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Supplement to "LAMN property for stable-Lévy SDEs with constant scale coefficient"

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Proofs of some technical results

Proof of Proposition 5.1. The first two inequalities are recalled from Clément and Gloter (2020), here, we only need to prove the last one. First, we note that

$$\begin{aligned}\partial_\mu \xi_{1/n}(x, \mu) &= \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) \partial_\mu \xi_s(x, \mu) ds + \int_0^{1/n} (\partial_\mu b)(\xi_s(x, \mu), \mu) ds, \\ \partial_\mu^2 \xi_{1/n}(x, \mu) &= \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^2 b(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^2 ds \\ &\quad + 2 \int_0^{1/n} (\partial_\mu \partial_x b)(\xi_s(x, \mu), \mu) \partial_\mu \xi_s(x, \mu) ds + \int_0^{1/n} (\partial_\mu^2 b)(\xi_s(x, \mu), \mu) ds, \\ \partial_\mu^3 \xi_{1/n}(x, \mu) &= \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) \partial_\mu^3 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^2 b(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds \\ &\quad + \int_0^{1/n} (\partial_\mu \partial_x^2 b)(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^3 b(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^3 ds \\ &\quad + 3 \int_0^{1/n} (\partial_\mu \partial_x^2 b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^2 ds \\ &\quad + 3 \int_0^{1/n} (\partial_\mu^2 \partial_x b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu)) ds + \int_0^{1/n} (\partial_\mu^3 b)(\xi_s(x, \mu), \mu) ds.\end{aligned}$$

Therefore, we get

$$|\partial_\mu^3 \xi_{1/n}(x, \mu) - \frac{1}{n} \partial_\mu^3 b(x, \mu)| \leq \int_0^{1/n} |\partial_x b(\xi_s(x, \mu), \mu)| |\partial_\mu^3 \xi_s(x, \mu) - \frac{1}{n} \partial_\mu^3 b(x, \mu)| ds + |\mathcal{R}_n(\mu)|,$$

where

$$\begin{aligned}\mathcal{R}_n(\mu) &= \frac{1}{n} \partial_\mu^3 b(x, \mu) \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) ds + \int_0^{1/n} \partial_x^2 b(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds\end{aligned}$$

$$\begin{aligned}
& + \int_0^{1/n} (\partial_\mu \partial_x b)(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^3 b(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^3 ds \\
& + 3 \int_0^{1/n} (\partial_\mu \partial_x^2 b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^2 ds + 3 \int_0^{1/n} (\partial_\mu^2 \partial_x b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu)) ds \\
& + \int_0^{1/n} [(\partial_\mu^3 b)(\xi_s(x, \mu), \mu) - \partial_\mu^3 b(x, \mu)] ds.
\end{aligned}$$

Now, for some $s \in (0, 1/n)$, since $b(\cdot)$ is Lipschitz, we have

$$\begin{aligned}
|\xi_s(x, \mu)| &= |x + \int_0^s b(\xi_v(x, \mu), \mu) dv| \leq |x| + \int_0^s C(1 + |\xi_v(x, \mu)|) dv, \\
|\xi_s(x, \mu) - x| &= \left| \int_0^s b(\xi_v(x, \mu), \mu) dv \right| \leq C(|x| + 1)s + \int_0^s C|\xi_v(x, \mu) - x| dv.
\end{aligned}$$

By Gronwall's lemma

$$|\xi_s(x, \mu)| \leq (|x| + Cs)e^{Cs}, \quad \text{and } |\xi_s(x, \mu) - x| \leq C(|x| + 1)se^{Cs}, \quad s \in (0, 1/n). \quad (5.18)$$

Then, from Assumption (A) and Gronwall's lemma again, we have

$$\sup_{\mu \in V_{\mu_0}} \sup_{k \in \{1, 2, 3\}} \partial_\mu^k \xi_{1/n}(x, \mu) \leq C(1 + |x|^p)/n \quad (5.19)$$

for some $p > 0$. Thus, from Assumption (A) and Jensen's inequality for convex function $x \in [1, \infty) \mapsto x^m$, $m > 1$, we easily deduce that $\sup_{\mu \in V_{\mu_0}} |\mathcal{R}_n(\mu)| \leq C(1 + |x|^p)/n^2$ for some $p > 0$. Finally, combined with Gronwall's lemma, we then obtain (5.2). \square

Proof of Theorem 5.2. First of all, let us denote $\eta_n(s) = \frac{\lfloor ns \rfloor}{n}$ for any $s \in [0, 1]$ and we introduce for any $q > 0$

$$\begin{aligned}
J_t^q &= J_t - \sum_{s \leq t} \Delta J_s \mathbb{1}_{\{|\Delta J_s| > q\}}, \\
T_q &= \inf\{t > 0 : |\Delta J_t| \geq q\}
\end{aligned}$$

and X^q the solution of the following SDE:

$$X_t^q = x_0 + \int_0^t b(X_s^q, \mu_0) ds + \int_0^t a(X_{s-}^q, \sigma_0) dJ_s^q, \quad t \in [0, 1]. \quad (5.20)$$

In a natural way, we denote by \bar{X}^q the associate scheme, for any $i \in \{0, 1, \dots, n-1\}$, we denote

$$\bar{X}_{t_{i+1}}^q = \xi_{t_{i+1}^n - t_i^n}(\bar{X}_{t_i^n}^q, \mu) + a(\bar{X}_{t_i^n}^q, \sigma)(J_{t_{i+1}^n}^q - J_{t_i^n}^q). \quad (5.21)$$

Let $q > 1$ and $p \geq 2$ be fixed. It is standard that

$$\mathbb{E}(\sup_{t \in [0, 1]} |X_t^q|^p) < +\infty. \quad (5.22)$$

Concerning the scheme, we also have for any $q > 1$ and n

$$\mathbb{E}(\sup_{i \in \{0,1,\dots,n\}} |\bar{X}_{t_i^n}^q|^p) \leq C_{q,p}. \quad (5.23)$$

Indeed, let $n \geq 1$ and $i \in \{1, 2, \dots, n-1\}$ then we have

$$\begin{aligned} \bar{X}_{t_{i+1}^n}^q &= \bar{X}_{t_i^n}^q + \int_0^{\frac{1}{n}} b(\xi_s(\bar{X}_{t_i^n}^q, \mu_0), \mu_0) ds + a(\bar{X}_{t_i^n}^q, \sigma_0)(J_{t_{i+1}^n}^q - J_{t_i^n}^q) \\ &= x_0 + \int_{t_i^n}^{t_{i+1}^n} b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) du + \int_{t_i^n}^{t_{i+1}^n} a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dJ_u^q \end{aligned}$$

Let us now introduce the process Z defined by

$$\forall t \in [0, 1], Z_t = x_0 + \int_0^t b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) du + \int_0^t a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dJ_u^q$$

so that for any i , $Z_{t_i^n} = \bar{X}_{t_i^n}^q$ and so

$$\forall t \in [0, 1], Z_t^* = \sup_{s \in [0, t]} |Z_s| \geq \sup_i |\bar{X}_{t_i^n}^q|. \quad (5.24)$$

Following for example [Protter \(1992\)](#) Section I.4, we know that J^q admits the following decomposition:

$$J_t^q = M_t^q + A_t^q, \quad (5.25)$$

with $M_t^q = \int_0^t \int_{|z| \leq q} z \tilde{N}(ds, dz)$, where \tilde{N} is the compensated Poisson measure with Lévy measure $\frac{1}{|z|^{1+\beta}} \mathbf{1}_{[-q, q]}(z) dz$. M^q is a martingale belonging to all the L^p spaces and A^q is a finite variation process such that its total variation satisfies

$$d|A|_t \leq \left(2 \int_1^q \frac{z}{z^{1+\beta}} dz \right) dt.$$

Then since coefficients a and b are Lipschitz w.r.t. variable x and thanks to estimates (5.18) and (5.24), we have:

$$\begin{aligned} |Z_t^*|^p &\leq C \left(|x_0|^p + \left(\int_0^t (Z_u^* + C/n) e^{C/n} du \right)^p + \left(\int_0^t (1 + Z_u^*) du \right)^p \right. \\ &\quad \left. + \sup_{s \in [0, t]} \left| \int_0^s a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dM_u^q \right|^p \right), \end{aligned} \quad (5.26)$$

where constant C above but also below denotes a constant which may vary from line to line but does not depend on n . Using ([Jacod and Protter, 2012](#), Inequality (2.1.36) in Lemma 2.1.5) (see also ([Kunita](#),

(2004, Theorem A.3)) we have:

$$\begin{aligned}
& \mathbb{E}_{\theta_0} \left(\sup_{s \in [0, t]} \left| \int_0^s a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dM_u^q \right|^p \right) \\
& \leq C \left(\mathbb{E}_{\theta_0} \left[\left(\int_0^t \int_{|z| \leq q} |a(\bar{X}_{\eta_n(u)}^q, \sigma_0)|^2 \frac{z^2}{|z|^{1+\beta}} dz ds \right)^{p/2} \right] + \mathbb{E}_{\theta_0} \left[\int_0^t \int_{|z| \leq q} |a(\bar{X}_{\eta_n(u)}^q, \sigma_0)|^p \frac{z^p}{|z|^{1+\beta}} dz ds \right] \right) \\
& \leq C \left(\mathbb{E}_{\theta_0} \left[\left(\int_0^t |a(\bar{X}_{\eta_n(u)}^q, \sigma_0)|^2 ds \right)^{p/2} \right] + \mathbb{E}_{\theta_0} \left[\int_0^t |a(\bar{X}_{\eta_n(u)}^q, \sigma_0)|^p ds \right] \right) \\
& \leq C \mathbb{E}_{\theta_0} \left[\int_0^t |a(\bar{X}_{\eta_n(u)}^q, \sigma_0)|^p du \right] \leq C \left(1 + \mathbb{E}_{\theta_0} \left[\int_0^t (Z_u^*)^p du \right] \right)
\end{aligned} \tag{5.27}$$

From this we easily get that

$$\forall t \in [0, 1], \mathbb{E}_{\theta_0} [(Z_t^*)^p] \leq C(1 + \int_0^t \mathbb{E}_{\theta_0} [(Z_s^*)^p] ds)$$

which ensures by Gronwall's Lemma that $\mathbb{E}_{\theta_0} [(Z_1^*)^p] \leq C_{q,p}$.

Clearly, on $N_q = \{T_q > 1\}$, $X_t = X_t^q$ a.s. for all $t \in [0, 1]$ and $\bar{X} = \bar{X}^q$. As $\lim_{q \rightarrow +\infty} \mathbb{P}_{\theta_0}(T_q \leq 1) = 0$, we only need to prove that for any $q > 0$,

$$\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i^n}^q, \mu, \sigma) g_\beta(z_n^i(\theta)) - \int_0^1 f(X_s^q, \mu_0, \sigma_0) ds \mathbb{E}_{\theta_0}(g_{\beta_0}(J_1)) \right| \mathbb{1}_{N_q} \xrightarrow[n \rightarrow \infty]{} 0, \tag{5.28}$$

in probability under \mathbb{P}_{θ_0} . This is implied by the following convergences in probability,

$$\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{X}_{t_i^n}^q, \mu, \sigma) - f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0)] g_\beta(z_n^i(\theta)) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0, \tag{5.29}$$

$$\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) [g_\beta(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta_0))] \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0, \tag{5.30}$$

$$n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) [g_{\beta_0}(z_n^i(\theta_0)) - \mathbb{E}_{\theta_0}(g_{\beta_0}(J_1))] \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0, \tag{5.31}$$

$$n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) - \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i^n}^q, \mu_0, \sigma_0) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0. \tag{5.32}$$

$$n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i^n}^q, \mu_0, \sigma_0) - \int_0^1 f(X_s^q, \mu_0, \sigma_0) ds \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0. \tag{5.33}$$

We remark then that

$$\begin{aligned}
z_n^i(\theta_0) &= n^{1/\beta_0} (J_{t_{i+1}^n} - J_{t_i^n}), \\
z_n^i(\theta) &= \frac{n^{1/\beta}}{a(\bar{X}_{t_i^n}^q, \sigma)} [\xi_{1/n}(\bar{X}_{t_i^n}^q, \mu_0) + a(\bar{X}_{t_i^n}^q, \sigma_0)(J_{t_{i+1}^n} - J_{t_i^n}) - \xi_{1/n}(\bar{X}_{t_i^n}^q, \mu)]
\end{aligned}$$

and that if $\theta \in V_n^{(\eta)}(\theta_0)$, then

$$|\mu - \mu_0| \leq \eta n^{1/2-1/\beta_0} (\log n)^2, \quad |\sigma - \sigma_0| \leq C \eta n^{-1/2} \log n \text{ and } |\beta - \beta_0| \leq C \eta n^{-1/2} \log n. \quad (5.34)$$

For (5.29) we remark that

$$\begin{aligned} & \mathbb{E}_{\theta_0} \left(\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{X}_{t_i^n}^q, \mu, \sigma) - f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0)] g_\beta(z_n^i(\theta)) \right| \mathbb{1}_{N_q} \right) \\ & \leq \mathbb{E}_{\theta_0} \left(C(1 + \sup_i |\bar{X}_{t_i^n}^q|^p) (\log n)^2 (n^{1/2-1/\beta_0+\varepsilon} + n^{-1/2+\varepsilon}) |g_\beta(z_n^i(\theta))| \mathbb{1}_{N_q} \right) \\ & \leq C(1 + \mathbb{E}_{\theta_0}(\sup_i |\bar{X}_{t_i^n}^q|^{2p}))^{1/2} (1 + \mathbb{E}_{\theta_0}(\sup_{\theta \in V_n^{(\eta)}(\theta_0)} |z_n^i(\theta)|^{2\kappa} \mathbb{1}_{N_q}))^{1/2} (\log n)^2 (n^{1/2-1/\beta_0+\varepsilon} + n^{-1/2+\varepsilon}) \\ & \leq C(\log n)^2 (n^{1/2-1/\beta_0+\varepsilon} + n^{-1/2+\varepsilon}) \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

where κ may be chosen in $]0, \beta/2[$ thanks to the assumptions made on g_β . The last line follows by adapting (Clément and Gloter, 2020, the proof of (4.11)) which ensures that $\mathbb{E}_{\theta_0}(|z_n^i(\theta)|^k | \mathcal{F}_{t_i^n}^n) \leq C$, for any $k < \beta_0$ and $\sup_{\theta \in V_n^{(\eta)}(\theta_0)} |z_n^i(\theta) - z_n^i(\theta_0)| \mathbb{1}_{N_q} \leq C(1 + |\bar{X}_{t_i^n}^q|^j)(1 + |z_n^i(\theta_0)|)(\log n)^2 / \sqrt{n}$, for some

$j > 0$. To prove (5.30), we proceed similarly by splitting the difference $g_\beta(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta_0))$ into two parts: $g_\beta(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta))$ and $g_{\beta_0}(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta_0))$ using the fact that f is Lipschitz w.r.t. variable x . For (5.31), let $\xi_i^n = n^{\varepsilon-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) [g_{\beta_0}(z_n^i(\theta_0)) - \mathbb{E}_{\theta_0}(g_{\beta_0}(J_1))]$, we simply use the fact that $z_n^i(\theta_0) = n^{1/\beta_0}(J_{t_{i+1}^n} - J_{t_i^n}) \stackrel{\mathcal{L}}{=} J_1$, using one more time the fact that f is Lipschitz we have:

$$n \mathbb{E}_{\theta_0}(\xi_i^n | \mathcal{F}_{t_i^n}^n) = 0, \quad n \mathbb{E}_{\theta_0}((\xi_i^n)^2 | \mathcal{F}_{t_i^n}^n) = n^{2\varepsilon-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0)^2 \text{Var}_{\theta_0}(g_{\beta_0}(J_1)) \rightarrow 0$$

in probability. We then conclude that $\sum_{i=1}^n \xi_i^n$ tends to 0 in probability using the results in Jacod and Protter (2012), Section (2.2.4) for triangular array. For (5.32), we note that

$$\begin{aligned} \bar{X}_{t_i^n}^q - X_{t_i^n}^q &= Z_{t_i^n} - X_{t_i^n}^q = \int_0^{t_i^n} (b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) - b(X_u^q, \mu_0)) du \\ &+ \int_0^{t_i^n} (a(\bar{X}_{\eta_n(u)}^q, \sigma_0) - a(X_{u-}^q, \sigma_0)) dJ_u^q. \end{aligned}$$

Then, we separate the first integral in the r.h.s. into three parts concerning the differences

$$\begin{aligned} & b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) - b(\bar{X}_{\eta_n(u)}^q, \mu_0) \\ & b(\bar{X}_{\eta_n(u)}^q, \mu_0) - b(X_{\eta_n(u)}^q, \mu_0) \\ & b(X_{\eta_n(u)}^q, \mu_0) - b(X_u^q, \mu_0) \end{aligned}$$

and we shall do similarly for the second integral. Now, thanks to the Lipschitz property of the coefficient functions $x \mapsto a(x, \sigma_0)$ and $x \mapsto b(x, \mu_0)$, we can easily recycle the arguments from (5.26) to obtain

$$\mathbb{E}_{\theta_0} \left(\sup_{s \in [0,1]} |\bar{X}_{\eta_n(s)}^q - X_s^q|^2 \right) = \mathcal{O}(n^{-1}).$$

Finally, using Lipschitz assumption on f and Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left(n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) - \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i^n}^q, \mu_0, \sigma_0) \right| \right) \xrightarrow{n \rightarrow +\infty} 0.$$

For (5.33), we rewrite the considering error as follows

$$n^\varepsilon \left| \int_0^1 (f(X_{\eta_n(s)}^q, \mu_0, \sigma_0) - f(X_s^q, \mu_0, \sigma_0)) ds \right|$$

and its convergence to zero in probability is directly using a part of the arguments for (5.32).

Finally, for (5.4), it is deduced from (5.3) when $\beta_0 \leq 1$ and by same arguments above similar to (Clement and Gloter, 2019, the proof of Theorem 3.1 ii)) when $\beta_0 > 1$. □

Proof of Lemma 3.8. To begin with, let us denote $\eta_n(s) = \frac{\lfloor ns \rfloor}{n}$ for any $s \in [0, 1]$ and

$$F_n(\sigma) = n^{\widehat{\beta}_n^0 - 1} V_n^1(p, X) - \mu_p(\widehat{\beta}_n^0) \int_0^1 |a(X_{\eta_n(s)}, \sigma)|^p ds.$$

Let us first prove that $(\sqrt{n}/\log n)F_n(\sigma_0)$ is tight. Indeed, we know that $\sup_{s \in [0, 1]} \sqrt{n}|X_{\eta_n(s)} - X_s| \xrightarrow{\mathbb{P}_{\theta_0}} 0$ and we rewrite

$$F_n(\sigma) = n^{\widehat{\beta}_n^0 - 1} V_n^1(p, X) - \mu_p(\widehat{\beta}_n^0) \int_0^1 |a(X_s, \sigma)|^p ds + o_P(1)$$

and we conclude thanks to (Todorov, 2013, (21)) its tightness with the rate $\sqrt{n}/\log n$. Then, we prove that any sequence $(\widehat{\sigma}_n^0, n \geq 1)$ belonging to V_{σ_0} (a neighborhood of σ_0 defined in assumption (A)) that solves $F_n(\sigma) = 0$ is consistent and eventually unique. To do so, using (Jacod and Sorensen, 2017, Theorem 2.7.a), since $F_n(\sigma_0)$ converges to zero in probability, we need to verify the following two conditions:

- (i) There exists F defined on a compact subset of V_{σ_0} , continuously differentiable, such that $F(\sigma_0) = 0$ and σ_0 is the unique root of $F(\sigma) = 0$.
- (ii) The following convergence in probability holds

$$\sup_{\sigma \in A} |\partial_\sigma F_n(\sigma) - \partial_\sigma F(\sigma)| \rightarrow 0,$$

and $\partial_\sigma F(\sigma)$ is non-singular with probability one.

For this proof, we choose

$$F(\sigma) = -p\mu_p(\beta_0) \int_{\sigma_0}^\sigma \int_0^1 \frac{\partial_\sigma a}{a^{1-p}}(X_s, u) ds du.$$

For this choice, the criteria (ii) is straightforward from the fact that $\widehat{\beta}_n^0 \xrightarrow{\mathbb{P}_{\theta_0}} \beta_0$. Considering (i), it is obvious that $F(\sigma_0) = 0$ and σ_0 is the unique solution of $F(\sigma) = 0$ since from those assumptions on the

function a ,

$$\int_{\sigma_0}^{\sigma} \int_0^1 \frac{\partial_{\sigma} a}{a^{1-p}}(X_s, u) ds du = 0 \Leftrightarrow \sigma = \sigma_0.$$

Moreover, we prove that $\frac{n^{1/2}}{\log n}(\widehat{\sigma}_n^0 - \sigma_0)$ is tight. From Taylor's formula and $F_n(\widehat{\sigma}_n^0) = 0$, we have

$$F_n(\sigma_0) = -p\mu_p(\widehat{\beta}_n^0) \int_0^1 \frac{\partial_{\sigma} a}{a^{1-p}}(X_{\eta_n(s)}, \tilde{\sigma}_n) ds (\widehat{\sigma}_n^0 - \sigma_0)$$

where $\tilde{\sigma}_n$ lies between $\widehat{\sigma}_n^0$ and σ_0 . Then, using the consistency of $\widehat{\sigma}_n^0$ proven above and the tightness of $(\sqrt{n}/\log n)F_n(\sigma_0)$, we complete the proof. \square

Proof of Theorem 3.9. Inspired by the work in Bayraktar and Clément (2024), using (Jacod and Sørensen, 2017, Theorem 2.7.a)), for the consistency and uniqueness, we need to verify the following two conditions:

- (i) There exists G defined on A , continuously differentiable, such that $G_n(\mu_0)$ converges to zero in probability, $G(\mu_0) = 0$ and μ_0 is the unique root of $G(\mu) = 0$.
- (ii) The following convergence in probability holds

$$\sup_{\mu \in A} |\partial_{\mu} G_n(\mu) - \partial_{\mu} G(\mu)| \rightarrow 0,$$

and $\partial_{\mu} G(\mu)$ is non-singular with probability one.

For some $\eta > 0$, we set

$$W_n^{(\eta)} = \left\{ (\sigma, \beta) : \left\| \frac{\sqrt{n}}{\log n} \begin{pmatrix} \sigma - \sigma_0 \\ \beta - \beta_0 \end{pmatrix} \right\| \leq \eta \right\}.$$

First, we prove the second assertion (ii) by setting

$$G(\mu) = \int_{\mu_0}^{\mu} \int_0^1 \frac{\partial_{\mu} b(X_s, z)^2}{a(X_s, \sigma_0)^2} ds dz \mathbb{E}(h_{\beta_0}^2(J_1)).$$

Here, we denote

$$\tilde{z}_n^i(\theta) = z_n(X_{t_i^n}, X_{t_{i+1}^n}, \theta) \text{ and } \tilde{I}_n(\theta) = -\partial_{\theta}^2 \ell_n(\theta; X).$$

Since $(\widehat{\sigma}_n^0, \widehat{\beta}_n^0) \in W_n^{(\eta)}$, we know that $\widehat{\beta}_n^0 > 1$ a.s. for any n large enough. We have $\partial_{\mu} G_n(\mu) = -n^{1-2/\widehat{\beta}_n^0} \tilde{I}_n^{1,1}(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)$ where

$$\begin{aligned} & \tilde{I}_n^{1,1}(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0) \\ &= n^{1/\widehat{\beta}_n^0} \sum_{i=0}^{n-1} \frac{\partial_{\mu}^2 \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} h_{\beta}(z_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - n^{2/\widehat{\beta}_n^0} \sum_{i=0}^{n-1} \frac{(\partial_{\mu} \xi_{1/n}(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} \partial_z h_{\beta}(z_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)). \end{aligned}$$

We rewrite $\partial_\mu G_n(\mu) - \partial_\mu G(\mu) = A_n(\mu) + B_n(\mu)$, where

$$A_n(\mu) = n \sum_{i=0}^{n-1} \frac{(\partial_\mu \xi_{1/n}(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} \partial_z h_\beta(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \int_0^1 \frac{\partial_\mu b(X_s, \mu)^2}{a(X_s, \sigma_0)^2} ds \mathbb{E}(h_{\beta_0}^2(J_1)),$$

$$B_n(\mu) = -n^{1-1/\beta_0} \sum_{i=0}^{n-1} \frac{\partial_\mu^2 \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} h_\beta(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)).$$

Our aim is to prove that $\sup_{\mu \in A} |A_n(\mu)| \xrightarrow{\mathbb{P}_{\theta_0}} 0$ and $\sup_{\mu \in A} |B_n(\mu)| \xrightarrow{\mathbb{P}_{\theta_0}} 0$. To do so, we first prove that

$$\sup_{\mu \in A} |\partial_\mu \xi_{1/n}(x, \mu) - \frac{1}{n} \partial_\mu b(x, \mu)| \leq C(1 + |x|^p)/n^2,$$

$$\sup_{\mu \in A} |\partial_\mu^2 \xi_{1/n}(x, \mu) - \frac{1}{n} \partial_\mu^2 b(x, \mu)| \leq C(1 + |x|^p)/n^2.$$

Indeed, under our assumption (A) and our additional assumptions on the regularity of the coefficient function b , the proof is classic and can be obtained by Gronwall's lemma, similarly to the proof of Proposition 5.1. Thanks to these results, we can replace $\partial_\mu^k \xi_{1/n}(x, \mu)$ by $\frac{1}{n} \partial_\mu^k b(x, \mu)$ for $k \in \{1, 2\}$ in the expressions of $A_n(\mu)$ and $B_n(\mu)$, the error for this replacement is negligible.

For $A_n(\mu)$: We rewrite this term as the sum of the two following terms

$$A_{n,1}(\mu) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{(\partial_\mu b(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} (\partial_z h_\beta(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\tilde{z}_n^i(\theta_0))),$$

$$A_{n,2}(\mu) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{(\partial_\mu b(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} \partial_z h_{\beta_0}(\tilde{z}_n^i(\theta_0)) - \int_0^1 \frac{\partial_\mu b(X_s, \mu)^2}{a(X_s, \sigma_0)^2} ds \mathbb{E}(h_{\beta_0}^2(J_1)).$$

The convergence to zero of $\sup_{\mu \in A} |A_{n,2}(\mu)|$ can be deduced using the assumptions on the regularity of functions a and b and similar arguments as for (5.29) and (5.31). Here, we only take into account the term $A_{n,1}(\mu)$. For this, we separate the difference into two parts: $\partial_z h_\beta(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0))$ and $\partial_z h_{\beta_0}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\tilde{z}_n^i(\theta_0))$. First, since $\partial_\beta \partial_z h_\beta(z)$ is bounded for any values of β and z large enough, by intermediate value theorem, we have

$$|\partial_z h_\beta(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0))| \leq C \frac{\log n}{\sqrt{n}}. \quad (5.35)$$

Second, since $\partial_z^2 h_\beta(z)$ is bounded for any values of β and z , again by intermediate value theorem,

$$\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0) - \tilde{z}_n^i(\theta_0) = \left(n^{1/\beta_0-1} \frac{a(X_{t_i^n}, \sigma_0)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} - 1 \right) \tilde{z}_n^i(\theta_0) + n^{1/\beta_0} \frac{\xi_{1/n}(X_{t_i^n}, \mu_0) - \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)}$$

and assumption (A), we have

$$\begin{aligned}
& |\partial_z h_{\beta_0}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\tilde{z}_n^i(\theta_0))| \leq C |\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0) - \tilde{z}_n^i(\theta_0)| \\
& \leq C \left(\left| n^{1/\widehat{\beta}_n^0 - 1/\beta_0} \frac{a(X_{t_i^n}, \sigma_0)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} - 1 \right| |\tilde{z}_n^i(\theta_0)| + n^{1/\widehat{\beta}_n^0} \left| \frac{\xi_{1/n}(X_{t_i^n}, \mu_0) - \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} \right| \right) \\
& \leq \frac{C}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} \left(\left| (n^{1/\widehat{\beta}_n^0 - 1/\beta_0} - 1) a(X_{t_i^n}, \sigma_0) + a(X_{t_i^n}, \sigma_0) - a(X_{t_i^n}, \widehat{\sigma}_n^0) \right| |\tilde{z}_n^i(\theta_0)| \right. \\
& \quad \left. + n^{1/\widehat{\beta}_n^0} \left| \frac{\xi_{1/n}(X_{t_i^n}, \mu_0) - \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} \right| \right) \\
& \leq C(1 + |X_{t_i^n}|^p) \left(\frac{(\log n)^2}{\sqrt{n}} |\tilde{z}_n^i(\theta_0)| + n^{1/\widehat{\beta}_n^0 - 1} \right), \tag{5.36}
\end{aligned}$$

for some $p > 0$. The last inequality is obtained by $n^{1/\widehat{\beta}_n^0 - 1/\beta_0} - 1 \sim \frac{|\widehat{\beta}_n^0 - \beta_0|}{\beta_0 \widehat{\beta}_n^0}$, the assumption (A), the intermediate value theorem using the Lipchitz property of a and an adapting result from (5.19) with our additional assumptions on the function b . Therefore, under our assumptions on the regularity of a and b , for triangular array combined with the fact that $\mathbb{E}_{\theta_0}(|\tilde{z}_n^i(\theta_0)|^\delta | \mathcal{F}_{t_i^n}) \leq C$ for $\delta < \beta_0$ by (Clément and Gloter, 2020, the proof of (4.11)), the convergence to zero of $\sup_{\mu \in A} |A_{n,1}(\mu)|$ is guaranteed from (5.35) and (5.36).

For $B_n(\mu)$: Since $h_\beta(z)$ is bounded for any values of β and z big enough, from our assumptions on the regularity of the coefficient functions, we have

$$\sup_{\mu \in A} |B_n(\mu)| \leq C n^{-1-1/\widehat{\beta}_n^0} \sum_{i=0}^{n-1} (1 + |X_{t_i^n}|^{p'}) \xrightarrow{\mathbb{P}_{\theta_0}} 0.$$

Thus, we obtain (ii). Considering the assertion (i), by the definition of G , we see immediately that $G(\mu_0) = 0$. In addition, μ_0 is the unique solution of $G(\mu) = 0$ thanks to the assumption non degeneracy that there exists $s \in [0, 1]$ such that $\partial_\mu b(X_s, \mu) \neq 0$. Now, it rests to prove that $G_n(\mu_0) \xrightarrow{\mathbb{P}_{\theta_0}} 0$. To do so, by Taylor's expansion, we have

$$\begin{aligned}
G_n(\mu_0) &= n^{1-2/\beta_0} \partial_\mu \ell_n(\mu_0, \sigma_0, \beta_0; X) \\
&+ n^{1-2/\beta_0} \int_0^1 \left(\tilde{\mathcal{I}}_n^{1,2}(\mu_0, \sigma_0 + t(\widehat{\sigma}_n^0 - \sigma_0), \beta_0 + t(\widehat{\beta}_n^0 - \beta_0)) \right)^\top \left(\widehat{\sigma}_n^0 - \sigma_0 \right) dt. \tag{5.37}
\end{aligned}$$

From this, on the one hand, from (Clément and Gloter, 2020, section 3.2.1.), $n^{1-2/\beta_0} \partial_\mu \ell_n(\mu_0, \sigma_0, \beta_0; X)$ is tight and converges to zero in probability. On the other hand, from (Clément and Gloter, 2020, section 3.2.2.), we have

$$\sup_{(\sigma, \beta) \in W_n^{(\eta)}} \left\| \frac{1}{n^{1/\beta} \log n} \begin{pmatrix} \tilde{\mathcal{I}}_n^{1,2}(\mu_0, \sigma, \beta) \\ \tilde{\mathcal{I}}_n^{1,3}(\mu_0, \sigma, \beta) \end{pmatrix} \right\|$$

is tight and since

$$\begin{aligned} \sup_{(\sigma, \beta) \in W_n^{(\eta)}} \left\| \frac{n^{1/\beta} \log n}{n^{2/\beta-1}} \left(\frac{\sigma - \sigma_0}{\beta - \beta_0} \right) \right\| &\leq C \frac{(\log n)^2}{n^{1/\beta_0-1/2}} \sup_{(\sigma, \beta) \in W_n^{(\eta)}} n^{\frac{1}{\beta_0} - \frac{1}{\beta}} \\ &\leq C \frac{(\log n)^2}{n^{1/\beta_0-1/2}} \exp((\log n)^2 / \sqrt{n}) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

the second term in the Taylor's expansion above converges to zero.

Now, for the tightness of $\frac{n^{1/\beta_0-1/2}}{(\log n)^2} (\hat{\mu}_n^0 - \mu_0)$, we proceed as follows. Since $G_n(\hat{\mu}_n^0) = 0$, by Taylor's expansion, we get

$$G_n(\mu_0) = -n^{1-2/\beta_n^0} \int_0^1 \partial_\mu^2 \ell_n(\mu_0 + t(\hat{\mu}_n^0 - \mu_0), \hat{\sigma}_n^0, \hat{\beta}_n^0; X) dt (\hat{\mu}_n^0 - \mu_0).$$

Then, reusing the decomposition of $G_n(\mu_0)$ in (5.37) and its arguments for its convergence above, combined with μ_0 -consistency of $\hat{\mu}_n^0$ proven above and the fact that $n^{1-2/\beta_0} \partial_\mu^2 \ell_n(\mu_0 + t(\hat{\mu}_n^0 - \mu_0), \hat{\sigma}_n^0, \hat{\beta}_n^0; X)$ converges to non-singular $\mathcal{I}^{1,1}(\theta_0)$ in probability uniformly from (Clément and Gloter, 2020, section 3.2.2.), the proof is completed. \square

Some more numerical results

Here, we give additional histograms for the same numerical tests discussed in Section 4 on the model (SR) (4.2) and the model (4.4).

For the square root model (SR):

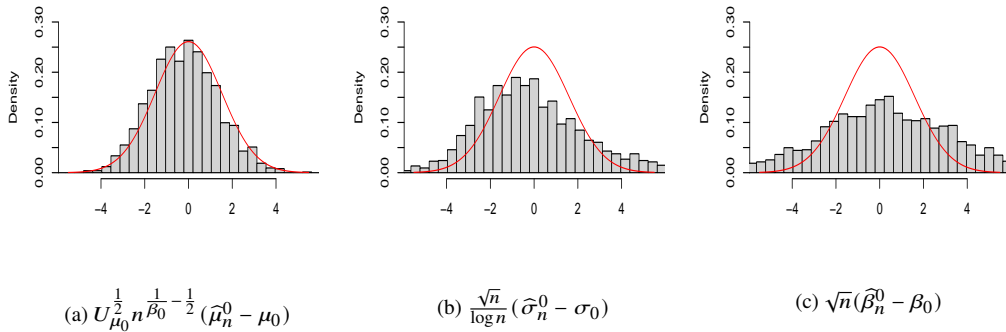


Figure 5: Distributions of the re-scaled errors of moment estimation (ME) and comparison with the asymptotic normal distribution with efficient variance given by (4.3)

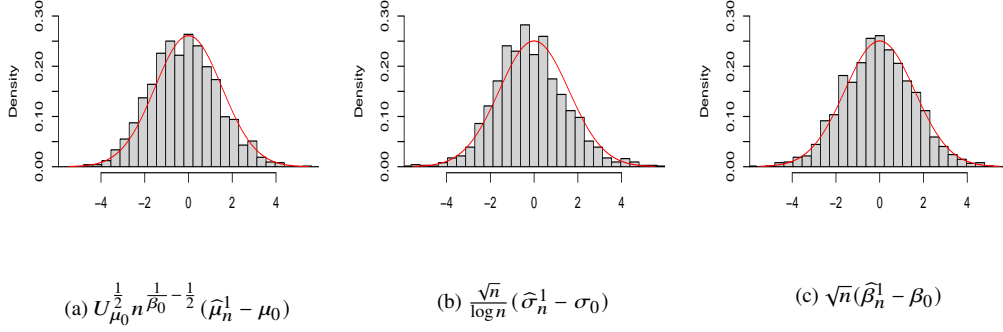


Figure 6: Distributions of the re-scaled errors of one-step estimation (OS) and comparison with their theoretical Gaussian limits

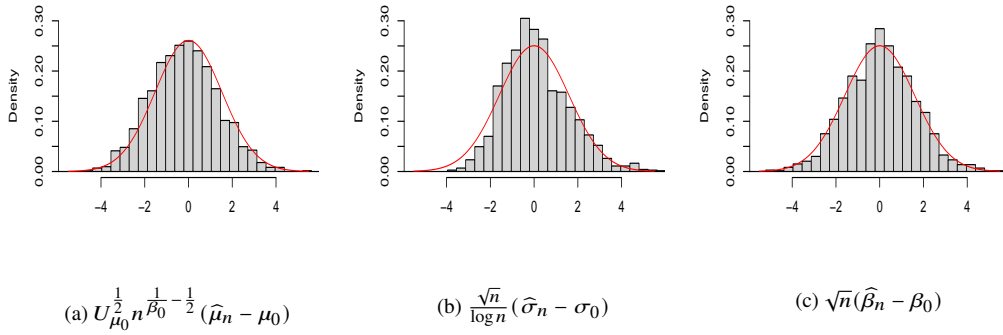


Figure 7: Distributions of the re-scaled errors of the maximum likelihood estimation (MLE) and comparison with their theoretical Gaussian limits

For the model (4.4):

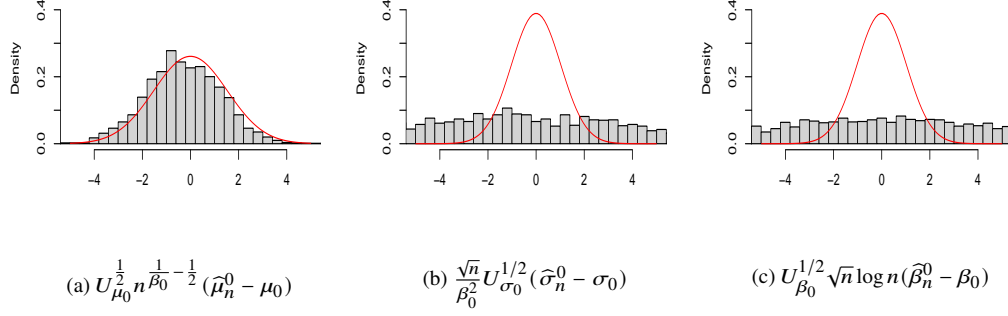


Figure 8: Histograms of the re-scaled errors of moment estimation (ME) and comparison with the asymptotic normal distribution with efficient variance

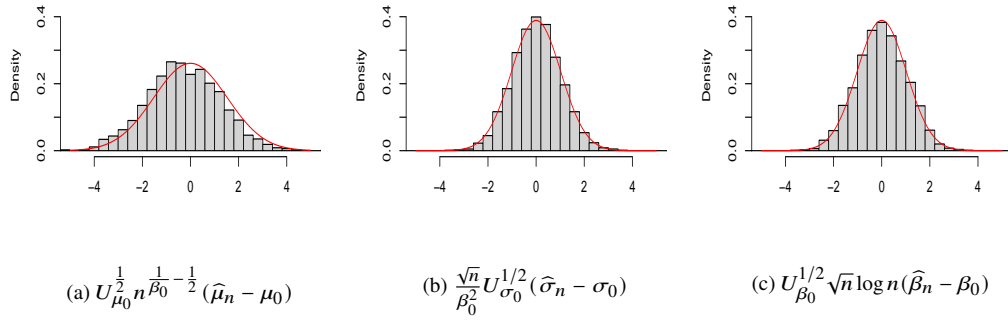


Figure 9: Histograms of the re-scaled errors of maximum-likelihood estimation (MLE) and comparison with their theoretical Gaussian limits