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# Supplement to "LAMN property for Euler type scheme of stochastic differential equations driven by a stable Lévy process and applications"

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## Proofs of some technical results

**Proof of Proposition 5.1.** The first two inequalities are recalled from Clément and Gloter (2020), here, we only need to prove the last one. First, we note that

$$\begin{aligned}
 \partial_\mu \xi_{1/n}(x, \mu) &= \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) \partial_\mu \xi_s(x, \mu) ds + \int_0^{1/n} (\partial_\mu b)(\xi_s(x, \mu), \mu) ds, \\
 \partial_\mu^2 \xi_{1/n}(x, \mu) &= \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^2 b(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^2 ds \\
 &\quad + 2 \int_0^{1/n} (\partial_\mu \partial_x b)(\xi_s(x, \mu), \mu) \partial_\mu \xi_s(x, \mu) ds + \int_0^{1/n} (\partial_\mu^2 b)(\xi_s(x, \mu), \mu) ds, \\
 \partial_\mu^3 \xi_{1/n}(x, \mu) &= \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) \partial_\mu^3 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^2 b(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds \\
 &\quad + \int_0^{1/n} (\partial_\mu \partial_x b)(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^3 b(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^3 ds \\
 &\quad + 3 \int_0^{1/n} (\partial_\mu \partial_x^2 b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^2 ds \\
 &\quad + 3 \int_0^{1/n} (\partial_\mu^2 \partial_x b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu)) ds + \int_0^{1/n} (\partial_\mu^3 b)(\xi_s(x, \mu), \mu) ds.
 \end{aligned}$$

Therefore, we get

$$\left| \partial_\mu^3 \xi_{1/n}(x, \mu) - \frac{1}{n} \partial_\mu^3 b(x, \mu) \right| \leq \int_0^{1/n} |\partial_x b(\xi_s(x, \mu), \mu)| |\partial_\mu^3 \xi_s(x, \mu) - \frac{1}{n} \partial_\mu^3 b(x, \mu)| ds + |\mathcal{R}_n(\mu)|,$$

where

$$\begin{aligned}
& \mathcal{R}_n(\mu) \\
&= \frac{1}{n} \partial_\mu^3 b(x, \mu) \int_0^{1/n} \partial_x b(\xi_s(x, \mu), \mu) ds + \int_0^{1/n} \partial_x^2 b(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds \\
&+ \int_0^{1/n} (\partial_\mu \partial_x b)(\xi_s(x, \mu), \mu) \partial_\mu^2 \xi_s(x, \mu) ds + \int_0^{1/n} \partial_x^3 b(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^3 ds \\
&+ 3 \int_0^{1/n} (\partial_\mu \partial_x^2 b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu))^2 ds + 3 \int_0^{1/n} (\partial_\mu^2 \partial_x b)(\xi_s(x, \mu), \mu) (\partial_\mu \xi_s(x, \mu)) ds \\
&+ \int_0^{1/n} [(\partial_\mu^3 b)(\xi_s(x, \mu), \mu) - \partial_\mu^3 b(x, \mu)] ds.
\end{aligned}$$

Now, for some  $s \in (0, 1/n)$ , since  $b(\cdot)$  is Lipschitz, we have

$$\begin{aligned}
|\xi_s(x, \mu)| &= |x + \int_0^s b(\xi_v(x, \mu), \mu) dv| \leq |x| + \int_0^s C(1 + |\xi_v(x, \mu)|) dv, \\
|\xi_s(x, \mu) - x| &= \left| \int_0^s b(\xi_v(x, \mu), \mu) dv \right| \leq C(|x| + 1)s + \int_0^s C|\xi_v(x, \mu) - x| dv.
\end{aligned}$$

By Gronwall's lemma

$$|\xi_s(x, \mu)| \leq (|x| + Cs)e^{Cs}, \quad \text{and } |\xi_s(x, \mu) - x| \leq C(|x| + 1)se^{Cs}, \quad s \in (0, 1/n). \quad (5.18)$$

Then, from Assumption (A) and Gronwall's lemma again, we have

$$\sup_{\mu \in V_{\mu_0}} \sup_{k \in \{1, 2, 3\}} \partial_\mu^k \xi_{1/n}(x, \mu) \leq C(1 + |x|^p)/n \quad (5.19)$$

for some  $p > 0$ . Thus, from Assumption (A) and Jensen's inequality for convex function  $x \in [1, \infty) \mapsto x^m$ ,  $m > 1$ , we easily deduce that  $\sup_{\mu \in V_{\mu_0}} |\mathcal{R}_n(\mu)| \leq C(1 + |x|^p)/n^2$  for some  $p > 0$ . Finally, combined with Gronwall's lemma, we then obtain (5.2).  $\square$

**Proof of Theorem 5.2.** First of all, let us denote  $\eta_n(s) = \frac{\lfloor ns \rfloor}{n}$  for any  $s \in [0, 1]$  and we introduce for any  $q > 0$

$$\begin{aligned}
J_t^q &= J_t - \sum_{s \leq t} \Delta J_s \mathbb{1}_{\{|\Delta J_s| > q\}}, \\
T_q &= \inf\{t > 0 : |\Delta J_t| \geq q\}
\end{aligned}$$

and  $X^q$  the solution of the following SDE:

$$X_t^q = x_0 + \int_0^t b(X_s^q, \mu_0) ds + \int_0^t a(X_{s-}^q, \sigma_0) dJ_s^q, \quad t \in [0, 1]. \quad (5.20)$$

In a natural way, we denote by  $\bar{X}^q$  the associate scheme, for any  $i \in \{0, 1, \dots, n-1\}$ , we denote

$$\bar{X}_{i+1}^q = \xi_{t_{i+1}^n - t_i^n}(\bar{X}_{t_i^n}^q, \mu) + a(\bar{X}_{t_i^n}^q, \sigma)(J_{t_{i+1}^n}^q - J_{t_i^n}^q). \quad (5.21)$$

Let  $q > 1$  and  $p \geq 2$  be fixed. It is standard that

$$\mathbb{E}(\sup_{t \in [0,1]} |X_t^q|^p) < +\infty. \quad (5.22)$$

Concerning the scheme, we also have for any  $q > 1$  and  $n$

$$\mathbb{E}(\sup_{i \in \{0,1,\dots,n\}} |\bar{X}_{t_i^n}^q|^p) \leq C_{q,p}. \quad (5.23)$$

Indeed, let  $n \geq 1$  and  $i \in \{1, 2, \dots, n-1\}$  then we have

$$\begin{aligned} \bar{X}_{t_{i+1}^n}^q &= \bar{X}_{t_i^n}^q + \int_0^{\frac{1}{n}} b(\xi_s(\bar{X}_{t_i^n}^q, \mu_0), \mu_0) ds + a(\bar{X}_{t_i^n}^q, \sigma_0)(J_{t_{i+1}^n}^q - J_{t_i^n}^q) \\ &= x_0 + \int_{t_i^n}^{t_{i+1}^n} b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) du + \int_{t_i^n}^{t_{i+1}^n} a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dJ_u^q \end{aligned}$$

Let us now introduce the process  $Z$  defined by

$$\forall t \in [0, 1], Z_t = x_0 + \int_0^t b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) du + \int_0^t a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dJ_u^q$$

so that for any  $i$ ,  $Z_{t_i^n} = \bar{X}_{t_i^n}^q$  and so

$$\forall t \in [0, 1], Z_t^* = \sup_{s \in [0,t]} |Z_s| \geq \sup_i |\bar{X}_{t_i^n}^q|. \quad (5.24)$$

Following for example [Protter \(1992\)](#) Section I.4, we know that  $J^q$  admits the following decomposition:

$$J_t^q = M_t^q + A_t^q, \quad (5.25)$$

with  $M_t^q = \int_0^t \int_{|z| \leq q} z \tilde{N}(ds, dz)$ , where  $\tilde{N}$  is the compensated Poisson measure with Lévy measure  $\frac{1}{|z|^{1+\beta}} \mathbf{1}_{[-q,q]}(z) dz$ .  $M^q$  is a martingale belonging to all the  $L^p$  spaces and  $A^q$  is a finite variation process such that its total variation satisfies

$$d|A|_t \leq \left( 2 \int_1^q \frac{z}{z^{1+\beta}} dz \right) dt.$$

Then since coefficients  $a$  and  $b$  are Lipschitz w.r.t. variable  $x$  and thanks to estimates (5.18) and (5.24), we have:

$$\begin{aligned} |Z_t^*|^p &\leq C \left( |x_0|^p + \left( \int_0^t (Z_u^* + C/n) e^{C/n} du \right)^p + \left( \int_0^t (1 + Z_u^*) du \right)^p \right. \\ &\quad \left. + \sup_{s \in [0,t]} \left| \int_0^s a(\bar{X}_{\eta_n(u)}^q, \sigma_0) dM_u^q \right|^p \right), \end{aligned} \quad (5.26)$$

where constant  $C$  above but also below denotes a constant which may vary from line to line but does not depend on  $n$ . Using ([Jacod and Protter, 2012](#), Inequality (2.1.36) in Lemma 2.1.5) (see also ([Kunita,](#)

2004, Theorem A.3)) we have:

$$\begin{aligned}
& \mathbb{E}_{\theta_0} \left( \sup_{s \in [0, t]} \left| \int_0^s a(\bar{X}_{\eta_n}^q(u), \sigma_0) dM_u^q \right|^p \right) \\
& \leq C \left( \mathbb{E}_{\theta_0} \left[ \left( \int_0^t \int_{|z| \leq q} |a(\bar{X}_{\eta_n}^q(u), \sigma_0)|^2 \frac{z^2}{|z|^{1+\beta}} dz ds \right)^{p/2} \right] + \mathbb{E}_{\theta_0} \left[ \int_0^t \int_{|z| \leq q} |a(\bar{X}_{\eta_n}^q(u), \sigma_0)|^p \frac{z^p}{|z|^{1+\beta}} dz ds \right] \right) \\
& \leq C \left( \mathbb{E}_{\theta_0} \left[ \left( \int_0^t |a(\bar{X}_{\eta_n}^q(u), \sigma_0)|^2 ds \right)^{p/2} \right] + \mathbb{E}_{\theta_0} \left[ \int_0^t |a(\bar{X}_{\eta_n}^q(u), \sigma_0)|^p ds \right] \right) \\
& \leq C \mathbb{E}_{\theta_0} \left[ \int_0^t |a(\bar{X}_{\eta_n}^q(u), \sigma_0)|^p du \right] \leq C \left( 1 + \mathbb{E}_{\theta_0} \left[ \int_0^t (Z_u^*)^p du \right] \right)
\end{aligned} \tag{5.27}$$

From this we easily get that

$$\forall t \in [0, 1], \mathbb{E}_{\theta_0} [(Z_t^*)^p] \leq C \left( 1 + \int_0^t \mathbb{E}_{\theta_0} [(Z_s^*)^p] ds \right)$$

which ensures by Gronwall's Lemma that  $\mathbb{E}_{\theta_0} [(Z_t^*)^p] \leq C_{q,p}$ .

Clearly, on  $N_q = \{T_q > 1\}$ ,  $X_t = X_t^q$  a.s. for all  $t \in [0, 1]$  and  $\bar{X} = \bar{X}^q$ . As  $\lim_{q \rightarrow +\infty} \mathbb{P}_{\theta_0}(T_q \leq 1) = 0$ , we only need to prove that for any  $q > 0$ ,

$$\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i}^q, \mu, \sigma) g_\beta(z_n^i(\theta)) - \int_0^1 f(X_s^q, \mu_0, \sigma) ds \mathbb{E}_{\theta_0}(g_{\beta_0}(J_1)) \right| \mathbb{1}_{N_q} \xrightarrow[n \rightarrow \infty]{} 0, \tag{5.28}$$

in probability under  $\mathbb{P}_{\theta_0}$ . This is implied by the following convergences in probability,

$$\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{X}_{t_i}^q, \mu, \sigma) - f(\bar{X}_{t_i}^q, \mu_0, \sigma)] g_\beta(z_n^i(\theta)) \right| \mathbb{1}_{N_q} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0, \tag{5.29}$$

$$\sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i}^q, \mu_0, \sigma) [g_\beta(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta_0))] \right| \mathbb{1}_{N_q} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0, \tag{5.30}$$

$$n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i}^q, \mu_0, \sigma) [g_{\beta_0}(z_n^i(\theta_0)) - \mathbb{E}_{\theta_0}(g_{\beta_0}(J_1))] \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0, \tag{5.31}$$

$$n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i}^q, \mu_0, \sigma) - \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i}^q, \mu_0, \sigma) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0. \tag{5.32}$$

$$n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i}^q, \mu_0, \sigma) - \int_0^1 f(X_s^q, \mu_0, \sigma) ds \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}} 0. \tag{5.33}$$

We remark then that

$$\begin{aligned}
z_n^i(\theta_0) &= n^{1/\beta_0} (J_{t_{i+1}^n} - J_{t_i^n}), \\
z_n^i(\theta) &= \frac{n^{1/\beta}}{a(\bar{X}_{t_i^n}, \sigma)} [\xi_{1/n}(\bar{X}_{t_i^n}, \mu_0) + a(\bar{X}_{t_i^n}, \sigma) (J_{t_{i+1}^n} - J_{t_i^n}) - \xi_{1/n}(\bar{X}_{t_i^n}, \mu)]
\end{aligned}$$

and that if  $\theta \in V_n^{(\eta)}(\theta_0)$ , then

$$|\mu - \mu_0| \leq \eta n^{1/2-1/\beta_0}(\log n)^2, \quad |\sigma - \sigma_0| \leq C \eta n^{-1/2} \log n \quad \text{and} \quad |\beta - \beta_0| \leq C \eta n^{-1/2} \log n. \quad (5.34)$$

For (5.29) we remark that

$$\begin{aligned} & \mathbb{E}_{\theta_0} \left( \sup_{\theta \in V_n^{(\eta)}(\theta_0)} n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{X}_{t_i^n}^q, \mu, \sigma) - f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0)] g_\beta(z_n^i(\theta)) \right| \mathbb{1}_{N_q} \right) \\ & \leq \mathbb{E}_{\theta_0} \left( C(1 + \sup_i |\bar{X}_{t_i^n}^q|^p) (\log n)^2 (n^{1/2-1/\beta_0+\varepsilon} + n^{-1/2+\varepsilon}) |g_\beta(z_n^i(\theta))| \mathbb{1}_{N_q} \right) \\ & \leq C(1 + \mathbb{E}_{\theta_0}(\sup_i |\bar{X}_{t_i^n}^q|^{2p}))^{1/2} (1 + \mathbb{E}_{\theta_0}(\sup_{\theta \in V_n^{(\eta)}(\theta_0)} |z_n^i(\theta)|^{2\kappa} \mathbb{1}_{N_q}))^{1/2} (\log n)^2 (n^{1/2-1/\beta_0+\varepsilon} + n^{-1/2+\varepsilon}) \\ & \leq C(\log n)^2 (n^{1/2-1/\beta_0+\varepsilon} + n^{-1/2+\varepsilon}) \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

where  $\kappa$  may be chosen in  $]0, \beta/2[$  thanks to the assumptions made on  $g_\beta$ . The last line follows by adapting (Clément and Gloter, 2020, the proof of (4.11)) which ensures that  $\mathbb{E}_{\theta_0}(|z_n^i(\theta)|^k | \mathcal{F}_{t_i^n}^n) \leq C$ , for any  $k < \beta_0$  and  $\sup_{\theta \in V_n^{(\eta)}(\theta_0)} |z_n^i(\theta) - z_n^i(\theta_0)| \mathbb{1}_{N_q} \leq C(1 + |\bar{X}_{t_i^n}^q|^j)(1 + |z_n^i(\theta_0)|)(\log n)^2 / \sqrt{n}$ , for some  $j > 0$ .

To prove (5.30), we proceed similarly by splitting the difference  $g_\beta(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta_0))$  into two parts:  $g_\beta(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta))$  and  $g_{\beta_0}(z_n^i(\theta)) - g_{\beta_0}(z_n^i(\theta_0))$  using the fact that  $f$  is Lipschitz w.r.t. variable  $x$ .

For (5.31), let  $\xi_i^n = n^{\varepsilon-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) [g_{\beta_0}(z_n^i(\theta_0)) - \mathbb{E}_{\theta_0}(g_{\beta_0}(J_1))]$ , we simply use the fact that  $z_n^i(\theta_0) = n^{1/\beta_0}(J_{t_{i+1}^n} - J_{t_i^n}) \stackrel{\mathcal{L}}{\cong} J_1$ , using one more time the fact that  $f$  is Lipschitz we have:

$$n \mathbb{E}_{\theta_0}(\xi_i^n | \mathcal{F}_{t_i^n}^n) = 0, \quad n \mathbb{E}_{\theta_0}((\xi_i^n)^2 | \mathcal{F}_{t_i^n}^n) = n^{2\varepsilon-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0)^2 \text{Var}_{\theta_0}(g_{\beta_0}(J_1)) \rightarrow 0$$

in probability. We then conclude that  $\sum_{i=1}^n \xi_i^n$  tends to 0 in probability using the results in Jacod and Protter (2012), Section (2.2.4) for triangular array. For (5.32), we note that

$$\begin{aligned} \bar{X}_{t_i^n}^q - X_{t_i^n}^q &= Z_{t_i^n} - X_{t_i^n}^q = \int_0^{t_i^n} (b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) - b(X_u^q, \mu_0)) du \\ & \quad + \int_0^{t_i^n} (a(\bar{X}_{\eta_n(u)}^q, \sigma_0) - a(X_{u-}^q, \sigma_0)) dJ_u^q. \end{aligned}$$

Then, we separate the first integral in the r.h.s. into three parts concerning the differences

$$\begin{aligned} & b(\xi_{u-\eta_n(u)}(\bar{X}_{\eta_n(u)}^q, \mu_0), \mu_0) - b(\bar{X}_{\eta_n(u)}^q, \mu_0) \\ & b(\bar{X}_{\eta_n(u)}^q, \mu_0) - b(X_{\eta_n(u)}^q, \mu_0) \\ & b(X_{\eta_n(u)}^q, \mu_0) - b(X_u^q, \mu_0) \end{aligned}$$

and we shall do similarly for the second integral. Now, thanks to the Lipschitz property of the coefficient functions  $x \mapsto a(x, \sigma_0)$  and  $x \mapsto b(x, \mu_0)$ , we can easily recycle the arguments from (5.26) to obtain

$$\mathbb{E}_{\theta_0} \left( \sup_{s \in [0,1]} |\bar{X}_{\eta_n(s)}^q - X_s^q|^2 \right) = \mathcal{O}(n^{-1}).$$

Finally, using Lipschitz assumption on  $f$  and Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left( n^\varepsilon \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\bar{X}_{t_i^n}^q, \mu_0, \sigma_0) - \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i^n}^q, \mu_0, \sigma_0) \right| \right) \xrightarrow{n \rightarrow +\infty} 0.$$

For (5.33), we rewrite the considering error as follows

$$n^\varepsilon \left| \int_0^1 (f(X_{\eta_n(s)}^q, \mu_0, \sigma_0) - f(X_s^q, \mu_0, \sigma_0)) ds \right|$$

and its convergence to zero in probability is directly using a part of the arguments for (5.32).

Finally, for (5.4), it is deduced from (5.3) when  $\beta_0 \leq 1$  and by same arguments above similar to (Clement and Gloter, 2019, the proof of Theorem 3.1 ii) when  $\beta_0 > 1$ .  $\square$

**Proof of Lemma 3.8.** To begin with, let us denote  $\eta_n(s) = \frac{\lfloor ns \rfloor}{n}$  for any  $s \in [0, 1]$  and

$$F_n(\sigma) = n^{\widehat{\beta}_n^0 - 1} V_n^1(p, X) - \mu_p(\widehat{\beta}_n^0) \int_0^1 |a(X_{\eta_n(s)}, \sigma)|^p ds.$$

Let us first prove that  $(\sqrt{n}/\log n)F_n(\sigma_0)$  is tight. Indeed, we know that  $\sup_{s \in [0,1]} \sqrt{n}|X_{\eta_n(s)} - X_s| \xrightarrow{\mathbb{P}_{\theta_0}} 0$

and we rewrite

$$F_n(\sigma) = n^{\widehat{\beta}_n^0 - 1} V_n^1(p, X) - \mu_p(\widehat{\beta}_n^0) \int_0^1 |a(X_s, \sigma)|^p ds + o_P(1)$$

and we conclude thanks to (Todorov, 2013, (21)) its tightness with the rate  $\sqrt{n}/\log n$ . Then, we prove that any sequence  $(\widehat{\sigma}_n^0, n \geq 1)$  belonging to  $V_{\sigma_0}$  (a neighborhood of  $\sigma_0$  defined in assumption (A)) that solves  $F_n(\sigma) = 0$  is consistent and eventually unique. To do so, using (Jacod and Sorensen, 2017, Theorem 2.7.a), since  $F_n(\sigma_0)$  converges to zero in probability, we need to verify the following two conditions:

- (i) There exists  $F$  defined on  $A$  compact subset of  $V_{\sigma_0}$ , continuously differentiable, such that  $F(\sigma_0) = 0$  and  $\sigma_0$  is the unique root of  $F(\sigma) = 0$ .
- (ii) The following convergence in probability holds

$$\sup_{\sigma \in A} |\partial_\sigma F_n(\sigma) - \partial_\sigma F(\sigma)| \rightarrow 0,$$

and  $\partial_\sigma F(\sigma)$  is non-singular with probability one.

For this proof, we choose

$$F(\sigma) = -p\mu_p(\beta_0) \int_{\sigma_0}^\sigma \int_0^1 \frac{\partial_\sigma a}{a^{1-p}}(X_s, u) ds du.$$

For this choice, the criteria (ii) is straightforward from the fact that  $\widehat{\beta}_n^0 \xrightarrow{\mathbb{P}^{\theta_0}} \beta_0$ . Considering (i), it is obvious that  $F(\sigma_0) = 0$  and  $\sigma_0$  is the unique solution of  $F(\sigma) = 0$  since from those assumptions on the function  $a$ ,

$$\int_{\sigma_0}^{\sigma} \int_0^1 \frac{\partial_{\sigma} a}{a^{1-p}}(X_s, u) ds du = 0 \Leftrightarrow \sigma = \sigma_0.$$

Moreover, we prove that  $\frac{n^{1/2}}{\log n}(\widehat{\sigma}_n^0 - \sigma_0)$  is tight. From Taylor's formula and  $F_n(\widehat{\sigma}_n^0) = 0$ , we have

$$F_n(\sigma_0) = -p\mu_p(\widehat{\beta}_n^0) \int_0^1 \frac{\partial_{\sigma} a}{a^{1-p}}(X_{\eta_n(s)}, \tilde{\sigma}_n) ds (\widehat{\sigma}_n^0 - \sigma_0)$$

where  $\tilde{\sigma}_n$  lies between  $\widehat{\sigma}_n^0$  and  $\sigma_0$ . Then, using the consistency of  $\widehat{\sigma}_n^0$  proven above and the tightness of  $(\sqrt{n}/\log n)F_n(\sigma_0)$ , we complete the proof.  $\square$

**Proof of Theorem 3.9.** Inspired by the work in [Bayraktar and Clément \(2024\)](#), using ([Jacod and Sorensen, 2017](#), Theorem 2.7.a)), for the consistency and uniqueness, we need to verify the following two conditions:

- (i) There exists  $G$  defined on  $A$ , continuously differentiable, such that  $G_n(\mu_0)$  converges to zero in probability,  $G(\mu_0) = 0$  and  $\mu_0$  is the unique root of  $G(\mu) = 0$ .
- (ii) The following convergence in probability holds

$$\sup_{\mu \in A} |\partial_{\mu} G_n(\mu) - \partial_{\mu} G(\mu)| \rightarrow 0,$$

and  $\partial_{\mu} G(\mu)$  is non-singular with probability one.

For some  $\eta > 0$ , we set

$$W_n^{(\eta)} = \left\{ (\sigma, \beta) : \left\| \frac{\sqrt{n}}{\log n} \begin{pmatrix} \sigma - \sigma_0 \\ \beta - \beta_0 \end{pmatrix} \right\| \leq \eta \right\}.$$

First, we prove the second assertion (ii) by setting

$$G(\mu) = \int_{\mu_0}^{\mu} \int_0^1 \frac{\partial_{\mu} b(X_s, z)^2}{a(X_s, \sigma_0)^2} ds dz \mathbb{E}(h_{\beta_0}^2(J_1)).$$

Here, we denote

$$\tilde{z}_n^i(\theta) = z_n(X_{t_i^n}, X_{t_{i+1}^n}, \theta) \text{ and } \tilde{I}_n(\theta) = -\partial_{\theta}^2 \ell_n(\theta; X).$$

Since  $(\widehat{\sigma}_n^0, \widehat{\beta}_n^0) \in W_n^{(\eta)}$ , we know that  $\widehat{\beta}_n^0 > 1$  a.s. for any  $n$  large enough. We have  $\partial_{\mu} G_n(\mu) = -n^{1-2/\widehat{\beta}_n^0} \tilde{I}_n^{1,1}(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)$  where

$$\begin{aligned} & \tilde{I}_n^{1,1}(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0) \\ &= n^{1/\widehat{\beta}_n^0} \sum_{i=0}^{n-1} \frac{\partial_{\mu}^2 \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} h_{\beta}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - n^{2/\widehat{\beta}_n^0} \sum_{i=0}^{n-1} \frac{(\partial_{\mu} \xi_{1/n}(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} \partial_z h_{\beta}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)). \end{aligned}$$



We rewrite  $\partial_\mu G_n(\mu) - \partial_\mu G(\mu) = A_n(\mu) + B_n(\mu)$ , where

$$A_n(\mu) = n \sum_{i=0}^{n-1} \frac{(\partial_\mu \xi_{1/n}(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} \partial_z h_\beta(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \int_0^1 \frac{\partial_\mu b(X_s, \mu)^2}{a(X_s, \sigma_0)^2} ds \mathbb{E}(h_{\beta_0}^2(J_1)),$$

$$B_n(\mu) = -n^{1-1/\beta_0} \sum_{i=0}^{n-1} \frac{\partial_\mu^2 \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} h_\beta(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)).$$

Our aim is to prove that  $\sup_{\mu \in A} |A_n(\mu)| \xrightarrow{\mathbb{P}^{\theta_0}} 0$  and  $\sup_{\mu \in A} |B_n(\mu)| \xrightarrow{\mathbb{P}^{\theta_0}} 0$ . To do so, we first prove that

$$\sup_{\mu \in A} |\partial_\mu \xi_{1/n}(x, \mu) - \frac{1}{n} \partial_\mu b(x, \mu)| \leq C(1 + |x|^p)/n^2,$$

$$\sup_{\mu \in A} |\partial_\mu^2 \xi_{1/n}(x, \mu) - \frac{1}{n} \partial_\mu^2 b(x, \mu)| \leq C(1 + |x|^p)/n^2.$$

Indeed, under our assumptions on the regularity of the coefficient function  $b(\cdot)$ , the proof is classic and can be obtained by Gronwall's lemma, similarly to the proof of Proposition 5.1. Thanks to these results, we can replace  $\partial_\mu^k \xi_{1/n}(x, \mu)$  by  $\frac{1}{n} \partial_\mu^k b(x, \mu)$  for  $k \in \{1, 2\}$  in the expressions of  $A_n(\mu)$  and  $B_n(\mu)$ , the error for this replacement is negligible.

For  $A_n(\mu)$ : We rewrite this term as the sum of the two following terms

$$A_{n,1}(\mu) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{(\partial_\mu b(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} (\partial_z h_\beta(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\bar{z}_n^i(\theta_0))),$$

$$A_{n,2}(\mu) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{(\partial_\mu b(X_{t_i^n}, \mu))^2}{a(X_{t_i^n}, \widehat{\sigma}_n^0)^2} \partial_z h_{\beta_0}(\bar{z}_n^i(\theta_0)) - \int_0^1 \frac{\partial_\mu b(X_s, \mu)^2}{a(X_s, \sigma_0)^2} ds \mathbb{E}(h_{\beta_0}^2(J_1)).$$

The convergence to zero of  $\sup_{\mu \in A} |A_{n,2}(\mu)|$  can be deduced using the assumptions on the regularity of functions  $a$  and  $b$  and similar arguments as for (5.29) and (5.31). Here, we only take into account the term  $A_{n,1}(\mu)$ . For this, we separate the difference into two parts:  $\partial_z h_\beta(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0))$  and  $\partial_z h_{\beta_0}(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\bar{z}_n^i(\theta_0))$ . First, since  $\partial_\beta \partial_z h_\beta(z)$  is bounded for any values of  $\beta$  and  $z$  large enough, by intermediate value theorem, we have

$$|\partial_z h_\beta(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0))| \leq C \frac{\log n}{\sqrt{n}}. \quad (5.35)$$

Second, since  $\partial_z^2 h_\beta(z)$  is bounded for any values of  $\beta$  and  $z$ , again by intermediate value theorem,

$$\bar{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0) - \bar{z}_n^i(\theta_0) = \left( n^{1/\beta_0 - 1/\beta_0} \frac{a(X_{t_i^n}, \sigma_0)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} - 1 \right) \bar{z}_n^i(\theta_0) + n^{1/\beta_0} \frac{\xi_{1/n}(X_{t_i^n}, \mu_0) - \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)}$$

and assumption (A), we have

$$\begin{aligned}
& |\partial_z h_{\beta_0}(\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0)) - \partial_z h_{\beta_0}(\tilde{z}_n^i(\theta_0))| \leq C |\tilde{z}_n^i(\mu, \widehat{\sigma}_n^0, \widehat{\beta}_n^0) - \tilde{z}_n^i(\theta_0)| \\
& \leq C \left( \left| n^{1/\widehat{\beta}_n^0 - 1/\beta_0} \frac{a(X_{t_i^n}, \sigma_0)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} - 1 \right| |\tilde{z}_n^i(\theta_0)| + n^{1/\widehat{\beta}_n^0} \left| \frac{\xi_{1/n}(X_{t_i^n}, \mu_0) - \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} \right| \right) \\
& \leq \frac{C}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} \left( \left| (n^{1/\widehat{\beta}_n^0 - 1/\beta_0} - 1) a(X_{t_i^n}, \sigma_0) + a(X_{t_i^n}, \sigma_0) - a(X_{t_i^n}, \widehat{\sigma}_n^0) \right| |\tilde{z}_n^i(\theta_0)| \right. \\
& \quad \left. + n^{1/\widehat{\beta}_n^0} \left| \frac{\xi_{1/n}(X_{t_i^n}, \mu_0) - \xi_{1/n}(X_{t_i^n}, \mu)}{a(X_{t_i^n}, \widehat{\sigma}_n^0)} \right| \right) \\
& \leq C(1 + |X_{t_i^n}|^p) \left( \frac{(\log n)^2}{\sqrt{n}} |\tilde{z}_n^i(\theta_0)| + n^{1/\widehat{\beta}_n^0 - 1} \right), \tag{5.36}
\end{aligned}$$

for some  $p > 0$ . The last inequality is obtained by  $n^{1/\widehat{\beta}_n^0 - 1/\beta_0} - 1 \sim \frac{|\widehat{\beta}_n^0 - \beta_0|}{\beta_0 \widehat{\beta}_n^0}$ , the assumption (A), the intermediate value theorem using the Lipschitz property of  $a$  and (5.19). Therefore, under our assumptions on the regularity of  $a$  and  $b$ , for triangular array combined with the fact that  $\mathbb{E}_{\theta_0}(|\tilde{z}_n^i(\theta_0)|^\delta | \mathcal{F}_{t_i^n}) \leq C$  for  $\delta < \beta_0$  by (Clément and Gloter, 2020, the proof of (4.11)), the convergence to zero of  $\sup_{\mu \in A} |A_{n,1}(\mu)|$  is guaranteed from (5.35) and (5.36).

For  $B_n(\mu)$ : Since  $h_\beta(z)$  is bounded for any values of  $\beta$  and  $z$  big enough, from our assumptions on the regularity of the coefficient functions, we have

$$\sup_{\mu \in A} |B_n(\mu)| \leq C n^{-1-1/\widehat{\beta}_n^0} \sum_{i=0}^{n-1} (1 + |X_{t_i^n}|^{p'}) \xrightarrow{\mathbb{P}_{\theta_0}} 0.$$

Thus, we obtain (ii). Considering the assertion (i), by the definition of  $G$ , we see immediately that  $G(\mu_0) = 0$ . In addition,  $\mu_0$  is the unique solution of  $G(\mu) = 0$  thanks to the assumption non degeneracy that there exists  $s \in [0, 1]$  such that  $\partial_\mu b(X_s, \mu) \neq 0$ . Now, it rests to prove that  $G_n(\mu_0) \xrightarrow{\mathbb{P}_{\theta_0}} 0$ . To do so, by Taylor's expansion, we have

$$\begin{aligned}
G_n(\mu_0) &= n^{1-2/\beta_0} \partial_\mu \ell_n(\mu_0, \sigma_0, \beta_0; X) \\
&+ n^{1-2/\beta_0} \int_0^1 \begin{pmatrix} \tilde{\mathcal{I}}_n^{1,2}(\mu_0, \sigma_0 + t(\widehat{\sigma}_n^0 - \sigma_0), \beta_0 + t(\widehat{\beta}_n^0 - \beta_0)) \\ \tilde{\mathcal{I}}_n^{1,3}(\mu_0, \sigma_0 + t(\widehat{\sigma}_n^0 - \sigma_0), \beta_0 + t(\widehat{\beta}_n^0 - \beta_0)) \end{pmatrix}^\top \begin{pmatrix} \widehat{\sigma}_n^0 - \sigma_0 \\ \widehat{\beta}_n^0 - \beta_0 \end{pmatrix} dt. \tag{5.37}
\end{aligned}$$

From this, on the one hand, from (Clément and Gloter, 2020, section 3.2.1.),  $n^{1-2/\beta_0} \partial_\mu \ell_n(\mu_0, \sigma_0, \beta_0; X)$  is tight and converges to zero in probability. On the other hand, from (Clément and Gloter, 2020, section 3.2.2.), we have

$$\sup_{(\sigma, \beta) \in \mathcal{W}_n^{(\eta)}} \left\| \frac{1}{n^{1/\beta} \log n} \begin{pmatrix} \tilde{\mathcal{I}}_n^{1,2}(\mu_0, \sigma, \beta) \\ \tilde{\mathcal{I}}_n^{1,3}(\mu_0, \sigma, \beta) \end{pmatrix} \right\|$$

is tight and since

$$\begin{aligned} \sup_{(\sigma, \beta) \in W_n^{(\eta)}} \left\| \frac{n^{1/\beta} \log n}{n^{2/\beta-1}} \begin{pmatrix} \sigma - \sigma_0 \\ \beta - \beta_0 \end{pmatrix} \right\| &\leq C \frac{(\log n)^2}{n^{1/\beta_0-1/2}} \sup_{(\sigma, \beta) \in W_n^{(\eta)}} n^{\frac{1}{\beta_0} - \frac{1}{\beta}} \\ &\leq C \frac{(\log n)^2}{n^{1/\beta_0-1/2}} \exp((\log n)^2 / \sqrt{n}) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

the second term in the Taylor's expansion above converges to zero.

Now, for the tightness of  $\frac{n^{1/\beta_0-1/2}}{(\log n)^2} (\widehat{\mu}_n^0 - \mu_0)$ , we proceed as follows. Since  $G_n(\widehat{\mu}_n^0) = 0$ , by Taylor's expansion, we get

$$G_n(\mu_0) = -n^{1-2/\beta_n^0} \int_0^1 \partial_{\mu}^2 \ell_n(\mu_0 + t(\widehat{\mu}_n^0 - \mu_0), \widehat{\sigma}_n^0, \widehat{\beta}_n^0; X) dt (\widehat{\mu}_n^0 - \mu_0).$$

Then, reusing the decomposition of  $G_n(\mu_0)$  in (5.37) and its arguments for its convergence above, combined with  $\mu_0$ -consistency of  $\widehat{\mu}_n^0$  proven above and the fact that  $n^{1-2/\beta_0} \partial_{\mu}^2 \ell_n(\mu_0 + t(\widehat{\mu}_n^0 - \mu_0), \widehat{\sigma}_n^0, \widehat{\beta}_n^0; X)$  converges to non-singular  $\mathcal{I}^{1,1}(\theta_0)$  in probability uniformly from (Clément and Gloter, 2020, section 3.2.2.), the proof is completed.  $\square$

## Some more numerical results

Here, we give additional histograms for the same numerical tests discussed in Section 4 on the model (SR) (4.2) and the model (4.4).

For the square root model (SR):

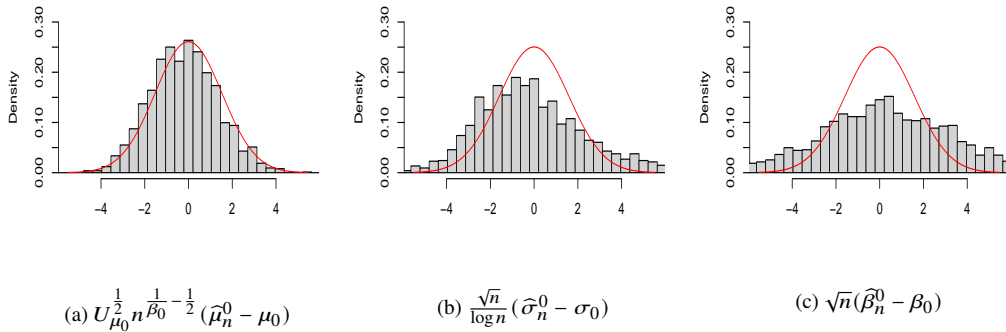


Figure 5: Distributions of the re-scaled errors of moment estimation (ME) and comparison with the asymptotic normal distribution with efficient variance given by (4.3)

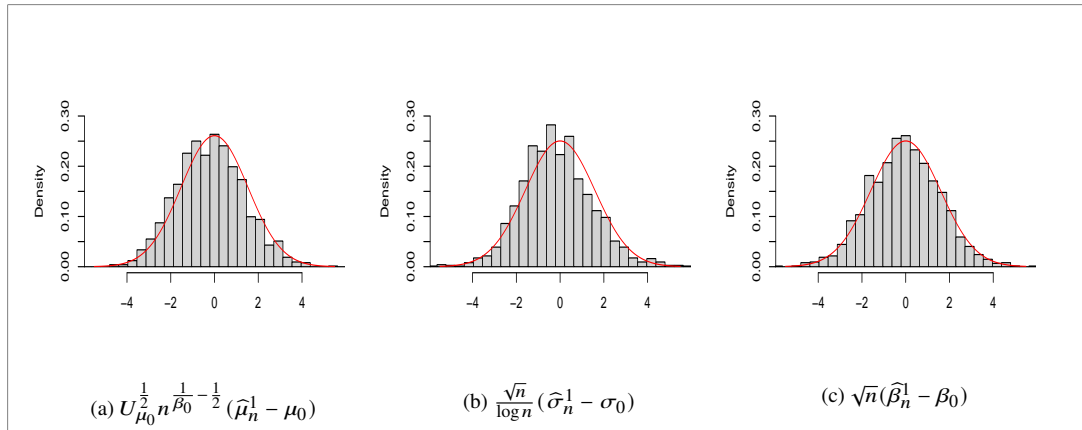


Figure 6: Distributions of the re-scaled errors of one-step estimation (OS) and comparison with their theoretical Gaussian limits

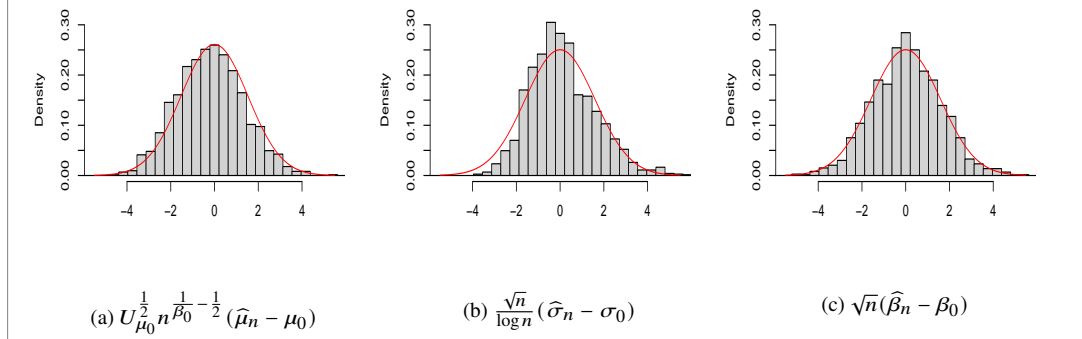


Figure 7: Distributions of the re-scaled errors of the maximum likelihood estimation (MLE) and comparison with their theoretical Gaussian limits

For the model (4.4):

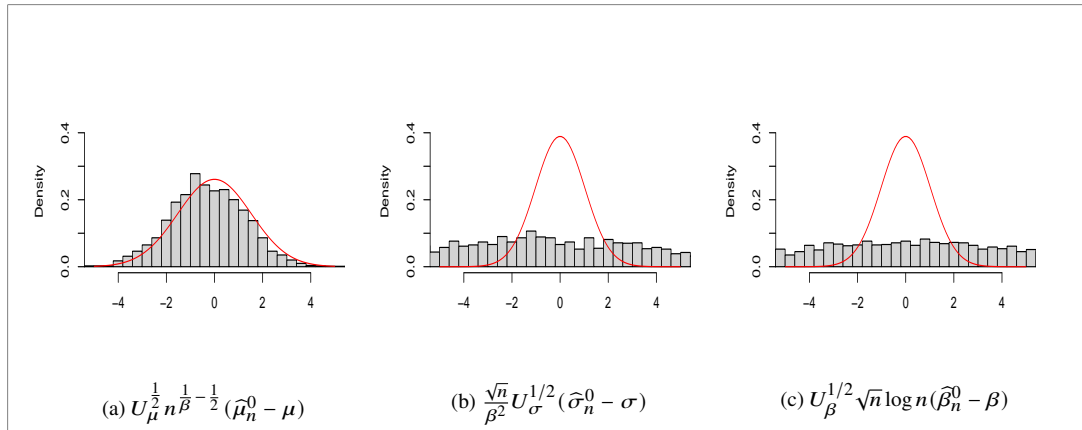


Figure 8: Histograms of the re-scaled errors of moment estimation (ME) and comparison with the asymptotic normal distribution with efficient variance

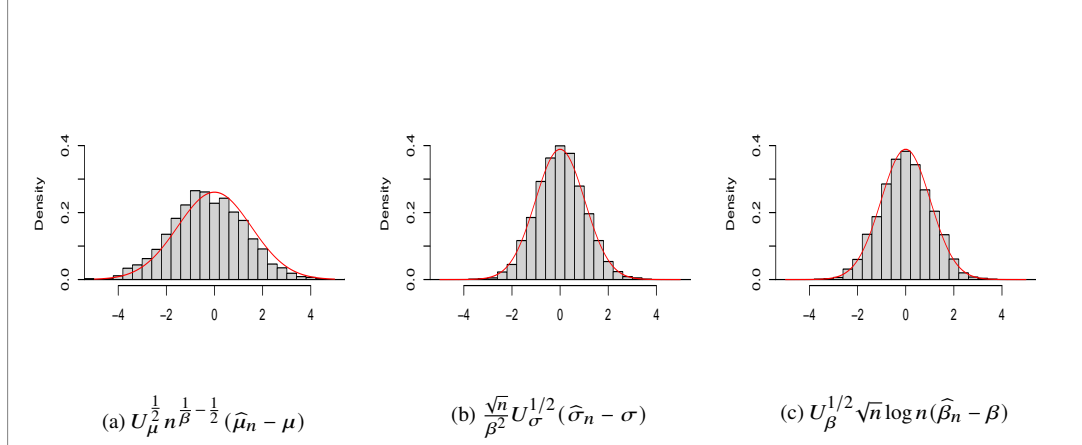


Figure 9: Histograms of the re-scaled errors of maximum-likelihood estimation (MLE) and comparison with their theoretical Gaussian limits