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ON MEASURES OF FUZZINESS OF SOLUTIONS
OF COMPOSITE FUZZY RELATION EQUATIONS*

by

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Abstract. In this paper we study the solutions of a max-min fuzzy relation equation of Sanchez [7], by determining ones which have the greatest energy measure and the smallest entropy measure of fuzziness.

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1. Introduction.

Let X, Y, Z be finite sets and $F(X) = \{A: X \rightarrow [0, 1]\}$ the class of all fuzzy sets of X . Let $Q \in F(X \times Y)$, $R \in F(Y \times Z)$ and $T \in F(X \times Z)$ fuzzy relations correlated in such a way the following equation holds:

$$R \circ Q = T \quad (1)$$

where " \circ " is the max-min composition and R is unknown.

Let's denote with \mathcal{R} the set of solutions R of the equation (1). In [7], Sanchez gives a fundamental theorem for the existence and the determination of the greatest element of \mathcal{R} and in [8] points out that the study of \mathcal{R} can be enlarged by investigating the entropy measures of fuzziness of the relations $R \in \mathcal{R}$. Following this suggestion, Di Nola and Sessa [4] propose some algorithms which minimize the fuzziness measure of a given element of \mathcal{R} , whereas Di Nola and Pedrycz [2] establish a necessary and sufficient condition for the existence of boolean solution $R \in \mathcal{R}$ (viz. $R: X \times Y \rightarrow \{0, 1\}$) which has fuzziness measure equal to zero. In this note, we discuss about the existence of the relations $R \in \mathcal{R}$ which have

- (E) the greatest energy measure of fuzziness and
- (H) the smallest entropy measure of fuzziness.

In order to attain this, in Section 2 we recall the basic definitions and notations of composite max-min fuzzy relation equations theory [7]. Section 3 contains the resolution of problem (E). In section 4 we have furtherly developed the theory of Sanchez [8]. By using some results of di Nola - Ventre [6], in section 5, we give the resolution of the problem (H).

For simplicity of exposition, we identify fuzzy sets and fuzzy relations with their membership functions. Furthermore, we represent the fuzzy relations as real matrices and we assume that the fuzzy equation (1) has at least one solution, viz. $\mathcal{R} \neq \emptyset$.

2. Preliminaries and basic definitions.

Let X be a nonempty set and let's denote with $a \in [0,1]$ the fuzzy set constantly equal to a . It is wellknown that [9] the following operations defined pointwise for any $x \in X$:

$(A \wedge B)(x) = \min \{A(x), B(x)\}$, $(A \vee B)(x) = \max \{A(x), B(x)\}$, where $A, B \in F(X)$, induce in $F(X)$ a complete lattice structure with universal bounds $0, 1$ and a partial ordering given by

$$A \leq B \quad \text{iff} \quad A(x) \leq B(x) \quad \text{for any } x \in X.$$

From now on we suppose that $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$, $Z = \{z_1, z_2, \dots, z_p\}$ are finite sets and let be $I_n = \{1, 2, \dots, n\}$ the set of first n natural numbers.

In according to Sanchez [7], we recall the following definitions:

Definition 1. A fuzzy relation Q between X and Y is an element of $F(X \times Y)$. We call inverse or transpose of Q the fuzzy relation $Q^{-1} \in F(Y \times X)$ as $Q^{-1}(y_j, x_i) = Q(x_i, y_j)$ for every $i \in I_n, j \in I_m$.

Let be $Q \in F(X \times Y)$ and $R \in F(Y \times Z)$ two fuzzy relations. For brevity, since no misunderstanding can arise, we put:

$$Q(x_i, y_j) = Q_{ij} \quad \text{for every } i \in I_n, j \in I_m$$

and

$$R(y_j, z_k) = R_{jk} \quad \text{for every } j \in I_m, k \in I_p.$$

Definition 2. We define $T = R \circ Q$, $T \in F(X \times Z)$, the max-min composite fuzzy relation of R and Q , as

$$T_{ik} = T(x_i, z_k) = \bigvee_{j=1}^m (Q_{ij} \wedge R_{jk})$$

for every $i \in I_n, k \in I_p$.

Definition 3. We define $S = Q^{-1} \textcircled{a} T$, $S \in F(Y \times Z)$, the \textcircled{a} -composite fuzzy relation of Q^{-1} and T , as

$$S_{jk} = S(y_j, z_k) = \bigwedge_{i=1}^n (Q_{ji}^{-1} \alpha T_{ik}) = \bigwedge_{i=1}^n (Q_{ij} \alpha T_{ik})$$

for every $j \in I_m, k \in I_p$ where

$$Q_{ij} \alpha T_{ik} = \begin{cases} 1 & \text{if } Q_{ij} \leq T_{ik} \\ T_{ik} & \text{if } Q_{ij} > T_{ik} \end{cases}$$

Definition 4. L is called lower solution of \mathcal{R} if for every $R \in \mathcal{R}$ such that $R \leq L$, we have $R = L$.

In according to De Luca and Termini [1], we introduce the following definitions:

Definition 5. Let $f: [0, 1] \rightarrow [0, 1]$ be a real function such that $f(0) = 0, f(1) = 1$ and strictly increasing in $[0, 1]$. We define energy measure of fuzziness of $R \in F(Y \times Z)$ the number

$$E(R) = \frac{1}{p \cdot m} \sum_{k=1}^p \sum_{j=1}^m f(R_{jk})$$

Definition 6. Let $g: [0, 1] \rightarrow [0, 1]$ be a real function such that $g(0) = g(1) = 0, g(a) = g(1-a)$ for any $a \in [0, 1]$ and strictly increasing in $[0, 1/2]$. We define entropy measure of fuzziness of $R \in F(Y \times Z)$ the number

$$H(R) = \frac{1}{p \cdot m} \sum_{k=1}^p \sum_{j=1}^m g(R_{jk})$$

3. Resolution of the problem (E).

By considering the composite fuzzy equation (1), Sanchez [7] proved that \mathcal{R} is not empty iff S is the greatest element of \mathcal{R} . Then the problem (E) is immediately solved by the Knowledge of S .

4. Further theoretical results.

The following lemma, which is merely an observation, will be useful in the sequel.

Lemma. Let be $R', R'' \in \mathcal{R}$ and $R \in F(Y \times Z)$ such that $R' \leq R \leq R''$. Then $R \in \mathcal{R}$.

Proof. By def. 2, one can easily deduce that $R' \circ Q \leq R \circ Q \leq R'' \circ Q$ and then $R \in \mathcal{R}$ since $R' \circ Q = R'' \circ Q = T$.

Let's define, for every $i \in I_n$, the fuzzy sets $Q_i \in F(\{x_i\} \times Y)$ and $T_i \in F(\{x_i\} \times Z)$ as

$$Q_i(x_i, y_j) = Q_{ij}, \quad T_i(x_i, z_k) = T_{ik}$$

for any $j \in I_m, k \in I_p$.

Then the study of the fuzzy equation (1) is led on the following fuzzy system of n equations:

$$R \circ Q_i = T_i, \tag{2}$$

For any $i \in I_n$, let be \mathcal{R}_i the set of solutions $R \in F(Y \times Z)$ of the equation (2). Clearly, we have:

$$\mathcal{R} = \bigcap_{i=1}^n \mathcal{R}_i \tag{3}$$

A equation of type (2) is called \textcircled{O} -fuzzy relation equation.

About these equations, also we recall some results of Sanchez [8].

(σ_1) If $\mathcal{R}_i \neq \emptyset$, the fuzzy relation $W_i = Q_i \odot T_i \in F(Y \times Z)$ defined by

$$W_i(y_j, z_k) = (Q_i \odot T_i)(y_j, z_k) = Q_{ij} \sigma T_{ik}$$

for every $i \in I_n$, $j \in I_m$, $k \in I_p$ where

$$Q_{ij} \sigma T_{ik} = \begin{cases} 0 & \text{if } Q_{ij} < T_{ik} \\ T_{ik} & \text{if } Q_{ij} \geq T_{ik} \end{cases}$$

belongs to \mathcal{R}_i .

(σ_2) If $\mathcal{R} \neq \emptyset$, then \mathcal{R}_i has lower solutions $L_i \neq 0$, whose fuzzy union (in the sense of Zadeh) is just W_i .

In order to determine such relations L_i it suffices to keep a non zero element in each column of W_i .

The following theorems hold:

Theorem 1. If $\mathcal{R}_i \neq \emptyset$, for any $R \in \mathcal{R}_i$, there exists at least lower solution L_i such that $L_i \leq R$.

Proof. Let be $R \in \mathcal{R}_i$ and for any $k \in I_p$ we have:

$$\bigvee_{j=1}^m (Q_{ij} \wedge R_{jk}) = T_{ik}$$

Then, for any $k \in I_p$, there exists an index $j^* \in I_m$ such that $Q_{ij^*} \geq$

$\geq Q_{ij^*} \wedge R_{j^*k} = T_{ik}$. This implies $W_i(y_{j^*}, z_k) = T_{ik} \leq R_{j^*k}$. Therefore, if we define for any $k \in I_p$:

$$L_i(y_j, z_k) = \begin{cases} 0 & \text{if } j \neq j^* \\ T_{ik} & \text{if } j = j^* \end{cases}$$

we obtain a lower solution $L_i \in \mathcal{R}_i$ such that $L_i \leq R$.

Theorem 2. If $\mathcal{R}_i \neq \emptyset$, the set $\mathcal{L}_i = \{L_i \in \mathcal{R}_i \text{ such that } L_i \leq S\}$ is not empty for every $i \in I_n$.

Proof. By theorem 5 of [7], $S \in \mathcal{R}$ and by (3) we have $S \in \mathcal{R}_i$ for every $i \in I_n$. Then, by theorem 1, follows the thesis.

Theorem 3. If $\mathcal{R} \neq \emptyset$, the fuzzy relation $L = \bigvee_{i=1}^n L_i \in F(Y \times Z)$, where $L_i \in \mathcal{L}_i$ for every $i \in I_n$, belongs to \mathcal{R} .

Proof. Since $\mathcal{R} \neq \emptyset$, then $\mathcal{R}_i \neq \emptyset$ for every $i \in I_n$. By theorem 2, \mathcal{R}_i has lower solutions $L_i \in \mathcal{L}_i$. Furthermore we have for every $i \in I_n$:

$$L_i \leq \bigvee_{i=1}^n L_i \leq S$$

By lemma, the fuzzy relation $L = \bigvee_{i=1}^n L_i \in F(Y \times Z)$ belongs to \mathcal{R}_i for any $i \in I_n$ and therefore it is an element of \mathcal{R} since (3) holds.

In according to theorem 3, now we define:

$$\mathcal{L} = \left\{ L \in \mathcal{R} \text{ such that } L = \bigvee_{i=1}^n L_i, L_i \in \mathcal{L}_i \text{ for any } i \in I_n \right\}$$

5 Resolution of the problem (H)

By using a result of Di Nola and Ventre [6], we prove that exists at least an element of \mathcal{R} which have the smallest fuzzy

entropy measure, basing on the knowledge of S and on the elements of \mathcal{L} .

We recall [5] the following:

Definition 7. Let $g: [0,1] \rightarrow [0,1]$ as well as in the def. 6 for any $a, b \in [0,1]$, we define a partial ordering as

$$a \leq b \quad \text{iff} \quad g(a) \leq g(b)$$

and a meet-operation as

$$a \cap b = \begin{cases} a & \text{if } g(a) < g(b) \\ a \wedge b & \text{if } g(a) = g(b) \end{cases}$$

Like in [6], we can prove that the structure $([0,1], \leq, \cap)$ is a semilattice under \cap , which, as showed in Di Nola and Sessa [5], no depends by the choice of the function g . So we consider the function $g(a) = a \wedge \bar{a}$ where $\bar{a} = 1 - a$ for any $a \in [0,1]$.

We will use later the following

Theorem 4. Let be $a, b, c \in [0, 1]$ such that $a \leq b \leq c$.
Then $a \cap c = (a \cap c) \cap b$.

Proof: We distinguish two cases:

(I) $a \cap c = a$.

Then:

$$a \wedge \bar{a} \leq c \wedge \bar{c}. \quad (4)$$

We must prove $a \wedge \bar{a} \leq b \wedge \bar{b}$. For absurd, let be

$$a \wedge \bar{a} > b \wedge \bar{b}, \quad (5)$$

From (4), we have:

$$c \wedge \bar{c} \geq a \wedge \bar{a} > b \wedge \bar{b}. \quad (6)$$

We claim $b \wedge \bar{b} = b$ otherwise, from (6) follows:

$$\bar{c} \geq c \wedge \bar{c} > b \wedge \bar{b} = \bar{b},$$

which implies $c < b$, a contradiction to the hypothesis $b \leq c$. So,

from (5), we deduce $a \geq a \wedge \bar{a} > b$, a contradiction to the hypothesis $a \leq b$.

(II) $a \cap c = c$.

Then:

$$c \wedge \bar{c} \leq a \wedge \bar{a}. \quad (7)$$

We must prove $c \wedge \bar{c} \leq b \wedge \bar{b}$. For absurd, let be

$$c \wedge \bar{c} > b \wedge \bar{b}. \quad (8)$$

From (7), we have:

$$a \wedge \bar{a} > c \wedge \bar{c} > b \wedge \bar{b}. \quad (9)$$

(9) forces $b \wedge \bar{b} = \bar{b}$ otherwise we obtain $a > a \wedge \bar{a} > b \wedge \bar{b} = b$, a contradiction to the assumption $a \leq b$.

Then (8) implies $\bar{c} \geq c \wedge \bar{c} > b$, that is $c < b$, a contradiction to the hypothesis $b \leq c$.

In [4], we show the following:

Theorem 5. Let be $\mathcal{R} \neq \emptyset$ and $R, R' \in \mathcal{R}$ such that $R \leq R'$. Then the relation $R \cap R' \in \mathcal{F}(Y \times Z)$ so defined for every $j \in I_m, k \in I_p$:

$$(R \cap R')(y_j, z_k) = R_{jk} \cap R'_{jk}$$

is an element of \mathcal{R} .

Finally, we have the following:

Theorem 6. Let be $\mathcal{R} \neq \emptyset$. For any $L \in \mathcal{L}$, let be $L^* \in \mathcal{P}(Y \times Z)$ defined for every $j \in I_m, k \in I_p$ as

$$L_{jk}^* = L_{jk} \cap S_{jk}.$$

In the set $\mathcal{L}^* = \{L^*, L \in \mathcal{L}\}$ there is at least one element \bar{L}^* of \mathcal{R} such that $H(\bar{L}^*) \leq H(R)$ for any $R \in \mathcal{R}$.

Proof. By theorem 5, L^* belongs to \mathcal{R} for any $L \in \mathcal{L}$. Since \mathcal{L}^* is a finite set, let be \bar{L}^* such that

$$H(\bar{L}^*) = \min_{L^* \in \mathcal{L}^*} H(L^*) \quad (10)$$

Let be $R \in \mathcal{R}$. By theorem 5 of [7], $R \leq S$ and being $R \in \mathcal{R}_i$ for any $i \in I_n$, by theorem 1, there exists a lower solution $L_i \leq R$. Then $L_i \leq S$ and therefore $L_i \in \mathcal{L}_i$ for any $i \in I_n$. So $L = \bigvee_{i=1}^n L_i$, by theorem 3, is an element of \mathcal{L} such that $L \leq R \leq S$. By theorem 4, we have:

$$L^* = L \cap S = (L \cap S) \cap R = L^* \cap R \quad (11)$$

(11) implies $H(L^*) \leq H(R)$ and therefore the thesis because, for any $R \in \mathcal{R}$, we obtain from (10):

$$H(\bar{L}^*) \leq H(L^*) \leq H(R).$$

6. Concluding comments.

The problem (H) was anticipated by Di Nola and Sessa in [5] and there solved for \odot -fuzzy relation equations.

For a wide class of fuzzy relation equations, which have like limit case the \odot -fuzzy equations of Sanchez [8], the problems (E) and (H) have been solved by Di Nola, Pedrycz and Sessa [3].

In future papers, we carefully shall study further algebraic properties of the set \mathcal{R} and applications to fuzzy systems shall be attempted too.

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