\mathbb{P}_P, P \in \mathcal{P}\}$ be a family of probability measures indexed by \mathcal{P} . Assume that

$$\inf_{\tilde{P}} \sup_{P \in \mathcal{P}} \mathbb{E}_P v(\tilde{P}(Y), P) \geq as, \tag{47}$$

where $a > 0$, \mathbb{E}_P denotes the expectation with respect to random variable Y distributed according to \mathbb{P}_P , and $\inf_{\tilde{P}}$ is the infimum over all estimators \tilde{P} that take values in \mathcal{P}_0 . Then

$$\inf_{\tilde{P}} \sup_{P \in \mathcal{P}} \mathbb{P}_P(v(\tilde{P}(Y), P) \geq as/2) \geq a/6.$$

Proof. Consider the set $\mathcal{P}' = \{P' \in \mathcal{P}_0 : v(P', P) \leq 3s, \forall P \in \mathcal{P}\}$. This set is not empty since it contains U . Note that it is sufficient to consider estimators taking values in \mathcal{P}' , that is, for any estimator \tilde{P} there exists an estimator \bar{P} with values in \mathcal{P}' such that

$$v(\bar{P}, P) \leq v(\tilde{P}, P), \quad \forall P \in \mathcal{P}. \tag{48}$$

In fact, let \tilde{P} be any estimator. Define another estimator

$$\bar{P} = \begin{cases} \tilde{P} & \text{if } v(\tilde{P}, U) \leq 2s, \\ U & \text{if } v(\tilde{P}, U) > 2s. \end{cases}$$

Let us check that this estimator \tilde{P} satisfies (48) and $\tilde{P} \in \mathcal{P}'$. Indeed, if $v(\tilde{P}, U) \leq 2s$ then (48) obviously holds, and we have $v(\tilde{P}, P) \leq v(\tilde{P}, U) + v(P, U) \leq 3s$ for all $P \in \mathcal{P}$. Otherwise, if $v(\tilde{P}, U) > 2s$ then $v(\tilde{P}, P) = v(U, P) \leq 2s - s < v(\tilde{P}, U) - v(U, P) \leq v(\tilde{P}, P)$ for all $P \in \mathcal{P}$.

It follows that

$$\begin{aligned} \inf_{\tilde{P}} \sup_{P \in \mathcal{P}} \mathbb{E}_P v(\tilde{P}, P) &= \inf_{\tilde{P} \in \mathcal{P}'} \sup_{P \in \mathcal{P}} \mathbb{E}_P v(\tilde{P}, P), \\ \inf_{\tilde{P}} \sup_{P \in \mathcal{P}} \mathbb{P}_P(v(\tilde{P}, P) \geq as/2) &= \inf_{\tilde{P} \in \mathcal{P}'} \sup_{P \in \mathcal{P}} \mathbb{P}_P(v(\tilde{P}, P) \geq as/2). \end{aligned} \quad (49)$$

Thus, we have

$$\begin{aligned} as &\leq \inf_{\tilde{P} \in \mathcal{P}'} \sup_{P \in \mathcal{P}} \mathbb{E}_P v(\tilde{P}, P) \\ &= \inf_{\tilde{P} \in \mathcal{P}'} \sup_{P \in \mathcal{P}} \left\{ \mathbb{E}_P \left[v(\tilde{P}, P) \mathbf{1}_{\{v(\tilde{P}, P) \geq as/2\}} \right] + \mathbb{E}_P \left[v(\tilde{P}, P) \mathbf{1}_{\{v(\tilde{P}, P) < as/2\}} \right] \right\} \\ &\leq 3s \inf_{\tilde{P} \in \mathcal{P}'} \sup_{P \in \mathcal{P}} \mathbb{P}_P(v(\tilde{P}, P) \geq as/2) + as/2, \end{aligned}$$

which together with (49) implies the lemma. \square

Lemma 7. *Let Z_1, \dots, Z_m be iid random variables on a measurable space $(\mathcal{Z}, \mathcal{U})$. Let $\hat{p}_j = \frac{1}{m} \sum_{i=1}^m \mathbf{1}_{Z_i \in A_j}$ and $p_j = \mathbb{P}(Z_1 \in A_j)$, where A_j , $j = 1, \dots, \ell$, are disjoint subsets of \mathcal{X} . Then, for any $\delta \in (0, 1)$,*

$$\mathbb{P} \left(\sum_{j=1}^{\ell} |p_j - \hat{p}_j| > \sqrt{\frac{\ell}{m}} + \sqrt{\frac{2 \log(1/\delta)}{m}} \right) \leq \delta.$$

Proof. Set $G(Z_1, \dots, Z_m) := \sum_{j=1}^{\ell} |p_j - \hat{p}_j|$. We have

$$\mathbb{E}G(Z_1, \dots, Z_m) = \mathbb{E} \sum_{j=1}^{\ell} |p_j - \hat{p}_j| \leq \sum_{j=1}^{\ell} \left(\mathbb{E}[|p_j - \hat{p}_j|^2] \right)^{1/2} \leq \sum_{j=1}^{\ell} \sqrt{\frac{p_j}{m}} \leq \sqrt{\frac{\ell}{m}}.$$

Note that $G(Z_1, \dots, Z_m)$ changes its value by at most $1/m$ if we replace any single Z_i by another Z'_i . Therefore, by the bounded difference inequality (see, e.g., [DL01, Theorem 2.2]),

$$\mathbb{P}(G(Z_1, \dots, Z_m) > \mathbb{E}G(Z_1, \dots, Z_m) + t) \leq e^{-2t^2 m}, \quad \forall t > 0,$$

which yields the result. \square

Lemma 8 (Control of multinomial noise). *Let $\alpha > 1$, $N > 1$, $n \in \mathbb{N}^*$, let $P \in \mathbb{R}_+^{d_1 \times d_2}$ be a matrix such that $\sum_{i,j} P_{ij} = 1$, and let $Y \sim \mathcal{M}(P, n)$. Consider an extraction Q of P corresponding to two sets of indices $I \subseteq [d_1]$ and $J \subseteq [d_2]$, that is $Q = (P_{ij})_{i \in I, j \in J}$. Let $W_Q = (W_{ij})_{i \in I, j \in J}$, where $W = \frac{Y}{n} - P$ is the multinomial noise. Then*

$$\mathbb{P} \left(\|W_Q\|^2 \leq 9 \max \left\{ \frac{\alpha \|Q\|_{\square} \log N}{n}, \left(\frac{\alpha \log N}{n} \right)^2 \right\} \right) \geq 1 - \frac{|I| + |J|}{N^\alpha}.$$

Proof. We use matrix Bernstein inequality (see, for example, [Ver18, Exercise 5.4.15]), which yields that for any independent zero mean matrices $M_1, \dots, M_n \in \mathbb{R}^{|I| \times |J|}$ such that, almost surely, $\|M_i\| \leq K, \forall i \in [n]$, we have

$$\mathbb{P} \left(\left\| \sum_{i=1}^n M_i \right\| > t \right) \leq (|I| + |J|) \exp \left(-\frac{t^2/2}{\sigma^2 + Kt/3} \right), \quad \forall t > 0, \quad (50)$$

where

$$\sigma^2 = \max \left(\left\| \sum_{i=1}^n \mathbb{E}(M_i M_i^\top) \right\|, \left\| \sum_{i=1}^n \mathbb{E}(M_i^\top M_i) \right\| \right).$$

We apply (50) with $M_i = (X_i - P)_{I,J}$, where X_i 's are independent random matrices with distribution $\mathcal{M}(P, 1)$. In this case, the inequality $\|M_i\| \leq K$ holds with $K = 2$. To prove this, notice first that if Q is an extraction of a probability matrix P , then $\|Q\|^2 \leq \|Q\|_\square$. Indeed, letting $p_i = \sum_{j \in J} P_{ij}$, $\forall i \in I$, and $p_{\max} = \max_{i \in I} p_i$, we obtain

$$\|Q\|^2 \leq \|Q\|_F^2 = \sum_{i \in I, j \in J} P_{ij}^2 \leq \sum_{i \in I} p_i^2 \leq p_{\max} \sum_{i \in I} p_i \leq p_{\max} \leq \|Q\|_\square.$$

It follows that almost surely for all $i \in [n]$ we have

$$\|M_i\| \leq \|(X_i)_{IJ}\| + \|Q\| \leq 1 + \sqrt{\|P\|_\square} \leq 2.$$

Next,

$$\begin{aligned} \left\| \sum_{i=1}^n \mathbb{E}(M_i M_i^\top) \right\| &= \left\| \sum_{i=1}^n \mathbb{E}[(X_i)_{IJ}(X_i)_{IJ}^\top] - QQ^\top \right\| = \left\| n \text{Diag}(p_k)_{k \in I} - QQ^\top \right\| \\ &\leq np_{\max} \leq n\|Q\|_\square, \end{aligned}$$

and controlling $\left\| \sum_{i=1}^n \mathbb{E}(M_i^\top M_i) \right\|$ analogously yields

$$\sigma^2 \leq n\|Q\|_\square.$$

Now, we use the fact that nW_Q has the same distribution as $\sum_{i=1}^n M_i$. Therefore, applying inequality (50) and the above bounds on σ^2 and K we obtain that, for all $t > 0$,

$$\mathbb{P}(\|nW_Q\| > t) \leq (|I| + |J|) \exp\left(-\frac{t^2/2}{n\|Q\|_\square + 2t/3}\right).$$

Now, set

$$t = 3 \max\left(\sqrt{\alpha n\|Q\|_\square \log(N)}, \alpha \log(N)\right).$$

If $n\|Q\|_\square \geq \alpha \log(N)$ then $t = 3\sqrt{\alpha n\|Q\|_\square \log(N)} \leq 3n\|Q\|_\square$ and

$$\mathbb{P}(\|nW_Q\| > t) \leq (|I| + |J|) \exp\left(-\frac{t^2/2}{3n\|Q\|_\square}\right) \leq (|I| + |J|) \exp\left(-\frac{3}{2}\alpha \log(N)\right) \leq (|I| + |J|)N^{-\alpha}.$$

Otherwise, if $n\|Q\|_\square < \alpha \log(N)$ then $t = 3\alpha \log(N)$ and the same bound holds:

$$\mathbb{P}(\|nW_Q\| > t) \leq (|I| + |J|) \exp\left(-\frac{t^2/2}{3\alpha \log(N)}\right) \leq (|I| + |J|)N^{-\alpha}.$$

□

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