



HAL
open science

Parametric estimation and LAN property of the birth-death-move process with mutations

Lisa Balsollier, Frédéric Lavancier

► **To cite this version:**

Lisa Balsollier, Frédéric Lavancier. Parametric estimation and LAN property of the birth-death-move process with mutations. 2024. hal-04554652

HAL Id: hal-04554652

<https://hal.science/hal-04554652>

Preprint submitted on 26 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Parametric estimation and LAN property of the birth-death-move process with mutations

Lisa Balsollier^{1,2} and Frédéric Lavancier³

¹LMJL, UMR 6629 CNRS, Nantes Université, lisa.balsollier@cnrs.fr

²SERPICO Project-Team, Centre Inria de l'Université de Rennes

³Univ Rennes, Ensai, CNRS, CREST, frederic.lavancier@ensai.fr

Abstract

A birth-death-move process with mutations is a Markov model for a system of marked particles in interaction, that move over time, with births and deaths. In addition the mark of each particle may also change, which constitutes a mutation. Assuming a parametric form for this model, we derive its likelihood expression and prove its local asymptotic normality. The efficiency and asymptotic distribution of the maximum likelihood estimator, with an explicit expression of its covariance matrix, is deduced. The underlying technical assumptions are showed to be satisfied by several natural parametric specifications. As an application, we leverage this model to analyse the joint dynamics of two types of proteins in a living cell, that are involved in the exocytosis process. Our approach enables to quantify the so-called colocalization phenomenon, answering an important question in microbiology.

Keywords: Likelihood estimation, Jump move process, Colocalization, Microbiology.

1 Introduction

We address the parametric inference of birth-death-move processes with mutations, in short BDMM. These processes model the dynamics of a system

of particles in \mathbb{R}^d that move through time, while some new particles may appear and existing ones may disappear. In addition, each particle may be marked by a label, that can change over time, which we call a mutation. This type of dynamics is observed in epidemiology [21], in ecology [26, 24] and in microbiology [15, 2]. For example, the application we will consider later concerns the dynamics of proteins in a living cell, involved in the exocytosis process and observed near the plasma membrane of the cell: these proteins move within the cell, and for biological and photochemical reasons, some of them disappear while others appear. Moreover the motion of each protein can be of three types, the regime of which constitutes its label, and this regime may change over time, in line with what we call a mutation.

Formally, a BDMM process is defined through several characteristics: first, the intensity of births, of deaths, and of mutations, that are functions of the particle configurations and rule the waiting time before the next birth (or death, or mutation, respectively); second, the transition kernels that specify how a birth (or a death or a mutation) occurs when it happens; third, a continuous Markov process that drives the motion of the system of particles between two jumps, whether this jump is a birth, a death, or a mutation. Assuming a parametric form for each of these characteristics, we are interested by their inference given a single realisation of the process on a finite time interval. In this contribution, we address this question by maximum likelihood and we provide theoretical guarantees by proving the local asymptotic normality (LAN) property of the model, when the time interval increases, under some regularity conditions. The latter implies the efficiency and asymptotic normality of the maximum likelihood estimator, with an explicit and estimable asymptotic covariance matrix.

Birth-death-move processes (without mutations) have been introduced in [15] and further studied in [14], with a particular focus on their ergodic properties. These processes share close similarities with Markovian particle systems with killing and jumps as studied in [18, 17], branching processes [1] and spatially structured population models [3]. The introduction of possible mutations does not change much the probabilistic properties of a birth-death-move process, as we will justify it when needed in the paper. From a statistical point of view, non-parametric estimation of their intensity functions has been considered in [15]. When there is no move and no mutation, the process boils down to a spatial birth-death process, as introduced in [25]. Parametric maximum likelihood inference of a spatial birth-death process when $d = 1$ has been considered in [22], with an applications to the analy-

sis of the displacement of dunes. The same method has been employed in [27] for particular cases of a spatial birth-death process in $d = 2$, in a view to model openings and closures of shops in Tokyo. In these references, no theoretical study is provided and by definition the particles do not move. In [17], the LAN property of the closely related Markovian model of particle systems with killing and jumps is established, under a very general, quite abstract, setting. In our contribution, we specify the likelihood expression of the general birth-death-move process with mutations and we similarly show the LAN property under mild assumptions, that we prove to be satisfied for standard parametric specifications of the model.

The article is organised as followed. In Section 2, the formal definition of the BDMM process is provided, along with some specific examples for each of its characteristics. In Section 3, we derive the likelihood expression of the model and we prove its LAN property. We in particular show that the associated technical hypotheses are satisfied for the natural examples presented in Section 2, under mild assumptions. In Section 4, a simulation study is carried out, showing the performances of the maximum likelihood estimator of some parameters of the model, along with the estimation of the associated confidence region. The setting is close to that of the dataset analysed subsequently, in order to provide certain assurances regarding the reliability of the conclusions concerning it. This dataset is a video sequence showing Langerin proteins and Rab-11 proteins in a living cell, acquired by fluorescence microscopy [4]. As detailed in Section 4.2, their dynamics are consistent with the realisation of BDMM process. Leveraging on this model and its estimation, we prove that Langerin proteins are colocalized with Rab-11 proteins, and we quantify this phenomenon, answering an important biological question. Finally, an appendix gathers the proofs of our theoretical results.

The data, the Python code concerning the simulation study and the code for processing the data are available in our online GitHub repository at <https://github.com/balsollier-lisa/Parametric-estimation-of-the-BDMM-process>.

2 Definition of the BDMM process

2.1 State space

We denote by $(\mathbf{X}_t)_{t \geq 0}$ the BDMM process. At each time $t \geq 0$, \mathbf{X}_t describes the configuration of a set of marked particles. Each particle $x \in \mathbf{X}_t$ reads $x = (z, m)$ where $z \in \Lambda \subset \mathbb{R}^{p+q}$ encodes the spatial location of the particle in \mathbb{R}^p , $p \geq 1$, along with its possible continuous mark in \mathbb{R}^q , $q \geq 0$, and where $m \in \mathbb{M}$ is a discrete mark (or *label*), \mathbb{M} being the finite set of possible labels. In the following, we set $d = p + q$, so that $\Lambda \subset \mathbb{R}^d$.

At each time t , \mathbf{X}_t represents the set of all alive particles, where the ordering does not matter and the cardinality may change over time. For this reason, for any $n \geq 1$, we introduce the natural projection

$$\Pi_n : (x_1, \dots, x_n) \in (\Lambda \times \mathbb{M})^n \mapsto \{x_1, \dots, x_n\} \quad (1)$$

that identifies two elements (x_1, \dots, x_n) and (y_1, \dots, y_n) of $(\Lambda \times \mathbb{M})^n$ if there exists a permutation π of $\{1, \dots, n\}$ such that $x_i = y_{\pi(i)}$ for any $1 \leq i \leq n$.

We set for any $n \geq 1$, $E_n = \Pi_n((\Lambda \times \mathbb{M})^n)$. The state space of \mathbf{X}_t for $t \geq 0$ is then

$$E = \bigsqcup_{n \geq 0} E_n,$$

where $E_0 = \{\emptyset\}$ consists of the empty configuration.

The cardinality of a configuration $\mathbf{x} \in E$ will be denoted by $n(\mathbf{x})$, so that $\mathbf{x} \in E_{n(\mathbf{x})}$, and we write

$$\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\} = \{(z_1, m_1), \dots, (z_{n(\mathbf{x})}, m_{n(\mathbf{x})})\},$$

where $z_i \in \Lambda$ and $m_i \in \mathbb{M}$. In the following we will sometimes assimilate $(\Lambda \times \mathbb{M})^n$ with $\Lambda^n \times \mathbb{M}^n$ for convenience, so that we may also write $\mathbf{x} = \{(\mathbf{z}, \mathbf{m})\}$ where $\mathbf{z} = (z_1, \dots, z_{n(\mathbf{x})})$ and $\mathbf{m} = (m_1, \dots, m_{n(\mathbf{x})})$. Similarly, we will write the configuration at time t of the BDMM process in either way

$$\mathbf{X}_t = \{x_{1,t}, \dots, x_{n,t}\} = \{(z_{1,t}, m_{1,t}), \dots, (z_{n,t}, m_{n,t})\} \quad \text{or} \quad \mathbf{X}_t = \{(\mathbf{Z}_t, \mathbf{m}_t)\},$$

where $n = n(\mathbf{X}_t)$. Moreover, for $\mathbf{x} \in E_n$ and $(z, m) \in \Lambda \times \mathbb{M}$, we write $\mathbf{x} \cup (z, m)$ for $\{x_1, \dots, x_n, (z, m)\} \in E_{n+1}$, and for $n \geq 1$ and $1 \leq i \leq n$, we write $\mathbf{x} \setminus x_i$ for $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} \in E_{n-1}$.

We equip E with the Borel σ -algebra \mathcal{E} and we endow E with the distance d_1 defined for \mathbf{x} and \mathbf{y} in E such that $n(\mathbf{x}) \leq n(\mathbf{y})$ by

$$d_1(\mathbf{x}, \mathbf{y}) = \frac{1}{n(\mathbf{y})} \left(\min_{\pi \in S_{n(\mathbf{y})}} \sum_{i=1}^{n(\mathbf{x})} (\|x_i - y_{\pi(i)}\| \wedge 1) + (n(\mathbf{y}) - n(\mathbf{x})) \right),$$

with $d_1(\mathbf{x}, \emptyset) = 1$ and where S_n denotes the set of permutations of $\{1, \dots, n\}$. This distance makes in particular the function $\mathbf{x} \mapsto n(\mathbf{x})$ continuous on E , and further implies some nice topological properties for (E, d_1) , see [28] and [14] for details.

2.2 Dynamics of a BDMM process

The dynamics of a BDMM process alternates continuous motions and jumps. Inter-jumps motions concern the location and/or the continuous mark of all alive particles, that move in Λ . Jumps are of three types: either a birth (when a new particle appears), or a death (when an existing particle disappears), or a mutation (when an existing particle changes its label). Accordingly, the dynamics is based on the three following elements:

1. A continuous homogeneous Markov process $(\mathbf{Y}_t)_{t \geq 0}$ on E that drives the motion of all particles of $(\mathbf{X}_t)_{t \geq 0}$ between two jumps. This process will not affect the label of the particles, neither their cardinality. Given an initial condition $\mathbf{Y}_0 = \mathbf{x}$, $(\mathbf{Y}_t)_{t \geq 0}$ will be defined as the solution of a stochastic differential equation, as described and exemplified in Section 2.3.3.
2. Three continuous non-negative functions β , δ and τ on E , that refers respectively to the birth, death and mutation intensities. These functions govern the waiting times between the jumps of $(\mathbf{X}_t)_{t \geq 0}$. Heuristically, the probability that a birth occurs in the interval $[t, t + dt]$ given that the particles are in the configuration \mathbf{X}_t à time t is $\beta(\mathbf{X}_t)dt$, and similarly for δ and τ . We will denote $\alpha = \beta + \delta + \tau$ the total jump intensity of the process, that we assume to be bounded.
3. Three transition kernels K_β , K_δ , and K_τ from $E \times \mathcal{E}$ to $[0, 1]$, that specify how a jump occurs when it happens. For example, given that a birth occurs in a configuration \mathbf{x} , $K_\beta(\mathbf{x}, A)$ is the probability that the

new configuration $\mathbf{x} \cup (z, m)$ belongs to $A \in \mathcal{E}$, where (z, m) denotes the new particle, and similarly for K_δ and K_τ .

Details and examples about these three basic characteristics of the process are provided in the next section.

Let us specifically describe the algorithmic definition of the process. This iterative construction follows [15] and [14], and may serve as a simulation procedure, see [15] for details. Let $(\mathbf{Y}_t^{(i)})_{t \geq 0}, i \geq 0$, be a sequence of processes on E , identically distributed as $(\mathbf{Y}_t)_{t \geq 0}$. Starting from the initial configuration \mathbf{X}_0 at time $T_0 = 0$, we iteratively build the process as follows.

- i) Given \mathbf{X}_0 , generate the $n(\mathbf{X}_0)$ continuous trajectories of $(\mathbf{Y}_t^{(0)})_{t \geq 0}$.
- ii) Given \mathbf{X}_0 and $(\mathbf{Y}_t^{(0)})_{t \geq 0}$, generate the first inter-jump time $T_1 - T_0$ according to the cumulative distribution function

$$F_1(t) = 1 - \exp\left(-\int_0^t \alpha(\mathbf{Y}_u^0) du\right), \quad t > 0. \quad (2)$$

The process until time T_1 is given by the generated trajectories, i.e.

$$(\mathbf{X}_t)_{T_0 \leq t < T_1} = (\mathbf{Y}_{t-T_0}^0)_{T_0 \leq t < T_1}.$$

- iii) Given T_1 and $\mathbf{X}_{T_1^-} = \mathbf{Y}_{T_1-T_0}^0$, generate the first jump:
 - it is a birth with probability $\beta(\mathbf{X}_{T_1^-})/\alpha(\mathbf{X}_{T_1^-})$, in which case $\mathbf{X}_{T_1} \sim K_\beta(\mathbf{X}_{T_1^-}, \cdot)$;
 - it is a death with probability $\delta(\mathbf{X}_{T_1^-})/\alpha(\mathbf{X}_{T_1^-})$, in which case $\mathbf{X}_{T_1} \sim K_\delta(\mathbf{X}_{T_1^-}, \cdot)$;
 - it is a mutation with probability $\tau(\mathbf{X}_{T_1^-})/\alpha(\mathbf{X}_{T_1^-})$, in which case $\mathbf{X}_{T_1} \sim K_\tau(\mathbf{X}_{T_1^-}, \cdot)$.
- iv) Return to step i) with $T_0 \leftarrow T_1$ and $\mathbf{X}_0 \leftarrow \mathbf{X}_{T_1}$ to generate the new trajectories $(\mathbf{Y}_t^{(1)})_{t \geq 0}$ starting from \mathbf{X}_{T_1} , the next jumping time T_2 , and so on.

The sequence of jumping times of the BDMM process $(\mathbf{X}_t)_{t \geq 0}$ is $(T_n)_{n \geq 1}$. For $t > 0$, we denote by N_t the number of jumps on $[0, t]$, i.e.

$$N_t = \sum_{i \geq 1} \mathbf{1}_{T_i \leq t}.$$

Similarly, we denote by N_t^β , N_t^δ and N_t^τ the number of births, deaths and mutations, respectively, before t . By our assumption that the total jump intensity α is bounded, we have that $N_t < \infty$ for any $t \geq 0$, i.e. there is no explosion of the process.

The BDMM process, as defined by the above iterative construction, is a particular case of a jump-move process, as studied in [14]. We deduce that it is a homogeneous Markov process with respect to its natural filtration $(\mathcal{F}_t)_{t \geq 0}$. We refer to the latter reference for the expression of the infinitesimal generator of the process and further probabilistic properties. In the following, we will denote by $\mathbb{P}_{\mathbf{x}}$ and $\mathbb{E}_{\mathbf{x}}$ all probabilities and expectations given that $\mathbf{X}_0 = \mathbf{x}$.

2.3 Elements of the model and examples

In this section, we describe more precisely each characteristics of the process, that are the intensity functions, the transition kernels, and the move process $(\mathbf{Y}_t)_{t \geq 0}$. We also provide various examples.

2.3.1 Intensity functions

The birth, death and mutation intensity functions are such that, given $\mathbf{X}_t = \mathbf{x}$, a birth (resp. a death and a mutation) occurs in $(t, t+h]$ with probability $\beta(\mathbf{x})h+o(h)$ (resp. $\delta(\mathbf{x})h+o(h)$ and $\tau(\mathbf{x})h+o(h)$) as $h \rightarrow 0$. These intensities can also be seen in the following way: $\beta(\mathbf{X}_{t-})$ (resp. $\delta(\mathbf{X}_{t-})$ and $\tau(\mathbf{X}_{t-})$) is the intensity of the associated counting process N_t^β (resp. N_t^δ and N_t^τ). These interpretations are consequences of the specific form (2) of the waiting time before the next jump, see [15] for details.

To set some examples, let us denote by γ any of β , δ or τ . The most simple situation is when the intensity function is a constant rate, that is $\gamma(\mathbf{x}) = \gamma$ for any $\mathbf{x} \in E$, where $\gamma > 0$. Then there is in average $\gamma\Delta$ new events that occur in any time interval of size Δ , whatever the configuration of \mathbf{X}_t is. Another typical setting is when the intensity is proportional to the cardinality, that is $\gamma(\mathbf{x}) = \gamma n(\mathbf{x})$ for $\mathbf{x} \in E$ and $\gamma > 0$. This reflects the situation where each particle has its own rate $\gamma > 0$ and the particles do not interact, so that the total rate over all particles is $\gamma n(\mathbf{x})$. These two examples are observed in the biological applications studied in [15] and [2]. More complicated examples of intensity functions are also considered in [15], where $\gamma(\mathbf{x})$ depends on the underlying Voronoï tessellation induced by \mathbf{x} .

2.3.2 Transition kernels

Denoting γ for any of β , δ or τ , we assume that for any \mathbf{x} the transition kernel $K_\gamma(\mathbf{x}, \cdot)$ admits a density $k_\gamma(\mathbf{x}, \mathbf{y})$, $\mathbf{y} \in E$, with respect to a measure $\nu_\gamma(\mathbf{x}, \cdot)$, that is for all $\mathbf{x} \in E$ and $\mathbf{y} \in E$,

$$K_\gamma(\mathbf{x}, d\mathbf{y}) = k_\gamma(\mathbf{x}, \mathbf{y})\nu_\gamma(\mathbf{x}, d\mathbf{y}).$$

We now specify k_γ and ν_γ in each case, whether the transition concerns a birth, a death or a mutation.

For a birth transition, we assume that for any $F \in \mathcal{E}$, $\nu_\beta(\mathbf{x}, F) = 0$ except if there exist $A \subset \Lambda$ and $\mathcal{I} \subset \mathbb{M}$ such that

$$F = \mathbf{x} \cup (A \times \mathcal{I}) := \{\mathbf{y} \in E, \exists (z, m) \in A \times \mathcal{I}, \mathbf{y} = \mathbf{x} \cup (z, m)\}.$$

In this case we set $\nu_\beta(\mathbf{x}, \mathbf{x} \cup (A \times \mathcal{I})) = |A| \times |\mathcal{I}|$ to be the Lebesgue measure on $\Lambda \times \mathbb{M}$. This choice of ν_β ensures that the birth transition produces the addition of exactly one particle to the configuration \mathbf{x} , as expected for a birth. For the density $k_\beta(\mathbf{x}, \mathbf{y})$, we assume that

$$k_\beta(\mathbf{x}, \mathbf{y}) = \begin{cases} p_m k_\beta^m(\mathbf{x}, z) & \text{if } \exists (z, m) \in \Lambda \times \mathbb{M}, \mathbf{y} = \mathbf{x} \cup (z, m), \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where $p_m \in [0, 1]$ is the probability that the new particle has the label $m \in \mathbb{M}$, with $\sum_{m \in \mathbb{M}} p_m = 1$, and $k_\beta^m(\mathbf{x}, \cdot)$ is the density for the location of the new particle in Λ given that its label is m .

A simple example of birth transition is the uniform law for the label and the location, which corresponds to $p_m = 1/|\mathbb{M}|$ for all $m \in \mathbb{M}$ and $k_\beta^m(\mathbf{x}, z) = 1/|\Lambda|$, for all $z \in \Lambda$ and $m \in \mathbb{M}$. More sophisticated examples are provided below.

Example 1 (Birth kernel as a mixture of normal laws). In this example, a new particle is more likely to appear close to existing particles. The birth density is a mixture of isotropic normal distributions, centred at each existing particle, with deviation $\sigma > 0$. Specifically, for any configuration $\mathbf{x} = \{(z_1, m_1), \dots, (z_{n(\mathbf{x})}, m_{n(\mathbf{x})})\}$ and any $m \in \mathbb{M}$,

$$k_\beta^m(\mathbf{x}, z) = \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \frac{1}{v(z_i, z)} \exp\left(-\frac{\|z - z_i\|^2}{2\sigma^2}\right),$$

where $v(z_i, z) = \int_{\Lambda} \exp\left(-\frac{\|z-z_i\|^2}{2\sigma^2}\right) dz$. We may easily extend this example by requiring that a particle with label m can only appear close to particles with labels m' .

Example 2 (Birth kernel driven by a potential). In this example the birth kernel is given through a function $V : E \rightarrow \mathbb{R}$ by:

$$k_{\beta}^m(\mathbf{x}, z) = \frac{1}{a^m(\mathbf{x})} e^{-V(\mathbf{x} \cup (z, m)) - V(\mathbf{x})}$$

where $a^m(\mathbf{x}) = \int_{\Lambda} e^{-V(\mathbf{x} \cup (z, m)) - V(\mathbf{x})} dz$. Given a configuration \mathbf{x} , a new particle is more likely to appear in the vicinity of points $z \in \Lambda$ that make $V(\mathbf{x} \cup (z, m)) - V(\mathbf{x})$ minimal. Following the statistical physics terminology, the function V can then be seen as a potential that the system tends to minimizing at each birth. A typical instance, for $\mathbf{x} = \{(z_1, m_1), \dots, (z_{n(\mathbf{x})}, m_{n(\mathbf{x})})\}$, is $V(\mathbf{x}) = \sum_{i \neq j} \Phi_{m_i, m_j}(z_i - z_j)$ for some pair potential functions $\Phi_{m, m'}$, $m, m' \in \mathbb{M}$, see [14] for some examples.

Concerning the mutation transition kernel, we set for any $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$, where $x_i = (z_i, m_i)$, and any $F \in \mathcal{E}$,

$$\nu_{\tau}(\mathbf{x}, F) = \sum_{i=1}^{n(\mathbf{x})} \sum_{m \in \mathbb{M}} \mathbf{1}_{(\mathbf{x} \setminus (z_i, m_i)) \cup (z_i, m) \in F},$$

so that a transition only concerns a change of label of an existing particle. For the density k_{τ} , we assume that

$$k_{\tau}(\mathbf{x}, \mathbf{y}) = \begin{cases} s(x_i, \mathbf{x}) q_m(x_i, \mathbf{x}) & \text{if } \exists x_i \in \mathbf{x}, \exists m \in \mathbb{M}, \mathbf{y} = (\mathbf{x} \setminus (z_i, m_i)) \cup (z_i, m), \\ 0 & \text{otherwise.} \end{cases}$$

In this expression, $s(x_i, \mathbf{x}) \in [0, 1]$ is the probability that the particle x_i in \mathbf{x} changes his label and $q_m(x_i, \mathbf{x})$ is the probability that, given that the particle $x_i \in \mathbf{x}$ mutates, its label changes from m_i to m . We thus have $\sum_{i=1}^{n(\mathbf{x})} s(x_i, \mathbf{x}) = 1$ and $\sum_{m \in \mathbb{M}} q_m(x_i, \mathbf{x}) = 1$ with $q_{m_i}(x_i, \mathbf{x}) = 0$. An example is provided below.

Example 3 (Mutation kernel given by a transition matrix). A natural example is when the particle to modify is uniformly chosen among

all particles, and then the change of label is done according to a transition matrix with coefficients $(p_{m,m'})_{m,m' \in \mathbb{M}}$. This corresponds to the choices

$$s(x_i, \mathbf{x}) = \frac{1}{n(\mathbf{x})} \quad \text{and} \quad q_m(x_i, \mathbf{x}) = p_{m_i, m},$$

where $p_{m,m} = 0$ and for all $m \in \mathbb{M}$, $\sum_{m' \in \mathbb{M}} p_{m,m'} = 1$.

Finally, for the death transition kernel, we set for any $\mathbf{x} = \{x_1, \dots, x_{n(\mathbf{x})}\}$ and any $F \in \mathcal{E}$, $\nu_\delta(\mathbf{x}, F) = \sum_{i=1}^{n(\mathbf{x})} \mathbf{1}_{x_i \in F}$, so that this transition only concerns the death of an existing particle. For the density,

$$k_\delta(\mathbf{x}, \mathbf{y}) = \begin{cases} \omega(x_i, x) & \text{if } \mathbf{y} = \mathbf{x} \setminus x_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega(x_i, \mathbf{x}) \in [0, 1]$ is the probability that the particle x_i in \mathbf{x} dies, with $\sum_{i=1}^{n(\mathbf{x})} \omega(x_i, \mathbf{x}) = 1$. A simple example is the uniform death where $\omega(x_i, \mathbf{x}) = 1/n(\mathbf{x})$ for any i .

2.3.3 The inter-jumps motion

Remember that between two jumps, the BDMM process $(\mathbf{X}_t)_{t \geq 0}$ has a constant cardinality $n = n(\mathbf{X}_t)$ and constant labels in \mathbb{M} . To specify the inter-jumps motion of $(\mathbf{X}_t)_{t \geq 0}$, it is then enough to define the dynamics of a process $(\mathbf{Y}_t^n)_{t \geq 0}$ on each subspace E_n where

$$\mathbf{Y}_t^n = \{(z_{1,t}, m_1), \dots, (z_{n,t}, m_n)\}.$$

Accordingly, the position and the continuous mark of each particle may move, that is $z_{i,t} \in \Lambda$ may move, but the label m_i will remain constant. To define such dynamics, we first need to come back to a standard system of n ordered (labelled) particles in $(\Lambda \times \mathbb{M})^n$, that is

$$\tilde{\mathbf{Y}}_t^n = ((z_{1,t}, m_1), \dots, (z_{n,t}, m_n)).$$

The full definition of the move process $(\mathbf{Y}_t)_{t \geq 0}$ on E then follows the three steps:

1. Define the dynamics of $\tilde{\mathbf{Y}}_t^n$ on $(\Lambda \times \mathbb{M})^n$ thanks to a system of stochastic differential equations, as (M^n) specified below, where the motion acts only in Λ^n .

2. Provided this system yields a solution whose distribution satisfies the permutation equivariance property (see below), deduce $\mathbf{Y}_t^{|n} = \Pi_n(\tilde{\mathbf{Y}}_t^{|n})$ on E_n , where Π_n is given by (1).
3. Then define $(\mathbf{Y}_t)_{t \geq 0}$ on E by $\mathbf{Y}_t = \sum_{n \geq 0} \mathbf{Y}_t^{|n} \mathbf{1}_{\{\mathbf{Y}_0 \in E_n\}}$.

This construction and the following facts are detailed in [14]. The equivariance property means that for any permutation π of $\{1, \dots, n\}$, the law of $\tilde{\mathbf{Y}}_t^{|n} = ((z_{1,t}, m_1), \dots, (z_{n,t}, m_n))$ given $\tilde{\mathbf{Y}}_0^{|n} = ((z_{\pi(1)}, m_{\pi(1)}), \dots, (z_{\pi(n)}, m_{\pi(n)}))$ is the same as the distribution of $((z_{\pi(1),t}, m_{\pi(1)}), \dots, (z_{\pi(n),t}, m_{\pi(n)}))$ given $\tilde{\mathbf{Y}}_0^{|n} = ((z_1, m_1), \dots, (z_n, m_n))$. This property says that if we rearrange the ordering of the coordinates of the initial state, the ordering of the solution is rearranged in the same way, so that it makes sense to deduce $\mathbf{Y}_t^{|n}$ from $\tilde{\mathbf{Y}}_t^{|n}$. Under this assumption, the Markov process \mathbf{Y} is well-defined on E and its transition kernel $Q_t^{\mathbf{Y}}$, defined for any bounded and measurable function f on E by $Q_t^{\mathbf{Y}} f(x) = \mathbb{E}[f(\mathbf{Y}_t) | \mathbf{Y}_0 = x]$, reads

$$Q_t^{\mathbf{Y}} f(x) = \sum_{n \geq 0} Q_t^{\tilde{\mathbf{Y}}^{|n}} (f \circ \Pi_n)((x_1, \dots, x_n)) \mathbf{1}_{x \in E_n},$$

where $Q_t^{\tilde{\mathbf{Y}}^{|n}}$ denotes the transition kernel of $\tilde{\mathbf{Y}}^{|n}$ in $(\Lambda \times \mathbb{M})^n$. Moreover, \mathbf{Y} is continuous in (E, d_1) whenever for any n , $\tilde{\mathbf{Y}}^{|n}$ is continuous in $(\Lambda \times \mathbb{M})^n$ for the usual Euclidean distance.

As a consequence of this construction, the definition of $(\mathbf{Y}_t)_{t \geq 0}$ boils down to the first step above, that is the specification of the system of SDE $(M^{|n})$, provided the latter is permutation equivariant. We assume in this paper that $\tilde{\mathbf{Y}}_t^{|n} = (\mathbf{Z}_t, \mathbf{m})$, starting at $t = 0$ from (\mathbf{z}, \mathbf{m}) where $\mathbf{z} = (z_1, \dots, z_n)$, is the solution of

$$M^{|n}(\mathbf{z}, \mathbf{m}) : \begin{cases} dz_{i,t} = v_{i,n}(\mathbf{Z}_t, \mathbf{m})dt + \sigma_{i,n}(\mathbf{Z}_t, \mathbf{m})dB_{i,t}, & t \geq 0, \quad i = 1, \dots, n, \\ z_{i,0} = z_i, & i = 1, \dots, n. \end{cases}$$

Here the drift functions $v_{1,n}, \dots, v_{n,n}$ take their values in \mathbb{R}^d , the diffusion coefficients $\sigma_{1,n}, \dots, \sigma_{n,n}$ are (d, d) invertible matrices and $(B_{1,t})_{t \geq 0}, \dots, (B_{n,t})_{t \geq 0}$ are independent standard Brownian motions on \mathbb{R}^d . We assume that the functions $v_{i,n}$ and $\sigma_{i,n}$ are globally Lipschitzian, so that a strong solution exists. If $\Lambda \neq \mathbb{R}^d$, we further assume that edge conditions (reflective or periodic) are added to ensure that the solution stays in $(\Lambda \times \mathbb{M})^n$, see for instance [7].

The permutation equivariance property is ensured if for any $i = 1, \dots, n$ and any $\pi \in S_n$,

$$\begin{cases} v_{i,n}((z_{\pi(1)}, m_{\pi(1)}), \dots, (z_{\pi(n)}, m_{\pi(n)})) = v_{\pi(i),n}((z_1, m_1), \dots, (z_n, m_n)), \\ \sigma_{i,n}((z_{\pi(1)}, m_{\pi(1)}), \dots, (z_{\pi(n)}, m_{\pi(n)})) = \sigma_{\pi(i),n}((z_1, m_1), \dots, (z_n, m_n)). \end{cases}$$

This is for instance the case if $v_{i,n}(\mathbf{Z}, \mathbf{m}) = v(z_i, m_i)$ and $\sigma_{i,n}(\mathbf{Z}, \mathbf{m}) = \sigma(z_i, m_i)$ for some functions v and σ , a situation where there is no interaction between the particles, as considered in the microbiological applications in [6] and [2]. An example that includes interactions through a potential function is provided in the following example.

Example 4 (Langevin diffusion). In this example the motion of each particle depends on an interaction force with the other particles, driven by a pair potential functions $\Phi_{m,m'}$, $m, m' \in \mathbb{M}$, as in Example 2. Specifically, in this model $\sigma_i(\mathbf{Z}, \mathbf{m}) = \sigma_{m_i}$ and

$$v_{i,n}(\mathbf{Z}, \mathbf{m}) = - \sum_{j \neq i} \Phi_{m_i, m_j}(z_i - z_j).$$

For existence, each pair potential $\Phi_{m,m'}$ must be smooth enough. We refer to [7] and [14] for details and examples.

For later purposes, let us specify the likelihood of $\mathbf{Y}_t^{|n}$ constructed as above. Let us first rewrite $(M^{|n})$ so that it takes the form of a standard stochastic differential equation. Denote by $\bar{\sigma}_n$ the block diagonal matrix of size (nd, nd) formed by the $\sigma_{i,n}$ matrices, and by \bar{v}_n the vector of size nd formed by the concatenation of the $v_{i,n}$ vectors. Then, by denoting B_t the vector formed by the concatenation of the vectors $B_{i,t}$, we have

$$M^{|n}(\mathbf{z}, \mathbf{m}) : \begin{cases} d\mathbf{Z}_t = \bar{v}_n(\mathbf{Z}_t, \mathbf{m})dt + \bar{\sigma}_n(\mathbf{Z}_t, \mathbf{m})dB_t, & t \geq 0, \\ \mathbf{Z}_0 = \mathbf{z}. \end{cases} \quad (4)$$

It is well known, cf [16, 13], that the Radon-Nikodym density of the solution $(\mathbf{Z}_t)_{t \geq 0}$ of $M^{|n}(\mathbf{z}, \mathbf{m})$ with respect to the reference process $\mathbf{U}_t = \mathbf{z} + \int_0^t \bar{\sigma}_n(\mathbf{U}_s, \mathbf{m})dB_s$, on the interval $[0, t]$ reads

$$L(\mathbf{Z}_{[0,t]}, \mathbf{m}) = \exp \left(\int_0^t \bar{v}_n(\mathbf{Z}_s, \mathbf{m})^T a_n^{-1}(\mathbf{Z}_s, \mathbf{m}) d\mathbf{Z}_s - \frac{1}{2} \int_0^t \bar{v}_n(\mathbf{Z}_s, \mathbf{m})^T a_n^{-1}(\mathbf{Z}_s, \mathbf{m}) \bar{v}_n(\mathbf{Z}_s, \mathbf{m}) ds \right), \quad (5)$$

where $a_n(\mathbf{z}, \mathbf{m}) = \bar{\sigma}_n(\mathbf{z}, \mathbf{m})\bar{\sigma}_n^T(\mathbf{z}, \mathbf{m})$. This gives us the density of the process $\tilde{\mathbf{Y}}_t^{|n} = (\mathbf{Z}_t, \mathbf{m})$ on $[0, t[$ with respect to the reference process $(\mathbf{U}_t, \mathbf{m})_{t \geq 0}$ that belongs to $(\Lambda \times \mathbb{M})^n$. Finally, by our construction above, if $(\mathbf{Z}_t, \mathbf{m})$ is such that $\mathbf{Y}_t^{|n} = \Pi_n((\mathbf{Z}_t, \mathbf{m}))$, then the density of $\mathbf{Y}_t^{|n}$ with respect to $\Pi_n((\mathbf{U}_t, \mathbf{m}))$ on $[0, t]$ takes the same form and does not depend on the ordering chosen to define \mathbf{Z}_t , \mathbf{z} and \mathbf{m} from $\mathbf{Y}_t^{|n}$, thanks to the permutation equivariance property.

3 LAN property

3.1 Likelihood of the BDMM process

As described in the previous section, the BDMM process $(\mathbf{X}_t)_{t \geq 0}$ depends on the intensity functions β , δ , τ , assumed to be continuous on E , on kernel densities for the births, the deaths and the mutations, denoted by $k_\beta(\mathbf{x}, \mathbf{y})$, $k_\delta(\mathbf{x}, \mathbf{y})$ and $k_\tau(\mathbf{x}, \mathbf{y})$, as detailed in Section 2.3.2, and on a continuous Markov diffusion model on E that drives the inter-jump motion, see Section 2.3.3.

Remember that between two jumps T_i and T_{i+1} , $\mathbf{X}_t = \mathbf{Y}_{t-T_i}^{(i)}$ where $\mathbf{Y}^{(i)}$ has the same law as \mathbf{Y} . Write $\mathbf{Y}_t^{(i)} = \{(\mathbf{Z}_t^{(i)}, \mathbf{m}^{(i)})\}$ where $\mathbf{m}^{(i)}$ does not depend on t , as required for the labels of the continuous Markov process that drives the inter-jumps motion. As assumed in Section 2.3.3, $\mathbf{Z}^{(i)}$ is the solution of $M^{|n}(\mathbf{z}^{(i)}, \mathbf{m}^{(i)})$ given by (4), where $(\mathbf{z}^{(i)}, \mathbf{m}^{(i)})$ are such that $\mathbf{X}_{T_i} = \{(\mathbf{z}^{(i)}, \mathbf{m}^{(i)})\}$ and $n = n(\mathbf{X}_{T_i})$. The likelihood of the inter-jump motion between T_i and T_{i+1} , given T_i , T_{i+1} and \mathbf{X}_{T_i} , is thus $L(\mathbf{Z}_{[0, T_{i+1}-T_i]}^{(i)}, \mathbf{m}^{(i)})$ given by (5). For simplicity we will simply write the latter $L(\mathbf{X}_{[T_i, T_{i+1}[}$ in the following, since $\mathbf{X}_t = \{(\mathbf{Z}_{t-T_i}^{(i)}, \mathbf{m}^{(i)})\}$ for all $t \in [T_i, T_{i+1}[$.

Assume that all features of the BDMM process depend on a parameter $\vartheta \in \Theta \subset \mathbb{R}^\ell$ for a given $\ell > 0$. To emphasize this dependence, we introduce the argument ϑ into the notation of each of them. For example we will write $\beta(\vartheta, \mathbf{x})$ instead of $\beta(\mathbf{x})$, $k_\beta(\vartheta, \mathbf{x}, \mathbf{y})$ instead of $k_\beta(\mathbf{x}, \mathbf{y})$, $L(\vartheta, \mathbf{X}_{[T_i, T_{i+1}[}$ instead of $L(\mathbf{X}_{[T_i, T_{i+1}[}$), and so on. Note however that in (5), ϑ only appears in the drift coefficient \bar{v}_n , i.e. $\bar{v}_n(\mathbf{X}_s) = \bar{v}_n(\vartheta, \mathbf{X}_s)$, but not in the diffusion coefficient $\bar{\sigma}_n$ which is part of the reference process. This is indeed a common fact that for continuous time observations, the diffusion coefficient can be considered to be known [13].

Let us denote the set of birth times up to time t by

$$\mathcal{B}_t = \{i \in \mathbb{N}^*, T_i \leq t \text{ and } T_i \text{ is a birth time}\},$$

and similarly \mathcal{D}_t and \mathcal{T}_t for the set of death times and mutation times up to time t .

The following theorem provides the likelihood of the BDMM process. When there is no move and no mutation, we recover the formula given in [22] for their applications to the dynamics of dunes. It also has a consistent expression with the likelihood of a system of particles with killing and jumps, as derived in [18]. The justifications, detailed in Section A, are however different and simpler, as they only rely on successive backward applications of conditional expectations.

Theorem 1. *Let $(\mathbf{X}_t)_{t \geq 0}$ be a BDMM parameterized by $\vartheta \in \Theta$, according to the above conditions, and observed in continuous time on $[0, t]$, for $t > 0$. Then, the likelihood $\mathcal{L}_t(\vartheta)$ is expressed as follows:*

$$\mathcal{L}_t(\vartheta) = \mathcal{L}_t^{\mathcal{B}}(\vartheta) \mathcal{L}_t^{\mathcal{D}}(\vartheta) \mathcal{L}_t^{\mathcal{T}}(\vartheta) \mathcal{L}_t^{\text{move}}(\vartheta),$$

with

$$\mathcal{L}_t^{\mathcal{B}}(\vartheta) = \exp\left(-\int_0^t \beta(\vartheta, \mathbf{X}_s) ds\right) \prod_{i \in \mathcal{B}_t} k_\beta(\vartheta, \mathbf{X}_{T_{i-}}, \mathbf{X}_{T_i}) \beta(\vartheta, \mathbf{X}_{T_{i-}}),$$

$$\mathcal{L}_t^{\mathcal{D}}(\vartheta) = \exp\left(-\int_0^t \delta(\vartheta, \mathbf{X}_s) ds\right) \prod_{i \in \mathcal{D}_t} k_\delta(\vartheta, \mathbf{X}_{T_{i-}}, \mathbf{X}_{T_i}) \delta(\vartheta, \mathbf{X}_{T_{i-}}),$$

$$\mathcal{L}_t^{\mathcal{T}}(\vartheta) = \exp\left(-\int_0^t \tau(\vartheta, \mathbf{X}_s) ds\right) \prod_{i \in \mathcal{T}_t} k_\tau(\vartheta, \mathbf{X}_{T_{i-}}, \mathbf{X}_{T_i}) \tau(\vartheta, \mathbf{X}_{T_{i-}}),$$

$$\mathcal{L}_t^{\text{move}}(\vartheta) = L(\vartheta, \mathbf{X}_{[T_{N_t}, t]}) \prod_{i=0}^{N_t-1} L(\vartheta, \mathbf{X}_{[T_i, T_{i+1}]}).$$

Note that due to the Markov nature of the process, the likelihood is a simple product. As a consequence, inference on a specific aspect of the process can be done independently on the other characteristics, provided the parameter is different in each characteristic. As an illustration, we will infer the birth kernel k_β in our application in Section 4, without needing to specify or estimate the other terms.

3.2 LAN

Recall that all the features of the BDMM process depend on a parameter $\vartheta \in \Theta \subset \mathbb{R}^\ell$. In the following, for a multivariate function f , we denote by $\nabla_i f$ the gradient (or Jacobian matrix if f is multivariate) with respect to the i -th variable. We will also denote by $B(\vartheta, \rho)$ the ball centred at ϑ with radius $\rho > 0$.

We list below the assumptions we make to obtain the local asymptotic normality of our model. They are of two kinds: the first one **(H)** ensures that the process is non-explosive and geometric ergodic. Conditions implying this hypothesis concern the intensity functions: it basically holds if the total intensity function α is bounded and the death intensity function δ sufficiently compensates for the birth intensity β . Precise conditions are given in Appendix C.1, following [14]. The other assumptions are regularity hypotheses on the way each feature of the BDMM process is parametrised. Although quite technical at first sight, we show in Section 3.3 that they are satisfied in many situations, including the examples of Section 2.3.

(H) The process is non-explosive, i.e. $N_t < \infty$ for any $t \geq 0$, and there exists a measure μ_∞^ϑ on E , $r > 0$ and $c : E \rightarrow \mathbb{R}_+^*$ such that for any $t > 0$, any $\mathbf{x}_0 \in E$ and any measurable and bounded function g on E ,

$$\left| \frac{1}{t} \mathbb{E}_{\mathbf{x}_0}^\vartheta \left(\int_0^t g(\mathbf{X}_s) ds \right) - \int_E g(\mathbf{y}) \mu_\infty^\vartheta(d\mathbf{y}) \right| \leq \frac{c(\mathbf{x}_0) \|g\|_\infty}{rt}.$$

Concerning the intensity function γ (whether γ is β , δ or τ), we assume that it is differentiable with respect to ϑ and the two following hypotheses.

A(γ) For all $\vartheta \in \Theta$, $\mathbf{x} \mapsto \frac{\|\nabla_1 \gamma(\vartheta, \mathbf{x})\|^2}{\gamma(\vartheta, \mathbf{x})} \mathbf{1}_{\gamma(\vartheta, \mathbf{x}) > 0}$ is bounded.

B(γ) For all $\vartheta \in \Theta$, $\exists \rho(\vartheta) > 0$ such that $\mathbf{x} \mapsto f_\vartheta^\gamma(\mathbf{x}, \rho(\vartheta)) \gamma(\vartheta, \mathbf{x})$ is bounded, where

$$f_\vartheta^\gamma(\mathbf{x}, \rho) = \sup_{u \in B(\vartheta, \rho)} \left(\|\nabla_1 \gamma(u, \mathbf{x}) - \nabla_1 \gamma(\vartheta, \mathbf{x})\|^2 \right) \frac{\mathbf{1}_{\gamma(\vartheta, \mathbf{x}) > 0}}{(\gamma(\vartheta, \mathbf{x}))^2}. \quad (6)$$

Similarly, we assume that the transition kernel k_γ is differentiable with respect to ϑ and the two following hypotheses.

A(k_γ, γ) For all $\vartheta \in \Theta$, $\mathbf{x} \mapsto \gamma(\vartheta, \mathbf{x}) \int_E \frac{\|\nabla_1 k_\gamma(\vartheta, \mathbf{x}, \mathbf{y})\|^2}{k_\gamma(\vartheta, \mathbf{x}, \mathbf{y})} \mathbf{1}_{k_\gamma(\vartheta, \mathbf{x}, \mathbf{y}) > 0} \nu_\gamma(\mathbf{x}, d\mathbf{y})$ is bounded.

B(k_γ, γ) For all $\vartheta \in \Theta$, $\exists \rho(\vartheta) > 0$ such that $\mathbf{x} \mapsto f_\vartheta^{k_\gamma}(\mathbf{x}, \rho(\vartheta)) \gamma(\vartheta, \mathbf{x})$ is bounded, where

$$f_\vartheta^{k_\gamma}(\mathbf{x}, \rho) = \int_E \sup_{u \in B(\vartheta, \rho)} (\|\nabla_1 k_\gamma(u, \mathbf{x}, \mathbf{y}) - \nabla_1 k_\gamma(\vartheta, \mathbf{x}, \mathbf{y})\|^2) \frac{\mathbf{1}_{k_\gamma(\vartheta, \mathbf{x}, \mathbf{y}) > 0}}{k_\gamma(\vartheta, \mathbf{x}, \mathbf{y})} \nu_\gamma(\mathbf{x}, d\mathbf{y}). \quad (7)$$

Finally, concerning the inter-jump motion given as a solution of (4), we assume that its drift function \bar{v}_n is differentiable with respect to ϑ and:

A_{move} For all $n \geq 1$, $\forall \vartheta \in \Theta$, $\mathbf{x} \mapsto \|\bar{\sigma}_n(\mathbf{x})^{-1} \nabla_1 \bar{v}_n(\vartheta, \mathbf{x})\|$ is bounded.

B_{move} For all $n \geq 1$, $\forall \vartheta \in \Theta$, $\exists \rho(\vartheta) > 0$ such that $\mathbf{x} \mapsto \|\bar{\sigma}_n(\mathbf{x})^{-1}\|^2 f_\vartheta^{\text{move}}(\mathbf{x}, \rho(\vartheta))$ is bounded, where

$$f_\vartheta^{\text{move}}(\mathbf{x}, \rho) = \sup_{u \in B(\vartheta, \rho)} \|\nabla_1 \bar{v}_n(u, \mathbf{x}) - \nabla_1 \bar{v}_n(\vartheta, \mathbf{x})\|^2. \quad (8)$$

In order to state the LAN property, we introduce, for any $\vartheta \in \Theta$ and $N \in \mathbb{N}^*$, the following \mathcal{F}_{tN} -martingales, along with their associated angle brackets. The fact that they are truly martingales and the expression of their brackets are justified in the proof of the following theorem deferred to Appendix B. We recall that $a_n = \bar{\sigma}_n \bar{\sigma}_n^T$.

$$M_\gamma^{N, \vartheta}(t) = \frac{1}{\sqrt{N}} \int_0^{tN} \frac{\nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} dM_s^\gamma, \quad (9)$$

$$M_{k_\gamma}^{N, \vartheta}(t) = \frac{1}{\sqrt{N}} \left(\int_0^{tN} \frac{\nabla_1 k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} dN_s^\gamma - \int_0^{tN} \int_E \nabla_1 k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) \gamma(\vartheta, \mathbf{X}_s) ds \right), \quad (10)$$

$$M_L^{N, \vartheta}(t) = \frac{1}{\sqrt{N}} \sum_{i \geq 0} \int_0^{tN} \mathbf{1}_{[T_i; T_{i+1}[}(s) (\nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_{s^-}))^T a_n^{-1}(\mathbf{X}_{s^-}) dM_{s-T_i}^{(i)}, \quad (11)$$

where $M_t^\gamma = N_t^\gamma - \int_0^t \gamma(\vartheta, \mathbf{X}_s) ds$ and $M_s^{(i)} = \left(\mathbf{Z}_s^{(i)} - \mathbf{Z}_0^{(i)} - \int_0^s \bar{v}_n(\vartheta, \mathbf{Z}_u^{(i)}) du \right) \mathbf{1}_{s \geq T_i}$ if we agree that $\mathbf{X}_t = \{(\mathbf{Z}_{t-T_i}^{(i)}, \mathbf{m}^{(i)})\}$ when $t \in [T_i, T_{i+1}[$. Their associated angle brackets are given by

$$\langle M_\gamma^{N,\vartheta} \rangle(t) = \frac{1}{N} \int_0^{tN} \mathbf{1}_{\gamma>0} \frac{(\nabla_1 \gamma)(\nabla_1 \gamma)^T}{\gamma}(\vartheta, \mathbf{X}_s) ds, \quad (12)$$

$$\langle M_{k_\gamma}^{N,\vartheta} \rangle(t) = \frac{1}{N} \int_0^{tN} \int_E \mathbf{1}_{k_\gamma>0} \frac{\nabla_1 k_\gamma (\nabla_1 k_\gamma)^T}{k_\gamma}(\vartheta, \mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) \gamma(\vartheta, \mathbf{X}_s) ds, \quad (13)$$

$$\langle M_L^{N,\vartheta} \rangle(t) = \frac{1}{N} \int_0^{tN} (\nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) ds. \quad (14)$$

Finally denoting $\mathcal{S} = \{L, \beta, \delta, \tau, k_\beta, k_\delta, k_\tau\}$, we introduce

$$M^{N,\vartheta}(t) = \sum_{i \in \mathcal{S}} M_i^{N,\vartheta}(t), \quad (15)$$

the angle brackets of which reads

$$\langle M^{N,\vartheta} \rangle(t) = \sum_{i \in \mathcal{S}} \langle M_i^{N,\vartheta} \rangle(t). \quad (16)$$

Theorem 2. *Under the same conditions as in Theorem 1, suppose that (\mathbf{H}) , $\mathbf{A}(\gamma)$, $\mathbf{B}(\gamma)$, $\mathbf{A}(k_\gamma, \gamma)$, $\mathbf{B}(k_\gamma, \gamma)$, \mathbf{A}_{move} and \mathbf{B}_{move} are satisfied (whether γ is β , δ or τ). Then for all $\vartheta \in \mathbb{R}^\ell$, $t > 0$, $h \in \mathbb{R}^\ell$ and $N \in \mathbb{N}^*$, we have the following decomposition:*

$$\log \frac{\mathcal{L}_{tN}(\vartheta + \frac{h}{\sqrt{N}})}{\mathcal{L}_{tN}(\vartheta)} = h^T M^{N,\vartheta}(t) - \frac{1}{2} h^T \langle M^{N,\vartheta} \rangle(t) h + Rem_{N,\vartheta,h}(t), \quad (17)$$

where $Rem_{N,\vartheta,h}(t) \rightarrow 0$ in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability when $N \rightarrow +\infty$, for any $\mathbf{x}_0 \in E$, and where $M^{N,\vartheta}$ and $\langle M^{N,\vartheta} \rangle$ are given by (15) and (16). Moreover, for all $t > 0$ and $\vartheta \in \Theta$,

$$(M^{N,\vartheta}(t), \langle M^{N,\vartheta} \rangle(t)) \xrightarrow{N \rightarrow +\infty} (\mathcal{N}(0, tJ(\vartheta)), tJ(\vartheta)), \quad (18)$$

where the joint convergence takes place in distribution and where the matrix

$J(\vartheta)$, of size (ℓ, ℓ) , has for expression:

$$\begin{aligned}
J(\vartheta) &= \lim_{N \rightarrow +\infty} \frac{1}{t} \langle M^{N, \vartheta} \rangle(t) \\
&= \int_E (\nabla_1 \bar{v}_n(\vartheta, \mathbf{x}))^T a_n^{-1}(\mathbf{x}) \nabla_1 \bar{v}_n(\vartheta, \mathbf{x}) \mu_\infty^\vartheta(d\mathbf{x}) \\
&\quad + \sum_{\gamma \in \{\beta, \delta, \tau\}} \int_E \mathbf{1}_{\gamma > 0} \frac{(\nabla_1 \gamma)(\nabla_1 \gamma)^T}{\gamma}(\vartheta, \mathbf{x}) \mu_\infty^\vartheta(d\mathbf{x}) \\
&\quad + \sum_{\gamma \in \{\beta, \delta, \tau\}} \int_E \int_E \mathbf{1}_{k_\gamma > 0} \frac{\nabla_1 k_\gamma (\nabla_1 k_\gamma)^T}{k_\gamma}(\vartheta, \mathbf{x}, \mathbf{y}) \nu_\gamma(\mathbf{x}, d\mathbf{y}) \gamma(\vartheta, \mathbf{x}) \mu_\infty^\vartheta(d\mathbf{x}).
\end{aligned} \tag{19}$$

Under additional regularity assumptions on the model, as detailed for instance in [9], the LAN property implies that the maximum likelihood estimator is asymptotically efficient and asymptotically Gaussian with covariance $J(\vartheta)^{-1}$. The following corollary summarise this consequence, the proof of which may be found in [9].

Corollary 1. *In addition to the setting of Theorem 2, assume that the regularity conditions N2-N4 of [9, Chapter III] are satisfied. Then the maximum likelihood estimator $\hat{\vartheta}_N$, defined by $\mathcal{L}_N(\hat{\vartheta}_N) = \sup_{\vartheta} \mathcal{L}_N(\vartheta)$, where \mathcal{L}_N is given in Theorem 1, is asymptotically efficient and satisfies*

$$\sqrt{N}(\hat{\vartheta}_N - \vartheta) \xrightarrow[N \rightarrow +\infty]{} \mathcal{N}(0, J(\vartheta)^{-1}),$$

where the convergence is in distribution and $J(\vartheta)$ is given by (19).

Note that in Theorem 2, the length of the observed trajectory is Nt , and the asymptotic regime stands for N large and any $t > 0$. We set in the above corollary $t = 1$ for simplicity and considered a trajectory of size N .

In practice, the asymptotic covariance matrix $J(\vartheta)^{-1}$ is estimated by $J_N(\hat{\vartheta}_N)^{-1}$, where $\hat{\vartheta}_N$ is the maximum likelihood estimator and J_N is the empirical version of J given by (19), viz.

$$J_N(\vartheta) = \langle M^{N, \vartheta} \rangle(1),$$

where $\langle M^{N, \vartheta} \rangle$ is given by (16) and involves the terms (12), (13) and (14). All these quantities are computable from a single trajectory over $[0, N]$, as implemented in Section 4.

3.3 Examples

We show in this section that the regularity conditions $\mathbf{A}(\gamma)$, $\mathbf{B}(\gamma)$, $\mathbf{A}(k_\gamma, \gamma)$, $\mathbf{B}(k_\gamma, \gamma)$, \mathbf{A}_{move} and \mathbf{B}_{move} are satisfied under mild assumptions for the examples considered in Section 2.3.

For the intensity functions, whether $\gamma(\mathbf{x}) = \gamma$ is constant or $\gamma(\mathbf{x}) = \gamma n(\mathbf{x})$, for $\gamma > 0$, and the parameter of interest is $\vartheta = \gamma$, we have $f_\vartheta^\gamma = 0$ in (6), so that $\mathbf{B}(\gamma)$ is obviously satisfied. For $\mathbf{A}(\gamma)$, it clearly holds in the first case, while it is true in the second case if we assume that there is a maximal number of particles, i.e. $\sup_{\mathbf{x}} n(\mathbf{x}) < n^*$ for some $n^* > 0$. Note that the latter restriction is purely theoretical, but it is not a limitation in practice. It is also an assumption made in [15] and it amounts to the specific setting of condition (46) presented in appendix, that implies geometric ergodicity of the process, i.e. **(H)**.

Concerning the transition kernels, the parameters of interest are typically involved in the birth kernel and/or the mutation kernel. For the birth kernel, remember that it reads as in (3), i.e. $k_\beta(\mathbf{x}, \mathbf{y}) = p_m k_\beta^m(\mathbf{x}, z)$ and let us examine Examples 1 and 2 for k_β^m .

Example 1 (continued). In this example k_β^m is a mixture of Gaussian distributions with deviation $\sigma > 0$, so that the parameters of interest for the birth kernel are in this case p_m , $m \in \mathbb{M}$, and σ . It is not difficult to check that if Λ is a bounded set, then all partial derivatives of $k_\beta(\mathbf{x}, \mathbf{y})$ with respect to each parameter are uniformly bounded in \mathbf{x} and \mathbf{y} . On the other hand $k_\beta(\mathbf{x}, \mathbf{y})$ itself is lower bounded in \mathbf{x} and \mathbf{y} under the same setting. Consequently, $\mathbf{A}(k_\gamma, \gamma)$ holds true whenever Λ is a bounded set and the birth intensity $\beta(\mathbf{x})$ is bounded in \mathbf{x} . The latter is in particular true if β satisfies the setting discussed in the previous paragraph. For $\mathbf{B}(k_\gamma, \gamma)$, note that a rough control consists in upper-bounding the squared norm of the difference in (7) by twice the sum of each squared norm. Then the same bounds as previously can be used to check $\mathbf{B}(k_\gamma, \gamma)$ in the same setting.

Example 2 (continued). Assume that $k_\beta^m(\mathbf{x}, z) = e^{-\vartheta'_m S_m(\mathbf{x}, z)} / a^m(\mathbf{x})$, where $\vartheta_m = (\vartheta_{m, m'})_{m' \in \mathbb{M}}$, $a^m(\mathbf{x}) = \int_\Lambda e^{-\vartheta'_m S_m(\mathbf{x}, z)} dz$ and S_m is the vector

$$S_m(\mathbf{x}, z) = \left(\sum_{\substack{x_i = (z_i, m_i) \in \mathbf{x} \\ m_i = m'}} \Phi_{m, m'}(z - z_i) \right)_{m' \in \mathbb{M}}.$$

This corresponds to a potential of the form $V(\mathbf{x}) = \sum_{i \neq j} \vartheta_{m_i, m_j} \Phi_{m_i, m_j}(z_i - z_j)$ in Example 2. The parameters of interest here are all the $\vartheta_{m, m'}$, $m, m' \in \mathbb{M}$. Assume that for all $m, m' \in \mathbb{M}$, $\Phi_{m, m'}$ is bounded by L . Then if we denote by $n_{m'}(\mathbf{x})$ the number of particles in \mathbf{x} having label m' , we have

$$\|S_m(\mathbf{x}, z)\|^2 \leq \sum_{m' \in \mathbb{M}} L^2 n_{m'}^2(\mathbf{x}) \leq L^2 n(\mathbf{x}) \sum_{m' \in \mathbb{M}} n_{m'}(\mathbf{x}) = L^2 n^2(\mathbf{x}).$$

Assume in addition that there is a maximal number of particles n^* and that Λ is a bounded set, then using the previous upper-bound, we may prove by elementary inequalities that both $\mathbf{A}(k_\gamma, \gamma)$ and $\mathbf{B}(k_\gamma, \gamma)$ are satisfied.

Concerning the mutation kernel, let us inspect Example 3.

Example 3 (continued). Remember that for this example we have

$$k_\tau(\vartheta, \mathbf{x}, \mathbf{y}) = \frac{1}{n(\mathbf{x})} p_{m_i, m}$$

if there exists $x_i \in \mathbf{x}$ and $m \in \mathbb{M}$ such that $\mathbf{y} = (\mathbf{x} \setminus (z_i, m_i)) \cup (z_i, m)$. The parameter of interest ϑ in this example correspond to the non-null entries $p_{m, m'}$ of the transition matrix. Denote by p_* the minimal value of these non-null entries. We have $\|\nabla_1 k_\tau(\vartheta, \mathbf{x}, \mathbf{y})\|^2 = 1/n(\mathbf{x})^2$, so that

$$\begin{aligned} \int_E \frac{\|\nabla_1 k_\tau(\vartheta, \mathbf{x}, \mathbf{y})\|^2}{k_\tau(\vartheta, \mathbf{x}, \mathbf{y})} \mathbf{1}_{k_\tau(\vartheta, \mathbf{x}, \mathbf{y}) > 0} \nu_\tau(\mathbf{x}, d\mathbf{y}) \\ &= \int_E \frac{\|\nabla_1 k_\tau(\vartheta, \mathbf{x}, \mathbf{y})\|^2}{k_\tau(\vartheta, \mathbf{x}, \mathbf{y})^2} k_\tau(\vartheta, \mathbf{x}, \mathbf{y}) \mathbf{1}_{k_\tau(\vartheta, \mathbf{x}, \mathbf{y}) > 0} \nu_\tau(\mathbf{x}, d\mathbf{y}) \\ &\leq \frac{1}{p_*^2} \int_E k_\tau(\vartheta, \mathbf{x}, \mathbf{y}) \nu_\tau(\mathbf{x}, d\mathbf{y}) = \frac{1}{p_*^2}. \end{aligned}$$

Consequently $\mathbf{A}(k_\gamma, \gamma)$ (where $\gamma = \tau$) is satisfied in this setting whenever $\tau(\mathbf{x})$ is bounded in \mathbf{x} , which holds true in the setting discussed in the first paragraph of this section. On the other hand, since $\nabla_1 k_\tau(\vartheta, \mathbf{x}, \mathbf{y})$ does not depend on ϑ , $f_\vartheta^{k_\tau}(\mathbf{x}, \rho) = 0$ in (7), so that $\mathbf{B}(k_\gamma, \gamma)$ trivially holds.

Concerning the inter-jump motion and assumptions \mathbf{A}_{move} and \mathbf{B}_{move} , we consider as an example the Langevin diffusion introduced in Example 4.

Example 4 (continued). Assume the same kind of parametrisation of the potential as in Example 2 above, that is $v_{i, n}(\mathbf{Z}, \mathbf{m}) = -\sum_{j \neq i} \vartheta_{m_i, m_j} \Phi_{m_i, m_j}(z_i - z_j)$ in (4), where the parameters of interest are again all the $\vartheta_{m, m'}$, $m, m' \in \mathbb{M}$.

Remember that $\sigma_i(\mathbf{Z}, \mathbf{m}) = \sigma_{m_i}$ is known. Since the drift function is linear in the parameters, it is easily seen that \mathbf{A}_{move} is satisfied if $\Phi_{m,m'}$ is bounded for any $m, m' \in \mathbb{M}$ and there is a maximal number of particles n^* . As to \mathbf{B}_{move} , we clearly have $f_{\vartheta}^{\text{move}}(\mathbf{x}, \rho) = 0$ in (8), so that this condition holds trivially true.

4 Simulations and data analysis

4.1 Simulation study

We simulate in this section several trajectories of the BDMM process, for characteristics explained below, and we evaluate the quality of estimation of a parameter involved in the birth kernel. The chosen characteristics are motivated by the application that we will conduct in the next section, and are in line with previous studies by [6] and [15]. We present them below without precisely detailing each value.

The particles do not possess continuous marks, but can be of 6 different types, which correspond on one hand to 2 distinct particle natures (playing the role of Rab-11 type proteins and Langerin type proteins in the subsequent application), and on the other hand to 3 possible motion regimes: Brownian, sub-diffusive (according to an Ornstein-Uhlenbeck process), or super-diffusive (according to a Brownian motion with linear drift). Thus, the space \mathbb{M} contains 6 different labels, but mutations will only be possible between motion regimes and not between the type of particles, see below.

We assume that the movement of each particle is independent of the others and corresponds to one of the three aforementioned regimes. Birth rates are constant and differ only according to the Langerin or Rab-11 type. Death rates (also distinct for Langerin and Rab-11) are proportional to the number of particles present. Mutations correspond to a change in movement regime: their intensity is also constant. Regarding transitions, we assume that the death kernel is uniform: each particle has the same probability of disappearing when a death event occurs. For mutations, we follow the setting of Example 3, i.e., these occur for any particle uniformly, according to a fixed transition matrix. Finally, for births, we assume that they generate an equiprobable Langerin or Rab-11 particle, with a diffusion regime respecting predetermined proportions. The Rab-11 type particles are then generated uniformly in Λ . For the Langerin type particles, we assume that given the

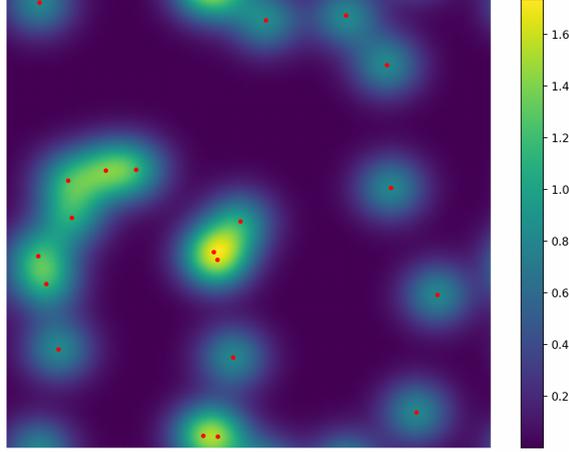


Figure 1: Example of the density (20) for the birth of a new Langerin particle, given the Rab-11 particles already present (shown as red dots).

pre-jump configuration $\mathbf{x} \in E$, they are generated according to the density $k_\beta^L(\mathbf{x}, \cdot)$ defined for all $z \in \Lambda$ by

$$k_\beta^L(\mathbf{x}, z) = \frac{p}{n_R(\mathbf{x})} \sum_{i=1}^{n_R(\mathbf{x})} \frac{1}{2\pi\sigma^2} \exp\left(\frac{-\|z - z_i^R\|^2}{2\sigma^2}\right) + \frac{(1-p)}{|\Lambda|} \mathbf{1}_\Lambda(z), \quad (20)$$

where $n_R(\mathbf{x})$ denotes the number of existing Rab-11 particles and z_i^R their spatial coordinates. This density is parametrized by $p \in [0, 1]$ and $\sigma > 0$. It is a convex combination between a mixture of normal distributions centered around the existing Rab-11 particles, with variance σ^2 , like in Example 1, and a uniform distribution over Λ . The closer the parameter p is to 1, the more likely Langerin and Rab-11 particles will tend to be colocalized, and the smaller the standard deviation σ is, the stronger the proximity between the two types of particles will be. An illustration of this density is shown in Figure 1, for a given configuration of Rab-11 particles represented by red dots.

In line with the application of the next section, we are not interested in the estimation of all features of the process, but only of the parameter $\vartheta = (p, \log \sigma)$ from the continuous time observation of a single trajectory of the process. We performed 500 simulations of trajectories for $p = 0.2$ and $\log \sigma =$

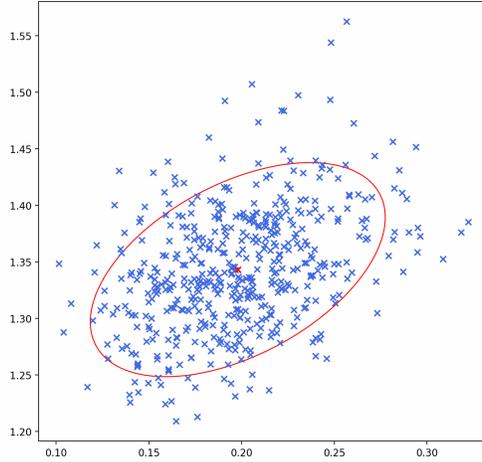


Figure 2: Estimation results for $(p, \log \sigma) = (0.2, 1.34)$ by MLE, based on 500 simulations of the BDMM process specified in Section 4.1. In red: 95% Gaussian confidence ellipsoid, based on the bivariate Gaussian distribution centred at the mean of all estimations, and with covariance matrix the mean of all estimates of the asymptotic covariance matrix $J^{-1}(\vartheta)$.

1.34, on a time interval similar to the data studied in Section 4.2. Figure 2 represents the set of all 500 maximum likelihood estimates of ϑ obtained for all 500 simulations, deduced from the likelihood established in Theorem 1. In red, the mean of the 95% confidence ellipsoid is plotted, more precisely it is the Gaussian confidence ellipsoid obtained from the bivariate Gaussian distribution centred at the mean of the estimates, and with covariance the mean of the 500 covariance matrices $J_N^{-1}(\hat{\vartheta}_N)$, obtained as explained at the end of Section 3.2. One can observe that most of the estimates provide a consistent value of the parameter and that the mean estimated confidence ellipsoid encodes the estimation uncertainty fairly well.

4.2 Data analysis

The dataset we consider comes from the observation by TIRF (Total Internal Reflection Fluorescence) microscopy of the intracellular traffic of some molecules near the membrane of a living cell [4]. This provides a video sequence showing two types of proteins observed simultaneously in the same

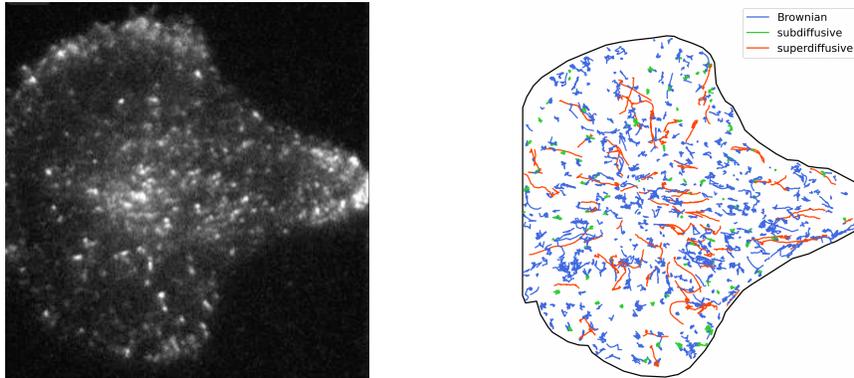


Figure 3: Left: a frame from the raw video sequence analysed in Section 4.2, showing Langerin proteins. Right: superposition of all trajectories of Langerin proteins obtained after post-processing (i.e. segmentation and tracking) of the whole raw video. The colors indicate the estimated motion regime.

cell: Langerin proteins and Rab-11 proteins. The image on the left of Figure 3 shows a frame obtained from the video of the Langerin proteins. After post-processing following [23], the proteins of interest are identified, represented by a point and tracked by the U-track algorithm [10] along the video sequence to provide trajectories, such as those visible in the right representation of Figure 3. These trajectories have been further analysed by the method developed in [5] to classify them into three diffusion regimes (these are the colors visible on the figure). The full dynamics is in line with a BDMM process: in addition to their displacement, some proteins disappear in the course of time, others appear, and finally some change their diffusion regime, which corresponds in our model to a mutation (even if it is not a mutation in the biological sense of the term).

An important biological question is to determine if Langerin proteins tend to appear close to existing Rab-11 proteins. This is the phenomenon of colocalization, indicating a strong interaction between these two types of proteins. To answer this question we estimate by maximum likelihood the birth kernel of the Langerin proteins, assuming that it is of the form (20), as in our simulation study. We obtained the estimate $(\hat{p}, \widehat{\log \sigma}) = (0.069, 1.1)$, where the unit of σ is the pixel (the image being of size 250×283 , each pixel representing an area of $160 \times 160 \text{ nm}^2$). Figure 4 shows the likelihood value for our dataset, with respect to p and $\log \sigma$, along with the empirical

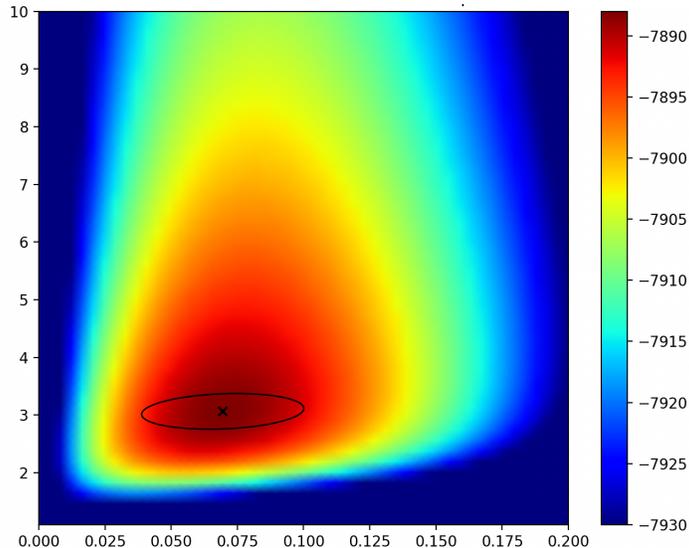


Figure 4: Likelihood of the BDMM model for the considered dataset, as a function of the parameters p and $\log \sigma$. The estimated 95% confidence ellipsoid, centred at the maximum likelihood, is overlaid in black.

95% confidence ellipsoid around the maximum, obtained from the estimation of the matrix $J^{-1}(\vartheta)$ explained at the end of Section 3.2. This estimate suggests that, for this dataset, about 7% of Langerin proteins are colocalized with Rab-11 ones, this proportion being significantly positive in view of the confidence ellipsoid, and each colocalized Langerin protein appears (with high probability) within $3\hat{\sigma} \times 160 = 528$ nm of a Rab-11 protein, where $\hat{\sigma} = \exp(\widehat{\log \sigma})$.

A Proof of Theorem 1

To simplify notation in the proof, we drop the dependence in ϑ of all terms. To get the likelihood, we view $(\mathbf{X}_t)_{t \geq 0}$ as a jump-move process, a more general Markov process than the BDMM process. A jump-move process alternates motions and jumps in E , see [14]. It is defined through an intensity function α , a jump kernel $K(\mathbf{x}, \cdot)$ and a continuous Markov process \mathbf{Y} on E that drives the motion of the process. For a BDMM, we have $\alpha(\mathbf{x}) = \beta(\mathbf{x}) + \delta(\mathbf{x}) + \tau(\mathbf{x})$

and

$$K(\mathbf{x}, \cdot) = \frac{\beta(\mathbf{x})}{\alpha(\mathbf{x})} K_\beta(\mathbf{x}, \cdot) + \frac{\delta(\mathbf{x})}{\alpha(\mathbf{x})} K_\delta(\mathbf{x}, \cdot) + \frac{\tau(\mathbf{x})}{\alpha(\mathbf{x})} K_\tau(\mathbf{x}, \cdot). \quad (21)$$

By our assumptions

$$K(\mathbf{x}, d\mathbf{y}) = k(\mathbf{x}, \mathbf{y}) \nu(\mathbf{x}, d\mathbf{y}), \quad (22)$$

where for any $F \in \mathcal{E}$, $\nu(\mathbf{x}, F) = \nu_\beta(\mathbf{x}, F)$ if there exist $A \subset \Lambda$ and $\mathcal{I} \subset \mathbb{M}$ such that $F = \mathbf{x} \cup (A \times \mathcal{I})$, $\nu(\mathbf{x}, F) = \nu_\delta(\mathbf{x}, F)$ if $\mathbf{x} \setminus x_i \in F$ for some i , and $\nu(\mathbf{x}, F) = \nu_\tau(\mathbf{x}, F)$ if there exist $m \in \mathbb{M}$ and i such that $(\mathbf{x} \setminus (z_i, m_i)) \cup (z_i, m) \in F$. In turn, $k(\mathbf{x}, \mathbf{y}) = k_\beta(\mathbf{x}, \mathbf{y}) \beta(\mathbf{x}) / \alpha(\mathbf{x})$ if there exists (z, m) such that $\mathbf{y} = \mathbf{x} \cup (z, m)$, $k(\mathbf{x}, \mathbf{y}) = k_\delta(\mathbf{x}, \mathbf{y}) \delta(\mathbf{x}) / \alpha(\mathbf{x})$ if $\mathbf{y} = \mathbf{x} \setminus x_i$ for some i , and $k(\mathbf{x}, \mathbf{y}) = k_\tau(\mathbf{x}, \mathbf{y}) \tau(\mathbf{x}) / \alpha(\mathbf{x})$ if there exist $m \in \mathbb{M}$ and i such that $\mathbf{y} = (\mathbf{x} \setminus (z_i, m_i)) \cup (z_i, m)$.

Note that a jump-move process observed continuously on $[0, t]$ has by definition a cadlag trajectory on E and can equivalently be described by the vector $\underline{\mathbf{X}}$ defined by:

$$\underline{\mathbf{X}} = \sum_{n \in \mathbb{N}} (\mathbf{X}_0, \mathbf{X}_{]0, T_1[}, T_1, \mathbf{X}_{T_1}, \mathbf{X}_{]T_1, T_2[}, \dots, T_n - T_{n-1}, \mathbf{X}_{T_n}, \mathbf{X}_{]T_n, t[}) \mathbf{1}_{\{N_t = n\}}. \quad (23)$$

We set for any $i \geq 1$, $S_i = T_i - T_{i-1}$, $\mathbf{V}_i = \mathbf{X}_{T_i}$, and we recall that $(\mathbf{X}_t)_{T_i \leq t < T_{i+1}} = (\mathbf{Y}_t^{(i)})_{0 \leq t \leq S_{i+1}}$, where $\mathbf{Y}^{(i)}$ is a continuous Markov process on E , identically distributed as \mathbf{Y} . With these notations $\underline{\mathbf{X}}$ is equivalent to $\tilde{\underline{\mathbf{X}}}$ with:

$$\tilde{\underline{\mathbf{X}}} = \sum_{n \in \mathbb{N}} (\mathbf{V}_0, \mathbf{Y}_{[0, S_1]}^{(0)}, S_1, \mathbf{V}_1, \mathbf{Y}_{[0, S_2]}^{(1)}, \dots, S_n, \mathbf{V}_n, \mathbf{Y}_{[0, t - \sum_{j=1}^n S_j]}^{(n)}) \mathbf{1}_{\{N_t = n\}}. \quad (24)$$

Note that with this formalism, $\mathbf{Y}_0^{(i)} = \mathbf{V}_i$ for all i , and $\tilde{\underline{\mathbf{X}}}$ takes values in the space $\bigcup_{n \geq 0} \mathcal{W}_n$ where

$$\mathcal{W}_n = \bigotimes_{i=0}^n (E \times \mathcal{C}_{[S_i]} \times \mathbb{R}_+) \times E \times \mathcal{C}_{[t - \sum_{i=0}^n S_i]},$$

and where $\mathcal{C}_{[s]}$ denotes the space of continuous functions from $[0, s]$ to E .

As detailed in Section 2.3.3, the continuous Markov process $\mathbf{Y}_s^n = \Pi_n((\mathbf{Z}_s, \mathbf{m}))$ defined on E_n and observed on $[0, t]$, given that $\mathbf{Z}_0 = \mathbf{z}$, admits the density $L(\mathbf{Z}_{[0, t]}, \mathbf{m})$ with respect to the reference process $\Pi_n((\mathbf{U}_t, \mathbf{m}))$ where

$\mathbf{U}_t = \mathbf{z} + \int_0^t \bar{\sigma}_n(\mathbf{U}_s, \mathbf{m}) dB_s$. Let us denote by $P_{[t]}^{\mathbf{z}}$ the distribution of this reference process observed on $[0, t]$. For any measurable and bounded function h defined on E_n , we thus have $\mathbb{E}(h(\mathbf{Y}_{[0,t]}^n)) = \int_{\mathcal{C}_{[t]}} h(\mathbf{y}) L(\mathbf{y}_{[t]}) dP_{[t]}^{\mathbf{Y}_0^n}(\mathbf{y})$, where $L(\mathbf{y}_{[t]})$ is a short notation for $L(\mathbf{z}_{[0,t]}, \mathbf{m})$ and where $\mathbf{y}_t = (\mathbf{z}_t, \mathbf{m})$ for any $t \geq 0$. Similarly, for $(\mathbf{Y}_t)_{t \geq 0}$ on E defined by $\mathbf{Y}_t = \sum_{n \geq 0} \mathbf{Y}_t^n \mathbf{1}_{\{\mathbf{Y}_0 \in E_n\}}$, we will write without ambiguity

$$\mathbb{E}(h(\mathbf{Y}_{[0,t]})) = \int_{\mathcal{C}_{[t]}} h(\mathbf{y}) L(\mathbf{y}_{[t]}) dP_{[t]}^{\mathbf{Y}_0}(\mathbf{y}),$$

where the cardinality n defining $P_{[t]}^{\mathbf{Y}_0}$ is implicitly given by $n = n(\mathbf{Y}_0)$.

Let h be a measurable and bounded function. For $n \in \mathbb{N}$, we use the conditional law of S_{n+1} given $(\mathbf{X}_t)_{t \leq T_n}$, T_n and $\mathbf{Y}^{(n)}$, see (2), to compute:

$$\begin{aligned} & \mathbb{E}[h(\tilde{\mathbf{X}}) \mathbf{1}_{\{N_t=n\}}] \\ &= \mathbb{E} \left[h(\tilde{\mathbf{X}}) \mathbf{1}_{\{S_{n+1} > t - \sum_{j=1}^n S_j\}} \mathbf{1}_{\{\sum_{j=1}^n S_j \leq t\}} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[h(\tilde{\mathbf{X}}) \mathbf{1}_{\{S_{n+1} > t - \sum_{j=1}^n S_j\}} \mathbf{1}_{\{\sum_{j=1}^n S_j \leq t\}} \mid (\mathbf{X}_t)_{t \leq T_n}, T_n, \mathbf{Y}^{(n)} \right] \right] \\ &= \mathbb{E} \left[h(\tilde{\mathbf{X}}) \mathbf{1}_{\{\sum_{j=1}^n S_j \leq t\}} \exp \left(- \int_0^{t - \sum_{j=1}^n S_j} \alpha(\mathbf{Y}_u^{(n)}) du \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[h(\tilde{\mathbf{X}}) \mathbf{1}_{\{\sum_{j=1}^n S_j \leq t\}} \exp \left(- \int_0^{t - \sum_{j=1}^n S_j} \alpha(\mathbf{Y}_u^{(n)}) du \right) \mid (\mathbf{X}_t)_{t \leq T_n}, T_n \right] \right]. \end{aligned}$$

Since given S_1, \dots, S_n and $\mathbf{Y}_0^{(n)} = \mathbf{V}_n$, the Markov process $\mathbf{Y}^{(n)}$ observed on the time interval $[0, t - \sum_{j=1}^n S_j]$ admits the density $L(\mathbf{y}_{[t - \sum_{j=1}^n S_j]})$ with respect to $P_{[t - \sum_{j=1}^n S_j]}^{\mathbf{V}_n}$, we have

$$\begin{aligned} & \mathbb{E}[h(\tilde{\mathbf{X}}) \mathbf{1}_{\{N_t=n\}}] \\ &= \mathbb{E} \left[\int_{\mathcal{C}_{[t - \sum_{i=1}^n S_j]}} h(\mathbf{V}_0, \mathbf{Y}_{[0, S_1]}^{(0)}, \dots, S_n, \mathbf{V}_n, \mathbf{y}_{[t - \sum_{i=1}^n S_j]}^{(n)}) \mathbf{1}_{\{\sum_{j=1}^n S_j \leq t\}} \right. \\ & \quad \left. \times \exp \left(- \int_0^{t - \sum_{j=1}^n S_j} \alpha(\mathbf{y}_u^{(n)}) du \right) L \left(\mathbf{y}_{[t - \sum_{i=1}^n S_j]}^{(n)} \right) dP_{[t - \sum_{i=1}^n S_j]}^{\mathbf{V}_n}(\mathbf{y}^{(n)}) \right]. \end{aligned}$$

Now, if we condition on $(\mathbf{X}_t)_{t \leq T_{n-1}}$, T_n and $\mathbf{Y}^{(n-1)}$ we know that the post-jump location \mathbf{V}_n admits the density $k(\mathbf{X}_{T_n^-}, \mathbf{v}) = k(\mathbf{Y}_{S_n}^{(n-1)}, \mathbf{v})$ with respect

to the measure $\nu(\mathbf{X}_{T_n^-}, \cdot) = \nu(\mathbf{Y}_{S_n}^{(n-1)}, \cdot)$ on E . Hence

$$\begin{aligned} & \mathbb{E}[h(\tilde{\mathbf{X}})\mathbf{1}_{\{N_t=n\}}] \\ &= \mathbb{E} \left[\int_E \int_{\mathcal{C}_{[t-\sum_{i=1}^n S_j]}} h(\mathbf{V}_0, \mathbf{Y}_{[0, S_1[}, \dots, S_n, \mathbf{v}_n, \mathbf{y}_{[t-\sum_{i=1}^n S_j]}^{(n)}) \mathbf{1}_{\{\sum_{j=1}^n S_j \leq t\}} \right. \\ & \quad \times \exp \left(- \int_0^{t-\sum_{j=1}^n S_j} \alpha(\mathbf{y}_u^{(n)}) du \right) L \left(\mathbf{y}_{[t-\sum_{i=1}^n S_j]}^{(n)} \right) dP_{[t-\sum_{i=1}^n S_j]}^{\mathbf{v}_n}(\mathbf{y}^{(n)}) \\ & \quad \left. \times k(\mathbf{Y}_{S_n}^{(n-1)}, \mathbf{v}_n) \nu(\mathbf{Y}_{S_n}^{(n-1)}, d\mathbf{v}_n) \right]. \end{aligned}$$

Conditioning on $(\mathbf{X}_t)_{t \leq T_{n-1}}$, T_{n-1} and $\mathbf{Y}^{(n-1)}$, we use again the law of S_n given by (2) to get

$$\begin{aligned} & \mathbb{E}[h(\tilde{\mathbf{X}})\mathbf{1}_{\{N_t=n\}}] \\ &= \mathbb{E} \left[\int_{\mathbb{R}_+} \int_E \int_{\mathcal{C}_{[t-\sum_{i=1}^{n-1} S_j - s_n]}} h(\mathbf{V}_0, \mathbf{Y}_{[0, S_1[}, \dots, S_n, \mathbf{v}_n, \mathbf{y}_{[t-\sum_{i=1}^{n-1} S_j - s_n]}^{(n)}) \mathbf{1}_{\{\sum_{j=1}^{n-1} S_j + s_n \leq t\}} \right. \\ & \quad \times \exp \left(- \int_0^{t-\sum_{j=1}^{n-1} S_j + s_n} \alpha(\mathbf{y}_u^{(n)}) du \right) L \left(\mathbf{y}_{[t-\sum_{i=1}^{n-1} S_j - s_n]}^{(n)} \right) dP_{[t-\sum_{i=1}^{n-1} S_j - s_n]}^{\mathbf{v}_n}(\mathbf{y}^{(n)}) \\ & \quad \left. \times k(\mathbf{Y}_{s_n}^{(n-1)}, \mathbf{v}_n) \nu(\mathbf{Y}_{s_n}^{(n-1)}, d\mathbf{v}_n) \alpha(\mathbf{Y}_{s_n}^{(n-1)}) \exp \left(- \int_0^{s_n} \alpha(\mathbf{Y}_u^{(n-1)}) du \right) ds_n \right]. \end{aligned}$$

We can continue successively this process, using the (conditional) density of each move process $\mathbf{Y}^{(i)}$ on $[0, S_i]$, of each post-jump location \mathbf{V}_i , and of each

inter-jump time S_i , and we obtain

$$\begin{aligned}
& \mathbb{E}[h(\tilde{\mathbf{X}})\mathbf{1}_{\{N_t=n\}}] \\
&= \int_{\mathcal{W}_n} h(\mathbf{v}_0, \mathbf{y}_{[s_1]}^{(0)}, \dots, s_n, \mathbf{v}_n, \mathbf{y}_{[t-\sum_{i=1}^n s_j]}^{(n)}) \mathbf{1}_{\{\sum_{j=1}^n s_j \leq t\}} \\
&\quad \times \exp\left(-\int_0^{t-\sum_{j=1}^n s_j} \alpha(\mathbf{y}_u^{(n)}) du\right) \times L(\mathbf{y}_{[t-\sum_{i=1}^n s_j]}^{(n)}) dP_{[t-\sum_{i=1}^n s_j]}^{\mathbf{v}_n}(\mathbf{y}^{(n)}) \\
&\quad \times k(\mathbf{y}_{s_n}^{(n-1)}, \mathbf{v}_n) \nu(\mathbf{y}_{s_n}^{(n-1)}, d\mathbf{v}_n) \alpha(\mathbf{y}_{s_n}^{(n-1)}) \exp\left(-\int_0^{s_n} \alpha(\mathbf{y}_u^{(n-1)}) du\right) ds_n \\
&\quad \times \prod_{i=1}^{n-1} \left[L(\mathbf{y}_{[s_{i+1}]}^{(i)}) dP_{[s_{i+1}]}^{\mathbf{v}_i}(\mathbf{y}^{(i)}) k(\mathbf{y}_{s_i}^{(i-1)}, \mathbf{v}_i) \nu(\mathbf{y}_{s_i}^{(i-1)}, d\mathbf{v}_i) \alpha(\mathbf{y}_{s_i}^{(i-1)}) \right. \\
&\quad \left. \times \exp\left(-\int_0^{s_i} \alpha(\mathbf{y}_u^{(i-1)}) du\right) ds_i \right] L(\mathbf{y}_{[s_1]}^{(0)}) dP_{[s_1]}^{\mathbf{v}_0}(\mathbf{y}^{(0)}) p(d\mathbf{v}_0),
\end{aligned}$$

where p denotes the distribution of the initial state on E . If we come back to the initial expression (23) of \mathbf{X} from that of $\tilde{\mathbf{X}}$, using the shortcut $L(\mathbf{x}_{[t_i, t_{i+1}[}$ for $L(\mathbf{y}_{[s_{i+1}]}^{(i)}) = L(\mathbf{z}_{[0, t_{i+1}-t_i]}^{(i)}, \mathbf{m}^{(i)})$ in view of $\mathbf{x}_t = \{(\mathbf{z}_{t-T_i}^{(i)}, \mathbf{m}^{(i)})\}$ for all $t \in [t_i, t_{i+1}[$, we obtain

$$\begin{aligned}
\mathbb{E}[h(\mathbf{X})\mathbf{1}_{\{N_t=n\}}] &= \int_{\mathcal{W}_n} h(\mathbf{x}_0, \mathbf{x}_{[0, t_1[}, \dots, t_n - t_{n-1}, \mathbf{x}_{t_n}, \mathbf{x}_{[t_n, t]}) \mathbf{1}_{t_1 \leq \dots \leq t_n \leq t} \\
&\quad \times \exp\left(-\int_0^t \alpha(\mathbf{x}_u) du\right) L(\mathbf{x}_{[t_n, t]}) \prod_{i=1}^n \left(L(\mathbf{x}_{[t_{i-1}, t_i[}) k(\mathbf{x}_{t_i^-}, \mathbf{x}_{t_i}) \alpha(\mathbf{x}_{t_i^-}) \right) \eta^{(n)}(d\mathbf{x}),
\end{aligned}$$

where $\eta^{(n)}$ is the measure given for $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_{[0, t_1[}, \dots, t_n - t_{n-1}, \mathbf{x}_{t_n}, \mathbf{x}_{[t_n, t]})$ by

$$\begin{aligned}
\eta^{(n)}(d\mathbf{x}) &= \mathbf{1}_{t_1 \leq \dots \leq t_n \leq t} dP_{[t_n, t]}^{\mathbf{x}_{t_n}}(\mathbf{x}) \nu(\mathbf{x}_{t_n^-}, d\mathbf{x}_{t_n}) dt_n \\
&\quad \times \prod_{i=1}^{n-1} \left(dP_{[t_i, t_{i+1}[}^{\mathbf{x}_{t_i}}(\mathbf{x}) \nu(\mathbf{x}_{t_i^-}, \mathbf{x}_{t_i}) dt_i \right) dP_{[0, t_1[}^{\mathbf{x}_0}(\mathbf{x}) p(d\mathbf{x}_0).
\end{aligned}$$

Here, just as we have used the natural notation $L(\mathbf{x}_{[t_i, t_{i+1}[}$ for $L(\mathbf{y}_{[s_{i+1}]}^{(i)})$, we use $dP_{[t_i, t_{i+1}[}^{\mathbf{x}_{t_i}}(\mathbf{x})$ for $dP_{[s_{i+1}]}^{\mathbf{x}_{t_i}}(\mathbf{y}^{(i)})$.

Since for any h , $\mathbb{E}(h(\mathbf{X})) = \sum_{n \geq 0} \mathbb{E}(h(\mathbf{X})\mathbf{1}_{\{N_t=n\}})$, we have proven that the likelihood of \mathbf{X} given by (23), with respect to the underlying measure

$\eta(\cdot) = \sum_{n \in \mathbb{N}} \eta^{(n)}(\cdot \cap \mathcal{W}_n)$ is given by

$$\mathcal{L}(\underline{\mathbf{X}}) = \exp\left(-\int_0^t \alpha(\mathbf{X}_u) du\right) L(\mathbf{X}_{[t_{N_t}, t]}) \prod_{i=1}^{N_t} \left(L(\mathbf{X}_{[T_{i-1}, T_i]}) k(\mathbf{X}_{T_i^-}, \mathbf{X}_{T_i}) \alpha(\mathbf{X}_{T_i^-})\right).$$

If we replace α and k by their specific expression in (22), we obtain the result in the case of a BDMM process. Given that there is a one to one correspondance between the representation (23) and the trajectory $(\mathbf{X}_s)_{0 \leq s \leq t}$, we can view the above likelihood as the likelihood of $(\mathbf{X}_s)_{0 \leq s \leq t}$, where η translates to a measure on the space of cadlag function from $[0, t]$ to E .

B Proof of Theorem 2

Let us fix $N \geq 1$ and $h \in \mathbb{R}^\ell$. We let $\vartheta_N = \vartheta + h/\sqrt{N}$ and we denote

$$\Lambda^{\vartheta_N/\vartheta}(t) = \log \frac{\mathcal{L}_t(\vartheta + \frac{h}{\sqrt{N}})}{\mathcal{L}_t(\vartheta)}.$$

By Theorem 1 the log-likelihood ratio process is given by:

$$\begin{aligned} \Lambda^{\vartheta_N/\vartheta}(t) = & \Lambda_L^{\vartheta_N/\vartheta}(t) + \Lambda_\beta^{\vartheta_N/\vartheta}(t) + \Lambda_\delta^{\vartheta_N/\vartheta}(t) + \Lambda_\tau^{\vartheta_N/\vartheta}(t) \\ & + \Lambda_{k_\beta}^{\vartheta_N/\vartheta}(t) + \Lambda_{k_\delta}^{\vartheta_N/\vartheta}(t) + \Lambda_{k_\tau}^{\vartheta_N/\vartheta}(t), \end{aligned} \quad (25)$$

where

$$\Lambda_\gamma^{\vartheta_N/\vartheta}(t) = \int_0^t \log \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} \right) dN_s^\gamma - \int_0^t (\gamma(\vartheta_N, \mathbf{X}_s) - \gamma(\vartheta, \mathbf{X}_s)) ds, \quad (26)$$

$$\Lambda_{k_\gamma}^{\vartheta_N/\vartheta}(t) = \int_0^t \log \left(\frac{k_\gamma(\vartheta_N, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} \right) dN_s^\gamma, \quad (27)$$

$$\Lambda_L^{\vartheta_N/\vartheta}(t) = \log \frac{L(\vartheta_N, \mathbf{X}_{[T_{N_t}, t]})}{L(\vartheta, \mathbf{X}_{[T_{N_t}, t]})} + \sum_{i=0}^{N_t-1} \log \frac{L(\vartheta_N, \mathbf{X}_{[T_i, T_{i+1}])}}{L(\vartheta, \mathbf{X}_{[T_i, T_{i+1}])}. \quad (28)$$

The proof of Theorem 2 follows a standard scheme as in [19] and [17]. The main step consists in proving the LAN decomposition for each of the three terms (26), (27) and (28) above, which is the statement of the three following lemmas. Then (17) is deduced by combining these decompositions, as carried out next.

Lemma 1. *Under the same assumptions as in Theorem 2, we have for any $t > 0$, $h \in \mathbb{R}^\ell$ and $\vartheta \in \Theta$,*

$$\Lambda_\gamma^{\vartheta N/\vartheta}(tN) = h^T M_\gamma^{N,\vartheta}(t) - \frac{1}{2} h^T \langle M_\gamma^{N,\vartheta} \rangle(t) h + \text{Rem}_{N,\vartheta,h}^\gamma(t), \quad (29)$$

where $\text{Rem}_{N,\vartheta,h}^\gamma(t)$ tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability when $N \rightarrow +\infty$, and $(M_\gamma^{N,\vartheta}(t))_{t>0}$ is a martingale with respect to \mathcal{F}_{tN} given by (9).

Lemma 2. *Under the same assumptions as in Theorem 2, we have for any $t > 0$, $h \in \mathbb{R}^\ell$ and $\vartheta \in \Theta$,*

$$\Lambda_{k_\gamma}^{\vartheta N/\vartheta}(tN) = h^T M_{k_\gamma}^{N,\vartheta}(t) - \frac{1}{2} h^T \langle M_{k_\gamma}^{N,\vartheta} \rangle(t) h + \text{Rem}_{N,\vartheta,h}^{k_\gamma}(t), \quad (30)$$

where $\text{Rem}_{N,\vartheta,h}^{k_\gamma}(t)$ tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability when $N \rightarrow +\infty$, and $(M_{k_\gamma}^{N,\vartheta}(t))_{t>0}$ is a martingale with respect to \mathcal{F}_{tN} given by (10).

Lemma 3. *Under the same assumptions as in Theorem 2, we have for any $t > 0$, $h \in \mathbb{R}^\ell$ and $\vartheta \in \Theta$,*

$$\Lambda_L^{\vartheta N/\vartheta}(tN) = h^T M_L^{N,\vartheta}(t) - \frac{1}{2} h^T \langle M_L^{N,\vartheta} \rangle(t) h + \text{Rem}_{N,\vartheta,h}^L(t), \quad (31)$$

where $\text{Rem}_{N,\vartheta,h}^L(t)$ tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability when $N \rightarrow +\infty$, and $(M_L^{N,\vartheta}(t))_{t>0}$ is a martingale with respect to \mathcal{F}_{tN} given by (11).

The proofs of these three lemmas are postponed to the end of this section. Let us deduce (17). From their statements and (25), we have

$$\Lambda^{\vartheta N/\vartheta}(tN) = h^T M^{N,\vartheta}(t) - \frac{1}{2} h^T \sum_{i \in \mathcal{S}} \langle M_i^{N,\vartheta} \rangle(t) h + \text{Rem}_{N,\vartheta,h}(t),$$

where $\mathcal{S} = \{L, \beta, \delta, \tau, k_\beta, k_\delta, k_\tau\}$, $M^{N,\vartheta}(t) = \sum_{i \in \mathcal{S}} M_i^{N,\vartheta}$ and $\text{Rem}_{N,\vartheta,h}(t) = \sum_{i \in \mathcal{S}} \text{Rem}_{N,\vartheta,h}^i(t)$. By Lemmas 1, 2 and 3, we know that $\text{Rem}_{N,\vartheta,h}(t)$ tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability. To show the decomposition (17), it remains to prove that

$$\langle M^{N,\vartheta} \rangle(t) = \sum_{i \in \mathcal{S}} \langle M_i^{N,\vartheta} \rangle(t),$$

which, given the definition of $M^{N,\vartheta}(t)$, boils down to proving that

$$\forall i, j \in \mathcal{S}, i \neq j, \quad \langle M_i^{N,\vartheta}, M_j^{N,\vartheta} \rangle(t) = 0. \quad (32)$$

First, for $i = L$, since $M_L^{N,\vartheta}(t)$ is a continuous martingale with finite variations, we have that $\langle M_L^{N,\vartheta}, M_j^{N,\vartheta} \rangle(t) = 0$ for any $j \neq L$, see [12]. Second, for all $i, j \in \mathcal{S} \setminus \{L\}$, $i \neq j$, we have

$$[M_i^{N,\vartheta}, M_j^{N,\vartheta}](t) = \sum_{s \leq t} \Delta M_i^{N,\vartheta}(s) \Delta M_j^{N,\vartheta}(s),$$

where $\Delta M_i^{N,\vartheta}(s) = M_i^{N,\vartheta}(s) - M_i^{N,\vartheta}(s^-)$. For $\gamma \neq \gamma'$, $M_\gamma^{N,\vartheta}(t)$ and $M_{\gamma'}^{N,\vartheta}(t)$ do not have any jump in common, and similarly for $M_\gamma^{N,\vartheta}(t)$ and $M_{k_{\gamma'}}^{N,\vartheta}(t)$, since a birth, a death or a mutation cannot occur at the same moment almost surely. Consequently $[M_\gamma^{N,\vartheta}, M_{\gamma'}^{N,\vartheta}](t) = [M_\gamma^{N,\vartheta}, M_{k_\gamma}^{N,\vartheta}](t) = 0$ for any $\gamma \neq \gamma'$. This implies (32) for $i = \gamma$ and $j \in \{\gamma', k_{\gamma'}\}$, $\gamma \neq \gamma'$. To complete the proof of (32), it remains to show that $\langle M_\gamma^{N,\vartheta}, M_{k_\gamma}^{N,\vartheta} \rangle(t) = 0$. We have that

$$[M_\gamma^{N,\vartheta}, M_{k_\gamma}^{N,\vartheta}](t) = \frac{1}{N} \int_0^{tN} \frac{\nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-}) (\nabla_1 k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s))^T}{\gamma(\vartheta, \mathbf{X}_{s^-}) k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} dN_s^\gamma,$$

so that by Lemma 4, the associated angle bracket (corresponding to the compensator of the square bracket, see [12]) is given by

$$\langle M_\gamma^{N,\vartheta}, M_{k_\gamma}^{N,\vartheta} \rangle(t) = \frac{1}{N} \int_0^{tN} \nabla_1 \gamma(\vartheta, \mathbf{X}_s) \int_E (\nabla_1 k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y}))^T \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) ds.$$

Remember that for any $\vartheta \in \Theta$ and any $\mathbf{x} \in E$, $\int_E k_\gamma(\vartheta, \mathbf{x}, \mathbf{y}) \nu(\mathbf{x}, d\mathbf{y}) = 1$. This implies $\int_E \nabla_1 k_\gamma(\vartheta, \mathbf{x}, \mathbf{y}) \nu_\gamma(\mathbf{x}, d\mathbf{y}) = 0$ if interchange of integration and differentiation is valid. In particular, it can be showed that the latter holds under Condition **B**(k_γ, γ). Consequently, $\langle M_\gamma^{N,\vartheta}, M_{k_\gamma}^{N,\vartheta} \rangle(t) = 0$ and (32) is proven, which completes the proof of (17).

The last statement of Theorem 2, that is the convergence (18), is a direct consequence of Theorems 4.12 and 4.22 of [8], thanks to our hypothesis **(H)** that implies that \emptyset is a recurrent atom for $(\mathbf{X}_t)_{t>0}$.

B.1 Proof of Lemma 1

Let us prove that $Rem_{N,\vartheta,h}^\gamma(t)$ converges to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability, where

$$Rem_{N,\vartheta,h}^\gamma(t) = \Lambda_\gamma^{\vartheta N/\vartheta}(tN) - h^T M_\gamma^{N,\vartheta}(t) + \frac{1}{2} h^T \langle M_\gamma^{N,\vartheta} \rangle(t) h,$$

and $M_\gamma^{N,\vartheta}(t)$ is given by (9). From (26), we get

$$\begin{aligned}\Lambda_\gamma^{\vartheta_N/\vartheta}(tN) &= \int_0^{tN} \left[\log \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} \right) - \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} + 1 \right] dN_s^\gamma \\ &\quad + \int_0^{tN} \left[\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right] dM_s^\gamma,\end{aligned}$$

where $M_t^\gamma = N_t^\gamma - \int_0^t \gamma(\vartheta, \mathbf{X}_s) ds$ is a \mathcal{F}_t -martingale. We have in particular $[M^\gamma]_t = N_t$ and $\langle M^\gamma \rangle_t = \int_0^t \gamma(\vartheta, \mathbf{X}_s) ds$, see for instance [12]. We deduce that

$$\begin{aligned}[M_\gamma^{N,\vartheta}](t) &= \frac{1}{N} \int_0^{tN} \frac{(\nabla_1 \gamma)(\nabla_1 \gamma)^T}{\gamma^2}(\vartheta, \mathbf{X}_{s^-}) d[M^\gamma]_s \\ &= \frac{1}{N} \int_0^{tN} \frac{(\nabla_1 \gamma)(\nabla_1 \gamma)^T}{\gamma^2}(\vartheta, \mathbf{X}_{s^-}) dN_s^\gamma.\end{aligned}$$

Note that $\gamma(\vartheta, \mathbf{X}_{T_i^-}) > 0$ for all γ -jump time T_i , so that the previous formula remains true with the addition of $\mathbf{1}_{\gamma > 0}$ in the integrand. So we get

$$\langle M_\gamma^{N,\vartheta} \rangle(t) = \frac{1}{N} \int_0^{tN} \mathbf{1}_{\gamma > 0} \frac{(\nabla_1 \gamma)(\nabla_1 \gamma)^T}{\gamma}(\vartheta, \mathbf{X}_s) ds,$$

as claimed in (12), which well satisfies the property that $[M_\gamma^{N,\vartheta}](t) - \langle M_\gamma^{N,\vartheta} \rangle(t)$ is a martingale. Using these representations, we may write

$$\begin{aligned}\text{Rem}_{N,\vartheta,h}^\gamma(t) &= \Lambda_\gamma^{\vartheta_N/\vartheta}(tN) - h^T M_\gamma^{N,\vartheta}(t) + \frac{1}{2} h^T (\langle M_\gamma^{N,\vartheta} \rangle(t) - [M_\gamma^{N,\vartheta}](t)) h + \frac{1}{2} h^T [M_\gamma^{N,\vartheta}](t) h \\ &= S_t^N + A_t^N + R_t^N,\end{aligned}$$

where

$$\begin{aligned}S_t^N &= \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})} \right) dM_s^\gamma, \\ A_t^N &= \frac{1}{2N} \int_0^{tN} \mathbf{1}_{\gamma > 0} \frac{(h^T \nabla_1 \gamma)^2}{\gamma}(\vartheta, \mathbf{X}_s) ds - \frac{1}{2N} \int_0^{tN} \frac{(h^T \nabla_1 \gamma)^2}{\gamma^2}(\vartheta, \mathbf{X}_{s^-}) dN_s^\gamma, \\ R_t^N &= \int_0^{tN} \left(\log \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} \right) - \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} + 1 + \frac{1}{2N} \frac{(h^T \nabla_1 \gamma)^2}{\gamma^2}(\vartheta, \mathbf{X}_{s^-}) \right) dN_s^\gamma.\end{aligned}$$

To complete the proof, we show below that each of these three terms tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability.

First step: *Convergence of S_t^N to 0 in probability.*

By the Markov inequality, we have for any $a > 0$,

$$\mathbb{P}_{\mathbf{x}_0}^\vartheta (|S_t^N| > a) \leq \frac{1}{a^2} \mathbb{E}_{\mathbf{x}_0}^\vartheta ((S_t^N)^2) = \frac{1}{a^2} \mathbb{E}_{\mathbf{x}_0}^\vartheta (\langle S^N \rangle_t),$$

where the last equality comes from the fact that $(S_t^N)^2 - \langle S^N \rangle_t$ is by definition a centred martingale (see [12]). We prove in the following that $\mathbb{E}_{\mathbf{x}_0}^\vartheta (\langle S^N \rangle_t)$ tends to 0. We have

$$\begin{aligned} [S^N]_t &= \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})} \right)^2 dN_s^\gamma \\ &= \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})} \right)^2 \mathbf{1}_{\gamma(\vartheta, \mathbf{X}_{s^-}) > 0} dN_s^\gamma, \end{aligned}$$

so that

$$\langle S^N \rangle_t = \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_s)}{\gamma(\vartheta, \mathbf{X}_s)} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_s)}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_s)} \right)^2 \mathbf{1}_{\gamma(\vartheta, \mathbf{X}_s) > 0} \gamma(\vartheta, \mathbf{X}_s) ds. \quad (33)$$

Note that for any $\vartheta, \vartheta' \in \Theta$ and any \mathbf{x} ,

$$\gamma(\vartheta', \mathbf{x}) - \gamma(\vartheta, \mathbf{x}) = \int_0^1 (\nabla_1 \gamma(\vartheta + t(\vartheta' - \vartheta), \mathbf{x}))^T (\vartheta' - \vartheta) dt.$$

Thus we have for all $\mathbf{x} \in E$ and $\vartheta, \vartheta' \in \Theta$ such that $\gamma(\vartheta, \mathbf{x}) > 0$:

$$\begin{aligned} &\left(\frac{\gamma(\vartheta', \mathbf{x})}{\gamma(\vartheta, \mathbf{x})} - 1 - \frac{(\nabla_1 \gamma(\vartheta, \mathbf{x}))^T (\vartheta' - \vartheta)}{\gamma(\vartheta, \mathbf{x})} \right)^2 \\ &= \left(\int_0^1 \frac{(\nabla_1 \gamma(\vartheta + t(\vartheta' - \vartheta), \mathbf{x}) - \nabla_1 \gamma(\vartheta, \mathbf{x}))^T (\vartheta' - \vartheta)}{\gamma(\vartheta, \mathbf{x})} dt \right)^2 \\ &\leq \|\vartheta' - \vartheta\|^2 f_\vartheta^\gamma(\mathbf{x}, \|\vartheta' - \vartheta\|), \end{aligned} \quad (34)$$

where $f_\vartheta^\gamma(\mathbf{x}, \rho)$ is given by (6). Applying this inequality in (33) with $\vartheta' = \vartheta_N = \vartheta + h/\sqrt{N}$, we obtain

$$\langle S^N \rangle_t \leq \frac{\|h\|^2}{N} \int_0^{tN} f_\vartheta^\gamma \left(\mathbf{X}_s, \frac{\|h\|}{\sqrt{N}} \right) \gamma(\vartheta, \mathbf{X}_s) ds. \quad (35)$$

Since $f_\vartheta^\gamma(\mathbf{x}, \cdot)$ is increasing, we have by condition **B**(γ) that for any $\vartheta \in \Theta$ and N large enough,

$$\sup_{\mathbf{x} \in E} \left| f_\vartheta^\gamma \left(\mathbf{x}, \frac{\|h\|}{\sqrt{N}} \right) \gamma(\vartheta, \mathbf{x}) \right| < d(\vartheta), \quad (36)$$

for some $d(\vartheta) > 0$. We can thus apply the ergodic assumption **(H)** to the right-hand side of (35) to get

$$\mathbb{E}_{\mathbf{x}_0}^\vartheta (|\langle S^N \rangle_t|) \leq \frac{\|h\|^2 c(\mathbf{x}_0) d(\vartheta)}{rN} + \|h\|^2 t \int_E \left| f_\vartheta^\gamma \left(\mathbf{x}, \frac{\|h\|}{\sqrt{N}} \right) \gamma(\vartheta, \mathbf{x}) \right| \mu_\infty^\vartheta(d\mathbf{x}).$$

In view of (36), and since for any $\mathbf{x} \in E$, $f_\vartheta^\gamma(\mathbf{x}, \|h\|/\sqrt{N}) \rightarrow 0$ when $N \rightarrow 0$, we can conclude by the dominated convergence theorem.

Second step: Convergence of A_t^N to 0 in probability.

Let $\epsilon > 0$. We have $2A_t^N = A_1 - A_2 - A_3$ where

$$\begin{aligned} A_1 &= \frac{1}{N} \int_0^{tN} \mathbf{1}_{\gamma > 0} \frac{(h^T \nabla_1 \gamma)^2}{\gamma}(\vartheta, \mathbf{X}_s) \mathbf{1}_{\left\{ \left| \frac{h^T \nabla_1 \gamma}{\gamma}(\vartheta, \mathbf{X}_s) \right| > \epsilon \sqrt{N} \right\}} ds, \\ A_2 &= \frac{1}{N} \int_0^{tN} \frac{(h^T \nabla_1 \gamma)^2}{\gamma^2}(\vartheta, \mathbf{X}_{s^-}) \mathbf{1}_{\left\{ \left| \frac{h^T \nabla_1 \gamma}{\gamma}(\vartheta, \mathbf{X}_{s^-}) \right| > \epsilon \sqrt{N} \right\}} dN_s^\gamma, \\ A_3 &= \frac{1}{N} \int_0^{tN} \mathbf{1}_{\gamma > 0} \frac{(h^T \nabla_1 \gamma)^2}{\gamma^2}(\vartheta, \mathbf{X}_{s^-}) \mathbf{1}_{\left\{ \left| \frac{h^T \nabla_1 \gamma}{\gamma}(\vartheta, \mathbf{X}_{s^-}) \right| \leq \epsilon \sqrt{N} \right\}} dM_s^\gamma. \end{aligned} \quad (37)$$

Note that $\mathbb{E}_{\mathbf{x}_0}^\vartheta (|A_1|) = \mathbb{E}_{\mathbf{x}_0}^\vartheta (A_1) = \mathbb{E}_{\mathbf{x}_0}^\vartheta (A_2) = \mathbb{E}_{\mathbf{x}_0}^\vartheta (|A_2|)$. We prove below that for any $\epsilon > 0$, $\mathbb{E}_{\mathbf{x}_0}^\vartheta (|A_1|) \rightarrow 0$ as $N \rightarrow \infty$ and then we deal with the convergence in probability of A_3 .

By condition **A**(γ), we know that given $\vartheta \in \Theta$, the integrand in A_1 is uniformly bounded in \mathbf{X}_s by some $d(\vartheta) > 0$. We can then apply **(H)** to get

$$\mathbb{E}_{\mathbf{x}_0}^\vartheta (|A_1|) \leq \frac{c(\mathbf{x}_0) d(\vartheta)}{N} + t \int_E \left| \mathbf{1}_{\gamma > 0} \frac{(h^T \nabla_1 \gamma)^2}{\gamma}(\vartheta, \mathbf{x}) \mathbf{1}_{\left\{ \left| \frac{h^T \nabla_1 \gamma}{\gamma}(\vartheta, \mathbf{x}) \right| > \epsilon \sqrt{N} \right\}} \right| \mu_\infty^\vartheta(d\mathbf{x}).$$

This integrand is bounded thanks to condition **A**(γ) and converges to 0 as $N \rightarrow \infty$ for any given \mathbf{x} and ϑ . We thus conclude by the dominated convergence theorem that $\mathbb{E}_{\mathbf{x}_0}^\vartheta (|A_1|) \rightarrow 0$ as $N \rightarrow 0$.

Concerning A_3 , since it is a centred martingale, we have for any $a > 0$,

$$\mathbb{P}_{\mathbf{x}_0}^\vartheta(|A_3| > a) \leq \frac{1}{a^2} \mathbb{E}_{\mathbf{x}_0}^\vartheta(|A_3|^2) = \frac{1}{a^2} \mathbb{E}_{\mathbf{x}_0}^\vartheta(\langle A_3 \rangle)$$

where

$$\begin{aligned} \langle A_3 \rangle &= \frac{1}{N^2} \int_0^{tN} \mathbf{1}_{\gamma>0} \frac{(h^T \nabla_1 \gamma)^4}{\gamma^3}(\vartheta, \mathbf{X}_s) \mathbf{1}_{\left\{ \left| \frac{h^T \nabla_1 \gamma}{\gamma}(\vartheta, \mathbf{X}_s) \right| \leq \epsilon \sqrt{N} \right\}} ds \\ &\leq \frac{\epsilon^2}{N} \int_0^{tN} \mathbf{1}_{\gamma>0} \frac{(h^T \nabla_1 \gamma)^2}{\gamma}(\vartheta, \mathbf{X}_s) ds. \end{aligned}$$

We can again apply **(H)** to the latest integral, thanks to condition **A**(γ), and we obtain

$$\mathbb{E}_{\mathbf{x}_0}^\vartheta(\langle A_3 \rangle) \leq \epsilon^2 C(\mathbf{x}_0, \vartheta, h)$$

where $C(\mathbf{x}_0, \vartheta, h) > 0$. So for any $a > 0$ and any $\eta > 0$, we can choose ϵ so that $\mathbb{P}_{\mathbf{x}_0}^\vartheta(|A_3| > a) \leq \eta$. For these choices, we can further choose N large enough so that $\mathbb{P}_{\mathbf{x}_0}^\vartheta(|A_1| > a) \leq \eta$ and $\mathbb{P}_{\mathbf{x}_0}^\vartheta(|A_2| > a) \leq \eta$, because for any $\epsilon > 0$, $\mathbb{E}_{\mathbf{x}_0}^\vartheta(|A_1|) = \mathbb{E}_{\mathbf{x}_0}^\vartheta(|A_2|) \rightarrow 0$ as $N \rightarrow \infty$. This entails the convergence of A_t^N to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability.

Third step: Convergence of R_t^N to 0 in probability.

Denoting by $\varphi : x > -1 \mapsto \log(1+x) - x + x^2/2$, we have $R_t^N = R_1 - R_2/2$ with

$$\begin{aligned} R_1 &= \int_0^{tN} \varphi \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right) dN_s^\gamma, \\ R_2 &= \int_0^{tN} \left(\left[\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right]^2 - \left[\frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})} \right]^2 \right) dN_s^\gamma. \end{aligned} \quad (38)$$

We start by proving that R_2 tends to 0 in probability. Write $R_2 = \int (f^2 - g^2) dN_s^\gamma$ with obvious notations. We have by the Cauchy-Schwartz inequality and the fact that $(f+g)^2 = (f-g+2g)^2 \leq 2(f-g)^2 + 8g^2$,

$$\begin{aligned} R_2^2 &\leq \int (f-g)^2 dN_s^\gamma \int (f+g)^2 dN_s^\gamma \\ &\leq \left(\int (f-g)^2 dN_s^\gamma \right) \left(8 \int (f-g)^2 dN_s^\gamma + 8 \int g^2 dN_s^\gamma \right). \end{aligned}$$

To prove the convergence of R_2 , it is then sufficient to show that $\int (f - g)^2 dN_s^\gamma = o_{\mathbb{P}_{\mathbf{x}_0}^\vartheta}(1)$ and $\int g^2 dN_s^\gamma = O_{\mathbb{P}_{\mathbf{x}_0}^\vartheta}(1)$. We have

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}_0}^\vartheta \int (f - g)^2 dN_s^\gamma \\ &= \mathbb{E}_{\mathbf{x}_0}^\vartheta \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})} \right)^2 dN_s^\gamma \quad (39) \\ &= \mathbb{E}_{\mathbf{x}_0}^\vartheta \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_s)}{\gamma(\vartheta, \mathbf{X}_s)} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_s)}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_s)} \right)^2 \mathbf{1}_{\gamma(\vartheta, \mathbf{X}_s) > 0} \gamma(\vartheta, \mathbf{X}_s) ds, \end{aligned}$$

which is exactly $\mathbb{E}_{\mathbf{x}_0}^\vartheta(\langle S^N \rangle_t)$, see (33). We have already proven that this term tends to 0, so $\int (f - g)^2 dN_s^\gamma = o_{\mathbb{P}_{\mathbf{x}_0}^\vartheta}(1)$. Now, for any $\eta > 0$, by the Markov's inequality, we can choose $M > 0$ such that

$$\mathbb{P}_{\mathbf{x}_0}^\vartheta \left(\int g^2 dN_s^\gamma > M \right) \leq \frac{1}{M} \mathbb{E}_{\mathbf{x}_0}^\vartheta \left(\frac{1}{N} \int_0^{tN} \frac{(h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_s))^2}{\gamma(\vartheta, \mathbf{X}_s)} ds \right) \leq \frac{t}{M} C(\vartheta, h) < \eta,$$

where $C(\vartheta, h)$ is a positive upper-bound deduced from condition **A**(γ). This proves that $\int g^2 dN_s^\gamma = O_{\mathbb{P}_{\mathbf{x}_0}^\vartheta}(1)$ and completes the proof that R_2 tends to 0 in probability.

It remains to address the convergence of R_1 . For any $\epsilon > 0$, we have $R_1 = R_{11} + R_{12}$ with

$$R_{11} = \int_0^{tN} \varphi \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right| > \epsilon \right\}} dN_s^\gamma, \quad (40)$$

$$R_{12} = \int_0^{tN} \varphi \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right| \leq \epsilon \right\}} dN_s^\gamma. \quad (41)$$

For R_{11} , note that for any $a > 0$,

$$\mathbb{P}_{\mathbf{x}_0}^\vartheta (R_{11} > a) \leq \mathbb{P}_{\mathbf{x}_0}^\vartheta \left(\int_0^{tN} \mathbf{1}_{\left\{ \left| \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right| > \epsilon \right\}} dN_s^\gamma > b \right), \quad (42)$$

for any $b < 1$. This is because when $b < 1$,

$$\begin{aligned} & \left(\int_0^{tN} \mathbf{1}_{\left\{ \left| \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right| > \epsilon \right\}} dN_s^\gamma \leq b \right) \\ & \implies \left(\text{for all } \gamma\text{-jump time } T_i \leq tN, \mathbf{1}_{\left\{ \left| \frac{\gamma(\vartheta_N, \mathbf{X}_{T_i^-})}{\gamma(\vartheta, \mathbf{X}_{T_i^-})} - 1 \right| > \epsilon \right\}} = 0 \right) \end{aligned}$$

which implies that $R_{11} = 0 \leq a$. Moreover

$$\mathbf{1}\left\{\left|\frac{\gamma(\vartheta_N, \mathbf{x})}{\gamma(\vartheta, \mathbf{x})} - 1\right| > \epsilon\right\} \leq \mathbf{1}\left\{\left|\frac{\gamma(\vartheta_N, \mathbf{x})}{\gamma(\vartheta, \mathbf{x})} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{x})}{\sqrt{N} \gamma(\vartheta, \mathbf{x})}\right| > \frac{\epsilon}{2}\right\} + \mathbf{1}\left\{\left|\frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{x})}{\sqrt{N} \gamma(\vartheta, \mathbf{x})}\right| > \frac{\epsilon}{2}\right\}.$$

Using this and the fact that for any z , $\mathbf{1}\{|z| > \frac{\epsilon}{2}\} \leq 4\epsilon^{-2} z^2 \mathbf{1}\{|z| > \frac{\epsilon}{2}\}$, we get

$$\begin{aligned} & \int_0^{tN} \mathbf{1}\left\{\left|\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1\right| > \epsilon\right\} dN_s^\gamma \\ & \leq 4\epsilon^{-2} \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 - \frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})}\right)^2 dN_s^\gamma \\ & \quad + 4\epsilon^{-2} \int_0^{tN} \left(\frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})}\right)^2 \mathbf{1}\left\{\left|\frac{h^T \nabla_1 \gamma(\vartheta, \mathbf{X}_{s^-})}{\sqrt{N} \gamma(\vartheta, \mathbf{X}_{s^-})}\right| > \frac{\epsilon}{2}\right\} dN_s^\gamma. \end{aligned}$$

The first term is exactly $4\epsilon^{-2} \int (f - g)^2 dN_s^\gamma$, as already studied in (39), and the second term is $4\epsilon^{-2} A_2$ (up to $\epsilon/2$ instead of ϵ), see (37). Both terms tend to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability whatever the value of $\epsilon > 0$. This proves that the right-hand side term in (42) tends to 0 for any $b < 1$, which yields the convergence of R_{11} in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability to 0.

For R_{12} given by (41), we choose $\epsilon < \frac{1}{2}$ and since $|\varphi(x)| \leq 2|x|^3$ for $|x| \leq \frac{1}{2}$, we have

$$\begin{aligned} R_{12} & \leq \int_0^{tN} 2 \left| \frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1 \right|^3 \mathbf{1}\left\{\left|\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1\right| \leq \epsilon\right\} dN_s^\gamma \\ & \leq 2\epsilon \int_0^{tN} \left(\frac{\gamma(\vartheta_N, \mathbf{X}_{s^-})}{\gamma(\vartheta, \mathbf{X}_{s^-})} - 1\right)^2 dN_s^\gamma. \end{aligned}$$

This last term is $2\epsilon(R_2 + \int g^2 dN_s^\gamma)$ if we use as previously the notation $R_2 = \int (f^2 - g^2) dN_s^\gamma$ for R_2 given by (38). We already know that $U := 2(R_2 + \int g^2 dN_s^\gamma)$ is a $O_{\mathbb{P}_{\mathbf{x}_0}^\vartheta}(1)$ as $N \rightarrow \infty$. This means that for any $\eta > 0$, there exists M such that for N sufficiently large $\mathbb{P}_{\mathbf{x}_0}^\vartheta(U > M) < \eta$. This implies that for any $a > 0$ and $\eta > 0$, we can choose ϵ small enough ($\epsilon < a/M$) so that $\mathbb{P}_{\mathbf{x}_0}^\vartheta(\epsilon U > a) < P_{\mathbf{x}_0}(U > M) < \eta$. So for any $a > 0$ and $\eta > 0$, we can choose ϵ so that $\mathbb{P}_{\mathbf{x}_0}^\vartheta(R_{12} > a) < \eta$ for N large enough. The same inequality holds true for R_{11} since we have proven that R_{11} tends to 0 in probability for any $\epsilon > 0$. So $R_1 = R_{11} + R_{12}$ tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability, which concludes the proof.

B.2 Proof of Lemma 2

The proof follows the same scheme as the proof of Lemma 1. Consider

$$Rem_{N,\vartheta,h}^{k_\gamma}(t) = \Lambda_{k_\gamma}^{\vartheta_N/\vartheta}(tN) - h^T M_{k_\gamma}^{N,\vartheta}(t) + \frac{1}{2} h^T \langle M_{k_\gamma}^{N,\vartheta} \rangle(t) h,$$

where $M_{k_\gamma}^{N,\vartheta}(t)$ is given by (10). The fact that $M_{k_\gamma}^{N,\vartheta}$ is a \mathcal{F}_{tN} -martingale is a consequence of Lemma 4. We deduce that

$$\begin{aligned} [M_{k_\gamma}^{N,\vartheta}] &= \frac{1}{N} \int_0^{tN} \frac{\nabla_1 k_\gamma (\nabla_1 k_\gamma)^T}{k_\gamma^2}(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s) dN_s^\gamma, \\ \langle M_{k_\gamma}^{N,\vartheta} \rangle &= \frac{1}{N} \int_0^{tN} \int_E \mathbf{1}_{k_\gamma > 0} \frac{\nabla_1 k_\gamma (\nabla_1 k_\gamma)^T}{k_\gamma}(\vartheta, \mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) \gamma(\vartheta, \mathbf{X}_s) ds. \end{aligned}$$

Starting from (27) and using the above representations we can decompose $Rem_{N,\vartheta,h}^{k_\gamma}(t)$ as in the proof of Lemma 1, that is $Rem_{N,\vartheta,h}^{k_\gamma}(t) = S_t^N + A_t^N + R_t^N$, where

$$\begin{aligned} S_t^N &= \int_0^{tN} \left(\frac{k_\gamma(\vartheta_N, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} - 1 - \frac{h^T \nabla_1 k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)}{\sqrt{N} k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} \right) dN_s^\gamma \\ &\quad + \frac{1}{\sqrt{N}} \int_0^{tN} \int_E h^T \nabla_1 k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) \gamma(\vartheta, \mathbf{X}_s) ds, \end{aligned}$$

$$\begin{aligned} A_t^N &= \frac{1}{2N} \int_0^{tN} \int_E \mathbf{1}_{k_\gamma > 0} \frac{(h^T \nabla_1 k_\gamma)^2}{k_\gamma}(\vartheta, \mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) \gamma(\vartheta, \mathbf{X}_s) ds \\ &\quad - \frac{1}{2N} \int_0^{tN} \frac{(h^T \nabla_1 k_\gamma)^2}{k_\gamma^2}(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s) dN_s^\gamma, \end{aligned}$$

$$\begin{aligned} R_t^N &= \int_0^{tN} \left(\log \left(\frac{k_\gamma(\vartheta_N, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} \right) - \frac{k_\gamma(\vartheta_N, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} + 1 \right. \\ &\quad \left. + \frac{1}{2N} \frac{(h^T \nabla_1 k_\gamma)^2}{k_\gamma^2}(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s) \right) dN_s^\gamma. \end{aligned}$$

We show that each of these three terms tends to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability as in the proof of Lemma 1.

First step: We have

$$[S_t^N] = \int_0^{tN} \left(\frac{k_\gamma(\vartheta_N, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} - 1 - \frac{h^T \nabla_1 k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)}{\sqrt{N} k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} \right)^2 dN_s^\gamma$$

so that by Lemma 4 in Appendix C.2,

$$\begin{aligned} \langle S^N \rangle_t &= \int_0^{tN} \int_E \left(\frac{k_\gamma(\vartheta_N, \mathbf{X}_s, \mathbf{y})}{k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y})} - 1 - \frac{h^T \nabla_1 k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y})}{\sqrt{N} k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y})} \right)^2 \\ &\quad \times k_\gamma(\vartheta, \mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) \gamma(\vartheta, \mathbf{X}_s) ds. \end{aligned}$$

Using the same inequalities as in (34) we obtain that

$$\langle S^N \rangle_t \leq \frac{\|h\|^2}{N} \int_0^{tN} f_\vartheta^{k_\gamma} \left(\mathbf{X}_s, \frac{\|h\|}{\sqrt{N}} \right) \gamma(\vartheta, \mathbf{X}_s) ds$$

where $f_\vartheta^{k_\gamma}$ is given by (7). We can then deduce that $\mathbb{E}_{\mathbf{x}_0}^\vartheta(\langle S^N \rangle_t)$ tends to 0 by use of Conditions **B**(k_γ, γ), **(H)** and the dominated convergence theorem, as in the proof of Lemma 1.

Second step: The proof is identical to the second step of the proof of Lemma 1.

It consists in splitting the integrals in A_t^N according to whether $\left| \frac{h^T \nabla_1 k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)}{k_\gamma(\vartheta, \mathbf{X}_{s^-}, \mathbf{X}_s)} \right| > \epsilon \sqrt{N}$ or not. The convergence of A_t^N is then proved similarly, by use of Conditions **A**(k_γ, γ), **(H)** and the dominated convergence theorem.

Third step: The proof follows exactly the same lines as the proof of the third step of Lemma 1 where $\gamma(\cdot, \cdot)$ is replaced by $k_\gamma(\cdot, \cdot, \cdot)$ and Condition **A**(k_γ, γ) is used instead of Condition **A**(γ).

B.3 Proof of Lemma 3

Remember that $\mathbf{X}_t = \{(\mathbf{Z}_{t-T_i}^{(i)}, \mathbf{m}^{(i)})\}$ for all $t \in [T_i, T_{i+1}[$, and that $L(\vartheta, \mathbf{X}_{[T_i, T_{i+1}[})$ is a shortcut for $L(\mathbf{Z}_{[0, T_{i+1}-T_i]}^{(i)}, \mathbf{m}^{(i)})$ given by (5). Accordingly, we have

$$\begin{aligned} \log L(\vartheta, \mathbf{X}_{[T_i, T_{i+1}[}) &= \int_{T_i}^{T_{i+1}} \bar{v}_n(\vartheta, \mathbf{X}_s)^T a_n^{-1}(\mathbf{X}_s) d\mathbf{Z}_{s-T_i}^{(i)} \\ &\quad - \frac{1}{2} \int_{T_i}^{T_{i+1}} \bar{v}_n(\vartheta, \mathbf{X}_s)^T a_n^{-1}(\mathbf{X}_s) \bar{v}_n(\vartheta, \mathbf{X}_s) ds, \end{aligned}$$

where $\bar{v}_n(\vartheta, \mathbf{X}_s)$ stands for $\bar{v}_n(\vartheta, \mathbf{Z}_{s-T_i}^{(i)}, \mathbf{m}^{(i)})$, with $n = n(\mathbf{X}_s)$, and similarly for $a_n^{-1}(\mathbf{X}_s)$. Then we have

$$\begin{aligned} & \log \frac{L(\vartheta_N, \mathbf{X}_{[T_i, T_{i+1}[})}{L(\vartheta, \mathbf{X}_{[T_i, T_{i+1}[})} \\ &= \int_{T_i}^{T_{i+1}} (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) (d\mathbf{Z}_{s-T_i}^{(i)} - \bar{v}_n(\vartheta, \mathbf{X}_s) ds) \\ & \quad - \frac{1}{2} \int_{T_i}^{T_{i+1}} (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s)) ds. \end{aligned}$$

Consequently, we may rewrite (28) as

$$\begin{aligned} \Lambda_L^{\vartheta_N/\vartheta}(t) &= \sum_{i \geq 0} \int_0^t \mathbf{1}_{[T_i, T_{i+1}[}(s) (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) dM_{s-T_i}^{(i)} \\ & \quad - \frac{1}{2} \int_0^t (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s)) ds. \quad (43) \end{aligned}$$

where $M_s^{(i)} = 0$ if $s < 0$ and

$$M_s^{(i)} = \mathbf{Z}_s^{(i)} - \mathbf{Z}_0^{(i)} - \int_0^s \bar{v}_n(\vartheta, \mathbf{Z}_u^{(i)}) du, \quad s \geq 0. \quad (44)$$

We know that $\mathbf{Z}^{(i)}$ is the solution of $M^n(\mathbf{z}^{(i)}, \mathbf{m}^{(i)})$ given by (4), where $(\mathbf{z}^{(i)}, \mathbf{m}^{(i)})$ are such that $\mathbf{X}_{T_i} = \{(\mathbf{z}^{(i)}, \mathbf{m}^{(i)})\}$ and $n = n(\mathbf{X}_{T_i})$. This implies that $(M_s^{(i)})$ is a martingale with respect to $\mathcal{F}_t^{\mathbf{Z}^{(i)}}$, the natural filtration associated to $\mathbf{Z}^{(i)}$. Note that $(M_s^{(i)})$ is independent from T_i and that \mathcal{F}_t is generated by $\left\{T_{N_t}, \mathbf{X}_{T_{N_t}}, \mathbf{Z}_{[0, t-T_{N_s}]}^{(N_t)}, (T_i, \mathbf{X}_{T_i}, \mathbf{Z}_{[0, T_{i+1}-T_i]}^{(i)})_{i=1, \dots, N_t-1}\right\}$. This implies that for $T_i \leq t < T_{i+1}$, if $s \in [T_i, t]$,

$$\mathbb{E}(M_{t-T_i}^{(i)} | \mathcal{F}_s) = \mathbb{E}(M_{t-T_i}^{(i)} | T_i, \mathcal{F}_{s-T_i}^{\mathbf{Z}^{(i)}}) = M_{s-T_i}^{(i)},$$

while if $s < T_i$, $\mathbb{E}(M_{t-T_i}^{(i)} | \mathcal{F}_s) = 0$.

Using these properties, we can easily verified that $M_L^{N, \vartheta}$ given by (11) is

a \mathcal{F}_{tN} -martingale. Similarly, since $\langle M^{(i)} \rangle_s = \int_0^s a_n(\mathbf{Z}_u^{(i)}) du$, we have

$$\begin{aligned} \langle M_L^{N,\vartheta} \rangle(t) &= \frac{1}{N} \sum_{i \geq 0} \int_0^{tN} \mathbf{1}_{[T_i; T_{i+1}[}(s) (\nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) \\ &\quad \times a_n(\mathbf{X}_s) a_n^{-1}(\mathbf{X}_s) (\mathbf{X}_s) \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) ds \\ &= \frac{1}{N} \int_0^{tN} (\nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) ds. \end{aligned}$$

From this representation, (11) and (43), we have

$$Rem_{N,\vartheta,h}^L(t) = \Lambda_L^{\vartheta N/\vartheta}(tN) - h^T M_L^{N,\vartheta}(t) + \frac{1}{2} h^T \langle M_L^{N,\vartheta} \rangle(t) h = S_t^N - \frac{1}{2} R_t^N,$$

where

$$\begin{aligned} S_t^N &= \sum_{i \geq 0} \int_0^{tN} \mathbf{1}_{[T_i; T_{i+1}[}(s) \left(\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s) \right. \\ &\quad \left. - \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) \frac{h}{\sqrt{N}} \right)^T a_n^{-1}(\mathbf{X}_s) dM_{s-T_i}^{(i)}, \end{aligned}$$

and

$$\begin{aligned} R_t^N &= \int_0^{tN} \left((\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s)) \right. \\ &\quad \left. - \frac{1}{N} h^T (\nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s))^T a_n^{-1}(\mathbf{X}_s) \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) h \right) ds. \end{aligned}$$

Let us prove that both S_t^N and R_t^N tend to 0 in $\mathbb{P}_{\mathbf{x}_0}^\vartheta$ -probability. For S_t^N , using the Markov inequality and the fact that $\mathbb{E}_{\mathbf{x}_0}^\vartheta((S_t^N)^2) = \mathbb{E}_{\mathbf{x}_0}^\vartheta(\langle S_t^N \rangle)$, this boils down to proving that $\mathbb{E}_{\mathbf{x}_0}^\vartheta(\langle S_t^N \rangle)$ tends to 0. We have, since $a_n = \bar{\sigma}_n \bar{\sigma}_n^T$,

$$\begin{aligned} \langle S_t^N \rangle &= \int_0^{tN} \left(\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s) - \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) \frac{h}{\sqrt{N}} \right)^T a_n^{-1}(\mathbf{X}_s) \\ &\quad \times \left(\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s) - \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) \frac{h}{\sqrt{N}} \right) ds \\ &= \int_0^{tN} \left\| \bar{\sigma}_n(\mathbf{X}_s)^{-1} \left(\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s) - \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) \frac{h}{\sqrt{N}} \right) \right\|^2 ds. \end{aligned}$$

Using the same argument as for (34), we get

$$\langle S_t^N \rangle \leq \frac{\|h\|^2}{N} \int_0^{tN} \|\bar{\sigma}_n(\mathbf{X}_s)^{-1}\|^2 f_{\vartheta}^{\text{move}} \left(\mathbf{X}_s, \frac{\|h\|}{\sqrt{N}} \right) ds.$$

where $f_{\vartheta}^{\text{move}}$ is given by (8). We then deduce that $\mathbb{E}_{\mathbf{x}_0}^{\vartheta}(\langle S^N \rangle_t)$ tends to 0 thanks to the dominated convergence theorem, using Conditions **B_{move}** and **(H)** as in the first step of the proof of Lemma 1.

Concerning R_t^N , note that

$$\begin{aligned} R_t^N &= \int_0^{tN} \left\| \bar{\sigma}_n(\mathbf{X}_s)^{-1} (\bar{v}_n(\vartheta_N, \mathbf{X}_s) - \bar{v}_n(\vartheta, \mathbf{X}_s)) \right\|^2 \\ &\quad - \frac{1}{N} \left\| \bar{\sigma}_n(\mathbf{X}_s)^{-1} \nabla_1 \bar{v}_n(\vartheta, \mathbf{X}_s) h \right\|^2 ds, \end{aligned}$$

that reads $\int (\|f\|^2 - \|g\|^2) ds$ with obvious notation, so that using the same argument as in the third step of the proof of Lemma 1 (term R_2), it suffices to show that $\int \|f - g\|^2 ds = o(1)$ and $\int \|g\|^2 ds = O(1)$ in $\mathbb{P}_{\mathbf{x}_0}^{\vartheta}$ -probability. But the former is exactly $\langle S_t^N \rangle$ studied above, that has been proven to converge to 0 in probability. And the expectation of the latter is bounded by use of Condition **A_{move}**, proving that it is $O(1)$ in $\mathbb{P}_{\mathbf{x}_0}^{\vartheta}$ -probability. This concludes the proof.

C Ancillary results

C.1 Conditions for geometric ergodicity

The following proposition provides some conditions ensuring the hypothesis **(H)**, namely non-explosion and geometric ergodicity of the BDMM process. It is based on the study conducted in [14] for birth-death-move processes (without mutations). The arguments in presence of mutations are similar and sketched below. The conditions include the technical assumption that the process is Feller. This property is discussed in [14] and proved for several examples of transition kernels that include all examples of this paper, provided the underlying spatial state Λ is compact. The other conditions deal with the intensity functions of the jumps. First, the total intensity function α is assumed to be bounded, to avoid explosion of the process. Second, the death intensity function δ must compensate in a proper way the birth intensity

function β . Notably, the ergodic properties do not depend on the inter-jump move process, neither on the mutation dynamics. We introduce the following notation:

$$\beta_n = \sup_{\mathbf{x} \in E_n} \beta(\mathbf{x}), \quad \delta_n = \inf_{\mathbf{x} \in E_n} \delta(\mathbf{x}) \quad \text{and} \quad \alpha_n = \beta_n + \delta_n. \quad (45)$$

In fact the conditions of geometric ergodicity below are exactly the same as the conditions of ergodicity for a simple birth-death process $(\eta_t)_{t \geq 0}$ on \mathbb{N} with birth rate (β_n) and death rate (δ_n) , as established in [11]. This is because to address the ergodic properties of a BDMM $(\mathbf{X}_t)_{t \geq 0}$, we construct a coupling between $(\mathbf{X}_t)_{t \geq 0}$ and $(\eta_t)_{t \geq 0}$, in such a way that $\eta_t = 0$ implies $\mathbf{X}_t = \emptyset$. Consequently \emptyset becomes a positive recurrent state for $(\mathbf{X}_t)_{t \geq 0}$ whenever 0 is a positive recurrent state for $(\eta_t)_{t \geq 0}$, which implies the geometric ergodicity of $(\mathbf{X}_t)_{t \geq 0}$.

Proposition 1. *Let $(\mathbf{X}_t)_{t \geq 0}$ be a Feller BDMM process with a bounded intensity function α . Suppose that $\delta_n > 0$ for all $n \geq 1$ and one of the following condition holds:*

$$(i) \text{ there exists } n_0 \geq 1 \text{ such that } \beta_n = 0 \text{ for any } n \geq n_0, \quad (46)$$

$$(ii) \beta_n > 0 \text{ for all } n \geq 1, \quad \sum_{n=2}^{\infty} \frac{\beta_1 \cdots \beta_{n-1}}{\delta_1 \cdots \delta_n} < \infty \text{ and } \sum_{n=1}^{\infty} \frac{\delta_1 \cdots \delta_n}{\beta_1 \cdots \beta_n} = \infty, \quad (47)$$

where β_n and δ_n are defined by (45). Assume moreover that

$$\sum_{n=2}^{\infty} \sqrt{\frac{\beta_1 \cdots \beta_{n-1}}{\delta_1 \cdots \delta_n}} < \infty \quad \text{and} \quad \exists N \geq 0, \text{ s.t. } \forall n \geq N, \beta_n \leq \delta_{n+1}.$$

Then **(H)** is satisfied.

As mentioned above, following [25] and [14], the proof of this proposition is based on a coupling between $(\mathbf{X}_t)_{t \geq 0}$ and $(\eta_t)_{t \geq 0}$, where the latter is a simple birth-death process with birth and death rates given by (45). We detail below how we construct the coupled process $\check{C}_t = (\mathbf{X}_t, \eta_t)$. This is a straightforward generalisation of the construction in [14], where we account for the presence of mutations. Specifically, \check{C} is a jump move process on

$E \times \mathbb{N}$ with intensity function $\check{\alpha}$ and transition kernel \check{K} defined as follows. The intensity function $\check{\alpha} : E \times \mathbb{N} \rightarrow \mathbb{R}_+$ is given by

$$\check{\alpha}(\mathbf{x}, n) = \begin{cases} \alpha(\mathbf{x}) + \alpha_n & \text{if } \mathbf{x} \in E_m, m \neq n, \\ \beta_n + \delta(\mathbf{x}) + \tau(\mathbf{x}) & \text{if } \mathbf{x} \in E_n. \end{cases}$$

Letting K be the kernel given by (21), the transition kernel \check{K} takes the form, for any $A \subset E$:

1. If $\mathbf{x} \in E_m, m \neq n$:

$$\check{K}((\mathbf{x}, n); A \times \{n\}) = \frac{\alpha(\mathbf{x})}{\check{\alpha}(\mathbf{x}, n)} K(\mathbf{x}, A);$$

$$\check{K}((\mathbf{x}, n); \{\mathbf{x}\} \times \{n+1\}) = \frac{\beta_n}{\check{\alpha}(\mathbf{x}, n)};$$

$$\check{K}((\mathbf{x}, n); \{\mathbf{x}\} \times \{n-1\}) = \frac{\delta_n}{\check{\alpha}(\mathbf{x}, n)}.$$

2. If $\mathbf{x} \in E_n$:

$$\check{K}((\mathbf{x}, n); A \times \{n+1\}) = \frac{\beta(\mathbf{x})}{\check{\alpha}(\mathbf{x}, n)} K_\beta(\mathbf{x}, A);$$

$$\check{K}((\mathbf{x}, n); \{\mathbf{x}\} \times \{n+1\}) = \frac{\beta_n - \beta(\mathbf{x})}{\check{\alpha}(\mathbf{x}, n)};$$

$$\check{K}((\mathbf{x}, n); A \times \{n-1\}) = \frac{\delta_n}{\check{\alpha}(\mathbf{x}, n)} K_\delta(\mathbf{x}, A);$$

$$\check{K}((\mathbf{x}, n); A \times \{n\}) = \frac{\delta(\mathbf{x}) - \delta_n}{\check{\alpha}(\mathbf{x}, n)} K_\delta(\mathbf{x}, A);$$

$$\check{K}((\mathbf{x}, n); A \times \{n\}) = \frac{\tau(\mathbf{x})}{\check{\alpha}(\mathbf{x}, n)} K_\tau(\mathbf{x}, A).$$

The inter-jump move process of \check{C} is in turn a simple independent coupling between the move \mathbf{Y} of \mathbf{X} and a constant move on \mathbb{N} (i.e. $y_t = y_0$ for all $t \geq 0$).

The fact that the above construction is a proper coupling, in the sense that \mathbf{X} and η do constitute the marginal distributions of \check{C} , can be proven

exactly as in [14] for the case without mutations. On the other hand, the key point is that if at some point, $C_t = (\mathbf{X}_t, \eta_t)$ is such that $n(\mathbf{X}_t) \leq \eta_t$, then by the above construction, $n(\mathbf{X}_s) \leq \eta_s$ for all $s \geq t$, almost surely. By this property, $\eta_t = 0$ implies $\mathbf{X}_t = \emptyset$ and the geometric ergodic conditions for η stated in the proposition, coming from [11], are sufficient for the geometric ergodicity of \mathbf{X} . The rigorous proof of these claims can be found in [14].

C.2 A useful martingale

The following result is useful in the proof of Theorem 2 in Section B. It is proven in Lemma 54 of [20], the BDMM being a particular case of a jump-move process (see the proof of Theorem 1 in Section A). It is also stated under a slightly different setting in Proposition 3.3 (b) of [18].

Lemma 4. *Let $(\mathbf{X}_t)_{t \geq 0}$ be a BDMM process on E , let g be a measurable bounded function defined on $E \times E$ and introduce for any $t > 0$*

$$M_t^* = \int_0^t g(\mathbf{X}_{s-}, \mathbf{X}_s) dN_s^\gamma - \int_0^t \gamma(\mathbf{X}_s) \int_E g(\mathbf{X}_s, \mathbf{y}) k_\gamma(\mathbf{X}_s, \mathbf{y}) \nu_\gamma(\mathbf{X}_s, d\mathbf{y}) ds.$$

If $N_t^\gamma < \infty$ for any $t \geq 0$, then $(M_t^)_{t \geq 0}$ is a \mathcal{F}_t -martingale.*

References

- [1] ATHREYA, K. B., AND NEY, P. E. *Branching Processes*, vol. 196. Springer Science & Business Media, 2012.
- [2] BALSOLLIER, L., LAVANCIER, F., SALAMERO, J., AND KERVRANN, C. A generative model to simulate spatiotemporal dynamics of biomolecules in cells. *Biological Imaging 3* (2023), e22.
- [3] BANSAYE, V., AND MÉLÉARD, S. *Stochastic models for structured populations*, vol. 16. Springer, 2015.
- [4] BOULANGER, J., GUEUDRY, C., MUNCH, D., CINQUIN, B., PAUL-GILLOTEAUX, P., BARDIN, S., GUÉRIN, C., SENGER, F., BLANCHON, L., AND SALAMERO, J. Fast high-resolution 3D total internal reflection fluorescence microscopy by incidence angle scanning and azimuthal averaging. *Proc Natl Acad Sci USA 111*, 48 (2014), 17164–17169.

- [5] BRIANE, V., KERVRANN, C., AND VIMOND, M. Statistical analysis of particle trajectories in living cells. *Phys Rev E*. 97, (6-1):062121 (2018).
- [6] BRIANE, V., VIMOND, M., AND KERVRANN, C. An overview of diffusion models for intracellular dynamics analysis. *Briefings in bioinformatics* 21, 4 (2020), 1136–1150.
- [7] FATTLER, T., AND GROTHAUS, M. Strong feller properties for distorted brownian motion with reflecting boundary condition and an application to continuous n-particle systems with singular interactions. *Journal of Functional Analysis* 246, 2 (2007), 217–241.
- [8] HOPFNER, R., AND LÖCHERBACH, E. *Limit theorems for null recurrent markov processes*. Univ., Fachbereich Mathematik, 2000.
- [9] IBRAGIMOV, H. M. *Statistical estimation: asymptotic theory*, vol. 16. Springer Science & Business Media, 2013.
- [10] JAQAMAN, K., LOERKE, D., METTLEN, M., KUWATA, H., GRINSTEIN, S., SCHMID, S. L., AND DANUSER, G. Robust single-particle tracking in live-cell time-lapse sequences. *Nature methods* 5, 8 (2008), 695.
- [11] KARLIN, S., AND MCGREGOR, J. The classification of birth and death processes. *Transactions of the American Mathematical Society* 86, 2 (1957), 366–400.
- [12] KLEBANER, F. C. *Introduction to stochastic calculus with applications*. World Scientific Publishing Company, 2012.
- [13] KUTOYANTS, Y. A. *Statistical inference for ergodic diffusion processes*. Springer Science & Business Media, 2004.
- [14] LAVANCIER, F., LE GUÉVEL, R., , AND MANENT, E. Feller and ergodic properties of jump-move processes with applications to interacting particle systems. *to appear in Journal of Applied Probability (arXiv:2204.02851)* (2024).
- [15] LAVANCIER, F., AND LE GUÉVEL, R. Spatial birth–death–move processes: Basic properties and estimation of their intensity functions. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 83, 4 (2021), 798–825.

- [16] LIPTSER, R. S., AND SHIRYAEV, A. N. *Statistics of random processes I: General Theory*. Springer Science & Business Media, 2001.
- [17] LOCHERBACH, E. LAN and LAMN for systems of interacting diffusions with branching and immigration. In *Annales de l'Institut Henri Poincaré (B) Probability and Statistics* (2002), vol. 38, Elsevier, pp. 59–90.
- [18] LOCHERBACH, E. Likelihood ratio processes for markovian particle systems with killing and jumps. *Statistical inference for stochastic processes* 5, 2 (2002), 153–177.
- [19] LUSCHGY, H. Local asymptotic mixed normality for semimartingale experiments. *Probability theory and related fields* 92, 2 (1992), 151–176.
- [20] MANENT, E. *Propriétés de Feller, d'ergodicité et inférence non-paramétrique du processus jump-move*. PhD thesis, Université Rennes 2, 2023.
- [21] MASUDA, N., AND HOLME, P. *Temporal network epidemiology*. Springer, 2017.
- [22] MØLLER, J., AND SØRENSEN, M. Statistical analysis of a spatial birth-and-death process model with a view to modelling linear dune fields. *Scandinavian journal of statistics* 21, 1 (1994), 1–19.
- [23] PÉCOT, T., BOUTHEMY, P., BOULANGER, J., CHESSEL, A., BARDIN, S., SALAMERO, J., AND KERVRANN, C. Background fluorescence estimation and vesicle segmentation in live cell imaging with conditional random fields. *IEEE Transactions on Image Processing* 24, 2 (2014), 667–680.
- [24] POMMERENING, A., AND GRABARNIK, P. *Individual-Based Methods in Forest Ecology and Management*. Springer, 2019.
- [25] PRESTON, C. Spatial birth and death processes. *Advances in applied probability* 7, 3 (1975), 371–391.
- [26] RENSHAW, E., AND SÄRKKÄ, A. Gibbs point processes for studying the development of spatial-temporal stochastic processes. *Computational statistics & data analysis* 36, 1 (2001), 85–105.

- [27] SADAHIRO, Y. Analysis of the appearance and disappearance of point objects over time. *International Journal of Geographical Information Science* 33, 2 (2019), 215–239.
- [28] SCHUHMACHER, D., AND XIA, A. A new metric between distributions of point processes. *Advances in applied probability* 40, 3 (2008), 651–672.