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# Boundary stabilization of a class of coupled reaction-diffusion system with one control\*

Gonzalo Arias<sup>†</sup>, Hugo Parada<sup>‡</sup>

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## Abstract

Controllability and stabilization of coupled reaction-diffusion systems has been widely studied in the last decades. In recent years, coupling reaction-diffusion systems to elliptic equations has been dragging attention. These systems arise as a simplified model for coupled reaction-diffusion systems when at least one diffusion coefficient is high. In this article, we study the eigenvalues of a coupled reaction-diffusion system with one high diffusion. As a consequence, when this system is unstable we are able to build a Backstepping-based controller to stabilize it. This is achieved due to the singular perturbation method (SPM), which provides a methodology to study the behaviour of the eigenvalues with respect to the singular perturbation parameter. Additionally, the SPM helps in building controller based on the reduced models, which are used to stabilize the coupled system when one diffusion is high enough.

## 1 Introduction

In this article, we consider the following  $2 \times 2$  reaction-diffusion system in singularly perturbed form

$$\begin{cases} u_t - u_{xx} + \lambda(\varepsilon)u = \alpha(\varepsilon)v, & (x, t) \in (0, L) \times (0, \infty), \\ \varepsilon v_t - v_{xx} + \gamma(\varepsilon)v = \beta(\varepsilon)u, & (x, t) \in (0, L) \times (0, \infty), \\ u(0, t) = 0, \quad u(L, t) = 0, & t \in (0, \infty), \\ v(0, t) = 0, \quad v(L, t) = 0, & t \in (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in (0, L), \end{cases} \quad (1)$$

i.e.,  $\varepsilon > 0$  is small enough, and the parameters  $\alpha, \beta, \gamma, \lambda$  are considered as smooth functions of  $\varepsilon$ . The topic of stabilization and observer design for coupled reaction-diffusion systems has been widely studied, see for instance [4, 28] and the references therein.

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Dividing by  $\varepsilon$  in (1) yields that the diffusion coefficient can be large for small values of  $\varepsilon$ . In [14,19] we can see many phenomena arising by considering a coupled system of reaction-diffusion PDEs when one of the components has a high diffusion coefficient term. The authors use an approximated system obtained by letting the diffusion parameter to go to infinity (i.e.,  $\varepsilon$  goes to zero from the right). This limit system is referred to as the *shadow system*. The controllability of the shadow system has been recently studied in [11,12]. On the other hand, letting  $\varepsilon \rightarrow 0^+$  in (1) yields a reaction-diffusion PDE coupled to a elliptic PDE.

Recently, there has been growing attention on the controllability of systems coupling parabolic and elliptic equations. In [7,8], null controllability is addressed for a linear system of this type. In [8,9,13], local null controllability is discussed when considering nonlinear parabolic-elliptic systems. In [10], boundary controllability is analyzed for a reaction-diffusion coupled to an elliptic equation with Dirichlet boundary conditions. However, stabilization and observer design remain relatively underexplored areas. We can mention the work [20] where the authors build a Backstepping-based controller and observer design to stabilize a parabolic-elliptic system with Dirichlet boundary conditions. They also study the stabilization with a finite-dimensional controller with input delay in [20,21]. Additionally, in [1], a Backstepping-based controller and observer design are developed for a parabolic-elliptic system with Neumann boundary conditions.

Since we are supposing that  $\varepsilon$  is small enough, we are tempted to use the singular perturbation method (SPM). The SPM, roughly speaking, aims to decouple a singularly perturbed full system into two approximated subsystems. These approximated subsystems are called the *reduced order system* and the *boundary-layer system*. The SPM was first developed in context of singularly perturbed ODEs. We encourage the reader to look at [15,16] for an overview of the SPM in a finite-dimensional setting and its connections with control theory.

As it will be explained in Section 2, we present the reduced order system corresponding to (1), which is given by

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} + \lambda_0 \bar{u} = \alpha_0 \bar{v}, & (x, t) \in (0, L) \times (0, \infty), \\ -\bar{v}_{xx} + \gamma_0 \bar{v} = \beta_0 \bar{u}, & (x, t) \in (0, L) \times (0, \infty), \\ \bar{v}(0, t) = 0, \quad \bar{v}(L, t) = 0, & t \in (0, \infty), \\ \bar{u}(0, t) = 0, \quad \bar{u}(L, t) = 0, & t \in (0, \infty), \\ \bar{u}(x, 0) = \bar{u}_0(x), & x \in (0, L). \end{cases} \quad (2)$$

On the other hand, the boundary-layer system obtained through the SPM is

$$\begin{cases} \bar{v}_\tau - \bar{v}_{xx} + \gamma_0 \bar{v} = 0, & (x, \tau) \in (0, L) \times (0, \infty), \\ \bar{v}(0, t) = 0, \quad \bar{v}(L, t) = 0, & t \in (0, \infty), \\ \bar{v}(x, 0) = \bar{v}_0(x), & x \in (0, L). \end{cases} \quad (3)$$

Here,  $\tau := \frac{t}{\varepsilon}$  is known as the stretched time-scale. When one have a singularly perturbed system like (1), it is said that the constituents live in different time scales. Thus, the variables  $u, v$  are called the slow dynamic and fast dynamic, respectively.

The main results of this article are as follows

**Theorem 1.** *Suppose that  $\gamma_0 \neq -\frac{n^2\pi^2}{L^2}$  for any  $n \in \mathbb{N}$ . If  $\alpha, \beta, \gamma, \lambda \in C^1([0, \infty))$  are such that*

$$\gamma'(0) = 0, \quad \gamma(\varepsilon) = \gamma_0 + O(\varepsilon^2), \quad \lambda(\varepsilon) = \lambda_0 + O(\varepsilon), \quad \alpha(\varepsilon) = \alpha_0 + O(\varepsilon), \quad \beta(\varepsilon) = \beta_0 + O(\varepsilon), \quad \alpha_0, \beta_0 \neq 0. \quad (4)$$

*Then, for sufficiently small  $\varepsilon$  the family of eigenvalues  $\{\mu_{s,n}(\varepsilon), \mu_{f,n}(\varepsilon)\}_{n \in \mathbb{N}}$  of (1) satisfies*

$$\mu_{s,n}(\varepsilon) = \sigma_n + O(\varepsilon), \quad \mu_{f,n}(\varepsilon) = \frac{1}{\varepsilon}(\theta_n + O(\varepsilon)),$$

*where  $\sigma_n$  and  $\theta_n$  are the eigenvalues of (2) and (3), respectively. As a consequence, for  $\varepsilon > 0$  small enough system (1) is exponentially stable if and only if  $\sigma_n, \theta_n < 0$  for any  $n \in \mathbb{N}$ .*

The SPM for infinite-dimensional coupled singularly perturbed systems has been receiving a lot of attention in recent years. Works concerning the SPM include [3, 5, 6, 22] for systems coupling an ODE with either a hyperbolic system or a wave equation, [23–25] for hyperbolic systems with different time scales, [2] for the stability of a linear coupled heat-wave system with different time-scales and [18] for systems coupling a KdV equation with an ODE. However, as shown in [5, 22, 24], this technique may fail for some infinite-dimensional systems.

A direct consequence of Theorem 1 is that if either (2) or (3) have at least one unstable eigenvalue, then (1) is also unstable. Thus, we are concerned with the following boundary stabilization problem

$$\begin{cases} u_t - u_{xx} + \lambda(\varepsilon)u = \alpha(\varepsilon)v, & (x, t) \in (0, L) \times (0, \infty), \\ \varepsilon v_t - v_{xx} + \gamma(\varepsilon)v = \beta(\varepsilon)u, & (x, t) \in (0, L) \times (0, \infty), \\ u(0, t) = 0, \quad u(L, t) = h(t), & t \in (0, \infty), \\ v(0, t) = 0, \quad v(L, t) = 0, & t \in (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in (0, L). \end{cases} \quad (5)$$

Here, the boundary control  $h$  acts on the right Dirichlet boundary condition of the  $u$  equation. In [20], a Backstepping controller is proposed to stabilize system (2) provided that there are some unstable eigenvalues. For instance, in the problem of stabilizing finite-dimensional singularly perturbed systems, [16, Corollary 2.1] states that if one is able to stabilize the reduced order system with a feedback law, then one should stabilize the full system with the same feedback law. Using singular perturbation and Backstepping methods to address this kind of problems is very useful but still not widely developed. We can mention [26, 27] where the authors stabilize the reduced order system given by the SPM. However, they did not build an explicit controller to stabilize its full system. In this work we use the SPM to build a single controller to stabilize system (1), whenever is unstable.

The following theorem gives a positive answer to this issue

**Theorem 2.** *Suppose that  $\gamma_0 > \frac{-\pi^2}{L^2}$ ,  $\alpha, \beta, \gamma, \lambda \in C^1([0, \infty))$  satisfies (4) and  $\sigma_n \geq 0$  for some  $n \in \mathbb{N}$ . Then, for  $\varepsilon$  small enough there exists a feedback law  $F_\varepsilon(u)$  such that (5) with  $h(t; \varepsilon) = F_\varepsilon(u)$  is exponentially stable.*

The paper is organized as follows. In Section 2, we derive the approximated subsystems (2), (3),

and we provide the proof of Theorem 1. In Section 3.1, we improve the results of [20] concerning the stabilization of (2). In Section 3.2, we give the prove of Theorem 2, which is based on the SPM. Finally, in Appendix A we prove some technical results regarding the backstepping transformation that will be useful throughout the paper.

## 2 Eigenvalue approximation

To start this section, we first define the operator  $F_{\gamma(\varepsilon)} = (-\partial_{xx} + \gamma(\varepsilon)Id_{L^2(0,L)})^{-1}$  with homogeneous Dirichlet boundary conditions. For a function  $g \in L^2(0,L)$ ,  $z = F_{\gamma(\varepsilon)}(g)$  is the unique solution to the following Sturm-Liouville problem

$$-z_{xx} + \gamma(\varepsilon)z = g, \quad z(0) = z(L) = 0.$$

This operator is linear and continuous provided that  $\gamma(\varepsilon) \neq -(k\pi)^2$  for any  $k \in \mathbb{N}$ . We set  $F_{\gamma_0}$  when  $\varepsilon = 0$ .

Now we are able to derive the subsystems of (1) given by the SPM. Setting  $\varepsilon = 0$  in (1) we have to solve

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} + \lambda_0 \bar{u} = \alpha_0 \bar{v}, & (x, t) \in (0, L) \times (0, \infty), \\ -\tilde{v}_{xx} + \gamma_0 \tilde{v} = \beta_0 \bar{u}, & (x, t) \in (0, L) \times (0, \infty), \\ \tilde{v}(0, t) = 0, \quad \tilde{v}(L, t) = 0, & t \in (0, \infty), \\ \bar{u}(0, t) = 0, \quad \bar{u}(L, t) = 0, & t \in (0, \infty), \\ \bar{u}(x, 0) = \bar{u}_0(x), & x \in (0, L). \end{cases} \quad (6)$$

By solving the Sturm-Liouville problem in (6), whose solution is given by  $\tilde{v} = \beta_0 F_{\gamma_0} \bar{u}$ , we get the following reduced order system

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} + \lambda_0 \bar{u} = \alpha_0 \beta_0 F_{\gamma_0}(\bar{u}), & (x, t) \in (0, L) \times (0, \infty), \\ \bar{u}(0, t) = 0, \quad \bar{u}(L, t) = 0, & t \in (0, \infty), \\ \bar{u}(x, 0) = \bar{u}_0(x), & x \in (0, L). \end{cases} \quad (7)$$

On the other hand, defining the stretched time scale  $\tau = \frac{t}{\varepsilon}$  we define

$$\bar{v}(x, \tau) = v(x, \tau) - \beta(\varepsilon)F_{\gamma(\varepsilon)}u(x, \tau) = v(x, \tau) - \beta(\varepsilon)z(x, \tau),$$

where  $z(\cdot, t) \in H^2(0, L) \cap H_0^1(0, L)$  solves  $-z_{xx}(x, t) + \gamma_0 z(x, t) = u(x, t)$ . We can see that

$$\bar{v}_\tau(x, \tau) = v_\tau(x, \tau) - \varepsilon \beta(\varepsilon)F_{\gamma(\varepsilon)}u_t(x, \tau),$$

thus, taking  $\varepsilon = 0$  we have that  $\bar{v}_\tau = v_\tau$ . Clearly, Dirichlet boundary conditions are null as well. Taking the derivatives respect to space we have

$$\bar{v}_{xx}(x, \tau) = v_{xx}(x, \tau) - \beta(\varepsilon)z_{xx}(x, \tau) = v_{xx}(x, \tau) - \beta(\varepsilon)(\gamma(\varepsilon)z(x, \tau) - u(x, \tau)),$$

where  $z = F_{\gamma(\varepsilon)}u$ . It follows that

$$\begin{aligned}\bar{v}_{xx}(x, \tau) &= v_{xx}(x, \tau) - \gamma(\varepsilon)\beta(\varepsilon)z(x, \tau) + \beta(\varepsilon)u(x, \tau) \\ &= v_{xx}(x, \tau) + \gamma(\varepsilon)\bar{v}(x, \tau) - \gamma(\varepsilon)v(x, \tau) + \beta(\varepsilon)u(x, \tau).\end{aligned}$$

Now note that

$$\bar{v}_\tau(x, \tau) = v_\tau(x, \tau) = v_{xx}(x, \tau) - \gamma(\varepsilon)v(x, \tau) + \beta(\varepsilon)u(x, \tau),$$

therefore, taking  $\varepsilon = 0$  we get the following boundary-layer system

$$\begin{cases} \bar{v}_\tau - \bar{v}_{xx} + \gamma_0\bar{v} = 0, & (x, \tau) \in (0, L) \times (0, \infty), \\ \bar{v}(0, t) = 0, \quad \bar{v}(L, t) = 0, & t \in (0, \infty), \\ \bar{v}(x, 0) = \bar{v}_0(x), & x \in (0, L). \end{cases} \quad (8)$$

**Proposition 3.** *The eigenvalues of (7) are  $\sigma_n = \frac{\beta_0\alpha_0}{(\frac{n\pi}{L})^2 + \gamma_0} - \lambda_0 - (\frac{n\pi}{L})^2$ . While, the eigenvalues of (8) are  $\theta_n = -\gamma_0 - \frac{n^2\pi^2}{L^2}$ .*

*Proof.* The computation of the eigenvalues of (8) is straightforward. The eigenvalues of (7) are derived in [20, Chapter 2].  $\square$

**Corollary 4.** *For any  $\bar{v}_0 \in L^2(0, L)$  it holds that the unique mild solution  $\bar{v}(x, \tau)$  to (8) satisfies*

$$\|\bar{v}(\cdot, \tau)\|_{L^2(0, L)} \leq C e^{-\left(\frac{\pi^2}{L^2} + \gamma_0\right)\tau} \|\bar{v}_0\|_{L^2(0, L)}, \quad \forall \tau \geq 0,$$

for a constant  $C \geq 1$ . Equivalently, in the  $t$  time-scale we have that

$$\|\bar{v}(\cdot, \tau)\|_{L^2(0, L)} \leq C e^{-\left(\frac{\pi^2}{L^2} + \gamma_0\right)\frac{t}{\varepsilon}} \|\bar{v}_0\|_{L^2(0, L)}, \quad \forall t \geq 0, \quad \forall \varepsilon > 0.$$

*Proof.* The proof is a direct consequence of Proposition 3 and the definition of  $\tau$ .  $\square$

Now our aim is to prove Theorem 1. To this end we recall the following fact about the Dirichlet Laplacian operator. Let  $A : H_0^1(0, L) \cap H^2(0, L) \rightarrow L^2(0, L)$  be the Laplacian with homogeneous boundary conditions, i.e.,  $Af = -\partial_{xx}f$ . We know that this operator has  $\lambda_n = \frac{n^2\pi^2}{L^2}$  as eigenvalues with correspondent eigenfunctions  $w_n = \sin\left(\frac{n\pi}{L}x\right)$ ,  $n \geq 1$ .

*Proof of Theorem 1.* Let  $\sigma$  be an eigenvalue of the spatial operator associated to (1), i.e., we have to solve

$$\begin{aligned}\partial_{xx}\psi_1 - (\lambda - \sigma)\psi_1 + \alpha\psi_2 &= 0, \\ \frac{1}{\varepsilon}\partial_{xx}\psi_2 - \left(\frac{\gamma}{\varepsilon} - \sigma\right)\psi_2 + \frac{\beta}{\varepsilon}\psi_1 &= 0,\end{aligned}$$

with homogeneous Dirichlet boundary conditions. Suppose that

$$(\psi_1, \psi_2) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)(a_{1,n}, a_{2,n}).$$

Therefore the coefficients  $(a_{1,n}, a_{2,n})$  must satisfy

$$\underbrace{\begin{bmatrix} -\frac{n^2\pi^2}{L^2} - \lambda & -\alpha \\ -\frac{\beta}{\varepsilon} & -\frac{n^2\pi^2}{\varepsilon L^2} - \frac{\gamma}{\varepsilon} \end{bmatrix}}_{:=L_n} \begin{bmatrix} a_{1,n} \\ a_{2,n} \end{bmatrix} = \sigma_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \end{bmatrix}$$

that is  $(a_{1,n}, a_{2,n})$  is an eigenvector associated to the eigenvalue  $\sigma_n$  of the matrix  $L_n$ . The characteristic polynomial of  $L_n$  is given therefore by

$$p_n(\sigma) = \sigma^2 + \left( \frac{n^2\pi^2}{L^2} + \lambda + \frac{1}{\varepsilon} \frac{n^2\pi^2}{L^2} + \frac{\gamma}{\varepsilon} \right) \sigma + \left[ \left( \frac{n^2\pi^2}{L^2} + \lambda \right) \left( \frac{n^2\pi^2}{\varepsilon L^2} + \frac{\gamma}{\varepsilon} \right) - \frac{\alpha\beta}{\varepsilon} \right].$$

For the ease on the notation we set  $\kappa_n = \frac{n^2\pi^2}{L^2} > 0$ , then  $p_n(\sigma)$  becomes

$$\begin{aligned} p_n(\sigma) &= \sigma^2 + \frac{1}{\varepsilon} (\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda)) \sigma + \left( (\kappa_n + \lambda) \left( \frac{\kappa_n}{\varepsilon} + \frac{\gamma}{\varepsilon} \right) - \frac{1}{\varepsilon} \alpha\beta \right) \\ &= \sigma^2 + \frac{1}{\varepsilon} (\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda)) \sigma + \frac{1}{\varepsilon} ((\kappa_n + \lambda)(\kappa_n + \gamma) - \alpha\beta). \end{aligned}$$

Since we are interested in the zeroes of  $p$  we multiply by  $\varepsilon$  and we look at its discriminant

$$\Delta(p_n) = (\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda))^2 - 4\varepsilon((\kappa_n + \lambda)(\kappa_n + \gamma) - \alpha\beta).$$

The roots of  $p_n$  are given by

$$\begin{aligned} \sigma_{\pm,n} &= \frac{-(\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda)) \pm \sqrt{(\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda))^2 - 4\varepsilon((\kappa_n + \lambda)(\kappa_n + \gamma) - \alpha\beta)}}{2\varepsilon} \\ &= \frac{-(\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda)) \pm \sqrt{(\kappa_n + \gamma - \varepsilon(\kappa_n + \lambda))^2 + 4\varepsilon\alpha\beta}}{2\varepsilon}. \end{aligned}$$

Using a Taylor approximation and supposing that  $\gamma_0 > \frac{-\pi^2}{L^2}$  we have that

$$\sigma_{\pm,n} = \frac{-(\kappa_n + \gamma + \varepsilon(\kappa_n + \lambda)) \pm \left( \kappa_n + \gamma_0 - (\kappa_n + \lambda_0)\varepsilon + 2\frac{\alpha_0\beta_0\varepsilon}{\kappa_n + \gamma_0} + O(\varepsilon^2) \right)}{2\varepsilon}.$$

Is easy to show that

$$\mu_{s,n} := \sigma_{+,n} = -\kappa_n - \lambda_0 + \frac{\beta_0\alpha_0}{\kappa_n + \gamma_0} + O(\varepsilon),$$

and that

$$2\varepsilon\sigma_{-,n} = -2(\kappa_n + \gamma_0) + O(\varepsilon).$$

We conclude the proof of Theorem 1 by setting  $\mu_{f,n} := \sigma_{-,n}$ . □

**Remark 5.** In the proof of Theorem 1 we have supposed that  $\gamma_0 > \frac{-\pi^2}{L^2}$  in the end of the proof. In the case where  $\gamma_0 < \frac{-\pi^2}{L^2}$ , but  $\gamma_0 \neq -\frac{n^2\pi^2}{L^2}$  for any  $n \in \mathbb{N}$  the proof follows the same path. The unique difference is that the roles of  $\sigma_{+,n}$  and  $\sigma_{-,n}$  are exchanged. ◦

### 3 Boundary stabilization

In this part, we analyse the boundary stabilization of (5). Using the Backstepping technique we built a feedback controller stabilizing the reduced system (11) which allow us to stabilize the full system.

We define  $T_\varepsilon : L^2(0, L) \rightarrow L^2(0, L)$  through

$$T_\varepsilon p(x) = p(x) - \int_0^x k(x, y; \varepsilon) p(y) dy, \quad (9)$$

where  $k(x, y; \varepsilon)$  is the unique solution to (35). This operator is linear and bounded, as well as its inverse  $T_\varepsilon^{-1} : L^2(0, L) \rightarrow L^2(0, L)$ , which is given by

$$T_\varepsilon^{-1} q(x) = q(x) + \int_0^x l(x, y; \varepsilon) q(y) dy, \quad (10)$$

where  $l$  is the unique solution to (36).

#### 3.1 Stabilization of the reduced system (11)

Recalling that the reduced order system with input control is given by

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} + \lambda_0 \bar{u} = \alpha_0 \beta_0 F_{\gamma_0}(\bar{u}), & (x, t) \in (0, L) \times (0, \infty), \\ \bar{u}(0, t) = 0, \quad \bar{u}(L, t) = h_s(t), & t \in (0, \infty), \\ \bar{u}(x, 0) = \bar{u}_0(x), & x \in (0, L), \end{cases} \quad (11)$$

where  $h_s(t) = h(t; 0)$ . From Proposition 3, we know that if

$$\lambda_0 \leq \frac{\alpha_0 \beta_0 L^2}{\pi^2 n^2 + \gamma_0 L^2} - \frac{\pi^2 n^2}{L^2},$$

for some  $n \in \mathbb{N}$ , then (11) is unstable if  $h_s \equiv 0$ . From [20] we know that  $\bar{w}(x, t) = T_0 \bar{u}(x, t)$  satisfies the following target system

$$\begin{cases} \bar{w}_t - \bar{w}_{xx} + \tilde{\lambda}_0 \bar{w} = \alpha_0 T_0 \bar{v}, & (x, t) \in (0, L) \times (0, \infty), \\ -\tilde{v}_{xx} + \gamma_0 \tilde{v} = \beta_0 T_0^{-1} \bar{w}, & (x, t) \in (0, L) \times (0, \infty), \\ \tilde{v}(0, t) = 0, \quad \tilde{v}(L, t) = 0, & t \in (0, \infty), \\ \bar{w}(0, t) = 0, \quad \bar{w}(L, t) = 0, & t \in (0, \infty), \\ \bar{w}(x, 0) = T_0 \bar{u}_0(x), & x \in (0, L), \end{cases} \quad (12)$$

provided that the controller is chosen as

$$h_s(t) = \int_0^L k(x, y; 0) \bar{u}(y) dy. \quad (13)$$

In particular, we have the following result.



**Theorem 6.** Let  $\alpha_0, \beta_0 \neq 0$ ,  $\gamma_0 > -\frac{\pi^2}{L^2}$  and  $\tilde{\lambda}_0 > \lambda_0$ . If

$$\mu := \tilde{\lambda}_0 + \frac{\pi^2}{L^2} - |\alpha_0\beta_0| \|F_{\gamma_0}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; 0)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; 0)\|_{L^2(\mathcal{T})}) > 0, \quad (14)$$

then (11) with controller (13) is exponentially stable with decay rate given by  $\mu$ .

*Proof.* First we define  $E(t) = \frac{1}{2} \|w\|_{L^2(0,L)}$  for any mild solution of (12). Taking the derivative of  $E$  along the strong solutions of (12) we get

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_0^L \bar{w} \bar{w}_t dx = \int_0^L w w_{xx} dx - \tilde{\lambda}_0 \int_0^L |w|^2 + \alpha_0 (\bar{w}, T\tilde{v})_{L^2(0,L)} \\ &\leq - \left( \tilde{\lambda}_0 + \frac{\pi^2}{L^2} \right) \int_0^L |w|^2 + \alpha_0 (\bar{w}, T\tilde{v})_{L^2(0,L)}. \end{aligned}$$

Using the fact that  $\tilde{v} = \beta_0 F_{\gamma_0} (T^{-1}\bar{w})$  we obtain

$$\begin{aligned} \frac{d}{dt} E(t) &\leq - \left( \tilde{\lambda}_0 + \frac{\pi^2}{L^2} \right) \int_0^L |w|^2 + \alpha_0 \beta_0 (\bar{w}, T F_{\gamma_0} T^{-1} \bar{w})_{L^2(0,L)} \\ &\leq -2 \left( \tilde{\lambda}_0 + \frac{\pi^2}{L^2} - |\alpha_0\beta_0| \|F_{\gamma_0}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot)\|_{L^2(\mathcal{T})}) \right) E(t), \end{aligned}$$

where the last inequality is justified by the definition of  $T$  and  $T^{-1}$ . Thus, if

$$\tilde{\lambda}_0 + \frac{\pi^2}{L^2} - |\alpha_0\beta_0| \|F_{\gamma_0}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot)\|_{L^2(\mathcal{T})}) > 0,$$

then the exponential stability yields with decay rate (14). We conclude the result by noting that  $\bar{u} = T_0^{-1}\bar{w}$  □

We end this section with the following corollary

**Corollary 7.** Let  $\mu$  defined by (14). Then, we have that

$$\|\bar{w}_t\|_{L^2(0,L)} \leq C e^{-\mu t} \|\bar{w}_0\|_{H^2(0,L)}.$$

*Proof.* Is enough to notice that taking  $u_0 \in H^2(0,L)$  we have that  $(\bar{w}_t, \tilde{v}_t)$  satisfies (12) with

$$\partial_x^2 T_0 \bar{u}_0 - \tilde{\lambda}_0 T_0 \bar{u}_0 + \alpha_0 \beta_0 T_0 F_{\gamma_0} \bar{u}_0,$$

as initial condition. □

### 3.2 Stabilization of the full system

Thanks to Theorem 1 we know that (1) is unstable if for some  $n \in \mathbb{N}$

$$\lambda_0 \leq \frac{\alpha_0 \beta_0 L^2}{\pi^2 n^2 + \gamma_0 L^2} - \frac{\pi^2 n^2}{L^2},$$

provided that  $\varepsilon > 0$  is small enough. Thus, we consider the following controlled system

$$\begin{cases} u_t - u_{xx} + \lambda(\varepsilon)u = \alpha(\varepsilon)v, & (x, t) \in (0, L) \times (0, \infty), \\ \varepsilon v_t - v_{xx} + \gamma(\varepsilon)v = \beta(\varepsilon)u, & (x, t) \in (0, L) \times (0, \infty), \\ u(0, t) = 0, \quad u(L, t) = h(t), & t \in (0, \infty), \\ v(0, t) = 0, \quad v(L, t) = 0, & t \in (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in (0, L). \end{cases} \quad (15)$$

Set  $w(x, t) = T_\varepsilon u(x, t)$ . Let us note that

$$w_{xx}(x, t) = u_{xx}(x, t) - \int_0^x k_{xx}(x, y; \varepsilon)u(y, t)dy - k_x(x, x; \varepsilon)u(x, t) - \frac{d}{dx}(k(x, x; \varepsilon))u(x, t) - k(x, x; \varepsilon)u_x(x, t).$$

On the other hand

$$\begin{aligned} w_t(x, t) = u_t(x, t) - \int_0^x k_{yy}(x, y; \varepsilon)(u(y, t) - \lambda(\varepsilon)u(y, t) + \alpha(\varepsilon)v(y, t))dy \\ - k(x, x; \varepsilon)u_x(x, t) + k_y(x, x; \varepsilon)u(x, t). \end{aligned}$$

It follows that  $w_t - w_{xx} + \tilde{\lambda}(\varepsilon)w = \alpha(\varepsilon)T_\varepsilon v$ . Moreover, taking the feedback law to be defined by

$$h(t, \varepsilon) = \int_0^L k(L, y; \varepsilon)u(y, t; \varepsilon)dy, \quad (16)$$

we end up with the following target system

$$\begin{cases} w_t - w_{xx} + \tilde{\lambda}(\varepsilon)w = \alpha(\varepsilon)T_\varepsilon v(x, t), & (x, t) \in (0, L) \times (0, \infty), \\ \varepsilon v_t - v_{xx} + \gamma(\varepsilon)v = \beta(\varepsilon)T_\varepsilon^{-1}w(x, t), & (x, t) \in (0, L) \times (0, \infty), \\ w(0, t) = 0, \quad w(L, t) = 0, & t \in (0, \infty), \\ v(0, t) = 0, \quad v(L, t) = 0, & t \in (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in (0, L). \end{cases} \quad (17)$$

Now we are able to prove Theorem 2.

*Proof of Theorem 2.* To show that the closed loop system (15) is exponentially stable let us consider the following energy

$$E_\varepsilon(t) = \frac{1}{2} \int_0^L (\varepsilon|v|^2 + |w|^2)dx, \quad \varepsilon \in (0, 1].$$

Taking its derivative along the strong solutions of (15) yields

$$\begin{aligned}
\frac{d}{dt}E_\varepsilon(w, v) &= \int_0^L (ww_t + \varepsilon vv_t) dx, \\
&= \int_0^L (ww_{xx} + vv_{xx}) - \gamma(\varepsilon) \int_0^L |v|^2 dx - \tilde{\lambda}(\varepsilon) \int_0^L w^2 dx + \alpha(\varepsilon)(w, T_\varepsilon v)_{L^2(0,L)} \\
&\quad + \beta(\varepsilon)(T_\varepsilon^{-1}w, v)_{L^2(0,L)} \\
&\leq -\frac{\pi^2}{L^2} \int_0^L (|v|^2 + |w|^2) dx - \gamma(\varepsilon) \int_0^L |v|^2 dx - \tilde{\lambda}(\varepsilon) \int_0^L w^2 dx + \alpha(\varepsilon)(w, T_\varepsilon v)_{L^2(0,L)} \\
&\quad + \beta(\varepsilon)(T_\varepsilon^{-1}w, v)_{L^2(0,L)}.
\end{aligned}$$

Now, for the crossed terms we have the following estimation

$$\begin{aligned}
\alpha(\varepsilon)(w, T_\varepsilon v)_{L^2(0,L)} &\leq |\alpha(\varepsilon)| \|T_\varepsilon\|_{\mathcal{L}(L^2(0,L))} \|w\| \|v\| \\
&\leq |\alpha(\varepsilon)| \|T_\varepsilon\|_{\mathcal{L}(L^2(0,L))} \left( \frac{|\beta(\varepsilon)| \|T_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,L))}}{2\delta} \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} \|w\|^2 \right. \\
&\quad \left. + \frac{\delta}{2|\beta(\varepsilon)| \|T_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,L))}} \frac{\|v\|^2}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \right),
\end{aligned}$$

and

$$\begin{aligned}
\beta(\varepsilon)(T_\varepsilon^{-1}w, v)_{L^2(0,L)} &\leq |\beta(\varepsilon)| \|T_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,L))} \|w\| \|v\|, \\
&\leq |\beta(\varepsilon)| \|T_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,L))} \left( \frac{|\alpha(\varepsilon)| \|T_\varepsilon\|_{\mathcal{L}(L^2(0,L))}}{2\delta} \|w\|^2 \right. \\
&\quad \left. + \frac{\delta}{2|\alpha(\varepsilon)| \|T_\varepsilon\|_{\mathcal{L}(L^2(0,L))}} \frac{\|v\|^2}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \right),
\end{aligned}$$

which are obtained by using suitable Young's inequality and are valid for any  $\delta > 0$ . Therefore,

$$\begin{aligned}
\frac{d}{dt}E_\varepsilon(w, v) &\leq - \left( \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} |\alpha(\varepsilon)\beta(\varepsilon)| (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\delta} \right) \|w\|^2, \\
&\quad - \left( \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{\delta}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right) \right) \|v\|^2,
\end{aligned}$$

where

$$a(\varepsilon) = \frac{|\alpha(\varepsilon)| \|T_\varepsilon\|_{\mathcal{L}(L^2(0,L))} \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}}{|\beta(\varepsilon)| \|T_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,L))}} > 0$$

Note that if  $\delta > 0$  is chosen as

$$\delta \leq \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} \left( \frac{\pi^2}{L^2} + \gamma(\varepsilon) \right) \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right)^{-1}, \quad (18)$$

then (15) is exponentially stable whenever

$$\tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} > \frac{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} |\alpha(\varepsilon)\beta(\varepsilon)| (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\delta}. \quad (19)$$

Since  $\alpha_0, \beta_0 \neq 0$  we have that

$$a(\varepsilon) \rightarrow \frac{|\alpha_0| \|T_0\|_{\mathcal{L}(L^2(0,L))} \|F_{\gamma_0}\|_{\mathcal{L}(L^2(0,L))}}{|\beta_0| \|T_0^{-1}\|_{\mathcal{L}(L^2(0,L))}} > 0, \text{ as } \varepsilon \rightarrow 0^+.$$

This guarantees that (18) is non-zero and therefore (19) is fulfilled. To show the exponential stability, let us note that under (18) and (19) we have

$$\begin{aligned} & \frac{d}{dt} E_\varepsilon(w, v) \\ & \leq - \left( \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{|\alpha(\varepsilon)\beta(\varepsilon)| \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\delta} \right) \|w\|^2 \\ & \quad - \frac{1}{\varepsilon} \left( \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{\delta}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right) \right) \varepsilon \|v\|^2, \end{aligned}$$

thus we have that

$$\frac{d}{dt} E_\varepsilon(w, v) \leq -2 \min\{\nu_1(\varepsilon), \frac{1}{\varepsilon} \nu_2(\varepsilon)\} E_\varepsilon(t),$$

where

$$\nu_1(\varepsilon) = \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{|\alpha(\varepsilon)\beta(\varepsilon)| \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\delta},$$

and

$$\nu_2(\varepsilon) = \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{\delta}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right).$$

Since  $\nu_1(\varepsilon), \nu_2(\varepsilon) > 0$  we can choose  $\varepsilon \in \left(0, \frac{\nu_2(\varepsilon)}{\nu_1(\varepsilon)}\right]$ , which implies that

$$\frac{d}{dt} E_\varepsilon(w, v) \leq -2\nu_1(\varepsilon) E_\varepsilon(w, v).$$

Therefore our target system (12) is exponentially stable. The exponential stability of (15) with feedback law (16) follows from the invertibility of the Backstepping transformation used.  $\square$

We end up this section with the following remark.

**Remark 8.** Note that we can always select  $\delta$  in (18) such that  $\delta \in (0, 1]$ . This is due to the fact, that if

$$\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} \left( \frac{\pi^2}{L^2} + \gamma(\varepsilon) \right) \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right)^{-1} > 1,$$

then we can select  $\delta$  to be 1. Otherwise, we would already have  $\delta \leq 1$ .  $\circ$

## 4 Tikhonov approximation

In this section we provide an analogue to [16, Theorem 2.1], which roughly speaking give us an approximation of order  $O(\varepsilon)$  of the solution  $(u_\varepsilon, v_\varepsilon)$  of (15) by the solutions to (11) and (8). That is, we aim to prove the following theorem

**Theorem 9.** *Suppose that  $\gamma_0 > \frac{-\pi^2}{L^2}$ ,  $\alpha, \beta, \gamma, \lambda \in C^1([0, \infty))$  satisfies (4) and  $\sigma_n \geq 0$  for some  $n \in \mathbb{N}$ . Let  $u_0, v_0, \bar{v}_0 \in (L^2(0, L))^3$  and  $\bar{u}_0 \in H^2(0, L)$  be such that*

$$\|u_0 - \bar{u}_0\|_{L^2(0, L)} = O(\varepsilon), \quad \|v_0 - \bar{v}_0 - \beta_0 F_{\gamma_0} \bar{v}_0\|_{L^2(0, L)} = O(\varepsilon), \quad \|\bar{v}_0\|_{L^2(0, L)} = O(\varepsilon). \quad (20)$$

*Then, for  $\varepsilon$  small enough the solution  $(u_\varepsilon, v_\varepsilon)$  to (15) with control  $h(t, \varepsilon)$  (16) and initial condition  $(u_0, v_0)$  satisfies that*

$$\|u_\varepsilon(\cdot, t) - \bar{u}(\cdot, t)\|_{L^2(0, L)} = O(\varepsilon), \quad \|v_\varepsilon(\cdot, t) - \bar{v}(\cdot, \frac{t}{\varepsilon}) - \beta_0 F_{\gamma_0} \bar{u}(\cdot, t)\|_{L^2(0, L)} = O(\varepsilon). \quad (21)$$

*Here  $\bar{u}$  is the solution to (11) with controller  $h_s(t)$  (13) and initial condition  $\bar{u}_0$ . Also,  $\bar{v}$  is the solution to (8) with initial condition  $\bar{v}_0$ .*

In order to prove Theorem 9, let us consider the following error variables

$$\xi(x, t) = w(x, t) - \bar{w}(x, t), \quad \zeta(x, t) = v(x, t) - \bar{v}(x, \frac{t}{\varepsilon}) - \beta_0 F_{\gamma_0}(T_0^{-1} \bar{w}). \quad (22)$$

Is easy to note that  $\xi, \zeta$  satisfy homogeneous Dirichlet boundary conditions. Now we are tempted to derive the error system. To this end, let us compute

$$\begin{aligned} \xi_t &= w_t - \bar{w}_t = w_{xx} - \tilde{\lambda}(\varepsilon)w + \alpha(\varepsilon)T_\varepsilon v - (\bar{w}_{xx} - \tilde{\lambda}_0 \bar{w} + \alpha_0 \beta_0 T_0 F_{\gamma_0}(T_0^{-1} \bar{w})) \\ &= \xi_{xx} - \tilde{\lambda}(\varepsilon)\xi + O(\varepsilon)\bar{w} + \alpha(\varepsilon)T_\varepsilon v - \alpha_0 \beta_0 T_0 F_{\gamma_0}(T_0^{-1} \bar{w}), \end{aligned}$$

therefore  $\xi$  solves

$$\xi_t = \xi_{xx} - \tilde{\lambda}(\varepsilon)\xi + \alpha(\varepsilon)T_\varepsilon \zeta + O(\varepsilon)\bar{w} + \alpha(\varepsilon)T_\varepsilon \bar{v} + \alpha(\varepsilon)\beta_0 T_\varepsilon F_{\gamma_0}(T_0^{-1} \bar{w}) - \alpha_0 \beta_0 T_0 F_{\gamma_0}(T_0^{-1} \bar{w}). \quad (23)$$

On the other hand, let us note that

$$\zeta_t = v_t - \frac{1}{\varepsilon} \bar{v}_\tau - \beta_0 F_{\gamma_0}(T_0^{-1} \bar{w}_t),$$

thus

$$\varepsilon \zeta_t = (v_{xx} - \bar{v}_{xx} + \gamma_0 \bar{v}) - \lambda(\varepsilon)v + \beta(\varepsilon)T_\varepsilon^{-1} w - \varepsilon \beta_0 F_{\gamma_0}(T_0^{-1} \bar{w}_t). \quad (24)$$

For the space derivatives note that

$$\zeta_{xx} = v_{xx} - \bar{v}_{xx} - (\beta_0 F_{\gamma_0}(T_0^{-1} \bar{w}))_{xx},$$

or equivalently

$$\zeta_{xx} = (v_{xx} - \bar{v}_{xx}) - \beta_0(\gamma_0 F_{\gamma_0}(T_0^{-1}\bar{w}) - T_0^{-1}\bar{w}). \quad (25)$$

Using (23), (25) and (24), it follows that the error system is given by

$$\begin{cases} \xi_t - \xi_{xx} + \lambda(\varepsilon)\xi = \alpha(\varepsilon)T_\varepsilon\zeta + O(\varepsilon)\bar{w} + f(\bar{v}, \bar{w}, \varepsilon), & (x, t) \in (0, L) \times (0, \infty), \\ \varepsilon\zeta_t - \zeta_{xx} + \gamma(\varepsilon)\zeta = \beta(\varepsilon)T_\varepsilon^{-1}\xi + O(\varepsilon)(\beta_0 F_{\gamma_0}(T_0^{-1}\bar{w}_t) + \bar{v}) + g(\bar{w}, \varepsilon), & (x, t) \in (0, L) \times (0, \infty), \\ \xi(0, t) = 0, \quad \xi(L, t) = 0, & t \in (0, \infty), \\ \zeta(0, t) = 0, \quad \zeta(L, t) = 0, & t \in (0, \infty), \\ \xi(x, 0) = T_\varepsilon u_0(x) - T_0 \bar{u}_0(x), & x \in (0, L), \\ \zeta(x, 0) = v_0(x) - \bar{v}_0(x) - \beta_0 F_{\gamma_0} T_0^{-1} \bar{w}_0(x), & x \in (0, L), \end{cases} \quad (26)$$

where  $f, g$  are given by

$$f(\bar{v}, \bar{w}, \varepsilon) = \alpha(\varepsilon)T_\varepsilon\bar{v} + \alpha(\varepsilon)\beta_0 T_\varepsilon F_{\gamma_0}(T_0^{-1}\bar{w}) - \alpha_0\beta_0 T_0 F_{\gamma_0}(T_0^{-1}\bar{w}), \quad (27)$$

and

$$g(\bar{w}, \varepsilon) = \beta(\varepsilon)T_\varepsilon^{-1}\bar{w} - \beta_0 T_0^{-1}\bar{w}. \quad (28)$$

**Proposition 10.** *Let  $f, g$  be defined by (27) and (28), respectively. Then, we have that*

$$\|f\|_{L^2(0,L)} \leq O(1)e^{-\left(\frac{\pi^2}{L^2} + \gamma_0\right)\frac{t}{\varepsilon}} \|\bar{v}_0\|_{L^2(0,L)} + O(\varepsilon)e^{-\mu t} \|\bar{w}_0\|_{L^2(0,L)}, \quad (29)$$

and

$$\|g\|_{L^2(0,L)} \leq O(\varepsilon)e^{-\mu t} \|\bar{w}_0\|_{L^2(0,L)}, \quad (30)$$

where  $\mu$  is given by (11).

*Proof.* We start by analyzing  $f$ ,

$$\begin{aligned} \|f\|_{L^2(0,L)} &= \|\alpha(\varepsilon)T_\varepsilon\bar{v} + \alpha(\varepsilon)\beta_0 T_\varepsilon F_{\gamma_0}(T_0^{-1}\bar{w}) - \alpha_0\beta_0 T_0 F_{\gamma_0}(T_0^{-1}\bar{w})\|_{L^2(0,L)} \\ &\leq \|\alpha(\varepsilon)T_\varepsilon\bar{v}\|_{L^2(0,L)} + \|(\alpha(\varepsilon) - \alpha_0)\beta_0 T_\varepsilon F_{\gamma_0}(T_0^{-1}\bar{w})\|_{L^2(0,L)} + \|\alpha_0\beta_0 [T_\varepsilon - T_0] F_{\gamma_0}(T_0^{-1}\bar{w})\|_{L^2(0,L)}. \end{aligned}$$

For the first term, we use (4) and Corollary 4 to obtain

$$\|\alpha(\varepsilon)T_\varepsilon\bar{v}\|_{L^2(0,L)} \leq O(1)e^{-\left(\frac{\pi^2}{L^2} + \gamma_0\right)\frac{t}{\varepsilon}} \|\bar{v}_0\|_{L^2(0,L)}.$$

For the second one, it is enough to remember that  $\alpha(\varepsilon) = \alpha_0 + O(\varepsilon)$ . Therefore using Theorem 6 we deduce

$$\|(\alpha(\varepsilon) - \alpha_0)\beta_0 T_\varepsilon F_{\gamma_0}(T_0^{-1}\bar{w})\|_{L^2(0,L)} \leq O(\varepsilon)e^{-\mu t} \|\bar{w}_0\|_{L^2(0,L)},$$

where  $\mu$  is the decay rate of (11) with (13). Finally, by Proposition 13 we get

$$\|\alpha_0\beta_0 [T_\varepsilon - T_0] F_{\gamma_0}(T_0^{-1}\bar{w})\|_{L^2(0,L)} \leq O(\varepsilon)e^{-\mu t} \|\bar{w}_0\|_{L^2(0,L)}.$$

Joining all these estimates we conclude (29). For the estimate on  $g$ , observe that using (4) and Proposition 13 we see

$$\begin{aligned}\|g\|_{L^2(0,L)} &= \|\beta(\varepsilon)T_\varepsilon^{-1}\bar{w} - \beta_0T_0^{-1}\bar{w}\|_{L^2(0,L)} \\ &\leq \|(\beta(\varepsilon) - \beta_0)T_\varepsilon^{-1}\bar{w}\|_{L^2(0,L)} + \|\beta_0[T_\varepsilon^{-1} - T_0^{-1}]\bar{w}\|_{L^2(0,L)} \\ &\leq O(\varepsilon)e^{-\mu t}\|\bar{w}_0\|_{L^2(0,L)}.\end{aligned}$$

That finishes the proof of Proposition 10.  $\square$

We state the following proposition before proving Theorem 9.

**Proposition 11.** *Let  $\|\xi\|_{L^2(0,L)}$  be  $O(\varepsilon)$ . Then, we have that  $\|u_\varepsilon(\cdot, t) - \bar{u}(\cdot, t)\|_{L^2(0,L)}$  is  $O(\varepsilon)$ .*

*Proof.* Note that

$$\|u_\varepsilon(\cdot, t) - \bar{u}(\cdot, t)\|_{L^2(0,L)} = \|T_\varepsilon^{-1}w(\cdot, t) - T_0^{-1}\bar{w}(\cdot, t)\| \leq \|[T_\varepsilon^{-1} - T_0^{-1}]w(\cdot, t) - T_0^{-1}\xi\|_{L^2(0,L)}.$$

The proof follows by the fact that  $\|\xi\|_{L^2(0,L)}$  and  $\|T_\varepsilon^{-1} - T_0^{-1}\|_{\mathcal{L}(L^2(0,L))}$  are  $O(\varepsilon)$ .  $\square$

Now we are able to give the proof of Theorem 9.

*Proof of Theorem 9.* Let us consider the following energy for our system

$$E_\varepsilon(\xi, \zeta) = \frac{1}{2} \int_0^L |\xi|^2 + \varepsilon|\zeta|^2.$$

Taking the time derivative of  $E_\varepsilon$  along the strong solutions to (26) we get

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(\xi, \zeta) &\leq - \left( \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{|\alpha(\varepsilon)\beta(\varepsilon)|\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\bar{\delta}} \right) \|\xi\|^2 \\ &\quad - \left( \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{\bar{\delta}}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right) \right) \|\zeta\|^2 + \int_0^1 (O(\varepsilon)\bar{w} + f(\bar{v}, \bar{w}, \varepsilon))\xi dx \\ &\quad + \int_0^1 (O(\varepsilon)(\beta_0F_{\gamma_0}(T_0^{-1}\bar{w}_t) + \bar{v}) + g(\bar{w}, \varepsilon))\zeta dx.\end{aligned}$$

Using Cauchy-Schwarz inequality we get

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(\xi, \zeta) &\leq -\frac{1}{2} \left( \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{2|\alpha(\varepsilon)\beta(\varepsilon)|\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\bar{\delta}} \right) \|\xi\|^2 \\ &\quad - \frac{1}{2} \left( \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{2\bar{\delta}}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right) \right) \|\zeta\|^2 + \frac{4}{\tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2}} \left( O(\varepsilon^2)\|\bar{w}\|_{L^2(0,L)}^2 + \|f\|_{L^2(0,L)}^2 \right) \\ &\quad + O(\varepsilon^2) \frac{4}{\gamma(\varepsilon) + \frac{\pi^2}{L^2}} \left( \|\bar{w}_t\|_{L^2(0,L)}^2 + \|\bar{v}\|_{L^2(0,L)}^2 \right) + \frac{4}{\gamma(\varepsilon) + \frac{\pi^2}{L^2}} \|g\|_{L^2(0,L)}^2.\end{aligned}$$

We recall that  $\mu$  defined by (14) is the decay rate of (11). Using Theorem 6, Corollary 7 and Proposition 10 we get

$$\begin{aligned} & \frac{d}{dt} E_\varepsilon(\xi, \zeta) \\ & \leq -\frac{1}{2} \left( \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{2|\alpha(\varepsilon)\beta(\varepsilon)| \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\bar{\delta}} \right) \|\xi\|^2 \\ & - \frac{1}{2} \left( \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{2\bar{\delta}}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right) \right) \|\zeta\|^2 + O(\varepsilon^2)e^{-2\mu t} + O(1)e^{-2\left(\frac{\pi^2}{L^2} + \gamma_0\right)\frac{t}{\varepsilon}} \|\bar{v}_0\|_{L^2(0,L)}^2. \end{aligned}$$

Let us choose  $\bar{\delta} > 0$  as

$$\bar{\delta} \leq \frac{1}{2} \left( \frac{\pi^2}{L^2} + \gamma(\varepsilon) \right) \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right)^{-1}, \quad (31)$$

which can be chosen such that  $\bar{\delta} \in (0, 1]$ . This follows the same argument as in Remark 8. On the other hand, we ask  $\tilde{\lambda}(\varepsilon)$  to fulfill

$$\tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} > \frac{2}{\bar{\delta}} |\alpha(\varepsilon)\beta(\varepsilon)| \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}). \quad (32)$$

Satisfying conditions (31) and (32) yields that  $\tilde{\lambda}(\varepsilon)$  and  $\gamma(\varepsilon)$  still satisfies (18) and (19). Thus, guaranteeing the exponential stability of (15) with the controller designed in Section 3. Let us define

$$\bar{v}_1(\varepsilon) = \tilde{\lambda}(\varepsilon) + \frac{\pi^2}{L^2} - \frac{2|\alpha(\varepsilon)\beta(\varepsilon)| \|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})})}{\bar{\delta}},$$

and

$$\bar{v}_2(\varepsilon) = \gamma(\varepsilon) + \frac{\pi^2}{L^2} - \frac{2\bar{\delta}}{\|F_{\gamma(\varepsilon)}\|_{\mathcal{L}(L^2(0,L))}} \left( a(\varepsilon) + \frac{1}{a(\varepsilon)} \right).$$

Choosing  $\varepsilon \in \left( 0, \min \left\{ \frac{\nu_2(\varepsilon)}{\nu_1(\varepsilon)}, \frac{\bar{v}_2(\varepsilon)}{\bar{v}_1(\varepsilon)} \right\} \right]$  we have that

$$\frac{d}{dt} E_\varepsilon(\xi, \zeta) \leq -\bar{v}_1(\varepsilon) E_\varepsilon(\xi, \zeta) + O(\varepsilon^2)e^{-2\mu t} + O(1)e^{-2\left(\frac{\pi^2}{L^2} + \gamma_0\right)\frac{t}{\varepsilon}} \|\bar{v}_0\|_{L^2(0,L)}^2.$$

In order to apply (14) we need to know the sign of  $\bar{v}_1(\varepsilon) - 2\mu$ . To this end, notice that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \bar{v}_1(\varepsilon) - 2\mu \\ & = - \left( \tilde{\lambda}_0 + \frac{\pi^2}{L^2} - 2\frac{\bar{\delta} - 1}{\bar{\delta}} |\alpha_0\beta_0| \|F_{\gamma_0}\|_{\mathcal{L}(L^2(0,L))} (1 + \|k(\cdot, \cdot)\|_{L^2(\mathcal{T})}) (1 + \|l(\cdot, \cdot)\|_{L^2(\mathcal{T})}) \right). \end{aligned}$$

For  $\bar{\delta} \in (0, 1]$  we have that the right-hand side is negative. Therefore, we have that  $\bar{v}_1(\varepsilon) - 2\mu < 0$  for small enough  $\varepsilon$ . Thus, using Lemma 14 and Corollary 15 we get the following estimate

$$E_\varepsilon(\xi(\cdot, t), \zeta(\cdot, t)) \leq e^{-\bar{v}_1(\varepsilon)t} (E_\varepsilon(\xi(\cdot, 0), \zeta(\cdot, 0)) + O(\varepsilon^2) + O(\varepsilon)\|\bar{v}_0\|_{L^2(0,L)}).$$

By Proposition 11 together with (20) we conclude (21). The control convergence is trivial in this case. In



fact, let us note that

$$|h(t, \varepsilon) - h_s(t)| \leq \|k(L, \cdot; \varepsilon) - k(L, \cdot; 0)\|_{L^2(0,L)} \|u(\cdot, t)\|_{L^2(0,L)} + \|k(L, \cdot; 0)\|_{L^2(0,L)} \|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^2(0,L)}.$$

As in the proof of Proposition 13 and the approximation for the slow variable we get the desired convergence on the controls.  $\square$

## 5 Numerical simulations

In this section we want to give some illustrations of Theorem 1, Theorem 2 and Theorem 9. All the simulations used the value  $L = 1$  for the spatial domain and

$$u_0 = x(1 - x), \quad v_0 = -10(1 - x) \sin(x),$$

as initial condition. The simulation code is provided <sup>1</sup>. First to illustrate Theorem 1 we used the following set of parameters

$$\alpha(\varepsilon) = 2, \quad \beta(\varepsilon) = 2.5, \quad \gamma(\varepsilon) = 0.1, \quad \text{and } \lambda(\varepsilon) = 3. \quad (33)$$

Under this choice of parameters we can check that all the eigenvalues of (7) and (8) have negative real part. Thus, by Theorem 1 we know that (1) is exponentially stable for  $\varepsilon > 0$  small enough. This is illustrated as follows

Solution of the PDE with  $\varepsilon = 0.05$

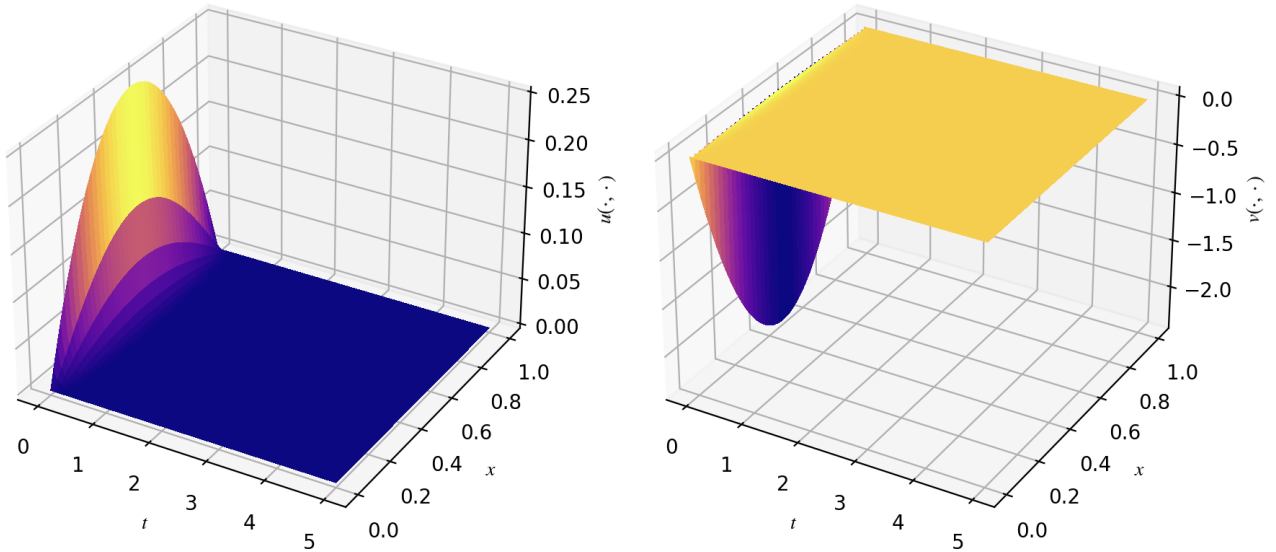


Figure 1: Solution to (1) for  $\varepsilon = 0.05$  with parameters given by (33).

<sup>1</sup>Simulations were done in Python, with center finite difference to approximate second order derivatives, and an explicit Euler scheme in time. The time step was adapted according to the value of  $\varepsilon$  and the space step was chosen in order to satisfy stability conditions for the numerical schemes. The simulation code is available in <https://github.com/5nzaolo/Boundary-stabilization-of-a-class-of-coupled-reaction-diffusion-system-with-one-control>.

On the other hand, by choosing

$$\alpha(\varepsilon) = 5, \quad \beta(\varepsilon) = 6, \quad \gamma(\varepsilon) = 0.1, \quad \text{and} \quad \lambda(\varepsilon) = -10, \quad (34)$$

we get from Theorem 1 that (1) is unstable. This is shown in the next figure

Solution of the PDE with  $\varepsilon=0.05$

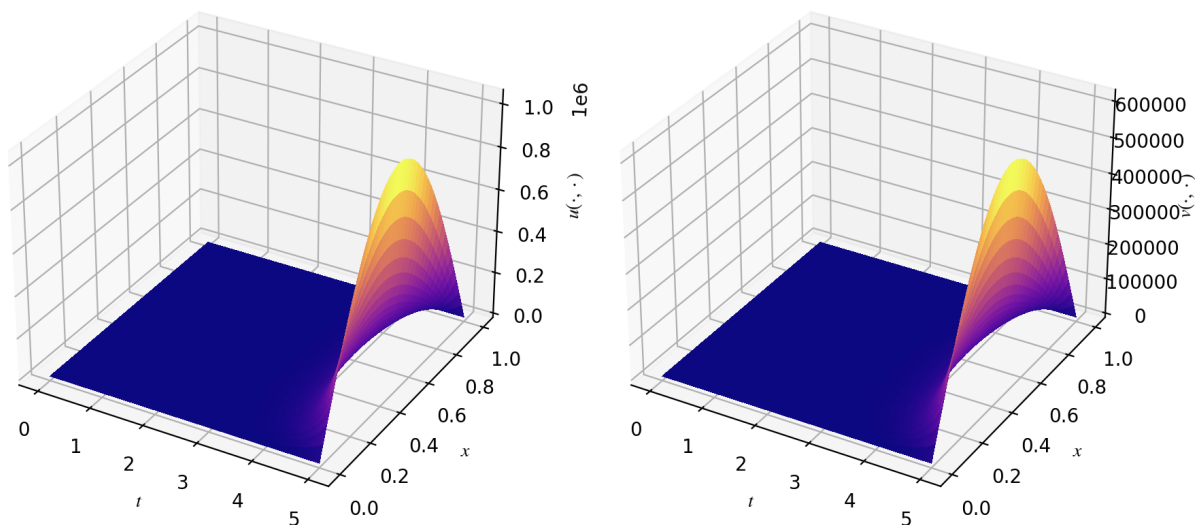


Figure 2: Solution to (1) for  $\varepsilon = 0.05$  with parameters given by (34).

However, using the control law (16) with parameter design  $\tilde{\lambda}(\varepsilon) = 60$  in (15) we achieve the exponential stabilization of the closed loop system (15) with (16) is illustrated as follows

Solution of the controlled PDE with  $\varepsilon=0.05$

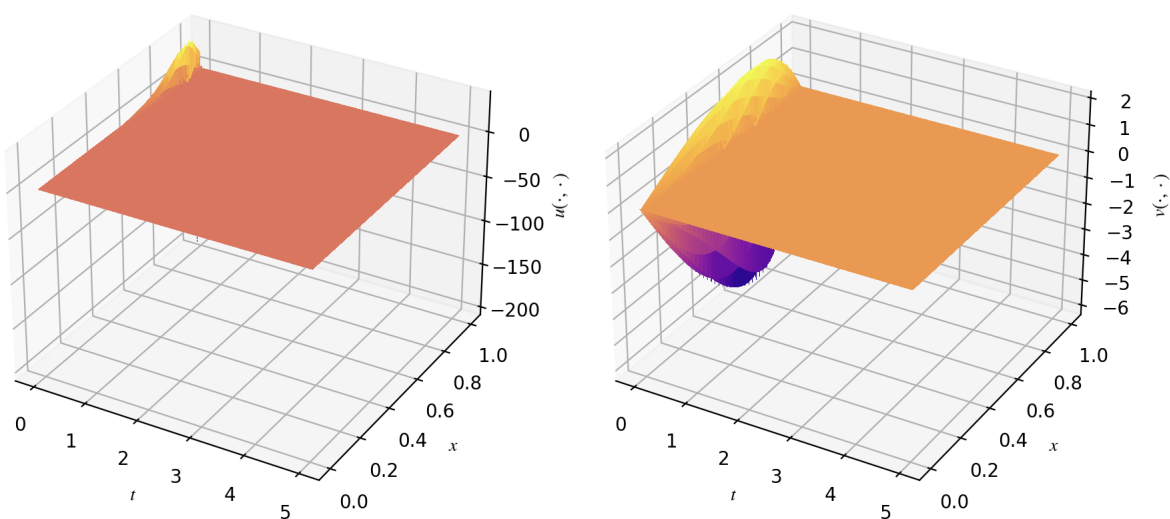


Figure 3: Solution to (15) for  $\varepsilon = 0.05$ , parameters (34) and control law (16).

Moreover, as stated in Theorem 9 we get the following behaviour of the error obtained by approximate

the solution  $(u_\varepsilon, v_\varepsilon)$  of (15) with control law (16) by  $\bar{u}$  and  $\bar{v} = \beta_0 F_{\gamma_0} \bar{u}(\cdot, t)$ . Here  $\bar{u}$  is the solution to (11) with (13). We dropped the term coming from the solution to (8), since its only important for a short period of time.

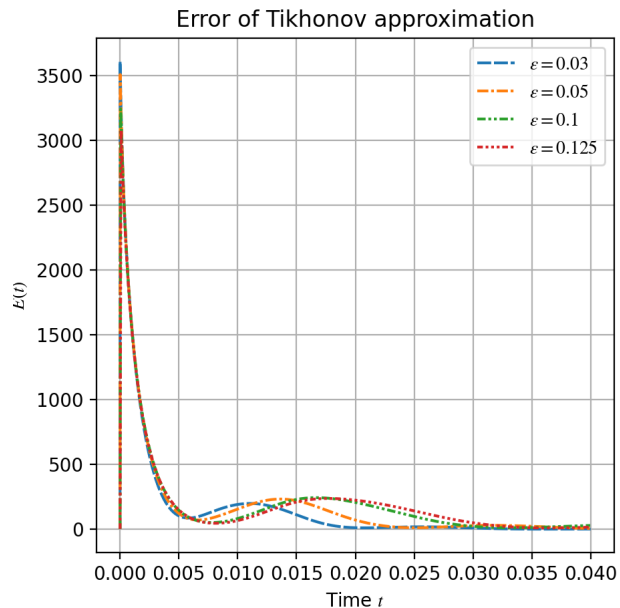


Figure 4: Sum of the errors given by Theorem 9 with varying  $\varepsilon$ .

## 6 Conclusions

In this paper we have studied the eigenvalue behaviour of a singularly perturbed  $2 \times 2$  reaction-diffusion system in terms of the singular perturbation parameter. As a consequence, we know whenever (1) is exponentially stable or not, by only looking at system parameters. Whenever (1) is unstable, we can build a feedback controller based on a suitable Backstepping transformation, which is inspired by the controller (13) designed to stabilize (11). This is achieved by using the SPM. This is the first time that a controller is built to stabilize singularly perturbed system through the SPM. Moreover, we gave an approximation result that, roughly speaking, approximates the state and control in (15) in terms of (11), (13) and (8). Numerical simulations are provided to illustrate the results obtained along the article.

## A Backstepping transformation and Gronwall type inequality

In this appendix we collect some results regarding the Backstepping transformations (9) and (10). We first define  $\mathcal{T} := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq L\}$  and  $\nu(\varepsilon) := \lambda(\varepsilon) - \tilde{\lambda}(\varepsilon)$ . So that, the kernel of (9) is given by the solution of

$$\begin{cases} k_{xx} + \nu(\varepsilon)k - k_{yy} = 0 & (x, y) \in \mathcal{T}, \\ k(x, 0) = 0 & x \in (0, L), \\ k(x, x) = \frac{\nu(\varepsilon)}{2}x & x \in (0, L), \end{cases} \quad (35)$$

and the kernel of (10) is the solution to

$$\begin{cases} l_{xx} - \nu(\varepsilon)l - l_{yy} = 0 & (x, y) \in \mathcal{T}, \\ l(x, 0) = 0 & x \in (0, L), \\ l(x, x) = \frac{\nu(\varepsilon)}{2}x & x \in (0, L). \end{cases} \quad (36)$$

As it was shown in [17, Chapter 4], the solutions to (35) and (36) are given respectively by

$$k(x, y, \varepsilon) = \nu(\varepsilon)y \frac{I_1(\sqrt{\nu(\varepsilon)(y^2 - x^2)})}{\sqrt{\nu(\varepsilon)(y^2 - x^2)}}, \quad l(x, y, \varepsilon) = \nu(\varepsilon)y \frac{J_1(\sqrt{\nu(\varepsilon)(y^2 - x^2)})}{\sqrt{\nu(\varepsilon)(y^2 - x^2)}}, \quad (37)$$

where  $I_1, J_1$  are Bessel Functions given by

$$I_1 = \sum_{m=0}^{\infty} \frac{(x/2)^{1+2m}}{m!(m+1)!}, \quad J_1 = - \sum_{m=0}^{\infty} \frac{(x/2)^{1+2m}}{m!(m+1)!}$$

We have the following bound for the  $L^2(\mathcal{T})$ -norm of the kernels

**Proposition 12.** *The  $L^2(\mathcal{T})$ -norm of the kernels (37)*

$$\begin{aligned} \|k(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})} &\leq \sqrt{\frac{\nu(\varepsilon)\pi}{8}} \left( \operatorname{erf}\left(\sqrt{\frac{\nu(\varepsilon)}{2}}\right) \operatorname{erfi}\left(\sqrt{\frac{\nu(\varepsilon)}{2}}\right) \right)^{1/2}, \\ \|l(\cdot, \cdot; \varepsilon)\|_{L^2(\mathcal{T})} &\leq \sqrt{\frac{\nu(\varepsilon)\pi}{8}} \left( \operatorname{erf}\left(\sqrt{\frac{\nu(\varepsilon)}{2}}\right) \operatorname{erfi}\left(\sqrt{\frac{\nu(\varepsilon)}{2}}\right) \right)^{1/2}. \end{aligned} \quad (38)$$

*Proof.* The proof is straightforward and it follows the same steps as in the proof of [1, Lemma 7].  $\square$

**Proposition 13.** *The Backstepping transformations (9) and (10) satisfies*

$$\|T_\varepsilon - T_0\|_{\mathcal{L}(L^2(0,L))}, \|T_\varepsilon^{-1} - T_0^{-1}\|_{\mathcal{L}(L^2(0,L))} = O(\varepsilon).$$

*Proof.* Note that

$$\begin{aligned} (T_\varepsilon - T_0)p &= \left( p(x) - \int_0^x k(x, y; \varepsilon)p(y)dy \right) - \left( p(x) - \int_0^x k(x, y; 0)p(y)dy \right), \\ &= - \int_0^x (k(x, y; \varepsilon) - k(x, y; 0))p(y)dy. \end{aligned}$$

so it is easy to note that

$$\|T_\varepsilon - T_0\|_{\mathcal{L}(L^2(0,L))} \leq \|k(\cdot, \cdot; \varepsilon) - k(\cdot, \cdot; 0)\|_{L^\infty(\mathcal{T})}.$$

We are going to show that  $\|T_\varepsilon - T_0\|_{\mathcal{L}(L^2(0,L))}$  provided that  $\nu(\varepsilon) = \nu_0 + O(\varepsilon)$  and  $\nu'(0) \neq 0$  with  $\nu_0 \neq 0$ .

Let us note that

$$k(x, y; \varepsilon) - k(x, y; 0) = \nu_0 y \left( \frac{I_1(\sqrt{\nu(\varepsilon)(y^2 - x^2)}}{\sqrt{\nu(\varepsilon)(y^2 - x^2)}} - \frac{I_1(\sqrt{\nu_0(y^2 - x^2)}}{\sqrt{\nu_0(y^2 - x^2)}} \right) + O(\varepsilon) y \frac{I_1(\sqrt{\nu(\varepsilon)(y^2 - x^2)}}{\sqrt{\nu(\varepsilon)(y^2 - x^2)}},$$

using an approximation of second order we have that

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{(\nu(\varepsilon)(x^2 - y^2))^m}{4^m m!(m+1)!} - \sum_{m=0}^{\infty} \frac{(\nu_0(x^2 - y^2))^m}{4^m m!(m+1)!} &= \sum_{m=1}^{\infty} \frac{(x^2 - y^2)^m}{4^m m!(m+1)!} (\nu(\varepsilon)^m - \nu_0^m), \\ &= \sum_{m=1}^{\infty} \frac{(x^2 - y^2)^m}{4^m m!(m+1)!} (m\nu_0^{m-1}\nu'(\varepsilon)\varepsilon + O(\varepsilon^2)), \end{aligned}$$

therefore  $\|T_\varepsilon - T_0\|_{\mathcal{L}(L^2(0,L))} = O(\varepsilon)$ . By following the same steps, we can conclude also that  $\|T_\varepsilon^{-1} - T_0^{-1}\|_{\mathcal{L}(L^2(0,L))} = O(\varepsilon)$ . So the statement is proved.  $\square$

**Lemma 14.** *Let  $V : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a Lyapunov function. Suppose that there exists  $\kappa > 0$  and  $\kappa \neq \eta > 0$  such that*

$$\frac{d}{ds} V(s) + \kappa V(s) \leq e^{-\eta s}, \quad \forall s \geq 0.$$

Then, we have that

$$V(t) \leq e^{-\kappa t} V(0) + C e^{-\min\{\kappa, \eta\}t}, \quad \forall t \geq 0,$$

for some constant  $C > 0$  that can be large whenever  $|\kappa - \eta|$  is small.

*Proof.* Multiplying by  $e^{\kappa s}$  and integrating over  $[0, t]$  we get

$$V(t)e^{\kappa t} - V(0) \leq \frac{1}{\kappa - \eta} \left( e^{(\kappa - \eta)t} - 1 \right).$$

If  $\kappa > \eta$ , then we have that

$$V(t) \leq e^{-\kappa t} V(0) + \frac{1}{\kappa - \eta} e^{-\eta t}.$$

On the other hand, if  $\kappa < \eta$  we get

$$V(t) \leq e^{-\kappa t} V(0) + \frac{1}{\eta - \kappa} e^{-\kappa t}.$$

This lead us to our desired result.  $\square$

**Corollary 15.** *Let  $V : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a Lyapunov function. Suppose that there exists  $\kappa > 0$  and  $\kappa \neq \eta > 0$  such that*

$$\frac{d}{ds} V(s) + \kappa V(s) \leq e^{-\eta \frac{s}{\varepsilon}}, \quad \forall s \geq 0,$$

for  $\varepsilon > 0$ . Then, we have that

$$V(t) \leq e^{-\kappa t} V(0) + O(\varepsilon) e^{-\kappa t}, \quad \forall t \geq 0,$$

for any  $\varepsilon \in (0, \eta/\kappa)$ .

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