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► **To cite this version:**

Henri Prade. Data bases with fuzzy information and approximate reasoning in expert systems. Bulletin pour les sous-ensembles flous et leurs applications, 1983, 14, pp.115–126. hal-04556639

HAL Id: hal-04556639

<https://hal.science/hal-04556639>

Submitted on 23 Apr 2024

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Data bases with fuzzy information
and
approximate reasoning in expert systems*

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This short paper deals successively with two important fields of applications of fuzzy set and possibility theory

- data bases with fuzzy information and fuzzy queries
- approximate/plausible reasoning in expert systems.

The main ideas of a work in progress concerning a very general approach to the management of relational databases with fuzzy attribute values are presented. This approach makes an extensive use of the dual concepts of possibility and necessity measures.

Besides the problem of the inference from uncertain/imprecise premises is discussed ; the unification of old and new results leads to the emergence of a coherent framework for dealing with this question ; the combination of uncertain or imprecise informations issued from different sources, which is another basic problem we have to deal with in production rule systems, is not addressed here.

Proof are omitted for sake of brevity.

* Prepared for the Seminar on "Fuzzy Sets and Knowledge - Based Systems", Queen Mary College, London, March 23, 1983.

Data bases with fuzzy information

In our approach [13], [14], [15] to relational data bases with fuzzy data we deal with ordinary relations on Cartesian product of sets of fuzzy sets as Umano in [20] rather than with fuzzy relations as in FRIL (see [2]). For example :

Student :	Name :	Age :	Level in Maths :	Sympathy: Name 1 :	Name 2 :	Type
:	:	:	:	:	:	:
:	Tom	20	Unknown	:	David	Tom : F
:	David	[20,25]	[10,12]	:	David	John : {I,H}
:	Maria	VERY-YOUNG	RATHER-GOOD	:	John	Tom : Unknown
:	:	:	:	:	:	:

The domain of the Type attribute is here
 VH: very hostile, H : hostile, I : Indifferent,
 F : friendly, VF: Very Friendly .

The use of labels of fuzzy sets or of intervals represented by possibility distributions on the attribute domains enables to represent attribute values which are only partially or imprecisely known. The null value "unknown" is represented by a possibility distribution equal to 1 for each value of the attribute domain while the null value not applicable" corresponds to a possibility distribution equal to 0 everywhere.

Given a predicate A(vague or not) represented by a possibility distribution, we can define two fuzzy sets of items in the data base : the fuzzy set of items which are possibly A defined by the membership function :

$$\mu_{\text{PosA}}(x) = \prod(A|T(x)) = \sup_d \min(\mu_A(d), \mu_{T(x)}(d))$$

and the fuzzy set of items which are necessarily A defined by the membership function

$$\mu_{\text{NesA}}(x) = N(A|T(x)) = \inf_d \max(\mu_A(d), 1 - \mu_{T(x)}(d))$$

where $\mu_{T(x)}$ is the possibility distribution modelling the knowledge available in the data base concerning x and where D is the domain on which A and T(x) are defined. We have

$$N(A|T(x)) \leq \prod(A|T(x)) \quad \text{i.e.}$$

Nes A \subseteq Pos A, which is in conformity with the intuition.

When our knowledge is complete, T(x) reduces to a singleton of D and we have $N(A|T(x)) = \prod(A|T(x))$ and $\text{NesA} = \text{PosA}$; however when A is vague and the knowledge is complete, the set of items which are necessarily A remains fuzzy. When A is non-vague, whatever T(x) is, we have $N(A|T(x)) > 0$ entails $\prod(A|T(x)) = 1$, which means that the items which are necessarily A to some degree are among the items whose possibility to be A is equal to 1; the necessity measure enables to discriminate when the possibility measure is equal to 1.

On each domain D which is a continuum, a tolerance R_D represented by a fuzzy relation μ_{RD} may be introduced for modelling the fact that two close values of D can be regarded as approximately equal (as in [4]). This tolerance relation may be used for making more flexible the notion of redundant tuples. Let (A_{k1}, \dots, A_{kn}) and (A_{l1}, \dots, A_{ln}) be two tuples of a relation; the two tuples may be considered as redundant if $\min_j (N(A_{kj} \text{ OR }_{D_j} | A_{lj}), N(A_{lj} \text{ OR }_{D_j} | A_{kj}))$ is greater than some threshold where R_{D_j} is the tolerance relation on domain D_j and where $\mu_{A \text{ OR } D}(d) = \sup_{d'} \min(\mu_A(d'), \mu_{RD}(d, d')) \geq \mu_A(d)$; $A \text{ OR }_D$ is larger than A due to the tolerance.

The usual operations of relational algebra (projection, selection, join, ...) can be extended. Thus, queries such that "Find the students whose level in maths is much greater than their level in physics" or "Find the students whose level in maths is at least good" can be easily dealt with. In the first example

$$\pi(\theta | T_1(x) \times T_2(x)) = \sup_{d, d'} \min(\mu_{T_1}(d), \mu_{T_2}(d'), \mu_\theta(d, d'))$$

$$\text{and } N(\theta | T_1(x) \times T_2(x)) = \inf_{d, d'} \max(\mu_\theta(d, d'), 1 - \mu_{T_1}(x)(d), 1 - \mu_{T_2}(x)(d'))$$

estimates the possibility and the necessity respectively that the level in maths of the student x (the available knowledge concerning this level is represented by $T_1(x)$) is in relation θ (here θ is a fuzzy relation which models "much greater than") with his level in physics (the available knowledge concerning the level in physics being represented by $T_2(x)$). Again these two evaluations induce the fuzzy set of items which possibly satisfy the query and the fuzzy set of items which necessarily satisfy the query. The second example is similarly dealt with by computing $\pi(\theta' | T_1(x) \times A)$ and $N(\theta' | T_1(x) \times A)$ where θ' is a relation modeling "greater or equal" and where A is a fuzzy set whose membership function (defined on a scale) represents "good".

N.B. This approach for the evaluation of queries involving comparisons such as (approximate) equalities or inequalities, is quite similar to the approach recently proposed for discussing the ranking of fuzzy numbers (see Dubois, D. Prade, H. Ranking fuzzy numbers in the setting of possibility theory. Submitted to Information Sciences).

Moreover, we are able to evaluate more complex queries such as "Find the students which are at least good in sciences" where the global level in sciences is defined (possibly in a fuzzy way) in terms of the levels in maths and in physics. This can be done either by a set of production rules "if level in maths is A and level in physics is B , then level in sciences is C " or using an analytical definition: for instance, the global level is equal to the arithmetic mean of the levels in maths and in physics (such an operation can be easily performed using results on fuzzy numbers).

Since $\pi(A | T(x))$ and $N(A | T(x))$ respectively represents the possibility and the necessity that the proposition " x is A " is true taking into account the available knowledge in the data base (see Dubois Prade [9] for a detailed discussion), "yes-or-no" queries can be considered. Such queries may involve universal or fuzzy quantifiers, for instance we can estimate the possibility and the necessity that a proposition such as "Most of the x 's which are A are also B " is true. See Prade [12], [14], [15] for details.

A general query language based on the extended relational algebra is going to be designed. The fuzzy pattern matching system implemented in LISP (see [5], [6] [7], see also [10] for a related application) can be regarded as a primitive version of such a query language. This system retrieves, in a database with fuzzy tuples, the items which possibly/necessarily satisfy a given requirement (possibly vague) represented by a pattern.

The investigation of possible extensions of the concepts of functional or multivalued dependencies is in progress. The approach which is very briefly presented here (for more details the reader is referred to [12], [13], [14], [15] and to

forthcoming papers) enables to deal in a very satisfying way with null values (in the sense of Codd , Extending the data base relational model to capture more meaning . ACM Trans. Database Systems ,Vol. 4 , pp 397-434, 1979) , with partial or incomplete information(in the sense of Lipski , W., Jr. On semantic issues connected with incomplete information data bases ? ACM Trans. Database Systems ,Vol 4 , pp. 262-296) and with uncertain information (as Wong ,E. A statistical approach to incomplete information in database systems. Tech. Rep. n°CCA-80-08 , Computer Corporation of America , Ma. USA , 1980) .

Approximate reasoning for expert systems

The two basic patterns of reasoning, which are needed in production rule systems when the knowledge is uncertain and/ or imprecise, are

- i) the extension of the rules of detachment (modus ponens, modus tollens,...)
- ii) the combination of more or less certain and more or less consonant informations given by different sources.

In the following we only deal with the first problem . For the second one, the reader is referred to [12] , [16] .

Extending the rules of detachment when the premises are uncertain

In classical logic, the two main rules of detachment are

the modus ponens
$$\frac{P \rightarrow Q \quad P}{Q}$$
 which corresponds to the first line of the

truth-table

Table 1	$P \rightarrow Q$	P	Q	
	1	1	1	modus ponens
	1	0	{0,1}	"denial of Q" (but the truth-value of Q remains indetermined)
	0	1	0	
	0	0	∅	no truth-value $\in \{0,1\}$ is possible

while the modus tollens
$$\frac{P \rightarrow Q \quad \neg Q}{\neg P}$$
 corresponds to the second line of

the truth-table

Table 2	$P \rightarrow Q$	Q	P	
	1	1	{0,1}	"confirmation of P" (but the truth-value of P remains indetermined)
	1	0	0	modus tollens
	0	1	∅	no truth-value $\in \{0,1\}$ is possible
	0	0	1	

N.B. In Table 2, knowing that $P \rightarrow Q$ is true, the truth of Q makes "P true" more credible (see Pólya, G. (1954) Mathematics and plausible reasoning. Vol. II : Patterns of plausible inference. Princeton University Press) since it is a necessary condition for the truth P ; similarly knowing that $P \rightarrow Q$ is true, the falsity of P makes "Q true" less plausible since it is a necessary condition for the falsity of Q (or if we prefer, knowing that $P \rightarrow Q$ is true, the truth of P is a sufficient condition for the truth of Q).

Now, let us consider the modus ponens and the modus tollens when certainty degrees or degrees of truth are attached to the premises.

If an event A has the probability 1, it can be regarded as certain since $\text{Prob}(A) = 1 - \text{Prob}(\bar{A}) = 0$. Similarly in possibility theory, if an event A has a necessity equal to 1, it can also be regarded as certain since $\text{Pos}(\bar{A}) = 1 - \text{Nec}(A) = 0$ where Pos is a possibility measure in the sense of Zadeh and Nec is the dual measure of necessity (note that $\forall A, \forall B, \text{Nec}(A \wedge B) = \min(\text{Nec}(A), \text{Nec}(B))$). Then, depending on the kind of certainty, we have, with $P \dashrightarrow Q = \neg P \vee Q$

	$\frac{\text{Prob}(P \dashrightarrow Q) \geq a}{\text{Prob}(P) \geq b}$		$\frac{\text{Prob}(P \dashrightarrow Q) \geq a}{\text{Prob}(Q) \leq b}$
(I)	$\text{Prob}(Q) \geq \max(0, a+b-1)$		$\text{Prob}(P) \leq \min(1, 1-a+b)$
	$\frac{\text{Prob}(Q P) \geq a}{\text{Prob}(P) \geq b}$		$\frac{\text{Prob}(Q P) \geq a}{\text{Prob}(Q) \leq b}$
(II)	$\text{Prob}(Q) \geq a \cdot b$		$\text{Prob}(P) \leq \min(1, \frac{b}{a}) \quad a \neq 0$
	$\frac{\text{Nec}(P \dashrightarrow Q) \geq a}{\text{Nec}(P) \geq b}$		$\frac{\text{Nec}(P \dashrightarrow Q) \geq a}{\text{Nec}(Q) \leq b}$
(III)	$\text{Nec}(Q) \geq \min(a, b) \quad ([42])$		$\text{Nec}(P) \leq \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$

N.B.1. It can also be shown that

	$\frac{\text{Nec}(P \dashrightarrow Q) \geq a}{\text{Pos}(P) \geq b}$		$\frac{\text{Pos}(P \dashrightarrow Q) \geq a}{\text{Nec}(P) \geq b}$
(IV)	$\text{Pos}(Q) \geq \begin{cases} 0 & \text{if } a+b \leq 1 \\ b & \text{if } a+b > 1 \end{cases}$		$\text{Pos}(Q) \geq \begin{cases} 0 & \text{if } a+b \leq 1 \\ a & \text{if } a+b > 1 \end{cases}$

However, if we only know that $\text{Pos}(P \dashrightarrow Q) \geq a$ and $\text{Pos}(P) \geq b$, nothing can be said on $\text{Pos}(Q)$.

N.B.2. Since we have $\max(0, p + q - 1) \leq \text{Prob}(P \wedge Q) \leq \min(p, q)$ where $\text{Prob}(P) = p$ and $\text{Prob}(Q) = q$, it can be easily checked with $P \dashrightarrow Q = \neg P \vee Q$ and taking into account $\text{Prob}(\neg P) = 1 - \text{Prob}(P)$, that

i) $\max(1 - p, q) \leq \text{Prob}(P \dashrightarrow Q) \leq \min(1, 1 - p + q)$

(V) ii) $\max(0, \frac{p+q-1}{p}) \leq \text{Prob}(Q|P) = \frac{\text{Prob}(P \wedge Q)}{\text{Prob}(P)} \leq \min(1, \frac{q}{p})$

iii) $\max(1-p, p+q-1) \leq \max(\text{Prob}(P \wedge Q), \text{Prob}(\neg Q)) \leq \max(1-p, \min(p, q)).$

Bandler and Kohout [3] have interpreted the inequalities (V) in the framework of a so-called "checklist paradigm" without explicitly referring to probability.

Multi-valued logics offer another framework for extending the modus ponens and the modus tollens to premises graded by degrees of truth ; $v(P) \in [0, 1]$ will denote the truth-value of a proposition P. A large collection of implication operators have been considered in the fuzzy reasoning literature recently. However, most of them can be derived from one of the two general formulae

(VI) $v(P \dashrightarrow Q) = \sup \{ t \in [0,1] , v(P) * t \leq v(Q) \}$
 where $*$ denotes a triangular norm¹ ;

(VII) $v(P \dashrightarrow Q) = n(v(P)) \perp v(Q)$

where n is a negation operator¹ and \perp a triangular co-norm¹. In the following, we use $n(x) = 1 - x$; however other choices are possible (see for instance [12]).

Implications defined by (VII) are such that $v(P \dashrightarrow Q) = v(\neg Q \dashrightarrow \neg P)$ with $v(\neg P) = n(v(P))$; many implications defined by (VI) does not satisfy $v(P \dashrightarrow Q) = v(\neg Q \dashrightarrow \neg P)$, then new implications can be defined by $\min(v(P \dashrightarrow Q), v(\neg Q \dashrightarrow \neg P))$ or by $\max(v(P \dashrightarrow Q), v(\neg Q \dashrightarrow \neg P))$. Definition VI has been motivated by the detachment problem, while definition VII extends the usual definition of implication in classical logic.

1- A triangular norm $*$ (see Schweitzer, B., Sklar, A. (1963) Associative functions and abstract semigroups. Publ. Math. Debrecen, Vol. 10, pp 69-81) is a two-place function from $[0,1] \times [0,1]$ to $[0,1]$ such that i) $a * b = b * a$, ii) $a * (b * c) = (a * b) * c$, iii) if $a \leq b$ and $c \leq d$, then $a * c \leq b * d$ and iv) $1 * a = a$. The greatest triangular norm is \min and the least one is defined by

$$a * b = T_w(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} ; \text{ thus we always have}$$

$T_w(a,b) \leq a * b \leq \min(a,b)$. Other noticeable triangular norms are

$$a * b = a \cdot b \text{ and } a * b = \max(0, a+b-1), \text{ moreover we have}$$

$$T_w(a,b) \leq \max(0, a + b - 1) \leq a \cdot b \leq \min(a,b) .$$

By duality each triangular norms $*$ is associated with a triangular co-norm \perp defined by

$$a \perp b = 1 - (1 - a) * (1 - b)$$

Triangular norms are conjunction operators while co-norms are disjunction operators (see [12] for instance).; the main co-norms are in increasing order

$$\max(a,b) \leq a+b-a \cdot b \leq \min(1, a+b) \leq \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$

There exist many parametered families of triangular norms and co-norms in the literature ; among them, the one studied and identified by Frank, M.J. (On the simultaneous associativity of $F(x,y)$ and $x + y - F(x,y)$. Aequat. Math. Vol. 19, pp 194-226, 1979) is specially remarkable since a triangular norm $*$ of this family is such that $a + b = a * b + a \perp b$ where \perp is the associated co-norm ; the members of this family range from $\max(0, a+b-1)$ to $\min(a,b)$ and are the only eligible operators compatible with an expression of $\text{Prob}(P \wedge Q)$ in terms of $\text{Prob}(P)$ and $\text{Prob}(Q)$ as

$$\text{Prob}(P \wedge Q) = \text{Prob}(P) * \text{Prob}(Q) .$$

See N.B.2.

A negation operator n is supposed^{to be} such that i) $n(0) = 1$, ii) $n(n(a)) = a$, iii) n is strictly decreasing ; iv) n is continuous ; See Alsina, C., Trillas, E., Valverde, L. (1980) On non-distributive logical connectives for fuzzy sets theory. BUSEFAL n° 3, pp 18-29.

$v(P \wedge Q)$ $= p * q$	$v(P \rightarrow Q)$	
	Formula VI	Formula VII
$\min(p, q)$	Gödel $\begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } p > q \end{cases}$	Dienes $\max(1-p, q)$
$p \cdot q$	Goguen $\begin{cases} 1 & \text{if } p = 0 \\ \min(1, q/p) & \text{if } p \neq 0 \end{cases}$	probabilistic $1 - p + p \cdot q$
$\max(0, p+q-1)$	Łukasiewicz $\min(1, 1-p+q)$	$\min(1, 1-p+q)$

Table 3

$$v(P) = p ; v(Q) = q$$

Table 3 gives the implication operators defined from formulae VI or VII for the main triangular norms ; note that the Łukasiewicz implication can be obtained by both approaches.

Depending on the kind of implication we use, we get as extensions of the modus ponens and of the modus tollens :

. implication type VI :

$$(VIII) \quad \begin{array}{l} v(P \rightarrow Q) \geq a \\ v(P) \geq b \\ \hline v(Q) \geq a * b \end{array} \quad \begin{array}{l} v(P \rightarrow Q) \geq a \\ v(Q) \leq b \\ \hline v(P) \leq a \rightarrow b \end{array}$$

where $a \rightarrow b = \sup \{ t, t \in [0, 1] , t * a \leq b \}$

. implication type VII :

$$(IX) \quad \begin{array}{l} v(P \rightarrow Q) \geq a \\ v(P) \geq b \\ \hline v(Q) \geq 1 - [b \rightarrow (1-a)] \end{array} \quad \begin{array}{l} v(P \rightarrow Q) \geq a \\ v(Q) \leq b \\ \hline v(P) \leq (1-b) \rightarrow (1-a) \end{array}$$

The following table gives the value of these different lower and upper bounds for the main triangular norms *.

$a * b$	$1 - [b \rightarrow (1-a)]$	$a \rightarrow b$	$(1-b) \rightarrow (1-a)$
$\min(a, b)$	$\begin{cases} 0 & \text{if } a+b \leq 1 \\ a & \text{if } a+b > 1 \end{cases}$	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$	$\begin{cases} 1 & \text{if } a \leq b \\ 1-a & \text{if } a > b \end{cases}$
$a \cdot b$	$\begin{cases} 0 & \text{if } b = 0 \\ \max(0, \frac{a+b-1}{b}) & \end{cases}$	$\begin{cases} 1 & \text{if } a = 0 \\ \min(1, \frac{b}{a}) & \text{if } a \neq 0 \end{cases}$	$\begin{cases} 1 & \text{if } b = 1 \\ \min(1, \frac{1-a}{1-b}) & \text{if } b \neq 1 \end{cases}$
$\max(0, a+b-1)$	$\max(0, a+b-1)$	$\min(1, 1-a+b)$	$\min(1, 1-a+b)$

Table 4

What is very striking is that the patterns (VIII), based on the family of implications defined by (VI), coincide with the patterns (I), (II) or (III) for different implications of this family ; the lower bound obtained in (IV) is a particular case of the lower bound given in (IX) where the implications defined by (VII) are used. Thus from a practical point of view, the multi-valued-logic-based approach and the (un)certainty measure - based approach coincide : moreover the differences which are introduced by the different implication functions or by the different (un)certainty measures which are used, are only nuances easy to compare, since the ordering of the main triangular norms induces an ordering on the $\star \rightarrow$ operations. The use of Łukasiewicz implication leads to the least lower bound and to the greatest upper bound ; it is the safest choice, particularly when we are not sure of the very nature of the "strength" of the rule $P \dashrightarrow Q$ (probability?, necessity?,....) or when viewing this strength as a truth-value, we have no reason to prefer a particular implication operation.

Mixing together the modus ponens and the modus tollens modes enables to obtain a lower but also an upper bound for $v(Q)$, we get from (VIII)

$$(X) \quad \begin{array}{l} v(P \dashrightarrow Q) \geq a \\ v(Q \dashrightarrow P) \geq a' \\ v(P) \in [b, B] \end{array} \quad \hline v(Q) \in [a \star b, a' \star B]$$

and from (IX)

$$(XI) \quad \begin{array}{l} v(P \dashrightarrow Q) \geq a \\ v(Q \dashrightarrow P) \geq a' \\ v(P) \in [b, B] \end{array} \quad \hline v(Q) \in [1 - (b \star (1-a)), (1-B) \star (1-a')]$$

Naturally, we are not obliged to use the same model of implication for $P \dashrightarrow Q$ and $Q \dashrightarrow P$. Note that in (X), the calculated interval for $v(Q)$ is never empty ; in (XI), this interval may be empty, for example in case of Dienes' implication it is impossible to have $v(P \dashrightarrow Q) = 1$, $v(Q \dashrightarrow P) = 1$ and $v(P) = 0.5$.

N.B.3. The pattern (X) is used with Łukasiewicz implication and $b = B = v(P)$ in the system PROTIS [18]. In practice, we may take for b and B the necessity and the possibility respectively that the proposition P is true (see Dubois Prade [9] and Prade (1982) Degree of truth : Matching statement against reality, BUSEFAL, n° 9, pp 88-92).

N.B.4. Reapplying the pattern (X) to $v(Q)$ gives for $v(P)$ an interval $[a * a * b, a' * \rightarrow (a' * \rightarrow B)]$ which clearly contains $[b, B]$; our knowledge concerning $v(P)$ cannot be improved in that way, which is natural. However, it is not always the case in (XI), for instance for Dienes implication, $a' = 1$ and $B = 0.5$ we have not $[1 - (1 - B) * \rightarrow (1 - a')] * \rightarrow (1 - a') \geq B!$ This fact tends to disqualify implications given by formulae (VII).

N.B.5. It is important to notice that the two statements

$$\begin{aligned} v(P \rightarrow Q) &\in [a, A] \\ v(P) &\in [b, B] \end{aligned}$$

are not always consistent. Indeed, $v(P \rightarrow Q) \leq A$ entails
 for Gödel's implication if $A < 1$, $v(Q) \leq \min(A, v(P))$ and $v(P) > v(Q) \geq 0$
 for Goguen's implication if $A < 1$, $v(Q) \leq A \cdot v(P)$ and $v(P) \neq 0$
 for Łukasiewicz implication if $A < 1$, $v(Q) \leq A + v(P) - 1$
 for the probabilistic implication $1 - A \leq v(P) \cdot (1 - v(Q))$
 for Dienes' implication $1 - A \leq \min(v(P), 1 - v(Q))$ } $v(P) \geq 1 - A$
 $v(Q) \leq A$

N.B.6. When $v(P \rightarrow Q)$ or $v(P)$ are fuzzily restricted by possibility distributions (e.g. fuzzy numbers) rather than by crisp intervals, the possibility distribution attached to $v(Q)$ can be easily computed as (see [8], [19])

$$(XII) \forall s \in [0, 1], \pi_{v(Q)}(s) = \sup_{r, t} \pi_{v(P)}(r) * \pi_{v(P \rightarrow Q)}(t) \\ t = \mu_I(r, s)$$

where $*$ is a triangular norm and μ_I the implication function. Note that, if $\forall r_0 \in \{r, \pi_{v(P)}(r) > 0\}$, $\forall t_0 \in \{t, \pi_{v(P \rightarrow Q)}(t) > 0\}$, $\{s, \mu_I(r_0, s) = t_0\} = \emptyset$, then we get $\pi_{v(Q)}(s) = 0$, $\forall s$, i.e. $v(Q) = \emptyset$, which generalizes the N.B.5.

Dealing with imprecise statements

Let us suppose that P expresses a restriction on the possible values of a variable X and Q a restriction on the possible values of a variable Y . A causal link from X to Y can be represented by a conditional possibility (probability resp.) distribution $\pi_{Y/X}$ ($p_{Y/X}$ resp.) which restricts the possible values of Y for a given value of X . From a possibility distribution π_X representing the proposition P , we compute the possibility distribution π_Y restricting Y ,

$$(XIII) \pi_Y(y) = \sup_x \pi_{Y/X}(y, x) * \pi_X(x)$$

where $*$ is a triangular norm; XIII is the analogous in possibility theory of

$$(XIV) p_Y(y) = \sum_x p_{Y/X}(y, x) \cdot p_X(x).$$

In this framework a rule such as "If X is A , then Y is B " is represented by the inequality

$$(XV) \forall y, \mu_B(y) \geq \sup_x \pi_{Y/X}(y, x) * \mu_A(x)$$

where μ_A and μ_B are the possibility distributions attached to X and Y respectively, $\pi_{Y/X}$ being unknown; the inequality stems from the entailment principle [21]; if Y is B , then Y is B' as soon as B' corresponds to a larger possibility distribution. The greatest solution of (XV) for a continuous

triangular norm is given by (see [17] , [12])

$$(XVI) \quad \widehat{\pi}_{Y/X}(y,x) = \mu_A(x) \star \longrightarrow \mu_B(y)$$

In case of several rules "if A is A_i, then Y is B_i", a possibility distribution is computed by aggregating (with min operator) the expressions (XVI) obtained for each rule. Under certain condition, it is possible to have also a lower bound of $\pi_{Y/X}$.

Now we are in position of considering the "generalized modus ponens" introduced by Zadeh [21]

$$(XVII) \quad \begin{array}{l} \text{If } X \text{ is } A, \text{ then } Y \text{ is } B \\ X \text{ is } A' \\ \hline \text{then } Y \text{ is } B' \end{array}$$

where $\mu_{B'} = \pi_{Y'}^*$ is computed from (XIII) and (XVI) as

$$(XVIII) \quad \forall y, \mu_{B'}(y) = \sup_x (\mu_A(x) \star \longrightarrow \mu_B(y)) \star \mu_{A'}(x)$$

For discussing (XVII) in terms of truth-values, Baldwin [1], has introduced the compatibilities² in the sense of Zadeh [21] , $CP(A/A')$ and $CP(B/B')$ which are nothing but the possibility distributions attached to the values of the degrees of truth of $P = X \text{ is } A$ and $Q = Y \text{ is } B$ knowing respectively that "X is A'" and that "Y is B'". Then (XVIII) becomes [1] , [21]

$$(XIX) \quad \begin{aligned} \mu_{CP(A/A')}(u) &= \sup_x \mu_{A'}(x) \\ &= 0 \text{ if } \mu_A^{-1}(u) = \emptyset \\ \mu_{CP(B/B')}(v) &= \sup_{u \in [0,1]} \mu_I(u,v) \star \mu_{CP(A/A')}(u) \\ \mu_{B'}(y) &= \mu_{CP(B/B')}(\mu_B(y)) \end{aligned}$$

where μ_I is the implication function which is used. (XIX) is equivalent to (XVIII). More generally, if a fuzzy truth-value τ is attached to "if X is A, then Y is B", it must be understood as

$$(XX) \quad \tau = CP(A \star \longrightarrow B/X \rightarrow Y)$$

$$\text{with } \mu_{X \rightarrow Y} = \pi_{Y/X}$$

$$\mu_A \star \longrightarrow B = \mu_A \star \longrightarrow \mu_B$$

Then, we have (see [21])

$$(XXI) \quad \widehat{\pi}_{Y/X}(y,x) = \mu_\tau (\mu_A(x) \star \longrightarrow \mu_B(y))$$

Thus (XIII) is generalized by

$$(XXII) \quad \mu_{B'}(y) = \sup_x \mu_\tau (\mu_A(x) \star \longrightarrow \mu_B(y)) \star \mu_{A'}(x)$$

2. The possibility and the necessity that X is A is true knowing that X is A' can be easily extracted from $CP(A/A')$. See [1] , [12] .

Using (XIX), it can be checked that (XXII) and (XII) are equivalent with

$$\begin{aligned} \pi_{\nu}(P) &= \mu_{CP(A/A')}, \quad \pi_{\nu}(Q) = \mu_{CP(B/B')} \quad , \quad \pi_{\nu}(P \rightarrow Q) = \mu_{\Sigma} \text{ and} \\ \mu_I(r, s) &= r \rightarrow s ; \text{ see [12] .} \end{aligned}$$

Thus the generalized modus ponens in Zadeh's or in Baldwin's form is consistent with the extended rule of detachment given by (XII).

N.B.7. It is important to remember that $CP(A/A')$ may be a fuzzy set of $[0,1]$ (i.e. corresponds to a non-crisp possibility distribution) only if A is a genuine fuzzy set. In other words, in the compatibility model a precise statement cannot have a fuzzy truth-value (see [11] , [12] for a discussion). However a truth-value may be fuzzy only because it is imprecisely specified, in that case the direct use of XII solves the problem.

N.B.8. Another important remark must be made concerning (XVIII). It can be shown that

$$\forall y, \mu_{B'}(y) \geq \sup_{x, \mu_A(x) = 0} \mu_{A'}(x) = \mu_{CP(A/A')}(0)$$

which means that a uniform level of indetermination appears as soon as a significant part of A' is not included in A , which intuitively seems natural. Thus the generalized modus ponens cannot enable to deduce from

"If a tomato is red, then the tomato is ripe"

and from "The tomato is very red"

that "This tomato is very ripe"

Such an inference presupposes we know that the ripeness degree is an increasing function of the color intensity. More generally if we know that " A' is not far from A " in the sense of some metrics, the generalized modus ponens is not a sufficient model for deducing from "if X is A , then Y is B " that " Y is not far from B ".

Acknowledgements . The author thanks Didier Dubois, Roger Martin-Clouaire and Claudette Testemale for the helpful discussions he had with them, concerning different points of this paper.

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