



HAL
open science

Explicit isomorphisms between a sub-class of expanding toral endomorphisms and one-sided Bernoulli shifts.

Christophe Leuridan

► To cite this version:

Christophe Leuridan. Explicit isomorphisms between a sub-class of expanding toral endomorphisms and one-sided Bernoulli shifts.. 2024. <hal-04556762v4>

HAL Id: hal-04556762

<https://hal.science/hal-04556762v4>

Preprint submitted on 13 Apr 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY-NC-ND 4.0 - Attribution - Non-commercial use - No Derivative Works - International License

Explicit isomorphisms between a sub-class of expanding toral endomorphisms and one-sided Bernoulli shifts.

Christophe Leuridan

April 8, 2026

Abstract

Let A be a $d \times d$ matrix with integer coefficients such that $r := |\det A| \geq 2$. The endomorphism T_A of the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ given by $T_A(x) := Ax$ preserves the Haar measure and is r -to-one, where $r := |\det A|$. When A is expanding (i.e. when $\rho(A^{-1}) < 1$, where ρ denotes the spectral radius), Mihailescu [13] proved that T_A is isomorphic to S_r , the one-sided uniform Bernoulli shift on $\llbracket 0, r-1 \rrbracket^{\mathbb{Z}^+}$. Using two different approaches, we provide explicit isomorphisms under the stronger assumption $\|A^{-1}\|_\infty < 1$, where $\|\cdot\|_\infty$ denotes the operator norm on $\mathcal{M}_d(\mathbb{R})$ associated to the usual $|\cdot|_\infty$ on \mathbb{R}^d . For arbitrary expanding endomorphisms of the torus, explicit constant-to-one morphisms transforming S_r into T_A were already known. By a probabilistic method involving Markov chains, we provide explicit constant-to-one morphisms transforming T_A into S_r .

MSC Classification : 37A05, 60J05.

Keywords : endomorphisms of the torus, Bernoulli shifts, iterated function systems, iterated random functions, self-affine tiles.

1 Introduction

Fix a positive integer d , and call η the Haar measure η on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Given a $d \times d$ matrix with integer coefficients $A \in \mathcal{M}_d(\mathbb{Z})$, the linear map $x \mapsto Ax$ from \mathbb{T}^d to \mathbb{T}^d is a surjective group morphism and preserves the measure η . Furthermore, T_A is r -to-one: given a uniform random variable X_0 with values in $\mathbb{R}^d/\mathbb{Z}^d$ and with law η , the distribution of $T_A(X_0)$ is still η , and for η -almost every $y \in \mathbb{R}^d/\mathbb{Z}^d$, the conditional law $\mathcal{L}(X_0|T_A(X_0) = y)$ is uniform on the set $T_A^{-1}(\{y\})$, which has exactly r elements.

Is T_A isomorphic to S_r , where S_r denotes the uniform one-sided Bernoulli shift on $\llbracket 0, r-1 \rrbracket^{\mathbb{Z}^+}$? Katznelson [8] proved in 1971 that if no eigenvalues of A is a root of unity, then the natural extensions of T_A and S_r are isomorphic. But this weaker property does not ensure that the endomorphism T_A itself is Bernoulli.

The exactness of T_A , i.e. the triviality of the tail σ -field $\bigcap_{n \geq 0} T_A^{-n} \mathcal{B}(\mathbb{R}^d)$, is a necessary condition for T_A to be isomorphic to a one-sided Bernoulli shift. In [10], Krzyżewski provides a necessary and sufficient for T_A to be exact: the only unimodular (i.e. monic, with integral coefficients and with constant term ± 1) factor of the characteristic polynomial χ_A must be 1. Otherwise, Krzyżewski shows that T_A – viewed as a dynamical system – is isomorphic to the direct product of a toral automorphism and an exact toral endomorphism.

Expanding toral endomorphisms form an important class among all non-invertible toral endomorphisms. By definition, A and T_A are expanding if the modulus of every eigenvalue of A is strictly greater than 1. Little is known on toral endomorphisms which are neither invertible nor expanding.

In 2013, Mihailescu [13] proved that every expanding toral endomorphism of the torus is one-sided Bernoulli. More generally, he proves that smooth toral endomorphisms (possibly not linear) are one-sided Bernoulli with regard to their measure of maximal entropy. This includes smooth perturbations of expanding linear endomorphisms. Mihailescu's proof is not constructive and relies on a criterion called *tree very weak Bernoulli*, introduced by Hoffman and Rudolph [6]. This criterion is a dynamic version of Vershik's standardness criterion [16, 17].

From now on, we assume that A is expanding. Equivalently, $\rho(A^{-1}) < 1$, where ρ denotes the spectral radius: for every $M \in \mathcal{M}_d(\mathbb{R})$,

$$\rho(M) := \max\{|\alpha| : \alpha \text{ eigenvalue of } M\}.$$

Recall that for any operator norm $\|\cdot\|$ on the linear space $\mathcal{M}_d(\mathbb{R})$,

$$\rho(M) = \lim_{n \rightarrow +\infty} \|M^n\|^{1/n} = \inf_{n \geq 1} \|M^n\|^{1/n}.$$

Given any basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{R}^d , we note $|\cdot|_{\mathbf{u}, \infty}$ the norm on \mathbb{R}^d defined by

$$|x|_{\mathbf{u}, \infty} := \max_{1 \leq i \leq d} |\xi_i| \text{ where } x = \sum_{i=1}^d \xi_i u_i,$$

and we note $\|\cdot\|_{\mathbf{u}, \infty}$ the associated operator norm of $\mathcal{M}_d(\mathbb{R})$.

In the present paper,

- we introduce constant-to-one factor maps transforming S_r into T_A , relying on radix expansions and already considered by Hacon, Saldanha and Veerman in [5].
- we construct constant-to-one factor maps transforming T_A into S_r , using Markov chains.
- under the stronger assumption $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis \mathbf{u} of \mathbb{Z}^d , we prove that both factor maps considered above are isomorphisms provided we choose the right digit system or fundamental domain.

Before giving more details, let us comment our assumption. One can check that $\rho(M)$ is the infimum of $\|M\|_{\mathbf{u}, \infty}$ over all basis \mathbf{u} of \mathbb{R}^d . Since $\|M\|_{\mathbf{u}, \infty}$ depends continuously on the vectors u_1, \dots, u_d , one may restrict ourselves to vectors with *rational* coordinates, or even with *integer* coordinates, by homogeneity. Yet, the determinant of (u_1, \dots, u_d) may be different from ± 1 , so *we may not* assume that \mathbf{u} is a \mathbb{Z} -basis of \mathbb{Z}^d (i.e. that $\mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_d$ is the whole group \mathbb{Z}^d). This distinction is important in what follows and explains why the condition $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis \mathbf{u} of \mathbb{Z}^d is a bit stronger than the condition $\rho(A^{-1}) < 1$.

Given any \mathbb{Z} -basis \mathbf{u} of \mathbb{Z}^d , the condition $\rho(A^{-1}) < 1$ yields $\|A^{-n}\|_{\mathbf{u}, \infty} < 1$ for every large enough n . Since $(T_A)^n = T_{A^n}$ for all $n \geq 1$, both methods above provide explicit isomorphisms between $(T_A)^n$ and the Bernoulli shift S_{r^n} , for every large enough n .

Note also that our problem is preserved by unimodular change of coordinates: given $U \in GL_d(\mathbb{Z})$, the isomorphism $x \mapsto Ux$ transforms the measure-preserving map T_A into the measure-preserving map $T_{UAU^{-1}}$. Therefore, it will be sufficient to consider the case where the \mathbb{Z} -basis \mathbf{u} of \mathbb{Z}^d is the canonical basis of \mathbb{R}^d .

1.1 The classical approach using radix expansions and self-affine tiles

A natural idea is to imitate the r -adic expansions of the real numbers: proving that the measure-preserving map $T_r : x \mapsto rx - \lfloor rx \rfloor$ from $[0, 1[$ (endowed with the Lebesgue measure) is isomorphic to the uniform one-sided Bernoulli shift on $\llbracket 0, r-1 \rrbracket^\infty$ is quite simple. Indeed, if x is uniformly chosen on $[0, 1[$, then in its r -adic expansion, its digits in are i.i.d. and uniform on $\llbracket 0, r-1 \rrbracket$. Moreover, if the r -adic expansion of x is $0.\xi_1\xi_2\dots$ the r -adic expansion of $T_r(x)$ is $0.\xi_2\xi_3\dots$

Now, let us replace the base r and the set $\llbracket 0, r-1 \rrbracket$ of possible digits by the matrix A and by some fixed set \mathbb{D} of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$, containing 0. The theory of lattices shows that $A\mathbb{Z}^d$ is a subgroup of index r in \mathbb{Z}^d , so $|\mathbb{D}| = r$.

It is convenient to introduce Minkovski sums. Given two subsets B and C of \mathbb{R}^d , their Minkovski sum is $B + C := \{b + c : (b, c) \in B \times C\}$. When every point of $B + C$ has a unique decomposition $b + c$ with $(b, c) \in B \times C$, we say that the sum is direct, and we may use the notation $B \oplus C$. Minkovski sums and direct sums are commutative and associative.

The assumption that \mathbb{D} is a complete system of representatives modulo $A\mathbb{Z}^d$, i.e. a system of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$, is equivalent to the equality $\mathbb{Z}^d = \mathbb{D} \oplus A\mathbb{Z}^d$. There is a natural way to produce systems of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$ from a *fundamental domain* of $\mathbb{R}^d/\mathbb{Z}^d$, i.e. any Borel subset of \mathbb{R}^d containing exactly one element in each equivalence class of $\mathbb{R}^d/\mathbb{Z}^d$. Indeed, if D is fundamental domain of $\mathbb{R}^d/\mathbb{Z}^d$, then $\mathbb{R}^d = D \oplus \mathbb{Z}^d$, so $\mathbb{R}^d = A\mathbb{R}^d = AD \oplus A\mathbb{Z}^d$ and $\mathbb{Z}^d = (AD \cap \mathbb{Z}^d) \oplus A\mathbb{Z}^d$. As a result, $AD \cap \mathbb{Z}^d$ is a set of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$.

Given any complete system of representatives \mathbb{D} modulo $A\mathbb{Z}^d$, containing 0, we get by induction

$$\forall n \geq 1, \quad \mathbb{Z}^d = \mathbb{D} \oplus \dots \oplus A^{n-1}\mathbb{D} \oplus A^n\mathbb{Z}^d.$$

so the set $\mathbb{D}_n := \mathbb{D} \oplus A\mathbb{D} + \dots \oplus A^{n-1}\mathbb{D}$ is a system of representatives of $\mathbb{Z}^d/A^n\mathbb{Z}^d$. Furthermore, the sequence $(\mathbb{D}_n)_{n \geq 1}$ is increasing since we assumed that $0 \in \mathbb{D}$. A natural question is whether the increasing union

$$\mathbb{D}_\infty := \bigcup_{n \geq 1} \mathbb{D}_n.$$

is the whole lattice \mathbb{Z}^d or not. If yes, one checks that any vector of \mathbb{Z}^d has a unique expansion ‘in base A ’ for the digit system \mathbb{D} , and any vector of \mathbb{R}^d has at least a fractional expansion, not necessarily unique.

Such generalizations of r -adic expansions have been considered independently by several authors for applications in various domains: computer science, wavelets analysis,... Matula [12] considered expansions of integers and of real numbers in a given base $\beta \in \mathbb{Z} \setminus \{-1, 0, 1\}$. The system of digits \mathbb{D} is assumed to contain 0 and to exactly one representative in each equivalence class of $\mathbb{Z}/|\beta|\mathbb{Z}$. Knuth [9] fixed an integer $n \geq 2$ and considered expansions of complex numbers in base $\sqrt{-n}$ with digits in $\llbracket 0, n-1 \rrbracket$. Later, W. Penney [14] considered expansions of complex numbers in base $-1 + i$ with digits

in $\{0, 1\}$. More generally, Duda [3] considered complex bases z which are solutions of some quadratic equation and have modulus > 1 . Duda also briefly considered possible generalisations in higher dimensions, replacing the base by some matrix with integer coefficients, of the form ρO , where $\rho > 1$ and O is an orthogonal matrix.

Expansions in which the base is replaced by any expansive matrix A with integer entries have been studied more recently. One important object of interest are the sets

$$K_{\mathbb{D}} := \left\{ \sum_{k=1}^{+\infty} A^{-k} v_k : (v_k)_{k \geq 1} \in \mathbb{D}^{\infty} \right\},$$

where \mathbb{D} is a set \mathbb{D} of representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$. Here, we omit the index A because the expansive matrix A is fixed.

Since $\rho(A^{-1}) < 1$, we can - and we do - endow \mathbb{R}^d with some norm $|\cdot|$ such that $\|A^{-1}\| < 1$, where $\|\cdot\|$ denotes the associated operator norm on $\mathcal{M}_d(\mathbb{R})$. Hence the maps $(g_v)_{v \in \mathbb{D}}$ defined on \mathbb{R}^d by $g_v(x) := A^{-1}x + A^{-1}v$ are contractions. Thus Hutchinson's theory on *iterated function systems* applies. The transformation $\tau_{\mathbb{D}}$ defined on Borel subsets of \mathbb{R}^d by

$$\tau_{\mathbb{D}}(B) := \bigcup_{v \in \mathbb{D}} g_v(B) = \bigcup_{v \in \mathbb{D}} (A^{-1}B + A^{-1}v) = A^{-1}B + A^{-1}\mathbb{D}$$

induces a contraction on the set of all non-empty compact subsets of \mathbb{R}^d , endowed with the Hausdorff metric, and $K_{\mathbb{D}}$ is the only fixed point of this contraction.

We now reformulate some facts stated in Hacon, Saldanha, Veerman in [5]. Since $K_{\mathbb{D}} + \mathbb{Z}^d = \mathbb{R}^d$, the compact set $K_{\mathbb{D}}$ has positive Lebesgue measure. Furthermore, $K_{\mathbb{D}}$ is the union of the $A^{-1}K_{\mathbb{D}} + A^{-1}v$ over all $v \in \mathbb{D}$, and this union is almost disjoint: given $v \neq v'$ in \mathbb{D} , the intersection of $A^{-1}K_{\mathbb{D}} + A^{-1}v$ and $A^{-1}K_{\mathbb{D}} + A^{-1}v'$ is a null set (for the Lebesgue measure λ). That is why the compact set $K_{\mathbb{D}}$ is called a *self affine-tile*.

Up to null sets, the subsets $(K_{\mathbb{D}} + v)_{v \in \mathbb{D}}$ form a partition of $AK_{\mathbb{D}}$. The map $T_{A, K_{\mathbb{D}}} : K_{\mathbb{D}} \rightarrow K_{\mathbb{D}}$ which sends each $x \in K_{\mathbb{D}}$ on the almost surely unique $y \in K_{\mathbb{D}}$ such that $Ax - y \in \mathbb{D}$ preserves the uniform measure $\mathcal{U}(K_{\mathbb{D}}) := \lambda(\cdot|_{K_{\mathbb{D}}})$ on $K_{\mathbb{D}}$, and the map $\Phi : \mathbb{D}^{\infty} \rightarrow K_{\mathbb{D}}$ defined by

$$\Phi((v_k)_{k \geq 1}) := \sum_{k=1}^{+\infty} A^{-k} v_k$$

is an isomorphism transforming the uniform Bernoulli shift on \mathbb{D}^{∞} into the measure-preserving map $T_{A, K_{\mathbb{D}}}$.

Call $p_{\mathbb{T}^d}$ the canonical projection from \mathbb{R}^d onto $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. By construction, $p_{\mathbb{T}^d}|_{K_{\mathbb{D}}}$ is a factor map transforming $T_{A, K_{\mathbb{D}}}$ into T_A . Corollary 1.7 in [5] states that the restriction $p_{\mathbb{T}^d}|_{K_{\mathbb{D}}}$ is m -to-one (up to null sets) for some positive integer m , and this integer m necessarily equals the Lebesgue measure of $K_{\mathbb{D}}$.

Hence, the endomorphism T_A is a factor of the uniform Bernoulli shift on \mathbb{D}^{∞} (or, equivalently, on $\llbracket 0, r-1 \rrbracket^{\mathbb{Z}^+}$) and the factor map $p_{\mathbb{T}^d}|_{K_{\mathbb{D}}} \circ \Phi$ is $\lambda(K_{\mathbb{D}})$ -to-one. In particular, $\lambda(K_{\mathbb{D}})$ is always a positive integer and $p_{\mathbb{T}^d}|_{K_{\mathbb{D}}} \circ \Phi$ is an isomorphism if and only if the Lebesgue measure of $K_{\mathbb{D}}$ is 1.

Actually, the volume $\lambda(K_{\mathbb{D}})$ does depend on the choice of the set \mathbb{D} of representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$. For example, when $d = 1$ and $A = 2$, the choice $\mathbb{D} = \{0, 1\}$ yields $K_{\mathbb{D}} = [0, 1]$, whereas the choice $\mathbb{D} = \{0, 3\}$ yields $K_{\mathbb{D}} = [0, 3]$.

The results given by Matula in [12] in the one-dimensional case can be adapted to our context and further pursued. This yields the theorem below.

Theorem 1. *Fix a norm $|\cdot|$ on \mathbb{R}^d such that the associated operator norm satisfies $\|A^{-1}\| < 1$. Set $M = \max\{|A^{-1}v| : v \in \mathbb{D}\}$. Then the statements below are equivalent:*

1. $\mathbb{D}_\infty = \mathbb{Z}^d$.
2. \mathbb{D}_∞ contains all vectors $x \in \mathbb{Z}^d$ such that $|x| \leq M/(1 - \|A^{-1}\|)$.
3. $\forall n \geq 1, \mathbb{D}_n \cap (A^n - I_d)\mathbb{Z}^d = \{0\}$.

Moreover, if the equivalent properties above hold, then $\lambda(K_{\mathbb{D}}) = 1$.

The equivalence between items 1,2,3 is a simple adaptation of Matula's work. That they imply $\lambda(K_{\mathbb{D}}) = 1$ was not written in Matula's paper, but it follows easily from the fact that $AK_{\mathbb{D}}$ is the almost disjoint union of the sets $K_{\mathbb{D}} + v$ over all $v \in \mathbb{D}$.

Theorem 1 provides a sufficient (but not necessary) conditions on A and \mathbb{D} to have $\lambda(K_{\mathbb{D}}) = 1$. For example, in the one-dimensional case considered by Matula,

- If $A = r$ and $\mathbb{D} = \llbracket 0, r - 1 \rrbracket$, then for every $n \geq 1$, $\mathbb{D}_n = \llbracket 0, r^n - 1 \rrbracket$. Therefore, $\mathbb{D}_n \cap (r^n - 1)\mathbb{Z} = \{0, r^n - 1\}$ and $\mathbb{D}_\infty = \mathbb{N}$. Yet, $K_{\mathbb{D}} = [0, 1]$ so $\lambda(K_{\mathbb{D}}) = 1$.
- If $A = -r$ and $\mathbb{D} = \llbracket 0, r - 1 \rrbracket$, then $\mathbb{D}_\infty = \mathbb{Z}$ and $K_{\mathbb{D}} = [-r/(r + 1), 1/(r + 1)]$, so $\lambda(K_{\mathbb{D}}) = 1$.

To get a less simple example, take $d = 2$, $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ so $r = 5$. One may choose $\mathbb{D} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$. The endomorphism of \mathbb{R}^2 canonically associated to A is a direct similitude with ratio $\sqrt{5}$. Thus theorem 1 applies with the canonical Euclidean norm on \mathbb{R}^2 , and $M = \|A^{-1}\| = 1/\sqrt{5}$, so $M/(1 - \|A^{-1}\|) = 1/(\sqrt{5} - 1) < 1$. Since the only point $x \in \mathbb{Z}^2$ such that $|x| \leq 1/(\sqrt{5} - 1)$ is 0, statement 2 holds trivially, therefore statements 1 and 3 also hold and $\lambda(K_{\mathbb{D}}) = 1$.

Let us mention more recent works which study the quantity $\lambda(K_{\mathbb{D}})$. In [4], Gabardo and Yu give an algorithm to compute this Lebesgue measure. In [11], Lagarias and Wang prove the existence of a lattice tiling with the tile $K_{\mathbb{D}}$: there exists some lattice $\Lambda \subset \mathbb{Z}^d$ such that $K_{\mathbb{D}} + \Lambda = \mathbb{R}^d$ and such that the translated sets $(K_{\mathbb{D}} + v)_{v \in \Lambda}$ do not overlap. Thus, the volume $\lambda(K_{\mathbb{D}})$ equals the index of Λ in \mathbb{Z}^d . Using this lattice tiling, Lagarias and Wang establish a highly non-trivial necessary and sufficient condition for $\lambda(K_{\mathbb{D}})$ to equal 1. This condition is involved and not very tractable, although it provides many simple corollaries.

Hence, many questions arise:

- Is it always possible to choose \mathbb{D} to have $\lambda(K_{\mathbb{D}})$ to equal 1 and how?
- Given $c \in \mathbb{R}^d$ and any \mathbb{Z} -basis (u_1, \dots, u_d) of \mathbb{Z}^d , one checks that the parallelotope $P = c + [0, 1[u_1 + \dots + [0, 1[u_d$ is a fundamental domain of $\mathbb{R}^d/\mathbb{Z}^d$, so $AP \cap \mathbb{Z}^d$ is system of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$. Does such a system of representatives always work?

The example above suggests that choosing a norm on \mathbb{R}^d such that the associated operator norm satisfies $\|A^{-1}\| < 1$ and choosing the vectors of \mathbb{D} having norms as small as possible could be a good strategy.

To answer these questions, we establish some remarkable consequences of Lagarias and Wang's condition in subsection 5.1. We provide counterexamples showing that the answer to both question is negative. Yet, we have a positive result when $d = 2$.

Theorem 2. *If $d = 2$, then it is always possible to choose the system \mathbb{D} of representatives so that $\lambda(K_{\mathbb{D}}) = 1$.*

And still when $d = 2$, the answer to the second question is positive provided an extra assumption of primitivity holds (see Corollary 41).

In subsection 4.4, we develop a variant of the approach followed by Hacon, Saldanha, Veerman and mentioned above. Given a bounded fundamental domain D , the set $\tau_{\mathbb{D}}(D)$ is still a fundamental domain, since

$$\mathbb{R}^d = D \oplus \mathbb{Z}^d = D \oplus (\mathbb{D} \oplus A\mathbb{Z}^d) = (D \oplus \mathbb{D}) \oplus A\mathbb{Z}^d = A\tau_{\mathbb{D}}(D) \oplus A\mathbb{Z}^d,$$

so $\mathbb{R}^d = A^{-1}\mathbb{R}^d = \tau_{\mathbb{D}}(D) \oplus \mathbb{Z}^d$. One checks that the sequence $(D_n)_{n \geq 1}$ of fundamental domains $(D_n)_{n \geq 1}$ defined by

$$D_n = \tau_{\mathbb{D}}^n(D) = A^{-n}D + A^{-n}\mathbb{D} + \dots + A^{-1}\mathbb{D}.$$

is uniformly bounded. Thus, the set

$$L = \limsup_n D_n := \{x \in \mathbb{R}^d : x \in D_n \text{ for infinitely many } n \geq 0\}$$

is bounded. One checks that $L + \mathbb{Z}^d = \mathbb{R}^d$, so $0 < \lambda(L) < +\infty$, and that L is invariant by the Hutchinson's map $\tau_{\mathbb{D}}$. Thus, its closure is compact and also invariant, so $\bar{L} = K_{\mathbb{D}}$.

Most of the arguments used for $K_{\mathbb{D}}$ still work with L . Up to null sets, the subsets $(L + v)_{v \in \mathbb{D}}$ form a partition of AL . We will find a full-measure subset $E \subset L$ such that for every $x \in E$, Ax has a unique decomposition $y + z$ with $(y, z) \in E \times \mathbb{D}$. Calling $T_{A,E}(x)$ the vector y in this decomposition, we get a measure-preserving map $T_{A,E}$ on $(E, \mathcal{B}(E), \mathcal{U}(E))$. By definition, for every $v \in \mathbb{D}$,

$$Ax - T_{A,E}(x) = v \iff Ax - v \in E \iff x \in A^{-1}v + A^{-1}E.$$

Consider the map $\Psi : E \rightarrow \mathbb{D}^{\infty}$ defined by $\Psi(x) := (AT_{A,E}^{n-1}(x) - T_{A,E}^n(x))_{n \geq 1}$. Concretely, the sequence $\Psi(x)$ begins with (v_1, \dots, v_n) if and only if x belongs to the set $A^{-1}v_1 + \dots + A^{-n}v_n + A^{-n}E$. We will check that Ψ is an isomorphism transforming $T_{A,E}$ into the uniform Bernoulli shift $S_{\mathbb{D}}$ on \mathbb{D}^{∞} .

Furthermore, we will see that $p_{\mathbb{T}^d}|_E$ is a factor map transforming $T_{A,E}$ into T_A , that the Lebesgue measure of E is a positive integer and the factor map $p_{\mathbb{T}^d}|_E$ is $\lambda(E)$ -to-one. Hence, when $\lambda(E)$ equals 1, $\Psi \circ p_{\mathbb{T}^d}|_E^{-1}$ is an isomorphism transforming T_A into $S_{\mathbb{D}}$.

Since $L \subset \bar{L} = K_{\mathbb{D}}$, we have $\lambda(K_{\mathbb{D}}) \geq \lambda(L) \geq 1$. Actually, we prove that $\lambda(K_{\mathbb{D}}) = \lambda(L)$. When $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{Z}^d , we are able to choose a fundamental set D and a system of representatives \mathbb{D} such that $\lambda(E) = \lambda(L) = 1$, so $p_{\mathbb{T}^d}|_E$ is an isomorphism.

Theorem 3. *Assume that $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{Z}^d . Let*

$$c := (1/2)A^{-1}(u_1 + \dots + u_d), \quad D := c + \sum_{i=1}^d [-1/2, 1/2[u_i, \quad \mathbb{D} := AD \cap \mathbb{Z}^d.$$

Consider the fundamental domains $(D_n)_{n \geq 1}$ defined by $D_n = A^{-n}D_0 + A^{-n}\mathbb{D} + \dots + A^{-1}\mathbb{D}$. Then $\lambda(\limsup_n D_n) = 1$ and $p_{\mathbb{T}^d}|_E$ is an isomorphism.

The proof of Theorem 3 does not use the theorem of Lagarias and Wang. It relies on the study of the external boundary of the sets $\mathbb{D}_n = \mathbb{D} \oplus \cdots \oplus A^{n-1}\mathbb{D}$ when \mathbb{Z}^d is endowed with some graph structure on \mathbb{Z}^d related to the set \mathbb{D} .

1.2 A new approach using a stationary Markov chain

An alternative approach relies on the study a stationary Markov chain on \mathbb{T}^d , indexed by \mathbb{Z} , whose evolution at each time consists in choosing uniformly one of the r preimages by T_A of the current position. This Markov chain arises naturally in the construction of the natural extension of T_A . We view it as the iteration of a random function. Iterated random functions is a fruitful method having many applications, see for example Diaconis and Freedman's paper [2].

Concretely, the idea is to fix r Borel maps F_0, \dots, F_{r-1} from \mathbb{T}^d to \mathbb{T}^d such that for every $y \in \mathbb{T}$, $T_A^{-1}(\{y\}) = \{F_0(y), \dots, F_{r-1}(y)\}$, and to use a stationary Markov chain $(X_n)_{n \in \mathbb{Z}}$ on \mathbb{T}^d , indexed by \mathbb{Z} , whose evolution is governed by an i.i.d. sequence $(I_n)_{n \in \mathbb{Z}}$ of uniform random variables in $\llbracket 0, r-1 \rrbracket$ through the recursion relation $X_n = F_{I_n}(X_{n-1})$, with I_n independent of $((X_k, I_k))_{k \leq n-1}$. Then for every $n \in \mathbb{Z}$ and $i \in \llbracket 0, r-1 \rrbracket$, $I_n = i$ if and only if $X_n = F_i(X_{n-1})$. Moreover, $X_{n-1} = T_A(X_n)$. Hence I_n and also $(I_k)_{k \leq n}$ are Borel functions of X_n . Those functions does not depend on n . By time reversal, the latter provides a factor map Ψ transforming T_A into the uniform shift S_r on $\llbracket 0, r-1 \rrbracket^{\mathbb{Z}_+}$.

The choice of the maps F_0, \dots, F_{r-1} has a crucial importance. By coupling methods, we give a general construction of F_0, \dots, F_{r-1} such that the conditional distribution of X_0 given $(I_k)_{k \leq 0}$ is almost surely uniform on some random finite set with constant size s . Equivalently, the factor map Ψ is s -to-one. The integer s does depend on the choice of F_0, \dots, F_{r-1} . Under the assumption $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{Z}^d , we provide an explicit choice of F_0, \dots, F_{r-1} such that $s = 1$, so Ψ is an isomorphism transforming T_A into the uniform shift S_r on $\llbracket 0, r-1 \rrbracket^{\mathbb{Z}_+}$.

Let us now explain how we build the maps F_0, \dots, F_{r-1} and state the main results. First, it is convenient to fix a set \mathbb{D} of representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$ and to use indexes vaying in \mathbb{D} instead of $\llbracket 0, r-1 \rrbracket$. Next, since the coefficients of A^{-1} are rational but not integral, one cannot apply A^{-1} to elements of \mathbb{T}^d . Therefore, to give a description of the preimages by T_A of elements of \mathbb{T}^d , it is convenient to fix a bounded fundamental domain D . The theory of lattices ensures that $\lambda(D) = 1$.

For every $x \in \mathbb{Z}^d$, call $p_D(x)$ the unique element of D in the coset $x + \mathbb{Z}^d$. We call *projection on D* the map $p_D : \mathbb{R}^d \rightarrow D$ thus defined. One checks that the discontinuity set of p_D is $\Delta_D := \partial D + \mathbb{Z}^d$. Moreover, if B is a Borel subset of \mathbb{R}^d , then

$$p_D(B) = \bigcup_{k \in \mathbb{Z}^d} (B - k) \cap D$$

is still a Borel subset.

Instead of T_A , we study the map $T_{A,D} : x \mapsto p_D(Ax)$ from D to D . One checks that $T_{A,D}$ preserves the uniform measure $\mathcal{U}_D := \lambda(\cdot|D)$ on D . Moreover, the restriction $p_{\mathbb{T}^d}|_D$ of the canonical projection from \mathbb{R}^d to \mathbb{T}^d is an isomorphism transforming the dynamical system $(D, \mathcal{B}(D), \mathcal{U}_D, T_{A,D})$ into $(\mathbb{R}^d/\mathbb{Z}^d, \mathcal{B}(\mathbb{R}^d/\mathbb{Z}^d), \eta, T_A)$. Hence, we will frequently omit the subscript D in $T_{A,D}$.

For each $v \in \mathbb{D}$, call $f_v : D \rightarrow D$ the map given by $f_v(y) := p_D(A^{-1}y + A^{-1}v)$ and set $P_v := f_v(D)$. Note that the maps $(f_v)_{v \in \mathbb{D}}$ and the subsets $(P_v)_{v \in \mathbb{D}}$ thus defined do not depend on the choice of \mathbb{D} . Indeed, if one replaces some vector $v \in \mathbb{D}$ by a vector v' belonging to the same class of $\mathbb{Z}^d/A\mathbb{Z}^d$, then for every $y \in \mathbb{R}^d$, $f_{v'}(y) = f_v(y)$ since $(A^{-1}y + A^{-1}v) - (A^{-1}y + A^{-1}v') \in \mathbb{Z}^d$.

The maps $(f_v)_{v \in \mathbb{D}}$ and the subsets $(P_v)_{v \in \mathbb{D}}$ have interesting properties.

Proposition 4. *Call $\Delta_D = \partial D + \mathbb{Z}^d$ the discontinuity set of p_D .*

- *For each $y \in D$, the vectors $(f_v(y))_{v \in \mathbb{D}}$ are pairwise distinct and are the preimages of y by T_A . Therefore, $\pi := (P_v)_{v \in \mathbb{D}}$ is a partition π of D . Moreover, for each $v \in \mathbb{D}$, the map f_v induces a bijection from D to P_v , whose inverse map is $T_A|_{P_v}$.*
- *For each $v \in \mathbb{D}$, the map $y \mapsto f_v(y) - A^{-1}y$ is constant on each subset $AD - v + Ak$ with $k \in \mathbb{Z}^d$ and takes only finitely many values on \mathbb{D} . Its discontinuity set is contained in $\Delta_v := A\Delta_D - v$.*
- *The sets $(P_v)_{v \in \mathbb{D}}$ are Borel and with Lebesgue measure $1/r$ each.*
- *The partitions $(T_A^{-n}(\pi))_{n \geq 0}$ are independent.*

For every $x \in D$, call $\pi(x) \in \mathbb{D}$ the index of the only block of the partition $\pi = (P_v)_{v \in \mathbb{D}}$ which contains x , i.e. $x \in P_{\pi(x)}$. By construction, the factor map $\Psi : x \mapsto (\pi(T_A^n(x)))_{n \geq 0}$ from D to $\mathbb{D}^{\mathbb{Z}^+}$ transforms T_A into the uniform Bernoulli shift S on $\mathbb{D}^{\mathbb{Z}^+}$.

We would like the factor map Ψ to be an isomorphism, i.e. the partition π to be an independent generator of T_A . It is not always true. The discontinuities of the maps $(f_v)_{v \in \mathbb{D}}$ cause difficulties. A strategy to avoid them is to look for a self-affine fundamental domain, as we saw in the previous subsection.

In the present approach, we do accept discontinuities, and we prove that Ψ is constant-to-one when D is bounded with ‘small’ boundary. To clarify this assumption, we need a few notations. Given a subset S of \mathbb{R}^d , the *upper box dimension* of S is defined by

$$\dim_{\text{upper box}} S = \limsup_{\varepsilon \rightarrow 0} \frac{\ln N_S(\varepsilon)}{|\ln \varepsilon|} = d - \liminf_{\varepsilon \rightarrow 0} \frac{|\ln \lambda(S_\varepsilon)|}{|\ln \varepsilon|},$$

where $N_S(\varepsilon)$ is the minimal number of open balls with radius ε required to cover S , and S_ε is the ε -neighbourhood of S , namely $S_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, S) < \varepsilon\}$.

Theorem 5. *Assume D is bounded and that $\dim_{\text{upper box}}(\partial D) < d$. Then the factor map $\Psi : x \mapsto (\pi(T_A^n(x)))_{n \geq 0}$ from D to $\mathbb{D}^{\mathbb{Z}^+}$ is s -to-one for some integer $s \geq 1$.*

In other words, if X_0 is a uniform random variable taking values in D , the conditional law of X_0 given $\Psi(X_0)$ is almost surely uniform on some random subset with constant size s . Equivalently, one can construct a uniform random variable U taking values in D and independent of $\Psi(X_0)$ such that $\sigma(X_0) = \sigma(\Psi(X_0)) \vee \sigma(U)$. The partition π is a generator if and only if $s = 1$.

Theorem 5 is very general, but it does not give indications on the value of s . We will be more specific in Theorems 7 and 8.

To illustrate that s does depend on the fundamental set D , we consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

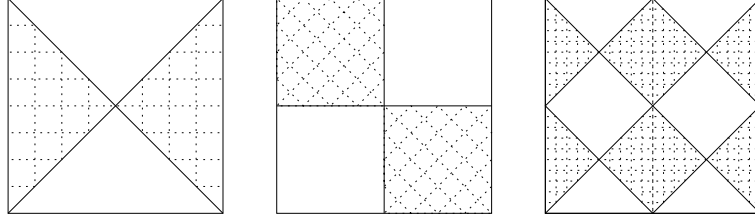


Figure 1: The partitions π , $T_A^{-1}\pi$, $T_A^{-2}\pi$ when $D = [0, 1]^2$. The shaded areas represent P_1 , $T_A^{-1}(P_1)$, $T_A^{-2}(P_1)$.

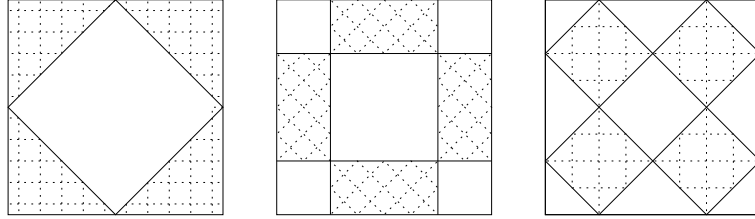


Figure 2: The partitions π , $T_A^{-1}\pi$, $T_A^{-2}\pi$ when $D = [-1/2, 1/2]^2$. The shaded areas represent the blocks P_1 , $T_A^{-1}(P_1)$, $T_A^{-2}(P_1)$.

Then $A^{-1} = (1/2)A$. The linear map $x \mapsto Ax$ from \mathbb{R}^2 to \mathbb{R}^2 is the composition of $\sqrt{2}\text{Id}$ with the reflexion with regard to the line $\mathbb{R}(\cos \pi/8, \sin \pi/8)$. Call (e_1, e_2) the canonical basis of \mathbb{R}^2 and take $\mathbb{D} = \{0, e_1\}$. Given a fundamental domain D , the corresponding maps f_0 and $f_1 := f_{e_1}$ from D to D are given by $f_0(y) = p_D(A^{-1}y)$ and $f_1(y) = p_D(A^{-1}y + (e_1 + e_2)/2)$. The associated partition is $\pi = \{P_0, P_1\}$ where $P_0 = f_0(D)$ and $P_1 := f_1(D)$. We consider three fundamental domains, all of which are parallelograms. In the first two cases, we get $s > 1$, because of the symmetries preserving the partitions $(T_A^{-n}\pi)_{n \geq 0}$, which prevent them from separating the points. See figures 1, 2, 3.

Example 6. *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

1. *If $D = [0, 1]^2$, then $s = 2$.*
2. *If $D = [-1/2, 1/2]^2$, then $s = 8$.*
3. *If $D = [0, 1]e_1 + [0, 1](e_1 + e_2)$, then $s = 1$.*

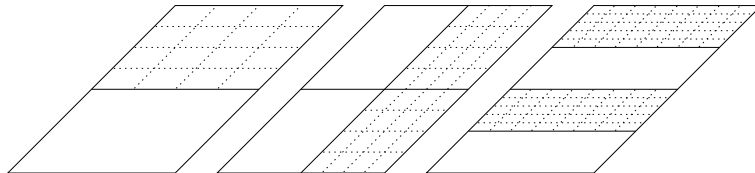


Figure 3: The partitions π , $T_A^{-1}\pi$, $T_A^{-2}\pi$ when $D = [0, 1]e_1 + [0, 1](e_1 + e_2)$. The shaded areas represent P_1 , $T_A^{-1}(P_1)$, $T_A^{-2}(P_1)$.

We now introduce an additional assumption which ensures that π is an independent generator of T_A . Observe that for every fundamental domain D and every $v_0, v_{-1}, v_{-2}, \dots$ in \mathbb{D} , the sets $(f_{v_0} \circ \dots \circ f_{v_{-n+1}})(D)$ form a decreasing sequence. Here, we use negative indexes because we couple from the past.

Theorem 7. *Keep the assumptions of Theorem 5. Fix any norm $|\cdot|$ on \mathbb{R}^d , and denote the associated open balls by the letter B . Set $Q := [0, 1[^d$. Assume that $D = c + UQ$ for some $c \in \mathbb{Z}^d$ and $U \in GL_d(\mathbb{Z})$. Assume furthermore that there exist $x_0 \in D$ which is a continuity point of all functions of the form $f_{v_n} \circ \dots \circ f_{v_1}$ and some sequence $v_0, v_{-1}, v_{-2}, \dots$ of elements of \mathbb{D} such that for every $\delta > 0$,*

$$(f_{v_0} \circ \dots \circ f_{v_{-n+1}})(D) \subset B(x_0, \delta) \text{ for every large enough } n.$$

Then the factor map $\Psi : D \rightarrow \mathbb{D}^{\mathbb{Z}^+}$ is an isomorphism. Moreover, Ψ is continuous almost everywhere on D and Ψ^{-1} is continuous almost everywhere on $\mathbb{D}^{\mathbb{Z}^+}$.

Actually, Theorem 7 applies in a lot of situations.

Theorem 8. *Assume that $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{Z}^d . Endow \mathbb{R}^d with the norm $|\cdot|_{\mathbf{u}, \infty}$, and consider the fundamental domain*

$$P_{\mathbf{u}} := \sum_{i=1}^d [-1/2, 1/2[u_i.$$

Let $c \in \mathbb{R}^d$ and $D = c + P_{\mathbf{u}}$. Assume that $0 \in \mathbb{D}$.

1. *If ($c = 0$ and r is odd), or if the real numbers $1, c_1, \dots, c_d$ are linearly independent over \mathbb{Q} , the point 0 is a continuity point of all functions of the form $f_{v_n} \circ \dots \circ f_{v_1}$.*
2. *If c is small enough, then $A^{-1}D \subset D$, so the map f_0 coincides with $y \mapsto A^{-1}y$, and for every $\delta > 0$, $f_0^n(D) \subset B(0, \delta)$ for every large enough n .*
3. *If conditions 1 and 2 hold, then Theorem 7 applies, so the factor map $\Psi : D \rightarrow \mathbb{D}^{\mathbb{Z}^+}$ is an isomorphism, Ψ is continuous almost everywhere on D and Ψ^{-1} is continuous almost everywhere on $\mathbb{D}^{\mathbb{Z}^+}$.*

1.3 Plan of the paper

Sections 2 and 3 are devoted to our new approach relying on Markov chains whereas Sections 4 and 5 are devoted to the classical approach via radix expansions.

In section 2, we prove preliminary results: the exactness of T_A , Proposition 4 and properties of maps $f_{v_n} \circ \dots \circ f_{v_1}$. We also establish a general result on Markov chains, Theorem 13. Given a (possibly non-stationary) Markov chain $(X_n)_{n \in \mathbb{Z}}$ governed by a recursion relation of the form $X_{n+1} = f(X_n, V_{n+1})$, this theorem compares the natural filtration of $((X_n, V_n))_{n \in \mathbb{Z}}$ with the smaller natural filtration of $(V_n)_{n \in \mathbb{Z}}$.

In section 3, we use Theorem 13 to prove Theorems 5, 7 and 8. We also set out the proof for Example 6.

In subsection 4.1, by adapting Matula's approach, we prove Theorem 1 which yields a sufficient condition to have $\lambda(K_{\mathbb{D}}) = 1$. Independently of this, we study the iteration of Hutchinson's map on a bounded fundamental domain (subsection 4.2), we adapt the

results of Hacon, Saldanha, Veerman to this context (subsection 4.3), and we introduce a graph structure on \mathbb{Z}^d to prove Theorem 8 (subsection 4.4). Last, when $\lambda(K_{\mathbb{D}}) = 1$, we show that the inverse of the isomorphism $p_{\mathbb{T}^d}|_{K_{\mathbb{D}}} \circ \Phi$ can be computed (subsection 4.5).

In subsection 5.1, we state Lagarias-Wang theorem about $\lambda(K_{\mathbb{D}})$ and derive many direct consequences. In particular, if A is \mathbb{Z} -similar to a companion matrix, we can choose \mathbb{D} so that $\lambda(K_{\mathbb{D}}) = 1$. In subsection 5.2, we give a self-contained proof of a corollary of Latimer-MacDuffee theorem, which ensures that A is \mathbb{Z} -similar to a companion matrix whenever the integral domain $\mathbb{Z}[X]/\chi_A\mathbb{Z}[X]$ is principal. In subsection 5.3, we apply this corollary and the results viewed at subsection 5.1 to establish results in low dimension, including Theorem 2 (if $d = 2$, one can always choose \mathbb{D} so that $\lambda(K_{\mathbb{D}}) = 1$) and an example of a 4×4 matrix A such that $\lambda(K_{\mathbb{D}}) \geq 2$ for every choice of \mathbb{D} .

2 Proof of preliminary facts

2.1 Exactness of T_A

In this section, we use the assumption $\rho(A^{-1}) < 1$ to prove directly in a straightforward manner the exactness of T_A , without using the necessary and sufficient condition of exactness established by Krzyżewski in [10]. We begin with a simple observation.

Lemma 9. *The subgroup $\bigcup_{n \geq 0} A^{-n}\mathbb{Z}^d$ is dense in \mathbb{R}^d .*

Proof. Let us endow \mathbb{R}^d with the usual norm $|\cdot|_{\infty}$ and $\mathcal{M}_d(\mathbb{R})$ with the associated operator norm. For every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, set $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$, where $\lfloor \cdot \rfloor$ denotes the integral part. Then, for every $n \geq 0$,

$$|x - A^{-n}\lfloor A^n x \rfloor|_{\infty} = |A^{-n}(A^n x - \lfloor A^n x \rfloor)|_{\infty} \leq \|A^{-n}\|_{\infty}.$$

Since $\|A^{-n}\|_{\infty}^{1/n} \rightarrow \rho(A^{-1}) < 1$ as $n \rightarrow +\infty$, we derive $A^{-n}\lfloor A^n x \rfloor \rightarrow x$ as $n \rightarrow +\infty$. \square

Proposition 10. *The transformation T_A is exact.*

Proof. Let B be in $\bigcap_{n \geq 0} T_A^{-n}\mathcal{B}(D)$. Call $f = \mathbf{1}_B \circ p_D$ the \mathbb{Z}^d -periodic extension of $\mathbf{1}_B$.

Fix $n \geq 1$ and $v \in A^{-n}\mathbb{Z}^d$. Let us check that v is a period of f . Since $B \in T_A^{-n}\mathcal{B}(D)$, there exists $C \in \mathcal{B}(D)$ such that $B = T_A^{-n}\mathcal{B}(C)$. For every $x \in \mathbb{R}^d$,

$$f(x+v) = \mathbf{1}_B(p_D(x+v)) = \mathbf{1}_C(T_A^n(p_D(x+v))).$$

Since $A\mathbb{Z}^d \subset \mathbb{Z}^d$ and $A^n v \in \mathbb{Z}^d$, we have

$$T_A^n(p_D(x+v)) \equiv A^n(p_D(x+v)) \equiv A^n(x+v) \equiv A^n x \equiv A^n(p_D x) \equiv T_A^n(p_D x) \pmod{\mathbb{Z}^d},$$

so $T_A^n(p_D(x+v)) = T_A^n(p_D x)$ since both sides of the equality belong to D . Thus

$$f(x+v) = \mathbf{1}_C(T_A^n(p_D(x+v))) = \mathbf{1}_C(T_A^n(p_D(x))) = f(x),$$

which shows that v is a period of f .

Hence, the set of all periods of f contains $\bigcup_{n \geq 0} A^{-n}\mathbb{Z}^d$, which it is dense in \mathbb{R}^d . Therefore, the continuous map $v \mapsto f(\cdot + v)|_D$ from \mathbb{R}^d to $L^1(D)$ is constant. Thus f is constant almost everywhere, namely $\lambda(B) \in \{0, 1\}$. \square

2.2 Study of the maps $(f_v)_{v \in \mathbb{D}}$

In this section, we establish useful properties of the maps $(f_v)_{v \in \mathbb{D}}$.

Let us prove Proposition 4.

Proof. Let x and y in D . Since $T_A(x) = p_D(Ax)$, $A^{-1}\mathbb{Z}^d = A^{-1}\mathbb{D} \oplus \mathbb{Z}^d$ and $f_v(y) = p_D(A^{-1}y + A^{-1}v)$ for every $v \in \mathbb{D}$, we have

$$\begin{aligned} T_A(x) = y &\iff Ax \in y + \mathbb{Z}^d \\ &\iff x \in A^{-1}y + A^{-1}\mathbb{Z}^d \\ &\iff \exists v \in \mathbb{D} : x \in A^{-1}y + A^{-1}v + \mathbb{Z}^d \\ &\iff \exists v \in \mathbb{D} : x = f_v(y). \end{aligned}$$

Item 1 follows.

Fix $v \in \mathbb{D}$. For every $k \in \mathbb{Z}^d$ and $y \in D$,

$$f_v(y) = A^{-1}y + A^{-1}v - k \iff A^{-1}y + A^{-1}v - k \in D \iff y \in AD - v + Ak.$$

Hence the map $y \mapsto f_v(y) - A^{-1}y$ is constant on $(AD - v + Ak) \cap D$.

Furthermore, if y is a discontinuity point of Δ_v , then $A^{-1}y + A^{-1}v$ is a discontinuity point of p_D , so $A^{-1}y + A^{-1}v \in \Delta_D$ and $y \in \Delta_v = A\Delta_D - v$. Item 2 follows.

Fix $v \in \mathbb{D}$ and $B \in \mathcal{B}(D)$. Then

$$P_v \cap T_A^{-1}(B) = f_v(B) = \bigcup_{k \in \mathbb{Z}^d} f_v(B \cap (AD - v + Ak)) = \bigcup_{k \in \mathbb{Z}^d} (A^{-1}B + A^{-1}v - k) \cap D.$$

This is a Borel subset, and since we have a disjoint union, we get

$$\begin{aligned} \lambda(P_v \cap T_A^{-1}(B)) &= \sum_{k \in \mathbb{Z}^d} \lambda((A^{-1}B + A^{-1}v - k) \cap D) \\ &= \sum_{k \in \mathbb{Z}^d} \lambda((A^{-1}B + A^{-1}v) \cap (D + k)) \\ &= \lambda(A^{-1}B + A^{-1}v) = r^{-1}\lambda(B). \end{aligned}$$

In particular, $\lambda(P_v) = \lambda(P_v \cap T_A^{-1}(D)) = r^{-1}$, so we get that π is independent of $T_A^{-1}\mathcal{B}$. A recursion yields the independence of $\pi, T_A^{-1}\pi, \dots, T_A^{-n}\pi, T_A^{-n-1}\mathcal{B}$ for every $n \geq 0$. Items 3 and 4 follow. \square

Before studying the discontinuity set of iterates $f_{v_n} \circ \dots \circ f_{v_1}$, we prove a simple lemma.

Lemma 11. *Let $v \in \mathbb{D}$. For every $B \in \mathcal{B}(D)$, $f_v^{-1}(B) \subset AB - v + AZ^d \subset AB + \mathbb{Z}^d$.*

Proof. Since f_v viewed from D to P_v is the inverse of the bijection $T_A|_{P_v}$, one has

$$f_v^{-1}(B) = T_A(B \cap P_v) = p_D(AB \cap AP_v).$$

By definition of P_v ,

$$AP_v = Ap_D(A^{-1}D + A^{-1}v) \subset Ap_D(A^{-1}D + A^{-1}v - \mathbb{Z}^d) = D + v - AZ^d.$$

On each set $D + v - Ak$ with $k \in \mathbb{Z}^d$, the projection p_D coincides with the translation of $-v + Ak$. Hence

$$f_v^{-1}(B) \subset p_D(AB \cap (D + v - AZ^d)) = AB - v + AZ^d.$$

This yields the first inclusion. The second one is immediate. \square

From Proposition 4 and Lemma 11, we get by recursion the next result.

Corollary 12. *Let $n \geq 1$ and $(v_1, \dots, v_n) \in \mathbb{D}^n$.*

1. *One has $(f_{v_2} \circ \dots \circ f_{v_n})^{-1}(\Delta_{v_1}) \subset A^n \Delta_D + \mathbb{Z}^d$.*

2. *The discontinuity set of $f_{v_n} \circ \dots \circ f_{v_1}$ is contained in*

$$\Delta_{v_1} \cup f_{v_1}^{-1}(\Delta_{v_2}) \cup \dots \cup (f_{v_{n-1}} \circ \dots \circ f_{v_1})^{-1}(\Delta_{v_n}),$$

which is contained in $(A\Delta_D + \mathbb{Z}^d) \cup \dots \cup (A^n \Delta_D + \mathbb{Z}^d)$.

3. *If ∂D is a null set, all those sets are null sets.*

2.3 General results on Markov chains

In the proof of Theorem 5, a key argument is the general theorem below, which is very close to Theorem 3 and Corollary 4 in [1] (paper written in French).

Theorem 13. *Let (E, \mathcal{E}) be a Polish space endowed with its Borel σ -field, (F, \mathcal{F}) be any measurable space, and $f : E \times F \rightarrow E$ be any measurable map. On probability space $(\Omega, \mathcal{A}, \mathbb{P})$, consider a (possibly non-stationary) Markov chain $X = (X_n)_{n \in \mathbb{Z}}$ indexed by the set of all integers, taking values in (E, \mathcal{E}) , whose evolution is governed by some sequence $V = (V_n)_{n \in \mathbb{Z}}$ of independent random variables taking values in (F, \mathcal{F}) as follows: for every $n \in \mathbb{Z}$,*

- V_{n+1} is independent of $\mathcal{F}_n^{X,V} := \sigma((X_k, V_k)_{k \leq n})$;
- $X_{n+1} = f(X_n, V_{n+1})$.

Assume that the tail σ -field $\mathcal{F}_{-\infty}^{X,V} := \bigcap_{n \in \mathbb{Z}} \mathcal{F}_n^{X,V}$ is trivial. Then for every $n \in \mathbb{Z}$, there exists some random variable U_n which is uniform on $[0, 1]$ or some finite set (not depending on n), independent of \mathcal{F}_n^V , and such that $\mathcal{F}_n^{X,V} = \mathcal{F}_n^V \vee \sigma(U_n)$.

Note $X_{\leq n} := (X_k)_{k \leq n}$ and $V_{\leq n} := (V_k)_{k \leq n}$. Assuming that E is a Polish space ensures the existence of the conditional law $\nu_n := \mathcal{L}(X_{\leq n} | V_{\leq n})$ and the possibility to order in a measurable way the atoms of ν_n (when ν_n has many atoms). Another way to state the conclusion is to say that the conditional laws ν_n are almost surely continuous (for all n simultaneously) or uniform on some random finite set with constant size (not depending on n). To keep the present paper self-contained, we give a proof now.

Proof. Let Y be a copy of X , such that X and Y are i.i.d. conditionally on V , namely

$$\mathcal{L}((X, Y) | V) = \mathcal{L}(X | V) \otimes \mathcal{L}(X | V).$$

Then the recursion relations $Y_{n+1} = f(Y_n, V_{n+1})$ hold almost surely; by restricting the probability space, we assume that they hold on the whole space Ω . Therefore, the event

$[X_{\leq n} = Y_{\leq n}]$ does not depend on the integer n , so it equals the event $X = Y$. By Lemma 14 below, for each $n \in \mathbb{Z}$, V_{n+1} is independent of $(X_{\leq n}, Y_{\leq n}, V_{\leq n})$.

$$\begin{aligned} \mathbb{P}[X_{\leq n+1} = Y_{\leq n+1} | \mathcal{F}_{n+1}^{X,V}] &= \mathbb{P}[X_{\leq n} = Y_{\leq n} | \mathcal{F}_{n+1}^{X,V}] \\ &= \mathbb{P}[X_{\leq n} = Y_{\leq n} | \mathcal{F}_n^{X,V} \vee \sigma(V_{n+1})] \\ &= \mathbb{P}[X_{\leq n} = Y_{\leq n} | \mathcal{F}_n^{X,V}] \text{ a.s.} \end{aligned}$$

Thus, the random variable $\mathbb{P}[X_{\leq n} = Y_{\leq n} | \mathcal{F}_n^{X,V}]$ does not depend on n (almost surely), so it is $\mathcal{F}_{-\infty}^{X,V}$ -measurable, hence almost surely constant, since $\mathcal{F}_{-\infty}^{X,V}$ is trivial.

Call p this constant (not depending on n). The random variables $X_{\leq n}$ and $Y_{\leq n}$ are i.i.d. conditionally on V , and also i.i.d. conditionally on $V_{\leq n}$, since the additional information given by $V_{\geq n+1}$ is independent of $(X_{\leq n}, Y_{\leq n}, V_{\leq n})$ (by Lemma 14 again). Thus $\mathcal{L}(Y_{\leq n} | (X_{\leq n}, V_{\leq n})) = \mathcal{L}(Y_{\leq n} | V_{\leq n}) = \nu_n$ almost surely and

$$p = \mathbb{E}[\mathbf{1}_{[X_{\leq n} = Y_{\leq n}]} | \mathcal{F}_n^{X,V}] = \nu_n\{X_{\leq n}\} \text{ a.s.}$$

Therefore, ν_n is almost surely continuous if $p = 0$, and almost surely purely atomic if $p > 0$. In the latter case, ν_n is almost surely uniform on some random finite set with size $1/p$ since all atoms have the same mass p . \square

Lemma 14. *Keep the notations and the assumptions of Theorem 13. Let $Y = (Y_n)_{n \in \mathbb{Z}}$ be a copy of X such that X and Y are i.i.d. conditionally to V . Then for every $n \in \mathbb{Z}$, V_{n+1} is independent of $\mathcal{F}_n^{X,Y,V}$.*

Moreover, if the process $(X_n, V_n)_{n \in \mathbb{Z}}$ is stationary, $(X_n, Y_n, V_n)_{n \in \mathbb{Z}}$ is also stationary.

Proof. Fix $v \in \mathbb{D}$ and consider an event of the form $A \cap B \cap C$, where A, B, C belong to $\mathcal{F}_n^X, \mathcal{F}_n^Y, \mathcal{F}_n^V$ respectively. Since X and Y are independent conditionally on $\sigma(V)$, and since $\sigma(V) = \mathcal{F}_n^V \vee \sigma(V_{n+1}, V_{n+2}, \dots)$ where $\sigma(V_{n+1}, V_{n+2}, \dots)$ is independent of $\mathcal{F}_n^{X,V}$ and of $\mathcal{F}_n^{Y,V}$, one has

$$\begin{aligned} \mathbb{E}[\mathbf{1}_A \mathbf{1}_B \mathbf{1}_C \mathbf{1}_{[V_{n+1}=v]} | \sigma(V)] &= \mathbb{E}[\mathbf{1}_A | \sigma(V)] \mathbb{E}[\mathbf{1}_B | \sigma(V)] \mathbf{1}_C \mathbf{1}_{[V_{n+1}=v]} \\ &= \mathbb{E}[\mathbf{1}_A | \mathcal{F}_n^V] \mathbb{E}[\mathbf{1}_B | \mathcal{F}_n^V] \mathbf{1}_C \mathbf{1}_{[V_{n+1}=v]} \end{aligned}$$

Taking expectations and using that V_{n+1} is independent of \mathcal{F}_n^V , we get

$$\mathbb{P}(A \cap B \cap C \cap [V_{n+1} = v]) = \mathbb{E}[\mathbb{E}[\mathbf{1}_A | \mathcal{F}_n^V] \mathbb{E}[\mathbf{1}_B | \mathcal{F}_n^V] \mathbf{1}_C] \mathbb{P}[V_{n+1} = v].$$

Summation over all $v \in \mathbb{D}$ yields $\mathbb{P}(A \cap B \cap C) = \mathbb{E}[\mathbb{E}[\mathbf{1}_A | \mathcal{F}_n^V] \mathbb{E}[\mathbf{1}_B | \mathcal{F}_n^V] \mathbf{1}_C]$. Hence

$$\mathbb{P}(A \cap B \cap C \cap [V_{n+1} = v]) = \mathbb{P}(A \cap B \cap C) \mathbb{P}[V_{n+1} = v],$$

which proves that V_{n+1} is independent of $\mathcal{F}_n^{X,Y,V}$.

Now, assume that the process $(X_n, V_n)_{n \in \mathbb{Z}}$ is stationary. Its law $\mathbb{P}_{X,V}$ is invariant by the shift S on $(E \times F)^{\mathbb{Z}}$. With an abuse of notation, call also S the shifts on the factors $E^{\mathbb{Z}}$ and $F^{\mathbb{Z}}$. Fix a regular version $(\nu_v)_{v \in F^{\mathbb{Z}}}$ of the conditional law of Y given V . For every $A \in \mathcal{E}^{\otimes \mathbb{Z}}, B \in \mathcal{E}^{\otimes \mathbb{Z}}, C \in \mathcal{F}^{\otimes \mathbb{Z}}$,

$$\begin{aligned} \mathbb{P}[(S(X), S(Y), S(V)) \in A \times B \times C] &= \mathbb{P}[X \in S^{-1}(A); Y \in S^{-1}(B); V \in S^{-1}(C)] \\ &= \int_{S^{-1}(C)} \nu_v(S^{-1}(A)) \nu_v(S^{-1}(B)) d\mathbb{P}_V(v) \\ &= \int_C \nu_{S^{-1}(u)}(S^{-1}(A)) \nu_{S^{-1}(u)}(S^{-1}(B)) d\mathbb{P}_V(u), \end{aligned}$$

since S preserves \mathbb{P}_V . The probability measures defined by $\nu'_u(A) := \nu_{S^{-1}(u)}(S^{-1}(A))$ still form a regular version of the conditional laws, since for every $A \in \mathcal{E}^{\otimes \mathbb{Z}}$ and $C \in \mathcal{F}^{\otimes \mathbb{Z}}$,

$$\begin{aligned} \int_C \nu'_u(A) d\mathbb{P}_V(u) &= \int_{S^{-1}(C)} \nu_v(S^{-1}(A)) d\mathbb{P}_V(v) \\ &= \mathbb{P}_{X,V}[S^{-1}(A) \times S^{-1}(C)] \\ &= \mathbb{P}_{X,V}[A \times C]. \end{aligned}$$

Hence $\nu'_u = \nu_u$ for \mathbb{P}_V -almost every $u \in F^{\mathbb{Z}}$, so the previous calculation yields

$$\begin{aligned} \mathbb{P}[(S(X), S(Y), S(Z)) \in A \times B \times C] &= \int_C \nu_u(A) \nu_u(B) d\mathbb{P}_V(u) \\ &= \mathbb{P}[(X, Y, Z) \in A \times B \times C]. \end{aligned}$$

We are done. □

3 Utilisation of Theorem 13

In this section, we prove Theorems 5, 7, 8 and Example 6.

3.1 Proof of Theorem 5

On some suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we consider a stationary Markov chain $X = (X_n)_{n \in \mathbb{Z}}$ taking values in D , whose evolution is governed by some sequence $V = (V_n)_{n \in \mathbb{Z}}$ of independent uniform random variables taking values in \mathbb{D} as follows. For every $n \in \mathbb{Z}$,

- X_n is uniform on D ;
- V_{n+1} is independent of $\mathcal{F}_n^{X,V} := \sigma((X_k, V_k))_{k \leq n}$;
- $X_{n+1} = f_{V_{n+1}}(X_n)$.

The last equality implies $T_A(X_{n+1}) = X_n$ and $\pi(X_{n+1}) = V_{n+1}$ by definition of the maps $(f_v)_{0 \leq v \leq r-1}$. Hence for every $n \in \mathbb{Z}$, $\mathcal{F}_n^{X,V} = \sigma(X_n)$. Moreover,

$$\Psi(X_0) = (\pi(T_A^n(X_0)))_{n \geq 0} = (\pi(X_{-n}))_{n \geq 0} = (V_{-n})_{n \geq 0}.$$

Hence, to prove that Ψ is almost surely s -to one for some positive integer s , it suffices to prove that $\mathcal{L}(X_0 | \mathcal{F}_0^V)$ is almost surely uniform on some random set with constant size. Note that $\mathcal{L}(X_0 | \mathcal{F}_0^V) = \mathcal{L}(X_0 | V)$ almost surely since $(V_n)_{n \geq 1}$ is independent of $\mathcal{F}_0^{X,V}$.

First, we prove that the tail σ -field $\mathcal{F}_{-\infty}^{X,V}$ is trivial. Let $E \in \mathcal{F}_{-\infty}^{X,V}$. For every $n \leq 0$, $E \in \sigma(X_n)$, so $E = [X_n \in B_n] = [X_0 \in T_A^{-n}(B_n)]$ for some $B_n \in \mathcal{B}(D)$. Hence

$$E = [X_0 \in B] \text{ where } B = \liminf_{n \rightarrow -\infty} T_A^{-n}(B_n)$$

Since T_A is exact (Proposition 10), we derive $\mathbb{P}(E) = \lambda(B) \in \{0, 1\}$.

Therefore, Theorem 13 applies and shows that the conditional law $\mathcal{L}(X_0 | V)$ is almost surely continuous or almost surely uniform on some finite set with constant size. It remains to rule out the first possibility. We do that by using a coupling argument.

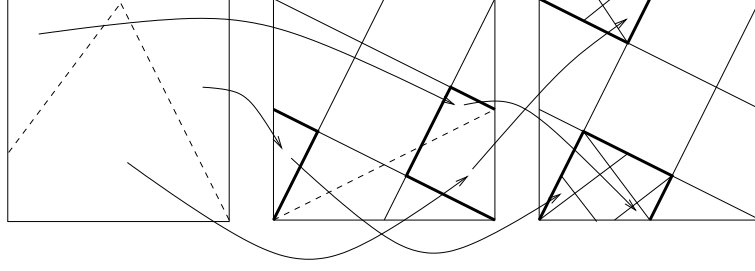


Figure 4: Example where $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, $D = [0, 1]^2$, $V_1 = (-1, 0)$ and $V_2 = (0, 0)$. The first square represents the partition $\Sigma_2 = \alpha_{V_1} \vee f_{V_1}^{-1}(\alpha_{V_2})$. The second square represents the subset $f_{V_1}(D)$ and the partition α_{V_2} . The third square represents the subset $f_{V_2}(D)$ when $V_2 = (1, 0)$. The arrows represent the effect of $f_{V_2} \circ f_{V_1}$ on each block of Σ_2 .

Consider another Markov chain $Y = (Y_n)_{n \in \mathbb{Z}}$ taking values in D such that X and Y are i.i.d. conditionally on V , namely $\mathcal{L}((X, Y)|V) = \mathcal{L}(X|V) \otimes \mathcal{L}(Y|V)$. In particular, the recursion relations $Y_{n+1} = f_{V_{n+1}}(Y_n)$ hold almost surely. By restricting the probability space, we may - and we do - assume that they hold on the whole space Ω . For each $v \in \mathbb{D}$, the map $y \mapsto f_v(y) - A^{-1}y$ takes finitely many values on D . Call α_v the partition of D associated to this map.

For every $n \geq 0$, let Σ_n be the random partition of D defined by

$$\Sigma_n = \alpha_{V_1} \vee f_{V_1}^{-1}(\alpha_{V_2}) \vee \cdots \vee (f_{V_{n-1}} \circ \cdots \circ f_{V_1})^{-1}(\alpha_{V_n}).$$

View figure 4. For every $y \in D$, call $\Sigma_n(y)$ the unique block of Σ_n which contains y .

On each block of Σ_n , the maps $y \mapsto f_{V_1}(y) - A^{-1}y$, $y \mapsto (f_{V_2} \circ f_{V_1})(y) - A^{-1}f_{V_1}(y), \dots$, $y \mapsto (f_{V_n} \circ \cdots \circ f_{V_1})(y) - A^{-1}(f_{V_{n-1}} \circ \cdots \circ f_{V_1})(y)$ are constant, hence the map $y \mapsto (f_{V_n} \circ \cdots \circ f_{V_1})(y) - A^{-n}y$ is constant. And since $(f_{V_n} \circ \cdots \circ f_{V_1})(X_0) = X_n$ we derive

$$(f_{V_n} \circ \cdots \circ f_{V_1})(\Sigma_n(X_0)) = X_n + A^{-n}(\Sigma_n(X_0) - X_0).$$

We have $\Sigma_n(X_0) \neq \Sigma_{n+1}(X_0)$ if and only if $\Sigma_n(X_0)$ is not contained in some block of the partition $(f_{V_n} \circ \cdots \circ f_{V_1})^{-1}(\alpha_{V_{n+1}})$, i.e. $(f_{V_n} \circ \cdots \circ f_{V_1})(\Sigma_n(X_0))$ is not contained in some block of the partition $\alpha_{V_{n+1}}$.

Let us endow \mathbb{R}^d with some norm $|\cdot|$ and $\mathcal{M}_d(\mathbb{R})$ with the associated operator norm. $(f_{V_n} \circ \cdots \circ f_{V_1})(\Sigma_n(X_0))$ is contained in the closed ball $\overline{B}(X_n, \|A^{-n}\| \text{diam}(D))$, where diam stands for diameter. We derive the implications

$$\begin{aligned} \Sigma_n(X_0) \neq \Sigma_{n+1}(X_0) &\implies f_{V_{n+1}} \text{ is not constant on } \overline{B}(X_n, \|A^{-n}\| \text{diam}(D)) \\ &\implies \overline{B}(X_n, \|A^{-n}\| \text{diam}(D)) \cap \Delta_{V_{n+1}} \neq \emptyset \\ &\implies \text{dist}(X_n, \Delta_{V_{n+1}}) \leq \|A^{-n}\| \text{diam}(D). \end{aligned}$$

But X_n and V_{n+1} are independent, X_n is uniform on D and V_{n+1} is uniform on D , so

$$\mathbb{P}[\Sigma_n(X_0) \neq \Sigma_{n+1}(X_0)] \leq \frac{1}{r} \sum_{v \in \mathbb{D}} \lambda(\{x \in D : \text{dist}(x, \Delta_v) \leq \|A^{-n}\| \text{diam}(D)\}).$$

For every subset S of \mathbb{R}^d , and every $\varepsilon > 0$, we note $S_\varepsilon := \{x \in \mathbb{R}^d : \text{dist}(x, S) < \varepsilon\}$ and $S_{\varepsilon+} := \{x \in \mathbb{R}^d : \text{dist}(x, S) \leq \varepsilon\}$. Given $v \in \mathbb{D}$, by proposition 4,

$$\Delta_v = A\Delta_D - v = \bigcup_{k \in \mathbb{Z}^d} A\partial D + Ak - v.$$

Since Δ_v is closed, the lower bound defining $\text{dist}(x, \Delta_v)$ is achieved, so for every $\varepsilon > 0$,

$$(\Delta_v)_{\varepsilon+} = \bigcup_{k \in \mathbb{Z}^d} (A\partial D - v + Ak)_{\varepsilon+}$$

and

$$\lambda(D \cap (\Delta_v)_{\varepsilon+}) \leq \sum_{k \in \mathbb{Z}^d} \lambda(D \cap (A\partial D - v + Ak)_{\varepsilon+}).$$

Since D and ∂D are bounded, the fundamental domain D intersects only finitely many, m_i say, subsets $(A\partial D - v + k)_1$ with $k \in \mathbb{Z}^d$. Hence, for $\varepsilon \in]0, 1[$,

$$\lambda(D \cap (\Delta_v)_{\varepsilon+}) \leq m_i \lambda((A\partial D)_{\varepsilon+}).$$

Let $\alpha > 0$ such that $\dim_{\text{upper box}}(\partial D) < d - \alpha$. Since $A\partial D$ and ∂D have the same upper box dimension,

$$\liminf_{\varepsilon \rightarrow 0} \frac{|\ln \lambda((A\partial D)_{\varepsilon+})|}{|\ln \varepsilon|} = \liminf_{\varepsilon \rightarrow 0} \frac{|\ln \lambda((A\partial D)_{\varepsilon})|}{|\ln \varepsilon|} > \alpha,$$

so $\lambda((A\partial D)_{\varepsilon}) \ll \varepsilon^{\alpha}$ as $\varepsilon \rightarrow 0$.

Putting things together yields

$$\mathbb{P}[\Sigma_n(X_0) \neq \Sigma_{n+1}(X_0)] \leq \frac{1}{r} \sum_{i=0}^{r-1} m_i \lambda((A\partial D)_{\|A^{-n}\| \text{diam}(D)+}) \ll \|A^{-n}\|^{\alpha} \text{ as } n \rightarrow +\infty.$$

Since $\|A^{-n}\|^{1/n} \rightarrow \rho(A^{-1}) < 1$ as $n \rightarrow +\infty$, the series $\sum_n \mathbb{P}[\Sigma_n(X_0) \neq \Sigma_{n+1}(X_0)]$ converges, so Borel-Cantelli Lemma applies. Almost surely, the sequence $(\Sigma_n(X_0))_{n \geq 0}$ is eventually constant.

Let Σ_{∞} be the random partition $\bigvee_{n \geq 1} \Sigma_n$ and Σ_{∞}^+ be the collection of all blocs of positive measure in Σ_{∞} . Then $\lambda(\Sigma_{\infty}(X_0)) > 0$ i.e. $\Sigma_{\infty}(X_0) \in \Sigma_{\infty}^+$ almost surely. Conditionally on V , the random variables X_0 and Y_0 are independent and equidistributed. Since Σ_{∞} is a function of $(V_n)_{n \geq 1}$, we have almost surely

$$\begin{aligned} \mathbb{P}[\Sigma_{\infty}(X_0) = \Sigma_{\infty}(Y_0) | V] &\geq \sum_{B \in \Sigma_{\infty}^+} \mathbb{P}[\Sigma_{\infty}(X_0) = \Sigma_{\infty}(Y_0) = B | V] \\ &= \sum_{B \in \Sigma_{\infty}^+} \mathbb{P}[X_0 \in B; Y_0 \in B | V] = \sum_{B \in \Sigma_{\infty}^+} \mathbb{P}[X_0 \in B | V]^2, \end{aligned}$$

whereas

$$\sum_{B \in \Sigma_{\infty}^+} \mathbb{P}[X_0 \in B | V] = \sum_{B \in \Sigma_{\infty}^+} \mathbb{P}[\Sigma_{\infty}(X_0) = B | V] = \mathbb{P}[\Sigma_{\infty}(X_0) \in \Sigma_{\infty}^+ | V] = 1.$$

Thus $\mathbb{P}[\Sigma_{\infty}(X_0) = \Sigma_{\infty}(Y_0) | V] > 0$ almost surely.

On the event $[\Sigma_{\infty}(X_0) = \Sigma_{\infty}(Y_0)]$, we have for every $n \geq 0$,

$$|Y_n - X_n| \leq \|A\|^{-n} |Y_0 - X_0| \leq \|A\|^{-n} \text{diam}(D).$$

By stationarity, we derive

$$\begin{aligned} \mathbb{P}[|Y_0 - X_0| \leq \|A\|^{-n} \text{diam}(D)] &= \mathbb{P}[|Y_n - X_n| \leq \|A\|^{-n} \text{diam}(D)] \\ &\geq \mathbb{P}[\Sigma_{\infty}(X_0) = \Sigma_{\infty}(Y_0)] \end{aligned}$$

Letting n go to infinity yields $\mathbb{P}[Y_0 = X_0] \geq \mathbb{P}[\Sigma_{\infty}(X_0) = \Sigma_{\infty}(Y_0)] > 0$. Therefore, $\mathbb{P}[Y_0 = X_0 | V] > 0$ with positive probability, so the conditional law $\mathcal{L}(X_0 | V)$ has atoms with positive probability. The proof is complete.

3.2 Proof of Theorem 7

Set $Q = [0, 1]^d$. We now assume that $D = c + UQ$ for some $c \in \mathbb{R}^d$ and some matrix $U \in \mathcal{M}_d(\mathbb{Z})$ with determinant ± 1 .

For each $v \in \mathbb{D}$, call α_v the partition of D associated to this map $y \mapsto f_v(y) - A^{-1}y$. Let $(V_n)_{n \geq 1}$ be a sequence of independent uniform random variables on \mathbb{D} . For every $n \geq 0$, let Σ_n be the random partition of D defined by

$$\Sigma_n = \alpha_{V_1} \vee f_{V_1}^{-1}(\alpha_{V_2}) \vee \cdots \vee (f_{V_{n-1}} \circ \cdots \circ f_{V_1})^{-1}(\alpha_{V_n}).$$

In the proof of Theorem 5, we viewed that given a random variable X_0 uniform on D and independent of $(V_n)_{n \geq 1}$, the sequence $(\Sigma_n(X_0))_{n \geq 0}$ is stationary. Thus for almost every $x_0 \in D$, the sequence $(\Sigma_n(x_0))_{n \geq 0}$ is stationary. We are going to prove that the restriction ‘almost’ can be removed.

Lemma 15. *For every $x_0 \in D$, the sequence $(\Sigma_n(x_0))_{n \geq 0}$ is stationary.*

Proof. Replacing A and D with $U^{-1}AU$ and $U^{-1}D$ preserves the assumptions, so we may and we do assume (without loss of generality) that $D = c + Q$. Then ∂D is the set of all $x \in \mathbb{R}^d$ such that $x - c$ has at least one coordinate in \mathbb{Z} .

Since A has integer entries, its adjugate matrix \tilde{A} also has integer entries. Call $\tilde{A}_{1,\cdot}, \dots, \tilde{A}_{d,\cdot}$ the rows of \tilde{A} . Using that $A^{-1} = (\det A)^{-1}\tilde{A} = \pm r^{-1}\tilde{A}$, we get that for every $v \in \mathbb{D}$,

$$\begin{aligned} \Delta_v &= A\partial D + AZ^d - v \\ &= \{y \in \mathbb{R}^d : \exists i \in \llbracket 1, d \rrbracket, (A^{-1}(y + v - c))_i \in \mathbb{Z}\} \\ &= \{y \in \mathbb{R}^d : \exists i \in \llbracket 1, d \rrbracket, (\tilde{A}(y + v - c))_i \in r\mathbb{Z}\} \\ &\subset \{y \in \mathbb{R}^d : \exists i \in \llbracket 1, d \rrbracket, \tilde{A}_{i,\cdot}(y - c) \in \mathbb{Z}\}. \end{aligned}$$

For every $n \geq 0$, $(f_{V_n} \circ \cdots \circ f_{V_1})(\Sigma_n(x_0)) = Z_n + A^{-n}(\Sigma_n(x_0) - x_0)$, where $Z_n := (f_{V_n} \circ \cdots \circ f_{V_1})(x_0)$. Adapting the arguments given in the proof of Theorem 5, we get

$$\begin{aligned} \Sigma_n(x_0) \neq \Sigma_{n+1}(x_0) &\iff f_{V_{n+1}} \text{ is not constant on } \Sigma_n(x_0) \\ &\implies Z_n + A^{-n}(\Sigma_n(x_0) - x_0) \cap \Delta_{V_{n+1}} \neq \emptyset \\ &\implies \exists i \in \llbracket 1, d \rrbracket, \tilde{A}_{i,\cdot}(Z_n + v - c + A^{-n}(\Sigma_n(x_0) - X_0)) \cap r\mathbb{Z} \neq \emptyset. \end{aligned}$$

Let us endow \mathbb{R}^d with the usual norm $\|\cdot\|_\infty$. The associated operator norm $\|\cdot\|_\infty$ on $\mathcal{M}_d(\mathbb{R})$ is given by

$$\|M\|_\infty = \max_{i \in \llbracket 1, d \rrbracket} \|M_{i,\cdot}\|_1, \text{ where } \|M_{i,\cdot}\|_1 := \sum_{j=1}^d |M_{i,j}|.$$

The diameter of $\Sigma_n(x_0)$ is at most $\text{diam}(D) = 1$. Since for every $i \in \llbracket 1, d \rrbracket$,

$$\|\tilde{A}_{i,\cdot}A^{-n}\|_1 = \|(\det A)(A^{-n-1})_{i,\cdot}\|_1 \leq r\|A^{-n-1}\|_\infty,$$

the diameter of $\tilde{A}_{i,\cdot}(Z_n - c + A^{-n}(\Sigma_n(x_0) - x_0))$ in \mathbb{R} is at most $r\|A^{-n-1}\|_\infty$. Therefore

$$\Sigma_n(x_0) \neq \Sigma_{n+1}(x_0) \implies \exists i \in \llbracket 1, d \rrbracket, \text{dist}(\tilde{A}_{i,\cdot}(Z_n - c), \mathbb{Z}) \leq r\|A^{-n-1}\|_\infty.$$

For every $x \in \mathbb{R}^d$ (respectively $x \in \mathbb{R}$), call \bar{x} its equivalence class in $\mathbb{R}^d/\mathbb{Z}^d$ (respectively \mathbb{R}/\mathbb{Z}). Since the matrix \tilde{A} has integer entries, the linear forms $\phi_i : x \mapsto \tilde{A}_{i,\cdot}x$ on \mathbb{R}^d yield group morphisms $\bar{\phi}_i : \mathbb{R}^d/\mathbb{Z}^d \rightarrow \mathbb{R}/\mathbb{Z}$ by taking quotients.

By proposition 4,

- the random variable Z_n is uniform on the set $T_{A,D}^{-n}(\{x_0\})$,
- the random variable $\overline{Z_n}$ is uniform on $T_A^{-n}(\{\overline{x_0}\}) = \overline{A^{-n}x_0} + (A^{-n}\mathbb{Z}^d)/\mathbb{Z}^d$,
- the random variable $\overline{Z_n - A^{-n}x_0}$ is uniform on the finite group $(A^{-n}\mathbb{Z}^d)/\mathbb{Z}^d$.
- the random variable $\overline{\phi_i(Z_n - A^{-n}x_0)} = \overline{\phi_i(Z_n A^{-n}x_0)}$ is uniform on the finite group $\overline{\phi_i}((A^{-n}\mathbb{Z}^d)/\mathbb{Z}^d) = \overline{\phi_i}(A^{-n}\mathbb{Z}^d)$.

The set $\phi_i(\mathbb{Z}^d)$ is subgroup of \mathbb{Z} , and also a subgroup of $\phi_i(A^{-n}\mathbb{Z}^d)$, with index $\leq r^n$, since \mathbb{Z}^d is a subgroup of $A^{-n}\mathbb{Z}^d$ with index r^n . Moreover, since A^{-n} is invertible, $\phi_i(A^{-n}e_j) \neq 0$ for at least one $j \in \llbracket 1, d \rrbracket$, where (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d . As $|\phi_i(A^{-n}e_j)| \leq \|\tilde{A}_i \cdot A^{-n}\|_1 \leq r\|A^{-n-1}\|_\infty$, we derive that

$$\phi_i(A^{-n}\mathbb{Z}^d) = \alpha_{n,i}\mathbb{Z} \text{ for some positive rational number } \alpha_{n,i} < r\|A^{-n-1}\|_\infty.$$

Set $\alpha_{n,i} = p_{n,i}/q_{n,i}$ where $p_{n,i}$ and $q_{n,i}$ are positive and relatively prime integers. Then

$$\overline{\phi_i(A^{-n}\mathbb{Z}^d)} = (q_{n,i}^{-1}\mathbb{Z})/\mathbb{Z}.$$

Note that any interval with length $2r\|A^{-n-1}\|_\infty$ contains at most $\lfloor q_{n,i}2r\|A^{-n-1}\|_\infty \rfloor + 1$ points of $q_{n,i}^{-1}\mathbb{Z}$. Putting things together, we derive

$$\begin{aligned} \mathbb{P}[\Sigma_n(x_0) \neq \Sigma_{n+1}(x_0)] &\leq \sum_{i=1}^d \mathbb{P}[\text{dist}(\tilde{A}_i \cdot (Z_n - c), \mathbb{Z}) \leq r\|A^{-n-1}\|_\infty] \\ &\leq d \frac{q_{n,i}2r\|A^{-n-1}\|_\infty + 1}{q_{n,i}} \\ &\leq d(2r\|A^{-n-1}\|_\infty + \alpha_{n,i}) \\ &\leq 3dr\|A^{-n-1}\|_\infty. \end{aligned}$$

The series $\sum_n \mathbb{P}[\Sigma_n(x_0) \neq \Sigma_{n+1}(x_0)]$ converges and Borel - Cantelli lemma yields the result. \square

The geometric decay of $\mathbb{P}[\Sigma_n(x_0) \neq \Sigma_{n+1}(x_0)]$ as $n \rightarrow +\infty$ leads us to conjecture a stronger result.

Conjecture 16. *For every $x_0 \in D$, the sequence $(\Sigma_n)_{n \geq 0}$ is stationary.*

Now, let us prove Theorem 7. Fix a sequence $v_0, v_{-1}, v_{-2}, \dots$ and x_0 as in the assumptions, and keep the notations of the proof of Theorem 5.

For every $n \geq 0$, the random map $f_{V_n} \circ \dots \circ f_{V_1}$ is continuous at x_0 , so x_0 is in the interior (in D) of the block $\Sigma_n(X_0)$. By lemma 15, almost surely, the sequence $(\Sigma_n(x_0))_{n \geq 1}$ is eventually constant equal to $\Sigma_\infty(x_0)$. Thus the random variable $R_0 := \text{dist}(x_0, D \setminus \Sigma_\infty(x_0))$ is positive, and the the open ball $B(x_0, R_0)$ is contained in $\Sigma_\infty(x_0)$.

Fix $\delta > 0$ such that $\mathbb{P}[R_0 \geq \delta] > 0$. Thus, on the event $[R_0 \geq \delta]$, the maps $y \mapsto f_{V_n} \circ \dots \circ f_{V_1}(y) - A^{-n}y$ are constant on the ball $B(x_0, \delta)$. By assumption, one can fix $n_0 \geq 1$ such that $f_{v_0} \circ \dots \circ f_{v_{-n_0+1}}(D) \subset B(x_0, \delta)$. Since the event $[R_0 \geq \delta]$ depends only on $(V_k)_{k \geq 1}$, one has

$$\mathbb{P}[R_0 \geq \delta ; (V_0, \dots, V_{-n_0+1}) = (v_0, \dots, v_{-n_0+1})] = r^{-n_0}\mathbb{P}[R_0 \geq \delta] > 0.$$

Moreover, on the event $[R_0 \geq \delta ; (V_0, \dots, V_{-n_0+1}) = (v_0, \dots, v_{-n_0+1})]$, one has

$$\forall k \geq 0, \quad |X_k - f_{V_k} \circ \dots \circ f_{V_1}(x_0)| \leq \|A^{-k}\| \times |X_0 - x_0| \leq \|A^{-k}\| \delta.$$

For each $n \in \mathbb{Z}$, one can define in a same way a random variable R_n depending only on $(V_{n+k})_{k \geq 1}$: R_n is the largest radius such that the open ball $B(x_0, R_n)$ is contained in the block containing x_0 in the partition

$$\bigvee_{k \geq 1} (f_{V_{n+k-1}} \circ \dots \circ f_{V_{n+1}})^{-1}(\alpha_{V_{n+k}}).$$

The ergodicity of the uniform Bernoulli shift on $\mathbb{D}^{\mathbb{Z}}$ shows that almost surely, the event $E_n := [R_n \geq \delta ; (V_n, \dots, V_{n-n_0+1}) = (v_0, \dots, v_{-n_0+1})]$ occurs for infinitely many $n \leq 0$.

To prove that the process $(V_n)_{n \leq 0}$ can be recovered as an almost everywhere continuous function of $(X_n)_{n \leq 0}$, fix a time interval $[-L, 0]$ and a precision level $\varepsilon > 0$. In the construction above, one may fix $\delta > 0$ such that $\delta \max\{\|A^{-n}\| : n \geq 0\} \leq \varepsilon$ and choose n_0 accordingly. Then the random variable $-M := \sup\{n \leq -L : E_n \text{ occurs}\}$ is almost surely finite, and the knowledge of $(V_n)_{-M-n_0 \leq n \leq 0}$ suffices to recover $(X_n)_{-L \leq n \leq 0}$ with an error whose norm is at most ε . Theorem 7 follows.

3.3 Proof of Theorem 8

Assume that $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{Z}^d . Call $\mathbf{u}^* = (u_1^*, \dots, u_d^*)$ its dual basis. Take $c \in \mathbb{R}^d$ and set $D = c + P_{\mathbf{u}}$. Then one checks that for every $x \in \mathbb{R}^d$,

$$p_D(x) = c + \sum_{i=1}^d \left(u_i^*(x - c) - \lfloor u_i^*(x - c) + 1/2 \rfloor \right) u_i.$$

The discontinuity set of p_D is $\Delta_D = \{x \in \mathbb{R}^d : \exists i \in \llbracket 1, d \rrbracket, u_i^*(x - c) + 1/2 \in \mathbb{Z}\}$.

By Corollary 12, for every $n \geq 1$ and (v_1, \dots, v_n) in $\llbracket 0, r-1 \rrbracket$, the discontinuity set of $f_{v_n} \circ \dots \circ f_{v_1}$ is contained in $(A\Delta_D + \mathbb{Z}^d) \cup \dots \cup (A^n \Delta_D + \mathbb{Z}^d)$. Thus for every $y \in D$,

$$\begin{aligned} y \in \text{Disc}(f_{v_n} \circ \dots \circ f_{v_1}) &\implies \exists j \in \llbracket 1, n \rrbracket, \exists k \in \mathbb{Z}^d : A^{-j}(y - k) \in \Delta_D \\ &\implies \exists i \in \llbracket 1, d \rrbracket, \exists j \in \llbracket 1, n \rrbracket, \exists k \in \mathbb{Z}^d : u_i^*(A^{-j}(y - k) - c) \in 1/2 + \mathbb{Z}. \end{aligned} \quad (1)$$

Since A has integer entries, its adjugate matrix \tilde{A} also has integer entries. The equality $A^{-1} = (\det A)^{-1} \tilde{A} = \pm r^{-1} \tilde{A}$, shows that $u_i^*(A^{-j}k) \in r^{-j} \mathbb{Z}$ for every $i \in \llbracket 1, d \rrbracket$ and $j \in \llbracket 1, n \rrbracket$. Hence implication (1) shows that 0 cannot be a discontinuity of $f_{v_n} \circ \dots \circ f_{v_1}$ when $(c = 0$ and r is odd) neither when $1, c_1, \dots, c_d$ are linearly independent over \mathbb{Q} . This proves item 1.

For every $y \in D$, we have $y - c \in P_{\mathbf{u}}$, so $|y - c|_{\mathbf{u}, \infty} \leq 1/2$ and

$$\begin{aligned} |A^{-1}y - c|_{\mathbf{u}, \infty} &\leq \|A^{-1}\|_{\mathbf{u}, \infty} |y|_{\mathbf{u}, \infty} + |c|_{\mathbf{u}, \infty} \\ &\leq \|A^{-1}\|_{\mathbf{u}, \infty} \times (|c|_{\mathbf{u}, \infty} + 1/2) + |c|_{\mathbf{u}, \infty} \\ &\leq (1/2) \|A^{-1}\|_{\mathbf{u}, \infty} + 2|c|_{\mathbf{u}, \infty}. \end{aligned}$$

If $|c|_{\mathbf{u}, \infty} < (1 - \|A^{-1}\|_{\mathbf{u}, \infty})/4$, we derive that for every $y \in D$, $|A^{-1}y - c|_{\mathbf{u}, \infty} < 1/2$, so $A^{-1}y \in D$ and $f_0(y) = A^{-1}y$. By recursion, for every $n \geq 1$, $f_0^n(y) = A^{-n}y$, so $f_0^n(D) \subset \overline{B}(0, \|A^{-n}\|R)$ where $R = \sup\{|y| : y \in D\}$. Item 2 follows.

Item 3 is a direct consequence of items 1 and 2 and of Theorem 7.

3.4 Proof of Example 6

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A^{-1} = (1/2)A$. The linear map $x \mapsto Ax$ from \mathbb{R}^2 to \mathbb{R}^2 is the composition of $\sqrt{2}\text{Id}$ with the reflexion with regard to the line $\mathbb{R}(\cos \pi/8, \sin \pi/8)$.

Call (e_1, e_2) the canonical basis of \mathbb{R}^2 . Take $\mathbb{D} = \{0, e_1\}$. Given a fundamental domain D , the corresponding maps f_0 and $f_1 := f_{e_1}$ from D to D are given by $f_0(y) = p_D(A^{-1}y)$ and $f_1(y) = p_D(A^{-1}y + (e_1 + e_2)/2)$. The associated partition is $\pi = \{P_0, P_1\}$ where $P_0 = f_0(D)$ and $P_1 := f_1(D)$. We want to describe the partitions $(T_A^{-k}\pi)_{k \geq 0}$. Given $x = (x_1, x_2) \in D$, we denote by $\pi(x)$ the index of the block of the partition π containing x . Since $\pi(x) \in \{0, 1\}$, congruences modulo 2 suffice to determine it.

Case where $D = [0, 1]^2$. One has $T_A(x) = (x_1 + x_2 - k, x_1 - x_2 - \ell)$, where $k := \lfloor x_1 + x_2 \rfloor$ and $\ell := \lfloor x_1 - x_2 \rfloor$. One checks that $\pi(x) \equiv \lfloor x_1 + x_2 \rfloor + \lfloor x_1 - x_2 \rfloor \pmod{2}$.

Given an integer $n \geq 0$, if $\pi(T_A^{2n}(x)) \equiv \lfloor 2^n(x_1 + x_2) \rfloor + \lfloor 2^n(x_1 - x_2) \rfloor \pmod{2}$, then

$$\begin{aligned} \pi(T_A^{2n+1}(x)) &\equiv \lfloor 2^n((x_1 + x_2 - k) + (x_1 - x_2 - \ell)) \rfloor \\ &\quad + \lfloor 2^n((x_1 + x_2 - k) - (x_1 - x_2 - \ell)) \rfloor \\ &\equiv \lfloor 2^{n+1}x_1 \rfloor - 2^n(k + \ell) + \lfloor 2^{n+1}x_2 \rfloor - 2^n(k - \ell) \\ &\equiv \lfloor 2^{n+1}x_1 \rfloor + \lfloor 2^{n+1}x_2 \rfloor \pmod{2}, \end{aligned}$$

since the integers $2^n(k + \ell)$ and $2^n(k - \ell)$ have the same parity. Thus

$$\begin{aligned} \pi(T_A^{2n+2}(x)) &\equiv \lfloor 2^{n+1}(x_1 + x_2 - k) \rfloor + \lfloor 2^{n+1}(x_1 - x_2 - \ell) \rfloor \\ &\equiv \lfloor 2^{n+1}(x_1 + x_2) \rfloor - 2^{n+1}k + \lfloor 2^{n+1}(x_1 - x_2) \rfloor - 2^{n+1}\ell \\ &\equiv \lfloor 2^{n+1}(x_1 + x_2) \rfloor + \lfloor 2^{n+1}(x_1 - x_2) \rfloor \pmod{2}, \end{aligned}$$

By recursion, those formulas hold for every $n \geq 0$. See figure 1 in the introduction.

As a result, for every $n \geq 1$, the partition $\pi \vee \dots \vee T_A^{-(n-1)}\pi$ has 2^n blocks, and each block is the union of two isosceles right triangles with small side $2^{-n/2}$, which are symmetric with regard to $(1/2, 1/2)$ (up to the boundaries). Hence, the missing information in the partitions $(T_A^{-k}\pi)_{k \geq 0}$ can be coded by a uniform random variable having two possible values.

Case where $D = [-1/2, 1/2]^2$. This time, one has $T_A(x) = (x_1 + x_2 - k, x_1 - x_2 - \ell)$, where $k := \lfloor x_1 + x_2 + 1/2 \rfloor$ and $\ell := \lfloor x_1 - x_2 + 1/2 \rfloor$. One checks that for every $n \geq 0$,

$$\begin{aligned} \pi(T_A^{2n}(x)) &\equiv \lfloor 2^n(x_1 + x_2) + 1/2 \rfloor + \lfloor 2^n(x_1 - x_2) + 1/2 \rfloor \pmod{2}, \\ \pi(T_A^{2n+1}(x)) &\equiv \lfloor 2^{n+1}x_1 + 1/2 \rfloor + \lfloor 2^{n+1}x_2 + 1/2 \rfloor \pmod{2}. \end{aligned}$$

See figure 2 in the introduction.

As a result, for every $n \geq 1$, the partition $\pi \vee \dots \vee T_A^{-(n-1)}\pi$ has 2^n blocks, and each block is the union of eight isosceles right triangles with small side $2^{-n/2-1}$. The eight isometries of the square $[-1/2, 1/2]^2$ preserves each block, up to the boundaries. Hence, the missing information in the partitions $(T_A^{-k}\pi)_{k \geq 0}$ can be coded by a uniform random variable having eight possible values. See figure 5.

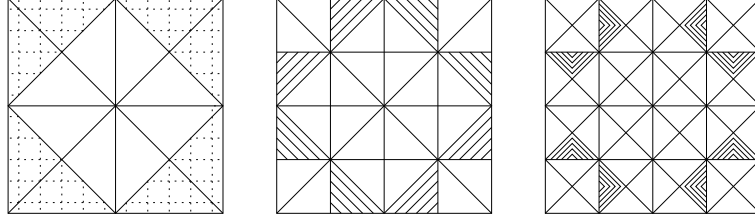


Figure 5: Refinement of the partitions π , $\pi \vee T_A^{-1}\pi$, $\pi \vee T_A^{-1}\pi \vee T_A^{-2}\pi$ when $D = [-1/2, 1/2]^2$. The shaded areas represent P_1 , $P_1 \cap T_A^{-1}(P_1)$, $P_1 \cap T_A^{-1}(P_1) \cap T_A^{-2}(P_1)$. Each one of this block is a union of eight isosceles right triangles.

Case where $D = [0, 1[e_1 + [0, 1[(e_1 + e_2)$. This parallelogram is a fundamental domain because the couple $(u_1, u_2) := (e_1, e_1 + e_2)$ is a \mathbb{Z} -basis of \mathbb{Z}^2 .

If $x \in D$, then $x = \xi_1 u_1 + \xi_2 u_2$ with ξ_1 and ξ_2 in $[0, 1[$. Since $Au_1 = u_2$ and $Au_2 = 2u_1$, we get $T_A(x) = (2\xi_2 - \lfloor 2\xi_2 \rfloor)u_1 + \xi_1 u_2$ and $T_A^2(x) = (2\xi_1 - \lfloor 2\xi_1 \rfloor)u_1 + (2\xi_2 - \lfloor 2\xi_2 \rfloor)u_2$. The parallelograms $A^{-1}D = [0, 1[u_1 + [0, 1/2[u_2$ and $A^{-1}D + A_1^{-1}u_1 = [0, 1[u_1 + [1/2, 1[u_2$ are contained in D , so those parallelograms are exactly the sets P_0 and P_1 .

One checks that for every $n \geq 1$, the partition $\pi \vee \dots \vee T_A^{-(2n-1)}\pi$ has 4^n blocks, namely the subsets $2^{-n}(D + k_1 u_1 + k_2 u_2)$ where k_1 and k_2 vary in $\llbracket 0, 2^n - 1 \rrbracket$. Hence, the partitions $(T_A^{-k}\pi)_{k \geq 0}$ provide the whole information. See figure 3 in the introduction.

4 The approach using radix expansions

In this section, we fix a subset \mathbb{D} of \mathbb{Z}^d containing exactly one element in each equivalence class modulo $A\mathbb{Z}^d$, i.e. such that $\mathbb{Z}^d = \mathbb{D} \oplus A\mathbb{Z}^d$. A recursion yields that for every $n \geq 1$,

$$\mathbb{Z}^d = \mathbb{D} \oplus \dots \oplus A^{n-1}\mathbb{D} \oplus A^n\mathbb{Z}^d.$$

By analogy with the decimal numeration system, we will view \mathbb{D} as the set of all possible digits. By convenience, we assume that $0 \in \mathbb{D}$.

4.1 Proof of Theorem 1

We define a map $\phi : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ as follows: for every $x \in \mathbb{Z}^d$, call $v(x)$ the only element of \mathbb{D} such that $x - v(x) \in A\mathbb{Z}^d$, and set $\phi(x) = A^{-1}(x - v_x)$. The vectors $\phi(x)$ and $v(x)$ can be viewed as the quotient and the remainder in the Euclidean division of x by A . Note that $\phi(0) = 0$ and

$$x \in \mathbb{D}_\infty \iff \exists n \geq 0, \phi^n(x) = 0 \iff \phi(x) \in \mathbb{D}_\infty.$$

Furthermore, for every $x \in \mathbb{Z}^d$, $|\phi(x)| \leq |A^{-1}x| + |A^{-1}v_x| \leq \|A^{-1}\| |x| + M$. Since $0 < \|A^{-1}\| < 1$, we can set $R_0 := M/(1 - \|A^{-1}\|)$ and we get

$$\begin{aligned} |\phi(x)| &< |x| \text{ whenever } |x| > R_0, \\ |\phi(x)| &\leq R_0 \text{ whenever } |x| \leq R_0. \end{aligned}$$

Since every ball of \mathbb{R}^d contains only finitely many vectors of \mathbb{Z}^d , for every $x \in \mathbb{R}^d$, the sequence $(\phi^n(x))_{n \geq 0}$ eventually reaches the absorbing set $\{y \in \mathbb{Z}^d : |y| \leq R_0\}$. The equivalence (1) \iff (2) follows.

Given $x \in \mathbb{Z}^d$, the points $(\phi^n(x))_{n \geq 0}$ cannot be all different since the absorbing set $\{y \in \mathbb{Z}^d : |y| \leq R_0\}$ is finite. Let $q \geq 0$ be the least integer such that the value $\phi^q(x)$ is reached twice and $p \geq 1$ the least integer such that $\phi^{q+p}(x) = \phi^q(x)$. Since

$$\begin{aligned} \phi^q(x) &= v(\phi^q(x)) + A\phi^{q+1}(x) \\ &= \dots \\ &= v(\phi^q(x)) + \dots + A^{p-1}v(\phi^{q+p-1}(x)) + A^p\phi^{q+p}(x), \end{aligned}$$

we derive $(I - A^q)\phi^q(x) = v(\phi^q(x)) + \dots + A^{p-1}v(\phi^{q+p-1}(x))$. If statement 3 holds, this vector is null, so $\phi^q(x)$ is null (because the matrix $I - A^q$ is invertible) and $x \in \mathbb{D}_\infty$. This shows the implication (3) \implies (2).

Conversely, if $\mathbb{D}_\infty = \mathbb{Z}^d$, then for every $x \in \mathbb{Z}^d$, the sequence $(\phi^n(x))_{n \geq 0}$ is eventually null. Therefore, for each $n \geq 1$, the only fixed point of ϕ^n is 0. If y is an element of $\mathbb{D}_n \cap (I - A^n)\mathbb{Z}^d$, then $y = v_0 + \dots + A^{n-1}v_{n-1}$ with $(v_0, \dots, v_{n-1}) \in \mathbb{D}^n$ and $y = x - A^n x$ for some $x \in \mathbb{Z}^d$. The equality $x = v_0 + \dots + A^{n-1}v_{n-1} + A^n x$ yields $\phi^n(x) = x$, so $x = 0$, so $y = 0$. This shows the implication (2) \implies (3).

Last, assume that $\mathbb{D}_\infty = \mathbb{Z}^d$ and fix a cube $[-R, R]^d$ containing the compact set $K_{\mathbb{D}}$. For every integer $N \geq 1$,

$$\bigcup_{z \in [-N, N]^d} (K_{\mathbb{D}} + z) \subset [-N - R, N + R]^d \quad (2)$$

and the finite set $[-N, N]^d$ is contained in \mathbb{D}_n for every large enough n . Since $AK_{\mathbb{D}}$ is the almost disjoint union of the sets $K_{\mathbb{D}} + v$ over all $v \in \mathbb{D}$, a recursion shows that for every $n \geq 1$, $A^n K_{\mathbb{D}}$ is the almost disjoint union of the sets $K_{\mathbb{D}} + z$ over all $z \in \mathbb{D}_n$. Thus the union in equation (2) is almost disjoint. Taking Lebesgue measures yields $(2N + 1)^d \lambda(K_{\mathbb{D}}) \leq (2N + 2R)^d$. Dividing by $(2N)^d$ and letting N go to infinity yields $\lambda(K_{\mathbb{D}}) \leq 1$, so $\lambda(K_{\mathbb{D}}) = 1$. The proof is complete.

4.2 Complements on Hutchinson's map $\tau_{\mathbb{D}}$

As in the introduction, we endow \mathbb{R}^d with a norm $|\cdot|$ such that $\|A^{-1}\| < 1$, we consider the contraction maps $(g_v)_{v \in \mathbb{D}}$ defined on \mathbb{R}^d by $g_v(x) := A^{-1}x + A^{-1}v$ and Hutchinson's map $\tau_{\mathbb{D}}$ defined on the Borel subsets B of \mathbb{R}^d by

$$\tau_{\mathbb{D}}(B) := \bigcup_{v \in \mathbb{D}} g_v(B) = \bigcup_{v \in \mathbb{D}} (A^{-1}B + A^{-1}v) = A^{-1}B + A^{-1}\mathbb{D}.$$

The map $\tau_{\mathbb{D}}$ preserves the inclusion order, and many properties: if B is bounded, open, closed, compact then map $\tau_{\mathbb{D}}(B)$ is still bounded, open, closed, compact. Since \mathbb{D} is finite, $\tau_{\mathbb{D}}$ commutes with the closure operator. Moreover, $\tau_{\mathbb{D}}$ also preserves the almost everywhere equality (for the Lebesgue measure).

We viewed that $K_{\mathbb{D}}$ is the only non-empty compact which is invariant by $\tau_{\mathbb{D}}$. Figure 6 represents $K_{\mathbb{D}}$ when $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, and $\mathbb{D} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$.

For every sequence $(B_n)_{n \geq 0}$ of Borel subsets of \mathbb{R}^d , we set

$$\begin{aligned} \limsup_{n \rightarrow +\infty} B_n &:= \{x \in \mathbb{R}^d : x \in B_n \text{ for infinitely many } n\}, \\ \liminf_{n \rightarrow +\infty} B_n &:= \{x \in \mathbb{R}^d : x \in B_n \text{ for every large enough } n\}. \end{aligned}$$

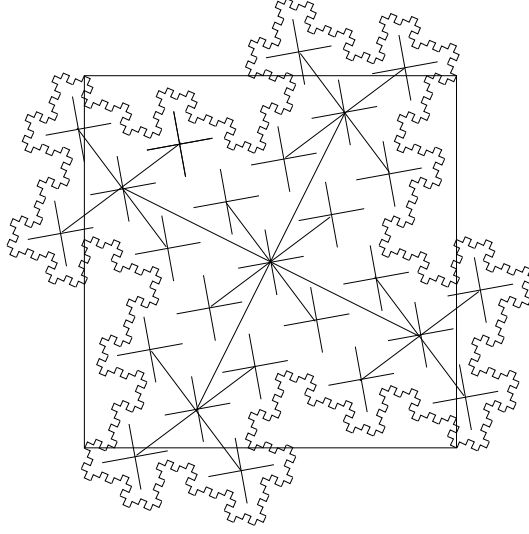


Figure 6: The square $[-1/2, 1/2]^2$ and the compact set $K_{\mathbb{D}}$ when $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbb{D} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$. The partial sums $A^{-1}v_1$, $A^{-1}v_1 + A^{-2}v_2$ and $A^{-1}v_1 + A^{-2}v_2 + A^{-3}v_3$ where $(v_1, v_2, v_3) \in \mathbb{D}^3$ are drawn.

Lemma 17. *For any Borel set B of \mathbb{R}^d , the set $\limsup_{n \rightarrow +\infty} \tau_{\mathbb{D}}^n(B)$ is invariant by $\tau_{\mathbb{D}}$.*

Proof. Let $L = \limsup_{n \rightarrow +\infty} \tau_{\mathbb{D}}^n(B)$. We prove that $\tau_{\mathbb{D}}(L) = L$ by double inclusion.

If $x \in \tau_{\mathbb{D}}(L)$, then $Ax = y + z$ with $y \in L$ and $z \in \mathbb{D}$. For infinitely many n , $y \in \tau_{\mathbb{D}}^n(B)$, so $x \in A^{-1}(\tau_{\mathbb{D}}^n(B) + \mathbb{D}) = \tau_{\mathbb{D}}^{n+1}(B)$. Thus $x \in L$.

Conversely, if $x \in L$, then Ax belongs to $A\tau_{\mathbb{D}}^n(B) = \tau_{\mathbb{D}}^{n-1}(B) + \mathbb{D}$ for infinitely many $n \geq 1$. For those n , let $Ax = y_n + z_n$ with $y_n \in \tau_{\mathbb{D}}^{n-1}(B)$ and $z_n \in \mathbb{D}$. Since \mathbb{D} is finite, the sequence $(z_n)_{n \geq 1}$ achieves some value z infinitely many times. Hence $y_n = Ax - z$ for infinitely many n , so $Ax - z$ belongs to L and x belongs to $\tau_{\mathbb{D}}(L)$. \square

When D is a bounded fundamental domain D , the successive images

$$D_n = \tau_{\mathbb{D}}^n(D) = A^{-n}D + A^{-n}\mathbb{D} + \dots + A^{-1}\mathbb{D}$$

are still bounded fundamental domains. Figure 7 represents D_0, D_1, D_2 when $d = 2$, $r = 5$, $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, $\mathbb{D} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$ and $D = [-1/2, 1/2]^2$.

Proposition 18. (First properties)

1. *The fundamental domains $(D_n)_{n \geq 0}$ are contained in a same closed ball $\overline{B}(0, R)$.*
2. *The Borel set $\limsup_n D_n$ is invariant.*
3. *$K_{\mathbb{D}}$ is the closure of $\limsup_n D_n$.*
4. *The set $\liminf_n D_n$ contains at most one point in each equivalence class of $\mathbf{R}^d / \mathbf{Z}^d$.*
5. *The set $\limsup_n D_n$ contains at least one point in each equivalence class of $\mathbf{R}^d / \mathbf{Z}^d$.*
6. *$\lambda(\liminf_n D_n) \leq 1 \leq \lambda(\limsup_n D_n)$.*

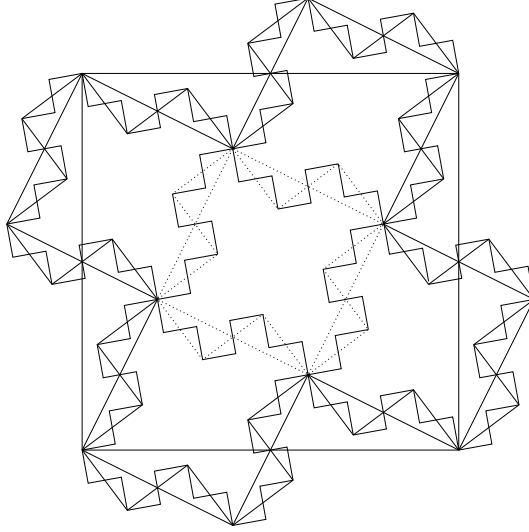


Figure 7: The fundamental domains D_0 , D_1 , D_2 , D_3 when $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbb{D} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$. The set D_0 is the square $[-1/2, 1/2]^2$, the set D_1 is a the big cross. The set D_2 is the union of five small copies of D_1 . The set D_3 is the union of five small copies of D_2 .

Proof. 1. Endow \mathbb{R}^d with the usual Euclidean metric. Let $M = \max\{|v| : v \in \mathbb{D}\}$. For every $n \geq 0$, $A^{-n}\mathbb{D} \subset \overline{B}(0, \|A^{-n}\|M)$ and $A^{-n}D \subset \overline{B}(0, \|A^{-n}\|\sqrt{d}/2)$. Since $D_n = A^{-n}D + A^{-n}\mathbb{D} + \dots + A^{-1}\mathbb{D}$, the triangle inequality yields

$$D_n \subset \overline{B}(0, R_n), \text{ where } R_n := \|A^{-n}\|\sqrt{d}/2 + \sum_{k=1}^n \|A^{-k}\|M.$$

Since $\|A^{-n}\|^{1/n} \rightarrow \rho(A^{-1}) < 1$ as $n \rightarrow +\infty$, the quantity $R := \sup_{n \geq 0} R_n$ is finite, and the ball $\overline{B}(0, R)$ contains D_n for every $n \geq 0$.

2. This item follows from Lemma 17.
3. Since $\limsup_n D_n$ is a non-empty bounded invariant subset, its closure is a non-empty compact invariant subset, hence it equals $K_{\mathbb{D}}$.
4. Let x and y be in $\liminf_n D_n$. Assume that x and y belong to the same equivalence class modulo \mathbb{Z}^d . Since x and y belong to D_n for every large enough n and since D_n is a fundamental domain, we derive that $x = y$. Item 4 follows.
5. Let $x \in \mathbb{Z}^d$. The sequence $(p_{D_n}(x))_{n \geq 0}$ varies in the finite set $(x + \mathbb{Z}^d) \cap B_{\mathbb{D}}(0, R)$, hence achieves infinitely many times some value a . This value a belongs to $\limsup_n D_n$ and to $x + \mathbb{Z}^d$. Item 5 follows.
6. Inequality $\lambda(\liminf_n D_n) \leq 1$ follows from Fatou's Lemma applied to the functions $(\mathbf{1}_{D_n})_{n \geq 1}$. Inequality $\lambda(\limsup_n D_n) \geq 1$ follows from Proposition 22.

The proof is complete. \square

4.3 Adaptating some results of Hacon, Saldanha, Veerman

The results and the arguments given in this subsection are very similar to those given by Hacon, Saldanha, Veerman in [5]. Yet, our presentation differs slightly since we focus

on the measure-preserving map T_A . Moreover, we show that many properties of the self-affine tile $K_{\mathbb{D}}$ also hold for sets of the form $\limsup \tau_{\mathbb{D}}^n(D)$, where D is a bounded fundamental domain.

Proposition 19. *Let $L = K_{\mathbb{D}}$ or $L = \limsup \tau_{\mathbb{D}}^n(D)$, where D is a bounded fundamental domain. Then*

1. *One has $\bar{L} = K_{\mathbb{D}}$.*
2. *One has $\mathbb{R}^d = L + \mathbb{Z}^d$.*
3. *One has $0 < \lambda(L) < +\infty$.*
4. *The set L is the almost disjoint union of the sets $(A^{-1}L + A^{-1}v)_{v \in \mathbb{D}}$.*
5. *The subsets $E_0 := \{x \in L : \exists!(y, z) \in L \times \mathbb{D}, x = A^{-1}y + A^{-1}z\}$, $(\tau^n(E_0))_{n \geq 0}$ and $E := \bigcap_{n \geq 0} \tau^n(E_0)$ have full measure.*
6. *For every $x \in E$, the vector Ax has unique decomposition $y + z$ with $(y, z) \in L \times \mathbb{D}$, and the vector $T_{A,E}(x) := y$ actually belongs to E .*

Proof. 1. Since L is non-empty bounded and invariant by $\tau_{\mathbb{D}}$, its closure is non-empty, compact and invariant by $\tau_{\mathbb{D}}$, so $\bar{L} = K_{\mathbb{D}}$.

2. Since $K_{\mathbb{D}}$ is compact, the set $K_{\mathbb{D}} + \mathbb{Z}^d$ is closed. But for all $n \geq 1$

$$K_{\mathbb{D}} + \mathbb{Z}^d \supset A^{-n}\mathbb{D} \oplus \cdots \oplus A^{-1}\mathbb{D} + \mathbb{Z}^d = A^{-n}\mathbb{Z}^d.$$

By Lemma 9, we derive $\mathbb{R}^d = K_{\mathbb{D}} + \mathbb{Z}^d$.

Now assume that $L = \limsup \tau_{\mathbb{D}}^n(D)$, where D is a bounded fundamental domain. We have to check that $\mathbb{R}^d \subset L + \mathbb{Z}^d$. Fix $x \in \mathbb{R}^d$. For every $n \geq 1$, the set $\tau_{\mathbb{D}}^n(D)$ is a fundamental domain, so $x = y_n + z_n$ for some (unique) $y_n \in \tau_{\mathbb{D}}^n(D)$ and $z_n \in \mathbb{Z}^d$. Since D and \mathbb{D} are bounded, one checks that the sequence $(y_n)_{n \geq 1}$ is bounded, so the sequence $(z_n)_{n \geq 1}$ varies in some finite set. It achieves some value z infinitely many times, so $y_n = x - z$ for infinitely many n , so $x - z \in L$ and $x \in L + \mathbb{Z}^d$.

3. Item 2 shows that \mathbb{R}^d is the union of the translated sets $L + z$ over all $z \in \mathbb{Z}^d$. Thus $\lambda(L) > 0$. Moreover, $0 < \lambda(L) < +\infty$ since L is bounded.
4. By Lemma 17, $L = \tau_{\mathbb{D}}(L)$ is the union of the sets $(A^{-1}L + A^{-1}v)_{v \in \mathbb{D}}$. This union is almost disjoint since

$$\sum_{v \in \mathbb{D}} \lambda(A^{-1}L + A^{-1}v) = \sum_{v \in \mathbb{D}} r^{-1} \lambda(L) = \lambda(L).$$

5. Item 5 directly follows from item 4.
6. Let $x \in E$ and $(y, z) \in L \times \mathbb{D}$ be the unique couple such that $Ax = y + z$. Then for infinitely many $n \geq 0$, we have $Ax \in \tau^{n+1}(E_0) = \tau^n(E_0) + \mathbb{D}$, so $y \in \tau^n(E_0)$ by uniqueness of the decomposition above, since $\tau^n(E_0) \subset \tau^n(L) = L$. Thus $y \in E$.

The proof is complete. \square

Last item of Proposition 19 provides a measurable map $T_{A,E} : E \rightarrow E$. Let us view some of its properties.

Proposition 20. *Let $L, E, T_{A,E}$ be like in proposition 19, define $\Phi : \mathbb{D}^\infty \rightarrow K_{\mathbb{D}}$ and $\Psi : E \rightarrow \mathbb{D}^\infty$ by*

$$\Psi((v_n)_{n \geq 1}) := \sum_{n=1}^{+\infty} A^{-n} v_n \text{ and } \Psi(x) := (AT_{A,E}^{n-1}(x) - T_{A,E}^n(x))_{n \geq 1}.$$

Then

1. The map $T_{A,E}$ preserves $\mathcal{U}_E := \lambda(\cdot|E)$, the uniform measure on E .
2. For every $x \in E$, $\Psi(T_{A,E}(x)) = S(\Psi(x))$, where S is the shift operator on \mathbb{D}^∞ .
3. Given $n \geq 1$ and $(v_1, \dots, v_n) \in \mathbb{D}^n$, the sequence $\Psi(x)$ begins by (v_1, \dots, v_n) if and only if x belongs to $A^{-n}E + A^{-n}v_n + \dots + A^{-1}v_1$. As a result, Ψ is Borel and the image of \mathcal{U}_E by Ψ is the uniform measure $\mathcal{U}_{\mathbb{D}^\infty}^{\otimes \infty}$ on \mathbb{D}^∞ .
4. For every $x \in E$, $\Phi(\Psi(x)) = x$. Therefore, the factor map Ψ is injective and the set $\Psi(E) = \{x \in \mathbb{D}^\infty : x = \Phi(\Psi(x))\}$ is Borel.
5. The image of $\mathcal{U}_{\mathbb{D}^\infty}^{\otimes \infty}$ by Φ is $\lambda(\cdot|K_{\mathbb{D}})$. As a result, $\lambda(E) = \lambda(L) = \lambda(K_{\mathbb{D}})$.

Hence, Ψ is an isomorphism transforming the measure-preserving map $T_{A,E}$ into the uniform Bernoulli shift S .

Proof. For every Borel subset $B \subset E$,

$$T_{A,E}^{-1}(B) = A^{-1}B \oplus A^{-1}\mathbb{D} = \bigcup_{v \in \mathbb{D}} (A^{-1}B + A^{-1}v),$$

and the union above is almost disjoint since $B \subset K_{\mathbb{D}}$. Thus $\lambda(T_{A,E}^{-1}(B)) = \lambda(B)$. Dividing by $\lambda(E)$ yields item 1.

Items 2 and 3 are immediate.

Let $x \in E$. Then $A^{-n}T_{A,E}^n(x) \rightarrow 0$ as $n \rightarrow +\infty$, because E is bounded. Thus

$$\begin{aligned} \Phi(\Psi(x)) &= \sum_{n=1}^{+\infty} A^{-n} (AT_{A,E}^{n-1}(x) - T_{A,E}^n(x)) \\ &= \sum_{n=1}^{+\infty} (A^{-(n-1)}T_{A,E}^{n-1}(x) - A^{-n}T_{A,E}^n(x)) = x. \end{aligned}$$

Item 4 follows.

Set $\nu := \Phi_*\mathcal{U}_{\mathbb{D}^\infty}^{\otimes \infty}$. Then $\nu = \frac{1}{r} \sum_{v \in \mathbb{D}} (g_v)_*\nu$ since for every $(v_n)_{n \geq 1} \in \mathbb{D}^\infty$,

$$\Phi((v_n)_{n \geq 1}) = A^{-1}v_1 + A^{-1}\Phi((v_{n+1})_{n \geq 1}) = g_{v_1}(\Phi((v_{n+1})_{n \geq 1})).$$

The measure $\mathcal{U}_{K_{\mathbb{D}}}$ satisfies the same equation since \mathbb{D} is the almost disjoint union of the sets $(g_v(K_{\mathbb{D}}))_{v \in \mathbb{D}}$. By Hutchinson's results (Theorem 1 in subsection 4.4 of [7]), we have $\nu = \mathcal{U}_{K_{\mathbb{D}}}$. By the last two items, we have also $\nu := \Phi_*(\Psi_*(\mathcal{U}_E)) = \mathcal{U}_E$. Hence $\mathcal{U}_E = \mathcal{U}_{K_{\mathbb{D}}}$. Since $E \subset L \subset K_{\mathbb{D}}$, we derive $\lambda(E) = \lambda(L) = \lambda(K_{\mathbb{D}})$, which proves item 5. \square

Let us recall a general lemma.

Lemma 21. *Let T be a measurable map from some measurable space (E, \mathcal{E}) to itself and η, ν be two probability measures on (E, \mathcal{E}) . Assume that η is T -ergodic, ν is T -invariant and that ν is absolutely continuous with regard to η . Then $\nu = \eta$.*

Proof. For every $A \in \mathcal{E}$, $\nu(T^{-1}A) = \nu(A)$. Subtracting $\nu(T^{-1}A \cap A)$ to both sides yields $\nu(T^{-1}A \setminus A) = \nu(A \setminus T^{-1}A)$, and the same equality holds for η .

Call f the density of ν with regard to η and let $A := [f < 1]$. Then $\eta(A) < 1$ (otherwise, we would have $\nu(E) < \eta(E) = 1$). If we had $\eta(A) > 0$, then the quantity $\eta(T^{-1}A \setminus A) = \eta(A \setminus T^{-1}A)$ would be positive by ergodicity of η . Since $f < 1$ on $A \setminus T^{-1}A$ whereas $f \geq 1$ on $T^{-1}A \setminus A$, we would have

$$\nu(A \setminus T^{-1}A) = \int_{A \setminus T^{-1}A} f d\eta < \eta(A \setminus T^{-1}A) = \eta(T^{-1}A \setminus A) \leq \int_{T^{-1}A \setminus A} f d\eta = \nu(T^{-1}A \setminus A),$$

which yields a contradiction. Hence $\eta[f < 1] = 0$, and in the same way, $\eta[f > 1] = 0$. As a result, $f = 1$ η -almost everywhere on E . The conclusion follows. \square

Recall that for every $x \in E$, $T_{A,E}(x)$ is the only element of $y \in E$ (and also the only element of $y \in L$) such that $Ax - y \in \mathbb{D}$. Calling $p_{\mathbb{T}^d}$ the canonical projection from \mathbb{R}^d onto $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, we derive $T_A(p_{\mathbb{T}^d}(x)) = p_{\mathbb{T}^d}(Ax) = p_{\mathbb{T}^d}(T_{A,E}(x))$. Thus

$$p_{\mathbb{T}^d}|_E \circ T_{A,E} = T_A \circ p_{\mathbb{T}^d}|_E,$$

so the measure $\mu := p_{\mathbb{T}^d}(\mathcal{U}_E)$ is invariant by T_A .

Call η is the Haar measure on $\mathbb{R}^d/\mathbb{Z}^d$. For every $z \in \mathbb{R}^d/\mathbb{Z}^d$, denote by $N(z)$ the size of $p_{\mathbb{T}^d}^{-1}\{z\} \cap E$. Then a computation yields $p_{\mathbb{T}^d}(\mathcal{U}_E) = (N/\lambda(E)) \cdot \eta$. In particular, $p_{\mathbb{T}^d}(\mathcal{U}_E)$ is absolutely continuous with regard to η .

Since η is ergodic for T_A , lemma 21 yields $p_{\mathbb{T}^d}(\mathcal{U}_E) = \eta$, i.e. $N/\lambda(E) = 1$ η -almost everywhere. This shows the next result.

Proposition 22. *Let L , E and $T_{A,E}$ be like in proposition 19. Then*

1. *The measure $\lambda(E) = \lambda(L) = \lambda(K_{\mathbb{D}})$ is a positive integer m .*
2. *The transformation T_A is a factor of $T_{A,E}$, hence a factor of the uniform Bernoulli shift on \mathbb{D}^∞ .*
3. *The factor map $p_{\mathbb{T}^d}|_E$ is m -to-one almost everywhere.*

Propositions 20 and 22 show that, up to null sets, $p_{\mathbb{T}^d}|_E \circ \Phi|_{\Psi(E)}$ is a $\lambda(L)$ -to-one factor map transforming the uniform Bernoulli shift on \mathbb{D}^∞ into T_A . If $\lambda(L) = 1$, we have thus an explicit isomorphism. When L equals $K_{\mathbb{D}}$, this fact was already observed by Hacon, Saldanha, Veerman at the end of the first section of [5].

Hence, we look for sufficient conditions A and \mathbb{D} to ensure that $\lambda(K_{\mathbb{D}}) = 1$, or (equivalently) conditions on A and \mathbb{D} and D to ensure that $\lambda(\limsup \tau_{\mathbb{D}}^n(D)) = 1$.

4.4 A symmetric graph structure on \mathbb{Z}^d

Fix a bounded fundamental domain D . For every $n \geq 0$, set

$$D_n = \tau_{\mathbb{D}}^n(D) = A^{-n}D + A^{-n}\mathbb{D} + \cdots + A^{-1}\mathbb{D}.$$

A strategy to prove - under appropriate assumptions - that $\lambda(\limsup D_n) = 1$ is to show that the sequence $(\lambda(D_n \Delta D_{n+1}))_{n \geq 0}$ decays exponentially fast, so the sequence of indicator functions $(\mathbf{1}_{D_n})_{n \geq 0}$ converges almost everywhere and in $L^1(\lambda)$. To do this, we use the equalities

$$\begin{aligned} A^n D_n &= D_0 + \mathbb{D} + \cdots + A^{n-1}\mathbb{D}, \\ A^n D_{n+1} &= D_1 + \mathbb{D} + \cdots + A^{n-1}\mathbb{D}, \end{aligned}$$

and we prove that the subset $\mathbb{D}_n := \mathbb{D} \oplus \cdots \oplus A^{n-1}\mathbb{D}$ of \mathbb{Z}^n , whose size is r^n , has a small boundary when n is large.

It is convenient to endow \mathbb{Z}^d with a graph structure, possibly different from the usual one. We fix a finite subset \mathbb{E} of \mathbb{Z}^d , containing 0, such that $-\mathbb{E} = \mathbb{E}$. The set \mathbb{E} provides a symmetric graph structure on \mathbb{Z}^d as follows: given x and y in \mathbb{Z}^d , we say that (x, y) is an edge if and only if $y - x \in \mathbb{E}$. We define the internal and external boundaries of any subset S of \mathbb{Z}^d by

$$\begin{aligned} \partial_{\text{int}} S &:= \{v \in S : \exists e \in \mathbb{E}, \quad v + e \notin S\}, \\ \partial_{\text{ext}} S &:= \{v \in S^c : \exists e \in \mathbb{E}, \quad v + e \in S\}. \end{aligned}$$

To simplify the notations, we do not indicate the dependence on \mathbb{E} . Observe that $\partial_{\text{ext}} S = (S - \mathbb{E}) \setminus S = (S + \mathbb{E}) \setminus S$.

We can now state sufficient conditions to have $\lambda(\limsup D_n) = 1$.

Theorem 23. *Assume that there exists $\ell \geq 1$ such that*

1. $D_1 \subset D_0 + \mathbb{E}$,
2. $\mathbb{E} \subset \mathbb{D}_\ell - \mathbb{D}_\ell$,
3. $\mathbb{D}_\ell + \mathbb{E} \subset \mathbb{D}_\ell + A^\ell \mathbb{E}$.

Then $\lambda(\limsup D_n) = 1$. Hence, the map $p_{\mathbb{T}^d}|_{\mathbb{E}} \circ \Phi|_{\Psi(\mathbb{E})}$ provided by propositions 19, 20 and 22 is an isomorphism transforming T_A into the uniform Bernoulli shift on \mathbb{D}^∞ .

Before proving Theorem 23, observe that the assumptions are fulfilled when $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbb{D} = \mathbb{E} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$ and $\ell = 1$. In this case, the graph structure on \mathbb{Z}^d is derived from the usual one by adding a loop at each vertex. More generally, under the assumption $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis \mathbf{u} of \mathbb{Z}^d , Proposition 28 below provides an explicit choice of \mathbb{D} , D and \mathbb{E} such that conditions 1, 2, 3 hold.

Let us comment those assumptions. First, the set \mathbb{E} must be sufficiently large for $D_0 + \mathbb{E}$ to cover D_1 , but not too large (because of condition 2). Next, if conditions 2 and 3 hold for some integer ℓ , they still hold for any larger integer, because $\mathbb{D}_\ell \subset \mathbb{D}_{\ell+1}$ and $\mathbb{D}_{\ell+1} = \mathbb{D}_\ell + A^\ell \mathbb{D}$.

Let $n \geq 1$, $x \in \mathbb{D}_{\ell n}$ and $e_0 \in \mathbb{E}$. Since $\mathbb{D}_{\ell n} = \mathbb{D}_\ell \oplus A^\ell \mathbb{D}_\ell \oplus \cdots \oplus A^{\ell(n-1)} \mathbb{D}_\ell$, the vector x has a unique decomposition

$$x = v_0 + A^\ell v_1 + \cdots + (A^\ell)^{n-1} v_{n-1} \quad \text{with } (v_0, \dots, v_{n-1}) \in \mathbb{D}_\ell^n.$$

By applying repeatedly assumption 3, we get the existence of $v'_0 \in \mathbb{D}_\ell$, $e_1 \in \mathbb{E}$, ... , $v'_{n-2} \in \mathbb{D}_\ell$, $e_{n-1} \in \mathbb{E}$ such that

$$\begin{aligned} x + e_0 &= (v_0 + e_0) + A^\ell v_1 + \cdots + (A^\ell)^{n-1} v_{n-1} \\ &= v'_0 + A^\ell(v_1 + e_1) + \cdots + (A^\ell)^{n-1} v_{n-1} \\ &= \cdots \\ &= v'_0 + A^\ell v'_1 + \cdots + (A^\ell)^{n-2} v'_{n-2} + (A^\ell)^{n-1} (v_{n-1} + e_{n-1}). \end{aligned}$$

By assumption 2, there exist vectors f_0, \dots, f_{n-1} in \mathbb{D}_ℓ such that $f_0 + e_0, \dots, f_{n-1} + e_{n-1}$ are still in D_n . Hence, most of the time, the sum $x + e_0$ belongs to $\mathbb{D}_{\ell n}$. Indeed,

$$x + e_0 \notin \mathbb{D}_{\ell n} \implies (v_0 \neq f_0 \text{ and } \cdots \text{ and } v_{n-1} \neq f_{n-1}).$$

As a result, $|\mathbb{D}_{\ell n} \setminus (\mathbb{D}_{\ell n} + e_0)| \leq (r^\ell - 1)^n$. The next lemma follows.

Lemma 24. *If the assumptions 2 and 3 hold, then for every $n \geq 1$ and every $e \in \mathbb{E}$,*

$$\begin{aligned} |\mathbb{D}_{\ell n} \setminus (\mathbb{D}_{\ell n} + e)| &= |(\mathbb{D}_{\ell n} + e) \setminus \mathbb{D}_{\ell n}| \leq (r^\ell - 1)^n, \\ |\partial_{\text{int}} \mathbb{D}_{\ell n}| &\leq |\mathbb{E}|(r^\ell - 1)^n \text{ and } |\partial_{\text{ext}} \mathbb{D}_{\ell n}| \leq |\mathbb{E}|(r^\ell - 1)^n. \end{aligned}$$

Proof. The equality and the first inequality follow from the equality $|\mathbb{D}_{\ell n} + e| = |\mathbb{D}_{\ell n}|$ and the discussion above. The last two inequalities follow by taking an union bound. \square

Lemma 25. *Let S be any subset of \mathbb{Z}^d . If $D_1 \subset D + \mathbb{E}$, then*

$$\begin{aligned} (S + D_1) \setminus (S + D) &\subset \partial_{\text{ext}} S + D, \\ \lambda((S + D_1) \setminus (S + D)) &\leq |\partial_{\text{ext}} S|. \end{aligned}$$

and provided S is finite,

$$\lambda((S + D) \setminus (S + D_1)) \leq |\partial_{\text{ext}} S|.$$

Proof. Note that for every finite subset S of \mathbb{Z}^d , and any fundamental domain D , the sum $S + D$ is direct, so $\lambda(S + D) = |S|\lambda(D) = |S|$. Since

$$(S + D_1) \setminus (S + D) \subset (S + \mathbb{E} + D) \setminus (S + D) \subset ((S + \mathbb{E}) \setminus S) + D = \partial_{\text{ext}} S + D,$$

we derive $\lambda((S + D_1) \setminus (S + D)) \leq |\partial_{\text{ext}} S|$.

If S is finite, then $\lambda(S + D) = \lambda(S + D_1) = |S|$ is finite, thus $(S + D_1) \setminus (S + D)$ and $(S + D) \setminus (S + D_1)$ have the same Lebesgue measure. The proof is complete. \square

Lemma 26. *For every Borel subsets B and C of \mathbb{R}^d ,*

$$\lambda(\tau_{\mathbb{D}}(B) \Delta \tau_{\mathbb{D}}(C)) \leq \lambda(B \Delta C).$$

Proof. It follows from the inclusion

$$\tau_{\mathbb{D}}(B) \Delta \tau_{\mathbb{D}}(C) \subset \bigcup_{v \in \mathbb{D}} (A^{-1}(B \Delta C) + A^{-1}v).$$

by taking an union bound. \square

Proposition 27. *If the assumptions 1, 2, 3 hold with $\ell \geq 1$, then*

$$\lambda(D_{n+1} \triangle D_n) = O((1 - r^{-\ell})^{n/\ell}) \text{ as } n \rightarrow +\infty.$$

Proof. By lemma 26, the sequence $(\lambda(D_{n+1} \triangle D_n))_{n \geq 0}$ is non-increasing. Hence, it is enough to bound above $\lambda(D_{\ell n+1} \triangle D_{\ell n})$ for $n \geq 0$.

We observe that for every $n \geq 0$,

$$A^{\ell n} D_{\ell n} = D + \mathbb{D} + \dots + A^{\ell n-1} \mathbb{D} = D + \mathbb{D}_{\ell n}$$

and

$$A^{\ell n} D_{\ell n+1} = A^{-1} D + A^{-1} \mathbb{D} + \mathbb{D} + \dots + A^{\ell n-1} \mathbb{D} = D_1 + \mathbb{D}_{\ell n}.$$

Hence, by Lemmas 25 and 24,

$$\lambda(A^{\ell n} D_{\ell n+1} \setminus A^{\ell n} D_{\ell n}) \leq |\partial_{\text{ext}} \mathbb{D}_{\ell n}| \leq |\mathbb{E}|(r^\ell - 1)^n.$$

Since $|\det A^{\ell n}| = r^{\ell n}$, we derive

$$\lambda(D_{\ell n+1} \triangle D_{\ell n}) = 2\lambda(D_{\ell n+1} \setminus D_{\ell n}) \leq 2|\mathbb{E}|(1 - r^{-\ell})^n.$$

We are done. \square

Theorem 23 directly follows. Indeed, the last proposition ensures that the series $\sum_n \|\mathbf{1}_{D_{n+1}} - \mathbf{1}_{D_n}\|_1 = \sum_n \lambda(D_{n+1} \triangle D_n)$ converges, so the series $\sum_n (\mathbf{1}_{D_{n+1}} - \mathbf{1}_{D_n})$ converges almost everywhere and in $L^1(\lambda)$. As a result, the sequence $(\mathbf{1}_{D_n})_{n \geq 0}$ converges almost everywhere and in $L^1(\lambda)$. In particular, $\lambda(\limsup D_n) = \lim \lambda(D_n) = 1$.

We viewed in the introduction that if D is any fundamental domain, then $AD \cap \mathbb{Z}^d$ is a system of representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$. Hence Theorem 3 follows from Theorem 23 and the Proposition below.

Proposition 28. *Assume that $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$ for some \mathbb{Z} -basis $\mathbf{u} = (u_1, \dots, u_d)$ of \mathbb{Z}^d . Let*

$$c := (1/2)A^{-1}(u_1 + \dots + u_d), \quad D := c + \sum_{i=1}^d [-1/2, 1/2[u_i,$$

$$\mathbb{D} := AD \cap \mathbb{Z}^d, \quad \mathbb{E} := \sum_{i=1}^d \{-1, 0, 1\}u_i.$$

Then conditions of Theorem 23 apply with $\ell = 1$.

Proof. By an unimodular change of coordinates, we may and we do assume that \mathbf{u} is the canonical basis. Thus we omit the index \mathbf{u} in the norms.

First, we observe that

$$|c|_\infty \leq (1/2)\|A^{-1}\|_\infty \times |(1, \dots, 1)|_\infty < 1/2.$$

By the triangle inequality, $D \subset]-1, 1[^d$, so $A^{-1}D \subset]-1, 1[^d$ since $\|A^{-1}\|_{\mathbf{u}, \infty} < 1$. Moreover, Since $\mathbb{D} \subset AD$,

$$\begin{aligned} D_1 = A^{-1}\mathbb{D} + A^{-1}D &\subset D + A^{-1}D \\ &\subset c + [-1/2, 1/2[^d +]-1, 1[^d \\ &\subset c + [-3/2, 3/2[^d. \end{aligned}$$

whereas

$$D + \mathbb{E} = c + [-1/2, 1/2]^d + \{-1, 0, 1\}^d = c + [-3/2, 3/2]^d.$$

Thus, condition 1 holds.

Next, by the definition of c and since $\|A^{-1}\|_\infty < 1$, we have

$$A^{-1}[0, 1]^d = c + A^{-1}[-1/2, 1/2]^d \subset c + [-1/2, 1/2]^d = D.$$

Thus, $[0, 1]^d \subset AD$ so $\mathbb{D} := AD \cap \mathbb{Z}^d$ contains $\{0, 1\}^d$ and $\mathbb{D} - \mathbb{D}$ contains $\mathbb{E} := \{-1, 0, 1\}^d$. Thus, condition 2 holds.

Last, since $\|A^{-1}\|_\infty < 1$, we have $A^{-1}\mathbb{E} \subset [-1, 1]^d$, so

$$D + A^{-1}\mathbb{E} \subset c + [-1/2, 1/2]^d + [-1, 1]^d = c + [-3/2, 3/2]^d = D + \mathbb{E}.$$

Thus

$$\mathbb{D} + \mathbb{E} \subset AD + \mathbb{E} \subset AD + A\mathbb{E}.$$

Since $\mathbb{D} + \mathbb{E}$ and $A\mathbb{E}$ contain only integer points, we derive

$$\mathbb{D} + \mathbb{E} \subset \mathbb{Z}^d \cap (AD + A\mathbb{E}) = (\mathbb{Z}^d \cap AD) + A\mathbb{E} = \mathbb{D} + A\mathbb{E}.$$

Thus, condition 3 holds. □

4.5 Remarks on the explicitness of the isomorphism

In this subsection, we assume that $\lambda(K_{\mathbb{D}}) = 1$ and we keep the notations of subsections 4.2 and 4.3. For every $n \geq 1$, we note $\mathbb{D}_n = \mathbb{D} \oplus \dots \oplus A^{n-1}\mathbb{D}$.

Proposition 20 applied to the set $L = K_{\mathbb{D}}$ shows that $p_{\mathbb{T}^d} \circ \Phi : \mathbb{D}^\infty \rightarrow \mathbb{T}$ is an isomorphism transforming the uniform Bernoulli shift on \mathbb{D}^∞ into T_A . Although this map is simple and explicit, its inverse is much more involved. To make it explicit, we have to answer the following questions: given an element of $\mathbb{R}^d/\mathbb{Z}^d$, how to find its (η -almost surely unique) representative in $K_{\mathbb{D}}$? given an element of $K_{\mathbb{D}}$, how to find its $\mathcal{U}(K_{\mathbb{D}})$ -almost surely unique (A, \mathbb{D}) expansion (in base A for the system of digits \mathbb{D})?

These questions can be answered for almost every point, once we effectively know two Borel bounded subsets B_1, B_2 with positive Lebesgue measure such that $B_1 \subset K_{\mathbb{D}} \subset B_2$. Indeed, each element of $\mathbb{R}^d/\mathbb{Z}^d$ has only finitely many representatives in B_2 , and the representative in $K_{\mathbb{D}}$ should be searched among them. Hence, the questions above can be reformulated as follows: given $x \in \mathbb{R}^d$, and v_1, v_2, \dots in \mathbb{D} prove or disprove that x belongs to $K_{\mathbb{D}}$, to $A^{-1}v_1 + A^{-1}K_{\mathbb{D}}$, to $A^{-1}v_1 + A^{-2}v_2 + A^{-2}K_{\mathbb{D}}$... Equivalently, prove or disprove that the vectors $x, Ax - v_1, A^2x - Av_1 - v_2$... belong to $K_{\mathbb{D}}$.

Fix a norm $|\cdot|$ on \mathbb{R}^d and denote by $\|\cdot\|$ the associated operator norm on $\mathcal{M}_d(\mathbb{R})$. For every $x \in \mathbb{R}^d$ and $n \geq 0$, since the assumption $0 \in \mathbb{D}$ yields $A^{-n}\mathbb{D}_n \subset K_{\mathbb{D}}$, we have

$$\text{dist}(x, K_{\mathbb{D}}) \leq \text{dist}(x, A^{-n}\mathbb{D}_n) \leq \|A^{-n}\| \text{dist}(A^n x, \mathbb{D}_n).$$

Since $K_{\mathbb{D}}$ is closed and $\|A^{-n}\| \rightarrow 0$ as $n \rightarrow +\infty$, we derive

$$x \notin K_{\mathbb{D}} \implies \text{dist}(x, K_{\mathbb{D}}) > 0 \implies \text{dist}(A^n x, \mathbb{D}_n) \rightarrow \infty \text{ as } n \rightarrow +\infty.$$

Thus, if $x \notin K_{\mathbb{D}}$, provided the integer n is sufficiently large, all differences $A^n x - s$ with $s \in \mathbb{D}_n$ are outside B_2 . This gives us the certainty that x does not belong to $K_{\mathbb{D}}$.

Indeed, when $x \in K_{\mathbb{D}}$, i.e. $x = \sum_{k \geq 1} A^{-k} v_k$ for some sequence $(v_k)_{k \geq 1} \in \mathbb{D}^{\infty}$, we have for every $n \geq 1$,

$$A^n x - \sum_{k=1}^n A^{n-k} v_k = \sum_{k \geq n+1} A^{-(k-n)} v_k \in K_{\mathbb{D}} \subset B_2.$$

Moreover, if $x \in E$, the (A, \mathbb{D}) -expansion of x is unique, and for every $n \geq 0$,

$$A^n x - \sum_{k=1}^n A^{n-k} v_k = T_{A,E}^n(x).$$

The ergodicity of $T_{A,E}$ for $\mathcal{U}(E)$ entails that for almost every $x \in E$ (hence for almost every $x \in K_{\mathbb{D}}$), the orbit $(T_{A,E}^n(x))_{n \geq 0}$ visits B_1 infinitely many times. Thus, by computing finitely many differences $A^n x - (A^{n-1} v_1 + \dots + v_n)$, we get at the same time the certainty that x belongs to $K_{\mathbb{D}}$ and the first digits of its (A, \mathbb{D}) -expansion.

Finding explicit bounded Borel subsets with positive Lebesgue measure such that $B_1 \subset K_{\mathbb{D}} \subset B_2$ is less simple for B_1 than for B_2 . Indeed, let $M = \max\{|v| : v \in \mathbb{D}\}$, $R := \sum_{k \geq 1} \|A^{-k}\| M$ and $R_0 := \text{dist}(0, \mathbb{Z}^d \setminus \{0\})$. Then $K_{\mathbb{D}} \subset \overline{B}(0, R)$ by the triangle inequality. If $R < R_0$, we derive $B(0, R_0 - R) \subset K_{\mathbb{D}}$, since

$$K_{\mathbb{D}}^c \subset \bigcup_{z \in \mathbb{Z}^d \setminus \{0\}} (K_{\mathbb{D}} + z) \subset \bigcup_{z \in \mathbb{Z}^d \setminus \{0\}} \overline{B}(z, R) \subset \{x \in \mathbb{R}^d : |x| \geq R_0 - R\}.$$

When $R \geq R_0$, sharper estimates are necessary.

Example 29. When $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbb{D} = \{(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)\}$ and when $|\cdot|$ is the usual Euclidean metric, we get $M = R_0 = 1$ and

$$R := \sum_{k=1}^{+\infty} (1/\sqrt{5})^k = \frac{1/\sqrt{5}}{1 - 1/\sqrt{5}} = \frac{1}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{4} < 1,$$

so our simple method applies.

5 Using Lagarias-Wang and Latimer-MacDuffee theorems

This section uses even more sophisticated tools.

In subsection 5.1, we view a theorem of Lagarias and Wang which provides a necessary and sufficient condition to have $\lambda(K_{\mathbb{D}}) = 1$. This theorem is profound but not easy to apply. We give corollaries which are easier to handle. In particular, if A is \mathbb{Z} -similar to a companion matrix, we view how we can choose \mathbb{D} so that $\lambda(K_{\mathbb{D}}) = 1$.

This brings us to the question: when can we be sure that A is \mathbb{Z} -similar to a companion matrix? In subsection 5.2, we state Latimer-MacDuffee theorem and give a self-contained proof of one of its corollaries: the matrix A is \mathbb{Z} -similar to a companion matrix whenever the integral domain $\mathbb{Z}[X]/\chi_A \mathbb{Z}[X]$ is principal.

In subsection 5.3, we apply this corollary and the results viewed at subsection 5.1 to establish results in low dimension, including Theorem 2 (if $d = 2$, one can always choose \mathbb{D} so that $\lambda(K_{\mathbb{D}}) = 1$) and an example of a 4×4 matrix A such that $\lambda(K_{\mathbb{D}}) \geq 2$ for every choice of \mathbb{D} .

5.1 Lagarias-Wang theorem and direct consequences

Call $\mathbb{Z}[A, \mathbb{D}]$ the smallest sublattice which contains $\mathbb{D} - \mathbb{D}$ and is stable under A . By Cayley - Hamilton theorem, A^d is a linear combination of I, A, \dots, A^{d-1} with integer coefficients. Thus $\mathbb{Z}[A, \mathbb{D}]$ is also the sublattice of \mathbb{Z}^d generated by $\mathbb{D} \cup A\mathbb{D} \dots \cup A^{d-1}\mathbb{D}$ since we assumed that $0 \in \mathbb{D}$.

Lemma 30. *The set $\mathbb{Z}[A, \mathbb{D}]$ is a lattice of \mathbb{R}^d , i.e. its rank is d .*

Proof. We argue by contradiction. Call d_1 the rank of $\mathbb{Z}[A, \mathbb{D}]$, assume that $d_1 < d$, and set $d_2 = d - d_1$. By the theory of lattices, there exist a \mathbb{Z} -basis (u_1, \dots, u_d) of \mathbb{Z}^d and positive integers $\alpha_1, \dots, \alpha_{d_1}$ such that $(\alpha_1 u_1, \dots, \alpha_{d_1} u_{d_1})$ is a \mathbb{Z} -basis of $\mathbb{Z}[A, \mathbb{D}]$. Call $U \in GL_d(\mathbb{Z})$ the matrix whose columns are u_1, \dots, u_d . Then $B := U^{-1}AU$ is the matrix of the endomorphism $x \mapsto Ax$ in the basis (u_1, \dots, u_d) . Since $\mathbb{Z}[A, \mathbb{D}]$ is the linear space spanned by (u_1, \dots, u_{d_1}) and is stable by the endomorphism $x \mapsto Ax$, one has

$$B = \begin{pmatrix} B_1 & C \\ 0 & B_2 \end{pmatrix},$$

with $B_1 \in \mathcal{M}_{d_1}(\mathbb{Z})$, $B_2 \in \mathcal{M}_{d_2}(\mathbb{Z})$, $C \in \mathcal{M}_{d_2, d_1}(\mathbb{Z})$. Denoting by χ the characteristic polynomial, we get $\chi_A = \chi_B = \chi_{B_1}\chi_{B_2}$, so the matrices B_1 and B_2 are expanding. In particular, the lattice $B_2\mathbb{Z}^{d_2}$ is strictly contained in \mathbb{Z}^{d_2} . Furthermore, on the one hand,

$$U^{-1}A\mathbb{Z}^d = BU^{-1}\mathbb{Z}^d = B\mathbb{Z}^d \subset \mathbb{Z}^{d_1} \times B_2\mathbb{Z}^{d_2}.$$

On the other hand, $\mathbb{D} \subset \mathbb{Z}[A, \mathbb{D}] \subset \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_{d_1} = U(\mathbb{Z}^{d_1} \times \{0_{d_2}\})$, so

$$U^{-1}\mathbb{D} \subset \mathbb{Z}^{d_1} \times \{0_{d_2}\}.$$

By addition

$$\mathbb{Z}^d = U^{-1}\mathbb{Z}^d = U^{-1}(\mathbb{D} \oplus A\mathbb{Z}^d) \subset \mathbb{Z}^{d_1} \times B_2\mathbb{Z}^{d_2},$$

which yields a contradiction. □

Definition 31. *One says that (A, \mathbb{D}) is primitive when $\mathbb{Z}[A, \mathbb{D}] = \mathbb{Z}^d$.*

Remark 32. *When $r = 2$, the set \mathbb{D} has exactly two elements, 0 and v , say. Then (A, \mathbb{D}) is primitive if and only if $(v, Av, \dots, A^{d-1}v)$ is a \mathbb{Z} -base of \mathbb{Z}^d .*

The next result enlightens the interest of the notion of primitivity.

Proposition 33. (Lagarias-Wang) *A necessary condition to have $\lambda(K_{\mathbb{D}}) = 1$ is that (A, \mathbb{D}) is primitive.*

Here is a proof of this fact which is more elementary than Lagarias and Wang's one.

Proof. By lemma 30 and by the theory of lattices, there exist a \mathbb{Z} -basis (u_1, \dots, u_d) of \mathbb{Z}^d and positive integers $\alpha_1, \dots, \alpha_d$ such that $(\alpha_1 u_1, \dots, \alpha_d u_d)$ is a \mathbb{Z} -basis of $\mathbb{Z}[A, \mathbb{D}]$. Call $Q \in \mathcal{M}_d(\mathbb{Z})$ the matrix whose columns are $\alpha_1 u_1, \dots, \alpha_d u_d$. Then $\det Q = \pm \alpha_1 \dots \alpha_d \neq 0$ and $\mathbb{D} \subset \mathbb{Z}[A, \mathbb{D}] = Q\mathbb{Z}^d$, so $\mathbb{D} = Q\mathbb{D}'$ for some $\mathbb{D}' \subset \mathbb{Z}^d$. Since $AQ\mathbb{Z}^d = A\mathbb{Z}[A, \mathbb{D}] \subset Q\mathbb{Z}^d$, there exists some matrix $A' \in \mathcal{M}_d(\mathbb{Z})$ such that $AQ = QA'$. The matrix $A' = Q^{-1}AQ$ has the same characteristic polynomial as A , hence it is expansive. Moreover,

$$\mathbb{R}^d = Q^{-1}\mathbb{R}^d = Q^{-1}(\mathbb{D} \oplus A\mathbb{Z}^d) = Q^{-1}\mathbb{D} \oplus Q^{-1}A\mathbb{Z}^d = \mathbb{D}' \oplus A'\mathbb{Z}^d,$$

since $\mathbb{Z}^d = Q\mathbb{Z}^d$. Thus \mathbb{D}' is a system of representatives of $\mathbb{Z}^d/A'\mathbb{Z}^d$. By proposition 22, the Lebesgue measure of the self-affine tile

$$K'_{\mathbb{D}'} := \left\{ \sum_{k \geq 1} (A')^{-k} v'_k : (v'_k)_{k \geq 1} \in \mathbb{D}'^{\infty} \right\}$$

is a positive integer. The equalities $\mathbb{D} = Q\mathbb{D}'$ and $A^{-k}Q = Q(A')^{-k}$ yield $K_{\mathbb{D}} = QK'_{\mathbb{D}'}$, so $\lambda(K_{\mathbb{D}}) = |\det(Q)|\lambda'(K'_{\mathbb{D}'})$ is a multiple of $|\det(Q)|$. Hence, if $\lambda(K_{\mathbb{D}}) = 1$, then $|\det(Q)| = 1$, so $\mathbb{Z}[A, \mathbb{D}] = \mathbb{Z}^d$. \square

Proposition 33 enables us to show that choosing $\mathbb{D} := AP \cap \mathbb{Z}^d$ where P is a fundamental paralleloptope (containing 0) does not guarantee that $\lambda(K_{\mathbb{D}})$ equals 1.

Example 34. Let $A := \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix}$, $P := [0, 1]^2$ and $\mathbb{D} := AP \cap \mathbb{Z}^d$. Then $\chi_A = X^2 - X + 2$, the matrix A is expanding, and $AP \cap \mathbb{Z}^d = \{0, v\}$ where $v = (1 \ 2)^\top$. Since $Av = (0 \ 2)^\top$, one has $\mathbb{Z}[A, \mathbb{D}] \subset \mathbb{Z} \times 2\mathbb{Z}$. Hence (A, \mathbb{D}) is not primitive, so $\lambda(K_{\mathbb{D}}) > 1$.

Actually, assuming that (A, \mathbb{D}) is primitive does not guarantee that $\lambda(K_{\mathbb{D}}) = 1$. Some other obstructions may occur, although they are quite rare. Lagarias and Wang give a necessary and sufficient condition to have $\lambda(K_{\mathbb{D}}) > 1$ when (A, \mathbb{D}) is primitive. This condition provides the existence of a unimodular matrix P such that PAP^{-1} is block-trigonalisable while $P\mathbb{D}$ (which is a complete system of representatives modulo $PAP^{-1}\mathbb{Z}^d$) has a ‘skew-product structure’. Hence the condition is rarely satisfied, although in some situations, it may be difficult to disprove it.

Theorem 35. (Theorem 6.1 of [11]). Assume that (A, \mathbb{D}) is primitive.

Then $\lambda(K_{\mathbb{D}}) > 1$ if and only if there exist

- an unimodular matrix $P \in GL_d(\mathbb{Z})$,
- two positive integers d_1, d_2 with sum d ,
- four matrices $B_1 \in \mathcal{M}_{d_1}(\mathbb{Z})$, $B_2 \in \mathcal{M}_{d_2}(\mathbb{Z})$, $C \in \mathcal{M}_{d_1, d_2}(\mathbb{Z})$, $Q \in \mathcal{M}_{d_2}(\mathbb{Z})$,
- three families $(a_i)_{1 \leq i \leq r_1} \subset \mathbb{R}^{d_1}$, $(b_i)_{1 \leq i \leq r_1} \subset \mathbb{R}^{d_2}$ and $(c_{i,j})_{1 \leq i \leq r_1, 1 \leq j \leq r_2} \subset \mathbb{R}^{d_2}$, where $r_1 := |\det(B_1)|$ and $r_2 := |\det(B_2)|$.

such that

1. $PAP^{-1} = \begin{pmatrix} B_1 & 0 \\ C & B_2 \end{pmatrix}$,
2. $|\det Q| \geq 2$ and $Q^{-1}B_2Q \in \mathcal{M}_{d_2}(\mathbb{Z})$,
3. $P\mathbb{D} = \left\{ \begin{pmatrix} a_i \\ b_i + Qc_{i,j} \end{pmatrix} : 1 \leq i \leq r_1, 1 \leq j \leq r_2 \right\}$,
4. The family $(a_i)_{1 \leq i \leq r_1}$ is a complete system of representatives modulo $B_1\mathbb{Z}^{d_1}$.
5. For each $i \in \llbracket 1, |\det(B_1)| \rrbracket$, the family $(Qc_{i,j})_{1 \leq j \leq r_2}$ is a complete system of representatives modulo $B_2\mathbb{Z}^{d_2}$.

In this statement, proving the ‘if’ part is rather simple and uses the same kind of arguments as Proposition 33, together with Fubini’s theorem. Actually, the case where (A, \mathbb{D}) is not primitive could be viewed as the case where $d_1 = 0$ in the conditions above. Yet, the ‘only if’ part is highly non-trivial, and relies on the existence of some lattice $\Lambda \subset \mathbb{Z}^d$ which is a tiling set for $K_{\mathbb{D}}$, i.e. such that \mathbb{R}^d is the almost disjoint union of the translated tiles $(K_{\mathbb{D}} + s)_{s \in S}$.

Note that condition 1 above implies $\chi_A = \chi_{B_1} \chi_{B_2}$ and $r = r_1 r_2$. Thus, the matrices B_1 and B_2 must also be expansive, and the integers $r_1 = |\det(B_1)|$, $r_2 = |\det(B_2)|$ must be larger than or equal to 2. Such a situation cannot occur when r is a prime number, or when the characteristic polynomial χ_A is irreducible in $\mathbb{Z}[X]$.

When $r_1 \geq 2$ and $r_2 \geq 2$, conditions 3, 4, 5 prevent the set \mathbb{D} from being contained in some line. This remark provides a notable consequence.

Corollary 36. *Assume that for some vector $v_1 \in \mathbb{Z}^d$, the family $(v_1, Av_1, \dots, A^{d-1}v_1)$ is a \mathbb{Z} -basis of \mathbb{Z}^d . Then $\mathbb{D} := \{0, v_1, \dots, (r-1)v_1\}$ is a system of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$. Furthermore, for this choice of \mathbb{D} , one has $\lambda(K_{\mathbb{D}}) = 1$.*

Proof. By assumption, $(v_1, Av_1, \dots, A^{d-1}v_1)$ is a \mathbb{Z} -basis of \mathbb{Z}^d , so $(Av_1, A^2v_1, \dots, A^d v_1)$ is a \mathbb{Z} -basis of $A\mathbb{Z}^d$. Let $X^d - (\alpha_0 + \alpha_1 X + \dots + \alpha_{d-1} X^{d-1})$ be the characteristic polynomial of A . Then $\alpha_0 = (-1)^{d-1} \det A = \pm r$, and $A^d v_1 = \alpha_0 v_1 + \alpha_1 Av_1 + \dots + \alpha_{d-1} A^{d-1} v_1$. As a result, $(Av_1, \dots, A^{d-1} v_1, r v_1)$ is a \mathbb{Z} -basis of $A\mathbb{Z}^d$ and $\mathbb{D} := \{0, v_1, \dots, (r-1)v_1\}$ is a system of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$.

The subgroup $\mathbb{Z}[A, \mathbb{D}]$ contains $\mathbb{Z}v_1 + \mathbb{Z}Av_1 + \dots + \mathbb{Z}A^{d-1}v_1 = \mathbb{Z}^d$. Hence (A, \mathbb{D}) is primitive. Moreover, for every $P \in GL_d(\mathbb{Z})$, the subset $P\mathbb{D}$ is contained in $\mathbb{Z}Pv_1$, so the conditions of Theorem 35 cannot be fulfilled. Thus $\lambda(K_{\mathbb{D}}) = 1$. \square

Corollary 37. *If A is similar in $\mathcal{M}_d(\mathbb{Z})$ to some block-diagonal matrix in which all diagonal blocks are companion matrices, then one can find a system \mathbb{D} of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$ such that $\lambda(K_{\mathbb{D}}) = 1$.*

Proof. By an unimodular change of coordinates, we may - and we do - assume that $A = \text{Diag}(C_1, \dots, C_n)$ where C_1, \dots, C_n are companion matrices. Call d_1, \dots, d_n the sizes of C_1, \dots, C_n and r_1, \dots, r_n the absolute value of their determinants.

For each $k \in \llbracket 1, n \rrbracket$, the set $\mathbb{D}_k := \llbracket 0, r_k - 1 \rrbracket (1, 0, \dots, 0)$ is a system of representatives of $\mathbb{Z}^{d_k}/C_k \mathbb{Z}^{d_k}$. The Cartesian product $\mathbb{D} := \mathbb{D}_1 \times \dots \times \mathbb{D}_n$ is a system of representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$ and $K_{\mathbb{D}}$ is the Cartesian product of the compact sets $K_{C_1, \mathbb{D}_1}, \dots, K_{C_n, \mathbb{D}_n}$ associated to $(C_1, \mathbb{D}_1), \dots, (C_n, \mathbb{D}_n)$. Hence $\lambda(K_{\mathbb{D}}) = \lambda(K_{C_1, \mathbb{D}_1}) \cdots \lambda(K_{C_n, \mathbb{D}_n}) = 1$. \square

5.2 Latimer-MacDuffee theorem and a corollary

Unfortunately, Corollary 37 does not cover all cases, since Frobenius reduction applies only to square matrices with entries in a field. Actually, the classification of all matrices of $\mathcal{M}_d(\mathbb{Z})$ up to \mathbb{Z} -similarity is a difficult and still open problem: no complete system of invariants is known. We now state an important result in this topic. Given any integral domain D , we define an equivalence relation on all non-zero ideals by

$$I \sim J \iff \exists a, b \in D \setminus \{0\}, \quad aI = bJ.$$

The equivalence classes for this relation are called *ideal classes*. For example, the ideal class of the ideal D is precisely the collection of all principal ideals.

Theorem 38. (Latimer-MacDuffee theorem) *Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial with integer coefficients and degree d . Then, among the matrices of $\mathcal{M}_d(\mathbb{Z})$ whose characteristic polynomial is f , the number of \mathbb{Z} -similarity classes equals the number of ideal classes in the integral domain $\mathbb{Z}[X]/f\mathbb{Z}[X]$.*

Taussky's paper [15] provides a simple proof via an explicit correspondence between the similarity classes and the ideal classes. We will only use the following corollary.

Corollary 39. *Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial with integer coefficients and degree d . Then, the two statements below are equivalent:*

1. *All matrices of $\mathcal{M}_d(\mathbb{Z})$ whose characteristic polynomial is f are \mathbb{Z} -similar.*
2. *The integral domain $\mathbb{Z}[X]/f\mathbb{Z}[X]$ is principal.*

To keep the paper self-contained, we give a proof of Corollary 39.

Proof. Fix a complex root α of f . Then α is a simple root of f . Indeed, f is irreducible in $\mathbb{Z}[X]$ and also in $\mathbb{Q}[X]$. Hence the greatest common divisor of f and f' in $\mathbb{Q}[X]$, and also in $\mathbb{C}[X]$ (by Bezout's identity) equals 1. Hence $(X - \alpha)^2$ cannot divide f .

The evaluation map $g \mapsto g(\alpha)$ from $\mathbb{Z}[X]$ to \mathbb{C} is a morphism of integral domains with range $\mathbb{Z}[\alpha]$ and with kernel $f\mathbb{Z}[X]$. Thus the integral domain $\mathbb{Z}[X]/f\mathbb{Z}[X]$ is isomorphic to $\mathbb{Z}[\alpha]$. Moreover, since f is irreducible, it is the minimal polynomial of α . Therefore, $(1, \alpha, \dots, \alpha^{d-1})$ is a \mathbb{Z} -basis of $\mathbb{Z}[\alpha]$.

Assume that condition 1 holds. Let I be a non-zero ideal of $\mathbb{Z}[\alpha]$. Then I is a subgroup of $\mathbb{Z}[\alpha]$, with same rank as $\mathbb{Z}[\alpha]$. Indeed, given a non-zero element $x \in I$, the elements $x, \alpha x, \dots, \alpha^{d-1}x$ are \mathbb{Z} -linearly independent, since for every $g \in \mathbb{Z}[X]$,

$$g(\alpha)x = 0 \iff g(\alpha) = 0 \iff f|g. \quad (3)$$

Thanks to equation (3), the minimal polynomial, and also the characteristic polynomial of the endomorphism $x \mapsto \alpha x$ on I is f . By condition 1, the matrix of this endomorphism in some \mathbb{Z} -basis (v_1, \dots, v_d) is a companion matrix. Hence $v_2 = \alpha v_1, \dots, v_d = \alpha v_{d-1}$, so

$$\mathbb{Z}[\alpha]v_1 \subset I = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_d \subset \mathbb{Z}[\alpha]v_1,$$

i.e. $I = \mathbb{Z}[\alpha]v_1$ is principal.

Conversely, assume that condition 2 holds. Let $A \in \mathcal{M}_d(\mathbb{Z})$ be any matrix whose minimal polynomial is f . Since α is a simple root of f , the matrix $A - \alpha I_d$ has rank $d - 1$ and its adjugate matrix M has at least one non-zero entry $M(i_0, j_0)$, say. Since $M(A - \alpha I_d) = \chi_A(\alpha)I_d = 0$, the row $w := M(i_0, \cdot)$ is a row eigenvector of A associated to α , whose coordinates belong to $\mathbb{Z}[\alpha]$.

Let $I = w_1\mathbb{Z} + \dots + w_d\mathbb{Z}$. Since the entries of A belong to \mathbb{Z} and since for all $j \in \llbracket 1, d \rrbracket$,

$$\alpha w_j = (wA)_j = \sum_{i=1}^d w_i A(i, j), \quad (4)$$

one has $\alpha I \subset I$. Thus I is an ideal of $\mathbb{Z}[\alpha]$. By condition 2, one can find $u \in I$ such that $I = u\mathbb{Z}[\alpha]$. Then $(u, \alpha u, \dots, \alpha^{d-1}u)$ is a \mathbb{Z} -basis of I . In this basis, the matrix of the endomorphism $x \mapsto \alpha x$ on I is a companion matrix. But (w_1, \dots, w_d) is also a \mathbb{Z} -basis of I and in this basis, the matrix on this endomorphism is A , thanks to equation (4). As a result, A is \mathbb{Z} -similar to a companion matrix. The proof is complete. \square

5.3 Applications to the low-dimensional case

In this subsection, we apply consequences of Lagarias-Wang theorem viewed at subsection 5.1 and Corollary 39 to get precise results in low dimension. We also introduce a notion of convexity in \mathbb{Z}^d , which will help us when $d = 2$.

Definition 40. *Let S be a subset S of \mathbb{Z}^d . One says that S is convex in \mathbb{Z}^d if the intersection of \mathbb{Z}^d with the convex hull of S in \mathbb{R}^d is contained in S . Equivalently, S is convex in \mathbb{Z}^d if and only if S is the intersection of \mathbb{Z}^d with some convex subset of \mathbb{R}^d .*

When $d = 2$, the conditions 1–5 of Theorem Lagarias and Wang condition imply $d_1 = d_2 = 1$, so $B_1 = r_1$, $B_2 = r_2$ and Q must be an integer relatively prime with r_2 , such that $|Q| \geq 2$. Moreover, the first components of $P\mathbb{D}$ can take r_1 different values and, when the first component a_i is known, the second can take r_2 different values, all of them belonging to the same coset $b_i + Q\mathbb{Z}$. Since $r_2 \geq 2$, this prevents $P\mathbb{D}$ and \mathbb{D} from being convex in \mathbb{Z}^d . We derive the following corollary.

Corollary 41. *Assume that $d = 2$. If \mathbb{D} is convex in \mathbb{Z}^2 (in particular if $\mathbb{D} := AD \cap \mathbb{Z}^2$ where D be a fundamental parallelogram containing 0), and if (A, \mathbb{D}) is primitive, then $\lambda(K_{\mathbb{D}}) = 1$.*

Remark 42. *The argument above does not work anymore when $d \geq 3$. Indeed, Lagarias and Wang conditions may hold with $d_2 = 2$, $B_2 = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, since $Q^{-1}B_2Q$ has integer coefficients. Then $\{(0, 0), (1, 1)\}$ is contained in $Q\mathbb{Z}^2$ and is a system of representatives of $\mathbb{Z}^2/B_2\mathbb{Z}^2$. Yet, it is convex in \mathbb{Z}^2 .*

We are now able to prove Theorem 2.

Proof. Assume that $d = 2$. We have to show that it is always possible to choose the system \mathbb{D} of representatives so that $\lambda(K_{\mathbb{D}}) = 1$. We distinguish two cases. Both of them split into subcases.

Case where $r \geq 3$. Since $A\mathbb{Z}^2$ is a lattice contained in \mathbb{Z}^2 , one can find a \mathbb{Z} -basis (u_1, u_2) of \mathbb{Z}^2 and two positive integers α_1, α_2 such that $(\alpha_1 u_1, \alpha_2 u_2)$ is a \mathbb{Z} -basis of $A\mathbb{Z}^2$. One has $\alpha_1 \alpha_2 = r$. Moreover, the set $\mathbb{D} := \llbracket 0, \alpha_1 - 1 \rrbracket u_1 \oplus \llbracket 0, \alpha_2 - 1 \rrbracket u_2$ is a system of representatives modulo $A\mathbb{Z}^2$ and is convex in \mathbb{Z}^2 .

If $\alpha_1 \geq 2$ and $\alpha_2 \geq 2$, then (A, \mathbb{D}) is primitive since $\mathbb{D} - \mathbb{D}$ contains u_1 and u_2 .

Otherwise, by symmetry, one may assume that $\alpha_2 = 1$ and $\alpha_1 = r \geq 3$, so $u_2 \in A\mathbb{Z}^2$. Then $\mathbb{D}' := \llbracket 0, \alpha_1 - 2 \rrbracket u_1 \cup \{(\alpha_1 - 1)u_1 + u_2\}$ is also a system of representatives modulo $A\mathbb{Z}^2$, and is also convex in \mathbb{Z}^2 . Moreover (A, \mathbb{D}) is primitive since $\mathbb{D} - \mathbb{D}$ contains u_1 and $u_1 + u_2$ (here we use the assumption $r \geq 3$).

Hence Corollary 41 applies in both subcases.

Case where $r = 2$. Call t the trace of A . Then $\chi_A = X^2 - tX + \det A$.

If $\det A = -2$, the eigenvalues of A are $(-t \pm \sqrt{t^2 + 8})/2$. Since A is expanding, one has $\sqrt{t^2 + 8} > |t| + 2$. Squaring both sides and simplifying yields $|t| < 1$, namely $t = 0$.

If $\det A = 2$, the discriminant of χ_A is $t^2 - 8$. If we had $|t| \geq 3$, we would have $|t| - 2 \leq \sqrt{t^2 - 8} \leq |t|$, which would contradict the assumption that A is expansive. Thus $|t| \leq 2$, the discriminant of χ_A is negative, and both eigenvalues have modulus $\sqrt{2}$.

As a result, χ_A is one of the polynomials $X^2 - 2$, $X^2 + 2$, $X^2 \pm X + 2$, $X^2 \pm 2X + 2$. One checks that the integral domain $\mathbb{Z}[X]/\chi_A\mathbb{Z}[X]$ is Euclidean hence principal. By Corollary 39, χ_A is similar to a companion matrix. Equivalently, one can find a vector $v_1 \in \mathbb{Z}^2$ such that (v_1, Av_1) is a \mathbb{Z} -basis of \mathbb{Z}^2 . Thus Corollary 36 applies.

The proof is complete. \square

Remark 43. *The coefficients of the polynomial $X^d\chi_A(1/X)$ are those of χ_A written in the reverse order, and its roots (i.e. the eigenvalues of A^{-1}) belong to the open unit disk. Thus the relations between the coefficients and the roots show that the coefficients of χ_A have only finitely many possible values when d and $|\det A| = r$ are fixed.*

For example, when $d = 3$ and $r = 2$, the possible characteristic polynomials are $X^3 - X^2 - X + 2$, $X^3 - 2X + 2$, $X^3 - X + 2$, $X^3 + 2$, $X^3 + X^2 + 2$, $X^3 + X^2 + X + 2$, $X^3 + 2X^2 + 2X + 2$, and the polynomials deduced from them by the transformation $P \mapsto -P(-X)$. One can check that the integer domain $\mathbb{Z}[X]/\chi_A\mathbb{Z}[X]$ is principal, and the same proof as in the two-dimensional case shows that it is possible to choose the system \mathbb{D} of representative such that $\lambda(K_{\mathbb{D}}) = 1$.

Unfortunately, it is not always possible to choose a system of representatives such that $\lambda(K_{\mathbb{D}}) = 1$. Here is a counterexample.

Example 44. *If z is a complex number such that $|z| \leq 1$, then $|z^4 + z^2| \leq |z|^4 + |z|^2 \leq 2$, and equality can hold only when $z^4 = z^2$ and $|z| = 1$, namely when $z = \pm 1$. Thus the polynomial $f := X^4 + X^2 + 2$ has no root in the closed unit disk. One checks that the integer domain $\mathbb{Z}[X]/f\mathbb{Z}[X]$ is not principal. By corollary 39, there exists some matrix $A \in \mathcal{M}_d(\mathbb{Z})$ with characteristic polynomial f which is not \mathbb{Z} -similar to a companion matrix. By Remark 32 and Proposition 33, we derive that $\lambda(K_{\mathbb{D}}) > 1$ for every system \mathbb{D} of representatives of $\mathbb{Z}^4/A\mathbb{Z}^4$. An example of such a matrix is*

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

Indeed, for every vector $v = (x \ y \ z \ t)^\top \in \mathbb{Z}^4$, the determinant

$$\det(v, Av, A^2v, A^3v) = \begin{vmatrix} x & -t & -2z + t & -2y + t \\ y & x & -t & 2z + t \\ z & y + z - t & x + y - z & x - y - z \\ t & 2z - t & 2y - t & 2x - 2z - t \end{vmatrix}$$

is always even, since the last two columns have the same parity. Hence (v, Av, A^2v, A^3v) cannot be a \mathbb{Z} -basis of \mathbb{Z}^4 .

Remark 45. *Similar arguments provide a counterexample whenever $d = 4n$ for some integer $n \geq 1$. The polynomial $f_n := X^{4n} + X^{2n} + 2 = f \circ X^n$ has no root in the closed unit disk and is irreducible in $\mathbb{Z}[X]$. Indeed, if $f_n = PQ$ with P and Q in $\mathbb{Z}[X]$, then up to a switch and a multiplication by -1 , the polynomials P and Q are monic, with constant terms ± 1 and ± 2 . The product of the roots of P have modulus 1, so it is an empty product since the modulus of every root of P is larger than 1. Hence P must be constant (equal to 1).*

Let α be a root of f_n and set $\beta := \alpha^n$. The integral domain $\mathbb{Z}[\alpha]$ is isomorphic to $\mathbb{Z}[X]/f_n\mathbb{Z}[X]$. By construction, the number $\gamma := (\beta^3 - \beta^2)/2$ belongs to the field of fractions of $\mathbb{Z}[\alpha]$, but not to $\mathbb{Z}[\alpha]$ itself. Yet

$$\begin{aligned} 4\gamma^2 = \beta^4(\beta - 1)^2 &= -(\beta^2 + 2)(\beta^2 - 2\beta + 1) \\ &= -\beta^4 + 2\beta^3 - 3\beta^2 + 4\beta - 2 \\ &= 2\beta^3 - 2\beta^2 + 4\beta \\ &= 4\gamma + 4\beta, \end{aligned}$$

so $\gamma^2 - \gamma = \beta$ is a root of $X^4 + X^2 + 2$ and γ is a root of the monic polynomial $(X^2 - X)^4 + (X^2 - X)^2 + 2$, which belongs to $\mathbb{Z}[X]$, therefore to $\mathbb{Z}[\alpha][X]$. Hence, the integral domain $\mathbb{Z}[\alpha]$ is not integrally closed, therefore it cannot be principal.

Acknowledgements

I thank my colleagues J. Brossard, A. Coquio, D. Piau, for their constructive remarks, and G. Berhuy, who simplified my original proof for the case where $d = 2$ and $r = 2$, drew my attention on Latimer-MacDuffee theorem and found Example 44.

References

- [1] Brossard J., Leuridan C., *Chaînes de Markov constructives indexées par \mathbb{Z}* , Annals of Probability **35-2**, 715–731 (2007).
- [2] Diaconis P., Freedman D., *Iterated random functions*, SIAM Review **41-1**, 45–76 (1999).
- [3] Duda J., *Complex base numeral systems*. arxiv:0712.1309
- [4] Gabardo J.P., Yu X., *Natural tiling, lattice tiling and Lebesgue measure of integral self-affine tiles*. Journal of London Mathematical Society (2) **74**, 184–204 (2006).
- [5] Hacon D., Saldanha C., Veerman J.J.P. *Remarks on Self-Affine Tilings*. Experimental mathematics 317–327 (1996).
- [6] Hoffman C., Rudolph D., *Uniform endomorphisms which are isomorphic to a Bernoulli shift*, Annals of Math., **156**, 79–101 (2002).
- [7] Hutchinson J., *Fractals and self similarity*, Indiana University Mathematical Journal **30**, 713–747 (1981).
- [8] Katznelson Y. *Ergodic automorphisms of \mathbb{T}^n are Bernoulli shifts*, Israel Journal of Mathematics, 10, 186–195 (1971).
- [9] Knuth D., *A imaginary number system*. Journal of the Association for Computing Machinery, **3-4**, 245–247 (1960).
- [10] Krzyżewski K. *On Exact Toral Endomorphisms*, Monatshefte für Mathematik **116**, 39–47 (1993).
- [11] Lagarias J., Wang Y. *Integral self-affine tiles in \mathbb{R}^n II. Lattice tilings*. Journal of Fourier Analysis and Applications. **3**, 83–102 (1997)

- [12] Matula D., *Basic digit sets for radix representation*. Journal of the Association for Computing Machinery, **29**-4, 1131–1143 (1982).
- [13] Mihailescu E., *Higher dimensional expanding maps and toral extensions*, Proceedings of the American Mathematical Society **141** n° 10, 3467–3475 (2013).
- [14] Penney W., *A ‘binary sytem’ for complex numbers*. Journal of the Association for Computing Machinery, **12**-2, 247–248 (1965).
- [15] Taussky O., *On a theorem of Latimer and MacDuffee*, Canadian Journal of Mathematics, bf 1-3, 300–302 (1949).
- [16] A.M. Vershik, *Decreasing sequences of measurable partitions, and their applications*. Dokl. Akad. Nauk SSSR, **193**, 748–751 (1970). English translation: Soviet mathematics - Doklady, **11**, 1007–1011 (1970).
- [17] A.M. Vershik, *The theory of decreasing sequences of measurable partitions*. Algebra i Analiz **6**:4, 1–68 (1994) (in Russian). English translation: St. Petersburg Mathematical Journal **6**:4, 705–761 (1995).