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Fuzzy sets - The approximation of Semisets, part 2

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► **To cite this version:**

Vilém Novák. Fuzzy sets - The approximation of Semisets, part 2. Bulletin pour les sous-ensembles flous et leurs applications, 1983, 14, pp.19–29. hal-04556061

HAL Id: hal-04556061

<https://hal.science/hal-04556061>

Submitted on 23 Apr 2024

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3.3. Up approximation

The relations (1) or (3) respectively will be dealt with now. The class of all the set formulas ψ_n from the relation (1) will be denoted

$$\Phi_\varphi = \{ \psi_n; \varphi \rightarrow \psi_n \mid n \in \text{FN} \ \& \ \psi_n \in \text{FL}_C \}.$$

Φ_φ will be supposed to be at most countable.

Proposition 7: Let $\Delta_n = \{ \psi_i \in \Phi_\varphi; i \leq n \}$ and $Y_n = \{ y; \psi_0(y) \& \dots \& \psi_n(y) \}$. Then $X_\varphi \subseteq a$ is dense in $Y_n \cap a$ with respect to $\sim_{\Delta_n}^C$.

Proof: $\text{Fig}(X_\varphi) = \{ y; (\exists x \in X_\varphi)(\forall \psi \in \Delta_n)(\psi(x) \equiv \psi(y)) \} = Y_n$, i.e. X_φ is dense in Y_n and $X_\varphi \subseteq Y_n \cap a \subseteq \bar{Y}_n = \bar{X}_\varphi$, i.e. X_φ is dense in $Y_n \cap a$.

Q.E.D.

Proposition 8: Put $P_\varphi = \bigcap \{ Y_n; n \in \text{FN} \ \& \ \Delta_n \subseteq \Phi_\varphi \}$. Then $X_\varphi \subseteq a$ is dense in $P_\varphi \cap a$ with respect to $\sim_{\Phi_\varphi}^C$.

Proof: Let $x \in P_\varphi$. Then $(\forall n)(\psi_0(x) \& \dots \& \psi_n(x))$ and $o(x, n) = \{ y; \psi_0(x) \& \dots \& \psi_n(x) \equiv \psi_0(y) \& \dots \& \psi_n(y) \} = \{ y; \psi_0(y) \& \dots \& \psi_n(y) \}$. Since each $x \in X_\varphi$ fulfils every conjunction of formulas from Φ_φ , $o(x, n) \cap X_\varphi \neq \emptyset$ for each n . Hence $X_\varphi \subseteq P_\varphi \subseteq \bar{X}$ and $X_\varphi \subseteq P_\varphi \cap a \subseteq \bar{X}_\varphi$.

Q.E.D.

Proposition 9: Let $\Gamma = \{ \varphi_n; n \in \text{FN} \}$ be a countable class of set formulas of FL_C and let $(\forall m)(\exists n > m)(\exists x, y)(\varphi_m(x) \& \varphi_m(y) \& \varphi_n(x) \& \neg \varphi_n(y))$. Then there does not exist set formula ψ of FL_C such that $\sim^\Gamma = \sim^\psi$.

Proof: Let x, y fulfilling the condition exist for $n > m$ and put $\beta_n(x, y) = (\varphi_n(x) \equiv \varphi_n(y))$. Then $\neg(\beta_m(x, y) \rightarrow \beta_n(x, y))$. Put $\alpha_n(x, y) = \beta_0(x, y) \& \dots \& \beta_m(x, y)$. Then $\alpha_m(x, y) \neq \alpha_n(x, y)$, i.e. $\{ X_n; n \in \text{FN} \}$ for $X_n = \{ \langle x, y \rangle; \alpha_n(x, y) \}$ is decreasing

sequence. Each X_n is closed and open (clopen) figure and $\sim^n = \bigcap \{X_n; n \in \mathbb{N}\}$ since there is only countably many of clopen figures.

Q. E. D.

Proposition 10: Let $\text{Ker}(X_\varphi)$ be the kernel of the semiset X_φ . Then $\text{SC}_{\Phi_\varphi}(\text{Ker}(X_\varphi))$ is set theoretically definable σ -class.

Proof: We shall write $\text{SC}(c)$ instead of $\text{SC}_{\Phi_\varphi}(\text{ker}(X_\varphi))$.
 $\text{SC}(c) = \{x; (\exists u)(\exists y \in c)(\psi_0(x) \& \dots \& \psi_n(x) \equiv \psi_0(y) \& \dots \psi_n(y))\} = \{x; (\exists n)(\psi_0(x) \& \dots \& \psi_n(x))\}$ since $y \in c$ fulfills any conjunction of formulas $\psi_0 \& \dots \& \psi_n$. If we put $Y_n = \{x; \psi_0(x) \& \dots \& \psi_n(x)\}$ then $Y_n \supseteq Y_{n+1}$ and $\text{SC}(c) = \bigcup \{Y_n; n \in \mathbb{N}\} = Y_0$.

Q.E.D.

The following definition can be stated on the basis of the Propositions 7 - 10.

Definition: Let $\text{Ker}(X_\varphi)$ be the kernel of the proper real semiset $X_\varphi = \{x; \varphi(x)\} \subseteq a$. Let $\Phi_\varphi = \{\psi_n; \varphi \rightarrow \psi_n\}$ be at most countable sequence of set formulas of the language FL_C . Put $a_n = Y_n \cap a$ where $Y_n = \{y; \psi_0(y) \& \dots \& \psi_n(y)\}$. Then the set $\text{SC}_{\Phi_\varphi}(\text{Ker}(X_\varphi)) \cap a = \bigcup \{a_n; n \in \mathbb{N}\} = a_0$ up approximates the semiset X_φ (in this case X is up approximable). This set will be denoted $\text{Fuz}(X_\varphi)$ and it will be called fuzzy set up approximating the semiset X_φ .

Since $\{a_n; n \in \mathbb{N}\}$ is a descending sequence, the semiset X_φ is dense in $\text{Fuz}(X_\varphi)$ according to $\sim^{\{\psi_0\}}$ and in $\bigcap \{a_n; n \in \mathbb{N}\}$

according to $\sim_{\mathcal{C}}^{\Phi_{\varphi}}$. If Φ_{φ} is countable then $\text{Fuz}(X_{\varphi}) = \cup \{a_{\alpha}; \alpha \leq \omega\}$ where $a_{\alpha} \supseteq a_{\alpha+1}$ and $a_{\omega} = \text{Ker}(X_{\varphi})$. In the case it is finite having n elements, $\text{Fuz}(X_{\varphi}) = \cup \{a_i; i \leq n\}$ and $a_n = \text{Ker}(X_{\varphi})$.

Proposition 11: Every proper real semiset is up approxi-
mable.

Proof: There exists Φ_{φ} containing at least the formula $\psi(x, a) = x \in a$ of the language $\text{FL}\{a\}$.

Q.E.D.

Proposition 12: If the semiset X_{φ} is completely approxi-
mable, i.e. $X_{\varphi} = \cap \{a_n; n \in \mathbb{N}\}$ then it
is uncountable.

Proof: X is a π -class and no π -class is countable.

Q.E.D.

(Let $n > m$, $x \in \text{Ker}(X_{\varphi})$ and $y \in a_m - a_{m+1}$, $z \in a_n - a_{n+1}$.
Then $x \sim_{\mathcal{C}}^{\Delta_m} y$ and $x \sim_{\mathcal{C}}^{\Delta_n} z$ and $\Delta_m \subseteq \Delta_n$, i.e. z is "more
similar" to x than y . The indiscernibility equivalence
 $\sim_{\mathcal{C}}^{\Delta_n}$ is a natural base for the definition of the grade of
membership of y into $\text{Fuz}(X_{\varphi})$).

Definition: The grade of membership of y into $\text{Fuz}(X_{\varphi})$
is the ordinal number of the set a_{α} such
that $y \in a_{\alpha} - a_{\alpha+1}$.

We shall also write $x \in_{\alpha} \text{Fuz}(X_{\varphi})$. It can be seen that
the grade of membership identifies the "number" of for-
mulas ψ_{α} which are fulfilled by y . If Φ_{φ} is not known it
gives restricted information only.

The grades of membership can be transformed on the
interval $\langle 0; 1 \rangle$. Define the membership function

$$F: V \rightarrow \langle 0; 1 \rangle$$

where

$$F(x) = \begin{cases} 0 & \text{if } x \notin a_0, \\ \frac{\alpha + 1}{\omega + 1} & \text{if } x \in a_\alpha - a_{\alpha+1}, \\ 1 & \text{if } x \in a_\omega = \text{Ker}(X_\varphi) \end{cases}$$

(for Φ_φ finite having n formulas ψ_i , ω is replaced by n). From [10] it follows that every two neighbouring grades of membership $\alpha/(\omega + 1)$ and $(\alpha + 1)/(\omega + 1)$ are indiscernible, i.e. their difference is infinitely small (it does not hold for Φ_φ finite).

There is a question now, how the operations on fuzzy sets can be realized. If X_φ and $X_{\varphi'}$ are two semisets then e.g. the approximating fuzzy set $\text{Fuz}(X_\varphi \cap X_{\varphi'})$ for $X_\varphi \cap X_{\varphi'}$ cannot be generally constructed from $\text{Fuz}(X_\varphi)$ and $\text{Fuz}(X_{\varphi'})$. It is, however, possible to find tolerably sound definitions for membership functions.

Proposition 13: Let $\text{Fuz}(X_\varphi)$, $\text{Fuz}(X_{\varphi'})$ be fuzzy sets up approximating X_φ , $X_{\varphi'}$ according to $\underline{\approx}$.

- $X_\varphi \cap X_{\varphi'}$ is dense in $\text{Fuz}(X_\varphi) \cap \text{Fuz}(X_{\varphi'})$ according to $\underset{C}{\sim} \{\psi_0, \psi_0'\}$
- $X_\varphi \cup X_{\varphi'}$ is dense in $\text{Fuz}(X_\varphi) \cup \text{Fuz}(X_{\varphi'})$ according to $\underset{C}{\approx} \{\psi_0, \psi_0'\}$
- $X_\varphi - X_{\varphi'}$ is dense in $\text{Fuz}(X_\varphi)$ according to $\underset{C}{\sim} \{\psi_0\}$

Proof: a) Evidently, $X_\varphi \cap X_{\varphi'} \subseteq \text{Fuz}(X_\varphi) \cap \text{Fuz}(X_{\varphi'})$.

Let $x \in \text{Fuz}(X_\varphi) \cap \text{Fuz}(X_{\varphi'})$. Then $\psi_0(x) \& \psi_0'(x)$

and each $y \in X_\varphi \cap X_{\varphi'}$ also fulfils these

formulas, i.e. $x \underset{C}{\sim} \{\psi_0, \psi_0'\} y$

- $X_\varphi \cup X_{\varphi'} \subseteq \text{Fuz}(X_\varphi) \cup \text{Fuz}(X_{\varphi'}) \subseteq \overline{X_\varphi \cup X_{\varphi'}} = \overline{X_\varphi \cup X_{\varphi'}}$

c) Let $x \in \text{Fuz}(X_\varphi)$. Then $\psi_0(x)$ and every $y \in X_\varphi - X_{\varphi'}$ fulfils $\psi_0(y)$, too. Hence $X_\varphi - X_{\varphi'} \subseteq \text{Fuz}(X_\varphi) \subseteq \overline{X_\varphi - X_{\varphi'}}$.

Q.E.D.

The definition of the operations on fuzzy sets can be introduced on the base of the above proposition.

Definition: Let $\text{Fuz}(X_\varphi)$, $\text{Fuz}(X_{\varphi'})$ be fuzzy sets up approximating semisets X_φ , $X_{\varphi'}$, and let $x \in_p \text{Fuz}(X_\varphi)$ and $x \in_{p'} \text{Fuz}(X_{\varphi'})$.

a) intersection

$$x \in_q \text{Fuz}(X_\varphi) \cap \text{Fuz}(X_{\varphi'}) \equiv q = p \wedge p'$$

b) union

$$x \in_q \text{Fuz}(X_\varphi) \cup \text{Fuz}(X_{\varphi'}) \equiv q = p \vee p'$$

c) difference

$$x \in_q \text{Fuz}(X_\varphi) \dot{-} \text{Fuz}(X_{\varphi'}) \equiv q = 0 \vee (p - p')$$

The definitions a), b) follow the common definitions of the intersection and union. The definition c) is weaker than the common definition of the difference. The reason is that it would "lose" many elements from $\text{Fuz}(X_\varphi)$. c) makes sure only that no element from the kernel $\text{Ker}(X_{\varphi'})$ belongs to $\text{Fuz}(X_\varphi) \dot{-} \text{Fuz}(X_{\varphi'})$.

3.4. Down approximation

This notion is dual to the notion of up approximation

in a certain sense. We shall start from the kernel. The farther from it the less is our certainty of the belongingness of an element to the semiset given, since we get closer to its edge. Therefore all the definitions are reversed.

Proposition 14: Let $\Phi_\varphi = \{\psi_n; \psi_n \rightarrow \varphi \& n \in \mathbb{N}\}$ be a class of set formulas FL_C and $X_\varphi = \{x; \varphi(x)\}$ a real proper semiset. Then the classes $Y_n = \{y; \psi_n(y)\}$ are sets. If they are finite then $Y = \cup\{Y_n; n \in \mathbb{N}\}$ is countable σ -class.

Proof: $Y_n \subseteq X_\varphi \subseteq a$, i.e. $a_n = Y_n \cap a$ is a set. The rest is obvious.

Q.E.D.

Proposition 15: The class X is countable iff $X = \cup\{a_n; n \in \mathbb{N}\}$ and a_n are finite sets.

Proof: The part "only if" follows from [10], page 42.

"If": there exists the ordering \leq in such a way that $\langle X, \leq \rangle$ is ordering of type ω . Let $x_0 \in X$. Put $z_0 = \{y; y \in X \& y \leq x_0\}$ and $z_{n+1} = z_n \cup \{y; y \leq x_{n+1} \& x_n \leq x_{n+1} \& x_n \neq x_{n+1}\}$. The sets z_n are finite and $X = \cup\{z_n; n \in \mathbb{N}\}$.

Q.E.D.

Let $c = \text{Ker}(X_\varphi)$ be the kernel of the semiset X_φ . Without loss of generality, the first formula $\psi_0 \in \Phi_\varphi$ can be supposed to have the form $x \in c$. Let the class Φ'_φ be constructed

$$(5) \quad \Phi'_\varphi = \{\psi'_n; \psi'_n = \psi_0 \vee \psi_1 \vee \dots \vee \psi_{n-1}\} .$$

Definition: Let $\Delta'_n = \{\psi'_i; i \geq n\}$. Then the elements x, y are Φ' -similar if there exists Δ'_n such that

$$x \underset{C}{\sim}^{\Delta'_n} y.$$

If $m < n$ and $x \underset{C}{\sim}^{\Delta'_m} y$ and $x \underset{C}{\sim}^{\Delta'_n} y$ then $\Delta'_n \subseteq \Delta'_m$ and y is "more similar" to x than z . Evidently, $x \underset{C}{\sim}^{\Delta'_n} y$ iff $\psi'_n(x) \equiv \psi'_n(y)$.

Proposition 16: Let $SC_{\Phi'}(c) = \{y; (\exists n)(\exists x \in c)(y \underset{C}{\sim}^{\Delta'_n} x)\}$. Then $SC_{\Phi'}(c)$ is a σ -class $\cup \{a_n; n \in \mathbb{N}\}$ for $a_n \subseteq a$.

Proof: Since $(\forall n)(\psi'_n(x))$, it follows that $SC_{\Phi'}(c) = \{y; (\exists n)(\psi'_n(y))\} = \cup \{Y_n; n \in \mathbb{N}\}$ where $Y_n = \{y; \psi'_n(y)\}$. The proposition follows from the Proposition 14.

Q.E.D.

Evidently, $SC_{\Phi'}(c) \subseteq X_\varphi$. The following definition can be stated now.

Definition: Let $X_\varphi \subseteq a$ be a real proper semiset,

$\text{Ker}(X_\varphi)$ be its kernel and

$\Phi_\varphi, \Phi'_\varphi$ the classes of formulas from (2) and (5) respectively. Then X_φ is down approximated by the σ -class $SC_{\Phi'_\varphi}(\text{Ker}(X_\varphi)) = \cup \{a_n; n \in \mathbb{N}\}$ where $a_n = \{y; \psi'_n(y) \& y \in a\}$ is ascending sequence of subsets of a and $a_0 = \text{Ker}(X_\varphi)$.

The Proposition 15 follows:

Proposition 17: Every countable class is completely down approximable.

Proof: It will do to start with $z_0 = c$ and to define $\psi_n(x) \equiv x \in z_n$ for z_n being sets from the Proposition 15.

Q.E.D.

Since Φ_φ is not necessarily infinite, it follows:

Proposition 18: Every real proper semiset is down approximable.

Definition: Let X_φ be down approximated by a σ -class $SC_{\Phi_\varphi}(\text{Ker}(X_\varphi)) = \cup \{a_n; n \in \mathbb{N}\}$. Then the grade of membership of y into X_φ is the ordinal number α such that $y \in a_{\alpha+1} - a_\alpha$. The lower is α the higher is the grade of membership.

In this case, the ordering of the grades of membership is reversed to the ordering of ordinal numbers. The membership function can be defined:

$$F: V \rightarrow \langle 0; 1 \rangle,$$

$$F(x) = \begin{cases} 1 & \text{if } x \in a_0 \\ 1 - \frac{\alpha + 1}{\omega + 1} & \text{if } x \in a_{\alpha+1} - a_\alpha \end{cases}$$

The grades of membership for the operations can be defined analogically to the up approximation.

4. Outline of the practical use

In the preceding chapter, the notion of approximation of a semiset by a class of sets was introduced. The analogy with representation theorem of the fuzzy sets theory is evident. The approximating class of sets is always immersed one into other and they are determined by one of the relations (1) or (2). It is a certain concretiza-

tion of the semiset property φ , either by the definition or by an example. From it follows the real basis for the application of the theory presented here. The set $a \in X_\varphi$ represents the context in the frame of which the semiset is defined. E.g. "warm" in the subtropical area is quite different from the "warm" near the pole. From the mathematical point of view, we are working in the witnessed universe (see [10]).

The sequence of definitions (examples) can be constructed each of which being more exact than the preceding one. We must realize that we work with finite numbers only, in the practice. Hence, the infinity presumption of $\tilde{\Phi}_\varphi$ is rather theoretical. It assures only that the number of definitions can unlimitedly increase. The above grades of membership become quite "exact" and "tangible".

The kernel determination represents a bit of problem. Every class contains a set. If the indiscernibility relation $\dot{=}$ is not known every "reasonably" defined set $c \subseteq X_\varphi$ can be considered to become the kernel. This is the way of introducing the theory into the practice.

There are many controversis concerning operations with fuzzy sets. This paper shows the reason for it. The classes $\tilde{\Phi}_\cap, \tilde{\Phi}_\cup, \tilde{\Phi}_-$ such that

$$\begin{aligned} \varphi \& \varphi' &\rightarrow \psi_i, & \psi_i \in \tilde{\Phi}_\cap \\ \varphi \cup \varphi' &\rightarrow \psi_j, & \psi_j \in \tilde{\Phi}_\cup \\ \varphi \& \neg \varphi' &\rightarrow \psi_k, & \psi_k \in \tilde{\Phi}_- \end{aligned}$$

cannot be generally found from the classes of formulas $\tilde{\Phi}_\varphi, \tilde{\Phi}_{\varphi'}$.

Therefore we must help ourselves by introducing approximate operations only, as are e.g. formulated in 3.4.

5. Conclusion

This paper deals with the relation of two mathematical notions - semiset and fuzzy set which model the same problem. I argue that semiset expresses the basis of vagueness better. From this point of view, fuzzy sets theory is reduced to the discipline being a technical means for the concrete work with semisets, and their elements. Fuzzy sets are introduced in the frame of AST serving for the semiset approximation. Two kinds of approximation are defined: up and down approximation. The basic properties of them are shown and the operations of intersection union and difference are defined. In the last part, the problems of the application of the theory are shortly discussed.

The grade of membership (membership function respectively) seems to be a very natural means for the characterization of the membership of an element in the semiset approximated. It is given by a number of formulas making the semiset property more exact. Without knowing these formulas, itself, it is a loss of information. An important feature of the theory presented is the fact that the approximation, and the membership function following from it are formulated in the frame of AST.

The definition of the intersection and difference of the approximating fuzzy sets remains a problem. A stron-

ger prescription assuring $X_{\varphi} \cap X_{\varphi'}$ to be dense in $Fuz(X_{\varphi}) \cap Fuz(X_{\varphi'})$ according to the finer indiscernibility equivalence must be looked for. It reminds the "bald conjunction(context)" formulated in [7]. Similarly for the difference.

The relation between both types of the approximation and the possibility of the imaginary semisets approximation must be further examined.

Acknowledgement

The author would like to express his thanks to RNDr. Josef Mlček, CSc. for his helpful and inspiring criticism which lead to improvements in this article.

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