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# Poisson imbedding meets the Clark-Ocone formula \*

Caroline Hillairet<sup>†</sup>      Thomas Peyrat<sup>‡</sup>      Anthony Réveillac<sup>§</sup>

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## Abstract

In this paper we develop a representation formula of Clark-Ocone type for any integrable Poisson functionals, which extends the Poisson imbedding for point processes. This representation formula differs from the classical Clark-Ocone formula on three accounts. First the representation holds with respect to the Poisson measure instead of the compensated one; second the representation holds true in  $L^1$  and not in  $L^2$ ; and finally contrary to the classical Clark-Ocone formula the integrand is defined as a pathwise operator and not as a  $L^2$ -limiting object. We make use of Malliavin's calculus and of the pseudo-chaotic decomposition with uncompensated iterated integrals to establish this Pseudo-Clark-Ocone representation formula and to characterize the integrand, which turns out to be a predictable integrable process.

**Keywords:** Hawkes processes; Poisson imbedding representation; Malliavin's calculus; Clark-Ocone formula.

**Mathematics Subject Classification (2020):** 60G55; 60G57; 60H07.

## 1 Introduction

Martingale representation formulas for Poisson functionals have been widely investigated in the literature (such as Løkka [?], Nualart and Schoutens [?]) in relation with stochastic analysis tools such as the chaotic expansion or the Malliavin calculus. At the crossroad of martingale representation and the Malliavin calculus lies the so-called Clark-Ocone<sup>1</sup> formula which allows one to provide a description of the integrand in terms of the Malliavin derivative. More precisely, on the Poisson space, the Clark-Ocone formula gives the following representation of a square-integrable random variable  $F$  :

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+} (D_t F)^p \tilde{N}(dt)$$

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<sup>1</sup>also referred as Clark-Ocone-Haussmann formula

where  $\tilde{N}$  is the compensated Poisson process and  $(D_t F)^p$  is the predictable projection of the Malliavin derivative of  $F$ . It is important to note that in nature the Clark-Ocone formula and the martingale representation slightly differ as the latter may exist when the former fails to hold (like for instance for the fractional Brownian motion which is not a semimartingale with respect to its natural filtration). In addition the Clark-Ocone formula is restricted to representation of random variables whereas martingale representation provides a dynamic representation of the integrand for a given martingale. Yet in a Poisson framework both representations are intricated. Various kinds of generalizations have been obtained. Using the Malliavin integration by parts formula given in Picard [?], Zhang [?] establishes the following Clark-Ocone formula: for any bounded Poisson functional  $F$

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+ \times \mathbb{X}} (\mathbf{D}_{(t,x)} F)^p \tilde{N}(dt, dx)$$

where  $\tilde{N}$  is the compensated Poisson random measure, and  $(\mathbf{D}_{(t,x)} F)^p$  is the predictable projection of the Malliavin derivative of  $F$ . Last and Penrose [?] also give a Clark-Ocone type martingale representation formula when the underlying filtration is generated by a Poisson process on a measurable space. Flint and Torrisi [?] provide a Clark-Ocone formula for point processes on a finite interval possessing a conditional intensity. Di Nunno and Vives [?] develop a Malliavin-Skorohod type calculus for additive processes and obtain a generalization of the Clark-Ocone formula for random variables in  $L^1$  whose integrand operator is supposed to be in  $L^1$  as well. At the same time, the so-called Poisson imbedding (see Jacod [?, Chapter 4] or Brémaud and Massoulié [?]) provides also a representation of any point process with respect to an uncompensated Poisson measure. More precisely, if  $H$  is a point process on  $\mathbb{R}_+$  with stochastic intensity  $\lambda$ , then these pair of processes  $(H, \lambda)$  can be represented on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  supporting a random Poisson measure  $\mathbf{N}$  on  $\mathbb{R}_+ \times \mathbb{R}_+$  and the following representation holds true :

$$H_T = \int_{(0,T] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t\}} \mathbf{N}(dt, d\theta); \quad \forall T > 0. \quad (1.1)$$

Yet Hillairet and Réveillac [?] investigate the so called pseudo-chaotic expansion which involves also non-compensated iterated integrals. This recent contribution leads the way to the study of a Clark-Ocone representation formula with respect to the non-compensated Poisson measure  $\mathbf{N}$ , which in a sense will generalize the Poisson imbedding relation (1.1). The aim of this paper is to investigate which Poisson-functionals admit such Pseudo-Clark-Ocone decomposition and to determine the integrand.

More precisely, we consider a Poisson random measure  $\mathbf{N}$  defined on  $(\mathbf{X}, \mathfrak{X}) := ([0, T] \times \mathbb{X}; \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{X})$  equipped with a non-atomic  $\sigma$ -finite measure  $\rho := dt \otimes \pi$ , and where  $(\mathbb{X}, \mathcal{X})$  is a complete separable metric space. The Pseudo-Clark-Ocone formula is stated in Theorem 3.8 for any  $F$  in  $L^1(\Omega)$  (under the assumption that  $\rho(\mathbf{X}) < +\infty$ ) :

$$F = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx), \quad \mathbb{P} - a.s.. \quad (1.2)$$

The operator  $\mathcal{H}F$  is defined as follows:

$$\mathcal{H} : \begin{array}{ll} L^0(\Omega) & \rightarrow L^0(\Omega \times \mathbf{X}) \\ F & \mapsto \mathcal{H}_{(t,x)} F := \mathbf{D}_{(t,x)} F \circ \tau_t \end{array}$$

where  $\mathbf{D}_{(t,x)}$  is the Malliavin derivative operator and

$$\tau_t : \boldsymbol{\omega} = \sum_{j=1}^n \delta_{(t_j, x_j)} \quad \rightarrow \quad \boldsymbol{\omega}_t := \sum_{j=1}^n \delta_{(t_j, x_j)} \mathbf{1}_{\{t_j < t\}}.$$

Let us emphasize that this Pseudo-Clark-Ocone formula holds for any  $F$  in  $L^1(\Omega)$  and its integrand  $\mathcal{H}F$  is well defined pathwise and is proved to belong to  $\mathbb{L}_p^1$  (see Theorems 3.8 and 3.9). This is a major difference with the standard Clark-Ocone formula that requires  $F \in L^2(\Omega)$  and for which the integrand  $(\mathbf{D}F)^p$  is only defined as a limit in  $L^2(\Omega \times \mathbf{X})$  (see Remark 3.3).

The paper is organized as follows. Notations and the description of the Poisson space and elements of Malliavin's calculus are presented in Section 2, as well as the operators and iterated integrals of the (pseudo)-chaotic expansions. The main contribution is stated in Section 3, which recalls the standard and pseudo chaotic expansions and then derives the Clark-Ocone formula with respect to the Poisson measure. Finally Section 4 gathers some technical lemmata.

## 2 Elements of Malliavin's calculus on the Poisson space

We introduce in this section some elements and notions of stochastic analysis on a general Poisson space. All the elements presented in this section are taken from Last [?, ?].

### 2.1 The Poisson space

$\mathbb{N}^* := \mathbb{N} \setminus \{0\}$  denotes the set of positive integers and for any finite set  $S$ ,  $|S|$  denotes its cardinal. We fix  $(\mathbb{X}, \mathcal{X})$  a complete separable metric space equipped with a non-atomic  $\sigma$ -finite measure  $\pi$  and we set  $(\mathbf{X}, \mathfrak{X}) := ([0, T] \times \mathbb{X}; \mathcal{B}([0, T]) \otimes \mathcal{X})$  equipped with the non-atomic  $\sigma$ -finite measure  $\boldsymbol{\rho} := dt \otimes \pi$  with  $dt$  the Lebesgue measure on  $\mathbb{R}_+$  and  $T$  is a fixed positive real number. Throughout this paper we will make use of the following notation :

**Notation 2.1.** We denote with bold letters  $\mathbf{x}$  elements in  $\mathbf{X}$  and for  $\mathbf{x} \in \mathbf{X}$  we set  $\mathbf{x} := (t, x)$  with  $t \in \mathbb{R}_+$  and  $x \in \mathbb{X}$ .

We define  $\Omega$  the space of configurations on  $\mathbf{X}$  as

$$\Omega := \left\{ \boldsymbol{\omega} = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathbf{X}, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\}. \quad (2.1)$$

**Notation 2.2.** We also denote by  $\boldsymbol{\omega}_\emptyset \in \Omega$  the unique element  $\boldsymbol{\omega}$  of  $\Omega$  such that  $\boldsymbol{\omega}(A \times B) = 0$  for any  $A \times B \in \mathfrak{X}$ .

Let  $\mathcal{F}$  be the  $\sigma$ -field associated to the vague topology on  $\Omega$ . Let  $\mathbb{P}$  the Poisson measure on  $\Omega$  under which the canonical evaluation  $\mathbf{N}$  defines a Poisson random measure with intensity measure  $\boldsymbol{\rho}$ . To be more precise given any element  $A \times B$  in  $\mathfrak{X}$  with  $\boldsymbol{\rho}(A \times B) > 0$ , the random variable

$$(\mathbf{N}(\boldsymbol{\omega}))(A \times B) := \boldsymbol{\omega}(A \times B)$$

is a Poisson random variable with intensity  $\boldsymbol{\rho}(A \times B)$  and  $\tilde{\mathbf{N}}$  defined as

$$(\tilde{\mathbf{N}}(\boldsymbol{\omega}))(A \times B) := \boldsymbol{\omega}(A \times B) - \boldsymbol{\rho}(A \times B)$$

is the compensated Poisson measure. We define the natural history associated to  $\mathbf{N}$

$$\mathcal{F}_t^{\mathbf{N}} := \sigma\{\mathbf{N}(A \times B); (A, B) \in \mathfrak{X}; A \subset [0, t]\}$$

and  $\mathcal{F}^{\mathbf{N}} := \sigma\{\mathbf{N}(A \times B); (A, B) \in \mathfrak{X}\}$ . For any  $F \in L^1(\Omega, \mathcal{F}^{\mathbf{N}})$  and any  $t \geq 0$  we denote  $\mathbb{E}_t[F] := \mathbb{E}[F | \mathcal{F}_t^{\mathbf{N}}]$ . We also set for any  $t > 0$ ,

$$\mathcal{F}_{t-}^{\mathbf{N}} := \bigvee_{0 < s < t} \mathcal{F}_s^{\mathbf{N}}.$$

We set  $\mathcal{P}^{\mathbf{N}}$  the predictable  $\sigma$ -field associated to  $\mathcal{F}^{\mathbf{N}}$  that is the  $\sigma$ -field on  $[0, T] \times \Omega$  generated by left-continuous  $\mathcal{F}^{\mathbf{N}}$ -adapted stochastic processes and we set  $\mathcal{P} := \mathcal{P}^{\mathbf{N}} \otimes \mathcal{B}(\mathbb{X})$  the set of real-valued predictable processes  $X : \Omega \times [0, T] \times \mathbb{X}$ . Let

$$\mathbb{L}_{\mathcal{P}}^0 := \{X : \Omega \times [0, T] \times \mathbb{X} \text{ which are } \mathcal{P} \text{-measurable}\},$$

and for  $r > 0$

$$\mathbb{L}_{\mathcal{P}}^r := \{X \in \mathbb{L}_{\mathcal{P}}^0, \|X\|_{\mathbb{L}_{\mathcal{P}}^r} < +\infty\}, \text{ with } \|X\|_{\mathbb{L}_{\mathcal{P}}^r} := \mathbb{E} \left[ \int_{[0, T] \times \mathbb{X}} |X_{(t, x)}|^r dt \pi(dx) \right]^{1/r}.$$

According to Jacod [?] (see also Protter [?, Corollary 3]), for any  $X \in \mathbb{L}_{\mathcal{P}}^2$ , the process  $t \mapsto \int_{[0, t] \times \mathbb{X}} X_{(r, x)} \tilde{\mathbf{N}}(dr, dx)$  is a  $\mathcal{F}^{\mathbf{N}}$ -martingale which satisfies the following  $L^2$ -isometry

$$\mathbb{E} \left[ \left| \int_{[0, t] \times \mathbb{X}} X_{(r, x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] = \mathbb{E} \left[ \int_{[0, t] \times \mathbb{X}} |X_{(r, x)}|^2 dr \pi(dx) \right], \quad \forall t \in [0, T]. \quad (2.2)$$

## 2.2 Add-points operators and the Malliavin derivative

We introduce some elements of Malliavin's calculus on Poisson processes. We set

$$L^0(\Omega) := \{F : \Omega \rightarrow \mathbb{R}, \mathcal{F}^{\mathbf{N}} \text{-measurable}\},$$

$$L^2(\Omega) := \{F \in L^0(\Omega), \mathbb{E}[|F|^2] < +\infty\}.$$

Similarly

$$L^0(\mathbf{X}^j) := \{f : \mathbf{X}^j \rightarrow \mathbb{R}, \mathfrak{X}^{\otimes j} \text{-measurable}\}$$

and for  $r \in \{1, 2\}$ , for  $j \in \mathbb{N}^*$

$$L^r(\mathbf{X}^j) := \left\{ f \in L^0(\mathbf{X}^j), \|f\|_{L^r(\mathbf{X}^j)}^r := \int_{\mathbf{X}^j} |f(\mathbf{x}_1, \dots, \mathbf{x}_j)|^r \rho^{\otimes j}(d\mathbf{x}_1 \cdots d\mathbf{x}_j) < +\infty \right\}. \quad (2.3)$$

Besides,

$$L_s^r(\mathbf{X}^j) := \{f \in L^r(\mathbf{X}^j) \text{ and } f \text{ is symmetric}\} \quad (2.4)$$

is the space of square-integrable symmetric mappings where we recall that  $f : \mathbf{X}^j \rightarrow \mathbb{R}$  is said to be symmetric if for any element  $\sigma$  in  $\mathcal{S}_j$  (the set of all permutations of  $\{1, \dots, j\}$ ),

$$f(\mathbf{x}_1, \dots, \mathbf{x}_j) = f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(j)}), \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_j) \in \mathbf{X}^j.$$

Given  $f : \mathbf{X}^j \rightarrow \mathbb{R}$ , we write  $\tilde{f}$  its symmetrization defined as :

$$\tilde{f}(\mathbf{x}_1, \dots, \mathbf{x}_j) := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_j} f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(j)}).$$

The main ingredient we will make use of are the add-points operators on the Poisson space  $\Omega$ .

**Definition 2.3.** [Add-points operators] Given  $n \in \mathbb{N}^*$ , and  $J := \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathbf{X}$  a subset of  $\mathbf{X}$  with  $|J| = n$ , we set the measurable mapping :

$$\begin{aligned} \varepsilon_J^{+,n} : \Omega &\longrightarrow \Omega \\ \omega &\longmapsto \omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}}. \end{aligned}$$

Note that by definition

$$\omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}} = \omega + \sum_{j=1}^n \delta_{\mathbf{x}_j} \mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$$

that is we add the atoms  $\mathbf{x}_j$  to the path  $\omega$  unless they already were part of it (which is the meaning of the term  $\mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$ ). Note that since  $\rho$  is assumed to be atomless, given a set  $J$  as above,  $\mathbb{P}[\mathbf{N}(J) = 0] = 1$  hence in what follows we will simply write  $\omega + \sum_{j=1}^n \delta_{\mathbf{x}_j}$  for  $\varepsilon_{\mathbf{x}}^{+,n}(\omega)$ .

We now recall the Malliavin derivative operator.

**Definition 2.4.** For  $F$  in  $L^0(\Omega)$ ,  $n \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$ , we set

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F \circ \varepsilon_J^{+,|J|}, \quad (2.5)$$

where we recall that  $\emptyset \subset \mathbf{X}$ . For instance when  $n = 1$ , we write  $\mathbf{D}_{\mathbf{x}} F := \mathbf{D}_{\mathbf{x}}^1 F = F(\cdot + \delta_{\mathbf{x}}) - F(\cdot)$  which is the difference operator (also called add-one cost operator<sup>2</sup>). Relation (2.5) rewrites as

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F(\omega) = \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} F \left( \omega + \sum_{j \in J} \delta_{\mathbf{x}_j} \right), \quad \text{for a.e. } \omega \in \Omega.$$

Note that with this definition, for any  $\omega$  in  $\Omega$ , the mapping

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F(\omega)$$

belongs to  $L_s^0(\mathbf{X}^j)$  defined as (2.4) and in addition the mapping

$$T_n F : (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbb{E}[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F] \quad (2.6)$$

is well-defined and belongs to  $L_s^2(\mathbf{X}^j)$  for any  $F$  in  $L^2(\Omega)$  (see [?, ?]).

We recall the following property (see *e.g.* [?, Relation (15)]) : for  $F$  in  $L^2(\Omega)$ ,  $n \in \mathbb{N}^*$ , and  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$ , the  $n$ th iterated Malliavin's derivative operator  $\mathbf{D}^n$  satisfies

$$\mathbf{D}^n F = \mathbf{D}(\mathbf{D}^{n-1} F), \quad n \geq 1; \quad \mathbf{D}^0 F := F. \quad (2.7)$$

**Remark 2.5.** If  $F$  is deterministic, then by definition  $\mathbf{D}^n F = 0$  for any  $n \geq 1$ .

<sup>2</sup>see [?, p. 5]

## 2.3 Factorial measures and iterated integrals

We first introduce a purely deterministic operator that will be at the core of the pseudo-chaotic expansion.

**Definition 2.6.** For  $F \in L^0(\Omega)$ , we define the deterministic operators:

$$\mathcal{T}_0 F := F(\omega_\emptyset),$$

and for  $n \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$ ,

$$\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n) := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F \left( \sum_{\mathbf{y} \in J} \delta_{\mathbf{y}} \right).$$

In particular, even though  $F$  is a random variable,  $\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is a real number as each term  $F(\varpi_J)$  is the evaluation of  $F$  at the outcome  $\varpi_J$  (where  $\varpi_J$  denotes  $\sum_{\mathbf{y} \in J} \delta_{\mathbf{y}}$ ). Besides, given the event  $\{\mathbf{N}(\mathbf{X}) = 0\}$ ,  $\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)$  coincides with  $\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F$  and  $\mathcal{T}_0 F$  coincides with  $F$ .

**Proposition 2.7.** (Factorial measures; See e.g. [?, Proposition 1]). There exists a unique sequence of counting random measures  $(\mathbf{N}^{(m)})_{m \in \mathbb{N}^*}$  where for any  $m$ ,  $\mathbf{N}^{(m)}$  is a counting random measure on  $(\mathbf{X}^m, \mathfrak{X}^{\otimes m})$  with

$$\begin{aligned} \mathbf{N}^{(1)} &:= \mathbf{N} \quad \text{and} \quad \text{for} \quad A \in \mathfrak{X}^{m+1}, \\ \mathbf{N}^{(m+1)}(A) &:= \int_{\mathfrak{X}^m} \left[ \int_{\mathfrak{X}} \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_{m+1}) \in A\}} \mathbf{N}(d\mathbf{x}_{m+1}) - \sum_{j=1}^m \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}_j) \in A\}} \mathbf{N}^{(m)}(d\mathbf{x}_1, \dots, d\mathbf{x}_m) \right] \end{aligned}$$

With this definition at hand we introduce the notion of iterated integrals. In particular for  $A \in \mathfrak{X}$ ,

$$\mathbf{N}^{(n)}(A^{\otimes n}) = \mathbf{N}(A)(\mathbf{N}(A) - 1) \times \dots \times (\mathbf{N}(A) - n + 1).$$

Note that by definition  $\mathbf{N}^{(n)}(A) \mathbf{1}_{\{\mathbf{N}(A) < n\}} = 0$ . We now turn to the definition of two types of iterated integrals.

**Definition 2.8.** Let  $n \in \mathbb{N}^*$  and  $f_n \in L^1(\mathbf{X}^n)$ .

- $\mathcal{I}_n(f_n)$  the  $n$ th iterated integral of  $f_n$  against the Poisson measure  $\mathbf{N}$  defined as

$$\mathcal{I}_n(f_n) := \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n),$$

where each of the integrals above is well-defined pathwise for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ .

- $\mathbf{I}_n(f_n)$  the  $n$ th iterated integral of  $f_n$  against the compensated Poisson measure  $\tilde{\mathbf{N}}$  defined as

$$\mathbf{I}_n(f_n) := \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \boldsymbol{\rho}^{\otimes(n-|J|)}(d\mathbf{x}_{J^c}),$$

where  $J^c := \{1, \dots, n\} \setminus J$  and  $d\mathbf{x}_J := (dx_j)_{j \in J}$  and where each of the integrals above is well-defined pathwise for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . We adopt the convention for  $J = \emptyset$  that  $\int \dots \mathbf{N}^{(0)} := 1$ . It is worth noting that in the Poisson framework the iterated integrals  $\mathbf{I}_n$  can be constructed in  $L^1$  as integrals with respect to the factorial measures in a Lebesgue-Stieltjes fashion; or as the elements of the chaotic expansion which gives a construction of  $L^2(\Omega)$  as an orthonormal sum of subsets named chaos. We refer *e.g.* to [?] for a description of this construction but we recall in particular the isometry  $\mathbb{E}[|\mathbf{I}_n(f_n)|^2] = \|f_n\|_{L^2(\mathbf{X}^n)}^2$  whenever  $f_n$  belongs to  $L^2(\mathbf{X}^n)$ .

Let us also remark that by definition  $\mathbf{I}_n(f_n) = \mathbf{I}_n(\tilde{f}_n)$  and  $\mathcal{I}_n(f_n) = \mathcal{I}_n(\tilde{f}_n)$  where  $\tilde{f}_n$  denotes the symmetrization of  $f_n$ .

To conclude this section, we recall a particular case of Mecke's formula (see *e.g.* [?, Relation (11)]).

**Lemma 2.9** (A particular case of Mecke's formula). *Let  $F \in L^0(\Omega)$ ,  $k \in \mathbb{N}$  and  $h \in L^0(\mathbf{X}^k)$  then*

$$\mathbb{E} \left[ F \int_{\mathbf{X}^k} h d\mathbf{N}^{(k)} \right] = \int_{\mathbf{X}^k} h(\mathbf{x}_1, \dots, \mathbf{x}_k) \mathbb{E} \left[ F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_k}^{+,k} \right] \rho^{\otimes k}(d\mathbf{x}_1, \dots, d\mathbf{x}_k),$$

*providing the right-hand side is well defined as an integral in  $L^1(\mathbf{X}^k)$ . The so-called integration by parts formula is then obtained with the same assumptions*

$$\mathbb{E} [F \mathbf{I}_k(h)] = \int_{\mathbf{X}^k} h(\mathbf{x}_1, \dots, \mathbf{x}_k) \mathbb{E} \left[ \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] \rho^{\otimes k}(d\mathbf{x}_1, \dots, d\mathbf{x}_k). \quad (2.8)$$

### 3 Standard/Pseudo Chaotic expansion and Clark-Ocone formula

In the literature, results on representations of Poisson functionals are mainly concentrated on the chaos expansion involving iterated compensated integrals operators  $\mathbf{I}_n$ . Similarly the classical Clark-Ocone formula is stated with respect to the compensated Poisson measure  $\tilde{\mathbf{N}}$ . Yet [?] investigates the so called pseudo-chaotic expansion which involves iterated non-compensated integrals operators  $\mathcal{I}_n$ . This recent contribution leads the way to the study of a Clark-Ocone representation formula with respect to the non-compensated Poisson measure  $\mathbf{N}$ .

#### 3.1 Chaotic expansion and the Clark-Ocone formula

Let us first recall the classical chaotic expansion and the Clark-Ocone formula with respect to the compensated Poisson measure  $\tilde{\mathbf{N}}$ .

**Theorem 3.1** (See *e.g.* Theorem 2 in [?]). *Let  $F$  in  $L^2(\Omega)$ . Then there exists a unique sequence  $(f_n^F)_{n \geq 1}$  with  $f_n^F \in L_s^2(\mathbf{X}^n)$  such that*

$$F = \mathbb{E}[F] + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(f_n^F),$$

*where the convergence of the series holds in  $L^2(\Omega)$ . In addition coefficients  $(f_n^F)_n$  are given as*

$$f_n^F = T_n F, \quad n \geq 1$$



where  $T_n F$  is defined by (2.6) and where the equality is understood in  $L^2(\mathbf{X}^n)$ . In addition Theorem 1 in [?] provides the convergence of the series

$$\sum_{n=1}^{+\infty} \int_{\mathbf{X}^n} |T_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)|^2 \rho^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n).$$

The other classical relation on the Poisson space is the Clark-Ocone formula, for which one can find in the literature different variants with different conditions, such as e.g. Privault [?], Zhang [?], Di Nunno and Vives [?]. In our setting it takes the following form.

**Theorem 3.2** (Clark-Ocone formula). *Let  $F$  in  $L^2(\Omega)$ . Then*

$$F = \mathbb{E}[F] + \int_{[0, T] \times \mathbb{X}} (\mathbf{D}_{(t,x)} F)^p \tilde{\mathbf{N}}(dt, dx)$$

where the process  $(\mathbf{D}F)^p$  belongs to  $\mathbb{L}_{\mathcal{P}}^2$  and is in this context the predictable projection of  $\mathbf{D}F$  which is properly defined in the Appendix.

**Remark 3.3.** Contrary to what it suggests, the meaning of the integrand in the stochastic integral has to be made precise. According to Definition 2.4,  $\mathbf{D}F$  belongs to  $L^0(\Omega \times \mathbf{X})$  whenever  $F$  belongs to  $L^0(\Omega)$ ; however, the definition of the operator  $(\mathbf{D}F)^p$  is not guaranteed. For instance  $F$  in  $L^2(\Omega \times \mathbf{X})$  does not entail  $\mathbf{D}F \in L^2(\Omega \times \mathbf{X})$ . Several sufficient conditions can be found in the literature (see Last [?]). In general as we will make it precise in the proof of Theorem 3.2 (see Section 5), it is possible to define  $(\mathbf{D}F)^p$  in  $L^2(\Omega \times \mathbf{X})$  in a limiting procedure and based on the continuous feature of the mapping  $F \mapsto (\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F])_{(t,x)}$  together with the fact that for fixed  $(t, x)$ ,  $\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F] = (\mathbf{D}_{(t,x)} F)^p \mathbb{P} - a.s.$  (see [?]). For sake of completeness, we reproduce the proof of this result in the Appendix (Section 5) following [?, Section 3.2]. Note that even in the case of the Clark-Ocone formula obtained in [?] where the  $L^2$  assumption on  $F$  is relaxed to  $L^1$ , the integrand operator is assumed to belong to  $L^1$  as well. In Theorem 3.8 below (Pseudo-Clark-Ocone formula) the integrability of the integrand is a consequence of  $F$  in  $L^1$ .

### 3.2 Pseudo-chaotic expansion and Pseudo-Clark-Ocone formula

We recall here the pseudo-chaotic expansion obtained with respect to the non-compensated iterated integrals operators  $\mathcal{I}_n$ .

**Theorem 3.4** (See Theorem 2 in [?]). *Assume  $\rho(\mathbf{X}) < +\infty$  and  $F$  in  $L^2(\Omega)$ . Then*

$$F = F(\omega_\emptyset) + \sum_{k=1}^{+\infty} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s. \quad (3.1)$$

Inspired by this pseudo-chaotic expansion, we introduce below the operator that will allow us to write a Clark-Ocone formula with respect to the non-compensated Poisson measure (that we called Pseudo-Clark-Ocone formula).

**Definition 3.5.** For fixed  $t > 0$ , we consider the measurable transformation  $\tau_t$

$$\tau_t : \begin{array}{l} \Omega \rightarrow \Omega \\ \omega \mapsto \omega_t \end{array}$$

where for  $\boldsymbol{\omega}$  of the form  $\boldsymbol{\omega} := \sum_{j=1}^n \delta_{(t_j, x_j)}$ ,  $(t_j, x_j) \in \mathbf{X}$ ,  $j = 1, \dots, n$ ,  $n \in \mathbb{N} \cup \{+\infty\}$ , we set

$$\boldsymbol{\omega}_t := \sum_{j=1}^n \delta_{(t_j, x_j)} \mathbf{1}_{\{t_j < t\}}.$$

We also set  $\tau_0$  as

$$\tau_0(\boldsymbol{\omega}) := \boldsymbol{\omega}_\emptyset; \quad \boldsymbol{\omega} \in \Omega$$

where we recall Notation 2.2.

Roughly speaking,  $\tau_t$  plays the role of a  $\mathcal{F}_t^{\mathbf{N}}$ -conditional expectation.

**Definition 3.6.** Fix  $(t, x)$  in  $\mathbf{X}$ . For  $F \in L^0(\Omega)$ , we define the operator  $\mathcal{H}$  as :

$$\begin{aligned} \mathcal{H} : L^0(\Omega) &\rightarrow L^0(\Omega \times \mathbf{X}) \\ F &\mapsto \mathcal{H}_{(t,x)} F := \mathbf{D}_{(t,x)} F \circ \tau_t \end{aligned}$$

that is for  $\boldsymbol{\omega} \in \Omega$

$$\mathcal{H}_{(t,x)} F(\boldsymbol{\omega}) := \mathbf{D}_{(t,x)} F(\boldsymbol{\omega}_t) = F(\boldsymbol{\omega}_t \circ \varepsilon_{(t,x)}^+) - F(\boldsymbol{\omega}_t).$$

As highlighted in the following lemma, it turns out that the operator  $\mathcal{H}$  can be written as the combination of the Malliavin derivative with a very specific Girsanov transformation  $L^{T,t}$ . This is exactly the (non-equivalent) Girsanov transformation which is at the core of the pseudo-chaotic expansion (see [?]) and under which the Poisson measure  $\mathbf{N}$  (with intensity  $\boldsymbol{\rho}$  on  $\mathbf{X}$ ) becomes a Poisson measure with intensity 0 on  $[t, T] \times \mathbb{X}$ .

**Lemma 3.7.** Assume  $\boldsymbol{\rho}(\mathbf{X}) < +\infty$  and  $F \in L^0(\Omega)$ . Fix  $(t, x) \in (0, T) \times \mathbb{X}$ . Then the operator  $\mathcal{H}_{(t,x)}$  re-writes as

$$\mathcal{H}_{(t,x)} F = \mathbb{E}_{t-}[L^{T,t} \mathbf{D}_{(t,x)} F], \quad \mathbb{P} - a.s.$$

$$\text{with } L^{T,t} := \exp((T-t)\pi(\mathbb{X})) \mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}}.$$

*Proof.* Let  $\boldsymbol{\omega} \in \Omega$ .

$$\begin{aligned} (L^{T,t} \mathbf{D}_{(t,x)} F)(\boldsymbol{\omega}) &= \exp((T-t)\pi(\mathbb{X})) \mathbf{1}_{\{\boldsymbol{\omega}([t,T] \times \mathbb{X})=0\}} \mathbf{D}_{(t,x)} F(\boldsymbol{\omega}) \\ &= (\exp((T-t)\pi(\mathbb{X})) \mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}} \mathbf{D}_{(t,x)} F \circ \tau_t)(\boldsymbol{\omega}). \end{aligned}$$

Hence since  $\mathbf{D}_{(t,x)} F \circ \tau_t$  is  $\mathcal{F}_{t-}^{\mathbf{N}}$ -measurable

$$\begin{aligned} \mathbb{E}_{t-}[L^{T,t} \mathbf{D}_{(t,x)} F] &= \mathbf{D}_{(t,x)} F \circ \tau_t \exp((T-t)\pi(\mathbb{X})) \mathbb{E}_{t-}[\mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}}] \\ &= (F(\boldsymbol{\omega}_t \circ \varepsilon_{(t,x)}^+) - F(\boldsymbol{\omega}_t)) \exp((T-t)\pi(\mathbb{X})) \mathbb{E}_{t-}[\mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}}] \\ &= \mathcal{H}_{(t,x)} F. \end{aligned}$$

□

We have now all the elements to state the Pseudo-Clark-Ocone formula for any  $F$  in  $L^1(\Omega)$ , first under the condition that  $\boldsymbol{\rho}(\mathbf{X}) < +\infty$ .

**Theorem 3.8.** Assume  $\rho(\mathbf{X}) < +\infty$ . Let  $F$  in  $L^1(\Omega)$ . Then  $\mathcal{H}F \in \mathbb{L}_{\mathcal{P}}^1$  and

$$F = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx), \mathbb{P} - a.s..$$

In addition this decomposition is unique in the sense that if there exists  $(c, Z) \in \mathbb{R} \times \mathbb{L}_{\mathcal{P}}^1$  such that

$$F = c + \int_{[0,T] \times \mathbb{X}} Z_{(t,x)} \mathbf{N}(dt, dx), \mathbb{P} - a.s.. \quad (3.2)$$

then  $c = F(\omega_\emptyset)$  and for a.-e.  $(t, x) \in [0, T] \times \mathbb{X}$ ,  $Z_{(t,x)} = \mathcal{H}_{(t,x)} F$ ,  $\mathbb{P}$ -a.s..

*Proof.* To highlight the main keys of the proof, some technical lemmata (namely Lemma 4.1 and Lemma 4.2) are postponed in Section 4.

**Uniqueness:** Assume Relation (3.2) holds. Then note that  $F(\omega_\emptyset) = F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} = c$ . Fix  $(t, x) \in \mathbf{X}$ ,  $t > 0$ . We have for  $\mathbb{P}$ -a.e.  $\omega$  in  $\Omega$ , since  $Z \in \mathbb{L}_{\mathcal{P}}^1$

$$\begin{aligned} \mathcal{H}_{(t,x)} F(\omega) &= \mathbf{D}_{(t,x)} F(\omega_t) \\ &= \int_{[0,T] \times \mathbb{X}} (Z_{(s,y)}(\omega_t + \delta_{(t,x)}) - Z_{(s,y)}(\omega_t)) \omega_t(ds, dy) + Z_{(t,x)}(\omega_t + \delta_{(t,x)}) \\ &= Z_{(t,x)}(\omega_t + \delta_{(t,x)}) = Z_{(t,x)}(\omega_t) = Z_{(t,x)}(\omega). \end{aligned}$$

**Existence:** We proceed in several parts.

Step 1. Since  $\rho(\mathbf{X}) < +\infty$ ,  $\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0] > 0$ . Let  $F \in L^2(\Omega)$ . Thus  $\lim_{p \rightarrow +\infty} \mathbb{E}[|F - F^k|^2] = 0$  with  $F^k := \mathbb{E}[F] + \sum_{n=1}^k \mathbf{I}_n(T_n F)$ . By Lemma 4.2 and the uniqueness of the pseudo-chaotic expansion, for any  $k$ ,

$$F^k = F^k(\omega_\emptyset) + \sum_{n=1}^k \mathcal{I}_n(\mathcal{T}_n F^k), \text{ and } F^k = F^k(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F^k \mathbf{N}(dt, dx),$$

and  $\mathcal{H}_{(t,x)} F^k$  belongs to  $\mathbb{L}_{\mathcal{P}}^1$ . In addition by Lemma 4.1,  $\mathcal{H}F$  is well-defined in  $\mathbb{L}_{\mathcal{P}}^1$  and

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[ \left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F^k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] = 0.$$

Hence

$$\begin{aligned} & \mathbb{E} \left[ \left| F - F(\omega_\emptyset) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] \\ & \leq \mathbb{E} \left[ \left| F^k - F^k(\omega_\emptyset) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F^k \mathbf{N}(dt, dx) \right| \right] \\ & + \mathbb{E} \left[ \left| \int_{[0,T] \times \mathbb{X}} (\mathcal{H}_{(t,x)} F^k - \mathcal{H}_{(t,x)} F) \mathbf{N}(dt, dx) \right| \right] \\ & + \mathbb{E} \left[ |F - F^k| \right] + |F(\omega_\emptyset) - F^k(\omega_\emptyset)| \rightarrow_{k \rightarrow +\infty} 0 \end{aligned}$$

as  $F(\omega_\emptyset) - F^p(\omega_\emptyset) = (\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0])^{-1} \mathbb{E}[|F - F^k| \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}] \rightarrow_{p \rightarrow +\infty} 0$ .

Step 2. Assume  $F \in L^1(\Omega)$ . Let for  $k \geq 1$ ,  $F^k := -k \vee F \wedge k$ . Then by monotone convergence theorem  $\lim_{k \rightarrow +\infty} \mathbb{E}[|F - F^k|] = 0$ . For any  $k$ , as  $F^k \in L^2(\Omega)$ , by Step 1,

$$F^k = F^k(\omega_\emptyset) + \int_{\mathbf{X}} \mathcal{H}_{(t,x)} F^k \mathbf{N}(dt, dx).$$

By definition,

$$\begin{aligned} |F^k(\omega_\emptyset) - F(\omega_\emptyset)| &= ((\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0])^{-1}) \mathbb{E}[|F^k(\omega_\emptyset) - F(\omega_\emptyset)|] \\ &= ((\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0])^{-1}) \mathbb{E}[|F(\omega_\emptyset)| \mathbf{1}_{\{|F(\omega_\emptyset)| > k\}}] \\ &\rightarrow_{k \rightarrow +\infty} 0. \end{aligned}$$

Finally by Lemma 4.1,  $\mathcal{H}F \in \mathbb{L}_{\mathcal{P}}^1$  and

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[ \left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F^k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] = 0$$

leading to  $F = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx)$ .  $\square$

We now extend the previous result to the case  $\rho(\mathbf{X}) = +\infty$ .

**Theorem 3.9.** *Assume  $\rho(\mathbf{X}) = +\infty$ . Let  $F$  in  $L^1(\Omega)$ . Then*

$$F = F(\omega_\emptyset) + \int_{\mathbb{R}_+ \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx), \mathbb{P} - a.s$$

and for any subset  $\mathbf{R} \subset \mathbf{X}$  with  $\rho(\mathbf{R}) < +\infty$ ,  $\mathcal{H}F \mathbf{1}_{\{\mathbf{R}\}} \in \mathbb{L}_{\mathcal{P}}^1$ . In addition if there exists  $Z \in \mathbb{L}_{\mathcal{P}}^0$  such that

$$\mathbb{E} \left[ \left| \int_{\mathbb{R}_+ \times \mathbb{X}} Z_{(t,x)} \mathbf{N}(dt, dx) \right| \right] < +\infty$$

and

$$F = F(\omega_\emptyset) + \int_{\mathbb{R}_+ \times \mathbb{X}} Z_{(t,x)} \mathbf{N}(dt, dx), \mathbb{P} - a.s.$$

then for a.-e.  $(t, x) \in \mathbb{R}_+ \times \mathbb{X}$ ,  $Z_{(t,x)} = \mathcal{H}_{(t,x)} F$ ,  $\mathbb{P}$ -a.s..

*Proof.* The proof is divided in two parts.

**Uniqueness:** First note that contrary to Theorem 3.8 the value of the random variable  $F$  on the set  $\mathbf{N}(\mathbb{R}_+ \times \mathbb{X}) = 0$  is not uniquely defined as  $\mathbb{P}[\mathbf{N}(\mathbb{R}_+ \times \mathbb{X}) = 0] = 0$ . The same proof as in Theorem 3.8 gives that for a.e.  $(t, x) \in \mathbb{R}_+ \times \mathbb{X}$ ,  $\mathcal{H}_{(t,x)} F = Z_{(t,x)}$ ,  $\mathbb{P}$ -a.s..

**Existence:** As  $\rho$  is a non-atomic  $\sigma$ -finite measure on  $(\mathbf{X}, \mathfrak{X})$ , there exists a family of sets  $(\mathbf{R}_j)_j$  such that for any  $j$ ,  $\rho(\mathbf{R}_j) < +\infty$ ,  $\mathbf{R}_j \subset \mathbf{R}_{j+1}$  and  $\cup_j \mathbf{R}_j = \mathbf{X}$ . For any  $j$  consider

$$\Omega_j := \left\{ \omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathbf{R}_j, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\} \subset \Omega$$

so that  $\Omega_j \nearrow \Omega$  when  $j$  tends to  $+\infty$ . Let  $F \in L^1(\Omega)$ . As  $\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0] = 0$ , once again we set  $F(\omega_\emptyset) := 0$ . Fix  $j \geq 1$  and consider  $F_j := F \mathbf{1}_{\{\Omega_j\}}$  so that  $F_j \in L^1(\Omega)$ . Writing  $\mathbf{N}^j$

the projection of  $\mathbf{N}$  on  $\mathbf{R}_j$  (that is  $\mathbf{N}^j(A \times B) := \mathbf{N}((A \times B) \cap \mathbf{R}_j)$ ) places ourselves in the framework of Theorem 3.8 and thus

$$F_j = F(\omega_\emptyset) + \int_{\mathbf{R}_j} \mathcal{H}_{(t,x)} F_j \mathbf{N}^j(dt, dx), \quad \mathbb{P} - a.s.$$

and  $\mathcal{H} F_j \mathbf{1}_{\{\mathbf{R}_j\}} \in \mathbb{L}_{\mathcal{P}}^1$ . By definition  $\mathbf{N}^j$  coincides with  $\mathbf{N}$  on  $\mathbf{R}_j$  and for  $(t, x)$  in  $\mathbf{R}_j$  we have for any  $\omega \in \Omega_j$

$$\begin{aligned} (\mathcal{H}_{(t,x)} F_j)(\omega) &= (\mathbf{D}_{(t,x)} F^j \circ \tau_t)(\omega) = F^j(\omega_t + \delta_{(t,x)}) - F^j(\omega_t) \\ &= F(\omega_t + \delta_{(t,x)}) \mathbf{1}_{\{\omega_t + \delta_{(t,x)} \in \Omega_j\}} - F(\omega_t) \mathbf{1}_{\{\omega_t \in \Omega_j\}} \\ &= (\mathbf{D}_{(t,x)} F \circ \tau_t)(\omega) \mathbf{1}_{\{\omega_t \in \Omega_j\}} = (\mathcal{H}_{(t,x)} F)(\omega) \mathbf{1}_{\{\omega_t \in \Omega_j\}}. \end{aligned}$$

So  $\mathbf{1}_{\{\Omega_j\}} \mathcal{H}_{(t,x)} F_j = \mathbf{1}_{\{\Omega_j\}} \mathcal{H}_{(t,x)} F$  and thus  $F_j = \int_{\mathbf{R}_j} \mathcal{H}_{(t,x)} F \mathbf{N}^j(dt, dx)$ ,  $\mathbb{P} - a.s.$

Note also that by definition of  $\mathbf{N}^j$ ,  $F_j = \int_{\mathbb{R}_+ \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}^j(dt, dx)$ ,  $\mathbb{P} - a.s.$

By monotone convergence,  $F^j$  converges to  $F$  in  $L^1(\Omega)$  so

$$\lim_{j \rightarrow +\infty} \mathbb{E} \left[ \left| \int_{\mathbb{R}_+ \times \mathbf{X}} \mathcal{H}_{(t,x)} F \mathbf{N}^j(dt, dx) - \int_{\mathbb{R}_+ \times \mathbf{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] = 0$$

proving that  $\int_{\mathbb{R}_+ \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \in L^1(\Omega)$ . □

**Remark 3.10.** For the case of point processes, the Pseudo-Clark-Ocone formula stated in Theorem 3.8 coincides with the Poisson imbedding representation (1.1). Indeed let  $H$  be a point process with predictable intensity  $\lambda$  defined through the thinning procedure

$$H_T = \int_{(0,T] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t\}} \mathbf{N}(dt, d\theta); \quad \forall T > 0.$$

Then  $H_T(\omega_\emptyset) = 0$ , and using the previous representation of  $H_T$ , the difference operator is

$$\begin{aligned} \mathbf{D}_{(t,x)} H_T &= H_T(\cdot + \delta_{(t,x)}) - H_T(\cdot) \\ &= \mathbf{1}_{\{x \leq \lambda_t\}} + \int_t^T (\mathbf{1}_{\{\theta \leq \lambda_r \circ \varepsilon_{(t,x)}^+\}} - \mathbf{1}_{\{\theta \leq \lambda_r\}}) \mathbf{N}(dr, d\theta). \end{aligned}$$

Therefore  $\mathbf{D}_{(t,x)} H_T \circ \tau_t = \mathbf{1}_{\{x \leq \lambda_t\}}$  which is  $\mathcal{F}_t$ -measurable. Thus (1.1) re-writes as

$$H_T = H_T(\omega_\emptyset) + \int_{[0,T] \times [0,\infty[} \mathcal{H}_{(t,x)} H_T \mathbf{N}(dt, dx), \quad \mathbb{P} - a.s..$$

Note that the standard Clark-Ocone formula is much more tricky to compute, since one has to compute the conditional expectation (given  $\mathcal{F}_{t-}$ ) of the integral term

$$\begin{aligned} H_T &= \mathbb{E}(H_T) + \int_{[0,T] \times [0,\infty[} \left( \mathbb{E}_{t-} \left[ \mathbf{1}_{\{x \leq \lambda_t\}} + \int_t^T (\mathbf{1}_{\{\theta \leq \lambda_r \circ \varepsilon_{(t,x)}^+\}} - \mathbf{1}_{\{\theta \leq \lambda_r\}}) \mathbf{N}(dr, d\theta) \right] \right) \tilde{\mathbf{N}}(dt, dx) \\ &= \mathbb{E} \left( \int_{[0,T]} \lambda_t dt \right) + \int_{[0,T] \times [0,\infty[} \mathbf{1}_{\{x \leq \lambda_t\}} \tilde{\mathbf{N}}(dt, dx) \\ &\quad + \int_{[0,T] \times [0,\infty[} \left( \mathbb{E}_{t-} \left[ \int_t^T (\mathbf{1}_{\{\theta \leq \lambda_r \circ \varepsilon_{(t,x)}^+\}} - \mathbf{1}_{\{\theta \leq \lambda_r\}}) \mathbf{N}(dr, d\theta) \right] \right) \tilde{\mathbf{N}}(dt, dx). \end{aligned}$$

The following section gathers some technical lemmata.

## 4 Technical lemmata

We first prove convergence results of the operator  $\mathcal{H}_{(t,x)}$ . More precisely:

**Lemma 4.1.** *Assume  $\rho(\mathbf{X}) < +\infty$ . Let  $r \in [1, 2]$  and  $F \in L^r(\Omega)$ . Assume there exists  $(G_k)_k \subset L^r(\Omega)$  converging in  $L^r(\Omega)$  to  $F$  such that for any  $k$ ,  $\mathcal{H}G_k$  belongs to  $\mathbb{L}_{\mathcal{P}}^r$ . Then*

$$\lim_{k \rightarrow +\infty} \left( \int_{[0,T] \times \mathbb{X}} \mathbb{E} [|\mathcal{H}_{(t,x)}G_k - \mathcal{H}_{(t,x)}F|^r] dt\pi(dx) \right)^{1/r} = 0.$$

In particular, the stochastic process  $\mathcal{H}F$  belongs to  $\mathbb{L}_{\mathcal{P}}^r$  and we have the following convergence in  $L^1$

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[ \left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}G_k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F \mathbf{N}(dt, dx) \right| \right] = 0.$$

*Proof.* We adapt the proof of [?, Lemma 2.3] and give only the main arguments. By definition of the Malliavin derivative (see Definition 2.4) and Lemma 3.7.

$$\begin{aligned} & \left( \int_{[0,T] \times \mathbb{X}} \mathbb{E} [|\mathcal{H}_{(t,x)}G_k - \mathcal{H}_{(t,x)}F|^r] dt\pi(dx) \right)^{1/r} \\ &= \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} \left| \mathbb{E}_{t-} [L^{T,t} \mathbf{D}_{(t,x)}G_k] - \mathbb{E}_{t-} [L^{T,t} \mathbf{D}_{(t,x)}F] \right|^r dt\pi(dx) \right]^{1/r} \\ &= \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} \left| \mathbb{E}_{t-} [L^{T,t} (\mathbf{D}_{(t,x)}G_k - \mathbf{D}_{(t,x)}F)] \right|^r dt\pi(dx) \right]^{1/r} \\ &\leq \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |L^{T,t} (\mathbf{D}_{(t,x)}G_k - \mathbf{D}_{(t,x)}F)|^r dt\pi(dx) \right]^{1/r} \\ &= \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} (L^{T,t})^r |\mathbf{D}_{(t,x)}G_k - \mathbf{D}_{(t,x)}F|^r dt\pi(dx) \right]^{1/r} \\ &\leq C_r \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} (L^{T,t})^r |G_k \circ \varepsilon_{(t,x)}^+ - F \circ \varepsilon_{(t,x)}^+|^r dt\pi(dx) \right]^{1/r} \\ &+ C_r \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} (L^{T,t})^r |G_k - F|^r dt\pi(dx) \right]^{1/r}, \end{aligned}$$

where  $C_r > 0$  is a combinatorial constant depending only on  $r$ . Using Mecke's formula (see Lemma 2.9)

$$\begin{aligned} & \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} (L^{T,t})^r |G_k \circ \varepsilon_{(t,x)}^+ - F \circ \varepsilon_{(t,x)}^+|^r dt\pi(dx) \right]^{1/r} \\ &= \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} \exp(r(T-t)\pi(\mathbb{X})) |F \circ \varepsilon_{(t,x)}^+ - G_k \circ \varepsilon_{(t,x)}^+|^r \mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}} dt\pi(dx) \right]^{1/r} \end{aligned}$$

$$\begin{aligned}
&\leq \exp(T\pi(\mathbb{X}))\mathbb{E}\left[\int_{[0,T]\times\mathbb{X}}|F\circ\varepsilon_{(t,x)}^+ - G_k\circ\varepsilon_{(t,x)}^+|^r\mathbf{1}_{\{\mathbf{N}((t,T]\times\mathbb{X})=0\}}dt\pi(dx)\right]^{1/r} \\
&= \exp(T\pi(\mathbb{X}))\mathbb{E}\left[\int_{[0,T]\times\mathbb{X}}|F\circ\varepsilon_{(t,x)}^+ - G_k\circ\varepsilon_{(t,x)}^+|^r\mathbf{1}_{\{\mathbf{N}((t,T]\times\mathbb{X})=0\}}\circ\varepsilon_{(t,x)}^+dt\pi(dx)\right]^{1/r} \\
&= \exp(T\pi(\mathbb{X}))\mathbb{E}\left[\int_{[0,T]\times\mathbb{X}}|F - G_k|^r\mathbf{1}_{\{\mathbf{N}((t,T]\times\mathbb{X})=0\}}\mathbf{N}(dt,dx)\right]^{1/r} \\
&= \exp(T\pi(\mathbb{X}))\mathbb{E}\left[|F - G_k|^r\int_{[0,T]\times\mathbb{X}}\mathbf{1}_{\{\mathbf{N}((t,T]\times\mathbb{X})=0\}}\mathbf{N}(dt,dx)\right]^{1/r}.
\end{aligned}$$

Fix  $\omega = \sum_{j=1}^n \delta_{(t_j, x_j)}$ ,  $(t_j, x_j) \in [0, T] \times \mathbb{X}$  and  $n \in \mathbb{N}$  (as  $\pi(\mathbb{X}) < +\infty$ ,  $\omega([0, T] \times \mathbb{X}) < +\infty$   $\mathbb{P}$ -a.s.). We have

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{X}}\mathbf{1}_{\{\mathbf{N}((t,T]\times\mathbb{X})=0\}}\mathbf{N}(dt,dx)(\omega) \\
&= \sum_{j=1}^n \mathbf{1}_{\{\omega((t_j, T] \times \mathbb{X})=0\}} \\
&= \mathbf{1}_{\{\omega((t_n, T] \times \mathbb{X})=0\}} = \mathbf{1}_{\{\omega((0, T] \times \mathbb{X})>0\}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times\mathbb{X}}(L^{T,t})^r|G_k\circ\varepsilon_{(t,x)}^+ - F\circ\varepsilon_{(t,x)}^+|^r dt\pi(dx)\right]^{1/r} \\
&\leq \exp(rT\pi(\mathbb{X}))\mathbb{E}[|F - G_k|^r\mathbf{1}_{\{\mathbf{N}((0,T]\times\mathbb{X})>0\}}]^{1/r} \\
&\leq \exp(rT\pi(\mathbb{X}))\mathbb{E}[|F - G_k|^r]^{1/r} \rightarrow_{k \rightarrow +\infty} 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times\mathbb{X}}(L^{T,t})^r|G_k - F|^r dt\pi(dx)\right]^{1/r} \\
&= \mathbb{E}\left[|G_k - F|^r\int_{[0,T]\times\mathbb{X}}(L^{T,t})^r dt\pi(dx)\right]^{1/r} \\
&\leq \exp(T\pi(\mathbb{X}))(T\pi(\mathbb{X}))^{1/r}\mathbb{E}[|G_k - F|^r]^{1/r} \rightarrow_{k \rightarrow +\infty} 0.
\end{aligned}$$

In addition as  $\mathcal{H}G_k$  converges in  $\mathbb{L}_{\mathcal{P}}^r$  it converges pointwise and thus  $\mathcal{H}F$  belongs to  $\mathbb{L}_{\mathcal{P}}^r$ . Furthermore

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{[0,T]\times\mathbb{X}}\mathcal{H}_{(t,x)}G_k\mathbf{N}(dt,dx) - \int_{[0,T]\times\mathbb{X}}\mathcal{H}_{(t,x)}F\mathbf{N}(dt,dx)\right|\right] \\
&\leq \mathbb{E}\left[\int_{[0,T]\times\mathbb{X}}|\mathcal{H}_{(t,x)}G_k - \mathcal{H}_{(t,x)}F|\mathbf{N}(dt,dx)\right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |\mathcal{H}_{(t,x)} G_k - \mathcal{H}_{(t,x)} F| dt \pi(dx) \right] \\
&\leq (T\pi(\mathbb{X}))^{\frac{r-1}{r}} \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |\mathcal{H}_{(t,x)} G_k - \mathcal{H}_{(t,x)} F|^r dt \pi(dx) \right]^{1/r} \xrightarrow{p \rightarrow +\infty} 0
\end{aligned}$$

using Hölder inequality.  $\square$

Finally, we prove that for any  $n \in \mathbb{N}$  the space generated by the iterated integrals  $\mathbf{I}_n(f_n)$  and  $\mathcal{I}_n(f_n)$  (for  $f_n \in L^r(\mathbf{X}^n)$ ) are identical.

**Lemma 4.2.** *Assume  $\rho(\mathbf{X}) < +\infty$  and let  $r \geq 1$ .*

$$\text{Span} \left\{ \mathbf{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \right\} = \text{Span} \left\{ \mathcal{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \right\}$$

where  $\mathbf{I}_0(f) := f$  and  $\mathcal{I}_0(f) := f$  for  $f \in \mathbb{R}$ . In addition for any  $n \in \mathbb{N}^*$  and  $f_n \in L^r(\mathbf{X}^n)$ ,

$$\mathbf{I}_n(f_n) = \int_{[0,T] \times \mathbb{X}} \mathbb{E}_{t-}[\mathbf{D}_{(t,x)} \mathbf{I}_n(f_n)] \tilde{\mathbf{N}}(dt, dx), \quad (4.1)$$

$$\mathcal{I}_n(f_n) = \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} \mathcal{I}_n(f_n) \mathbf{N}(dt, dx), \quad (4.2)$$

and  $\mathcal{H} \mathcal{I}_n(f_n)$  belongs to  $\mathbb{L}_{\mathcal{P}}^1$ . Finally if  $r = 2$ , then  $\mathbb{E}_{-}[\mathbf{D} \mathbf{I}_n(f_n)] = (\mathbf{D} \mathbf{I}_n(f_n))^p$  and it belongs to  $\mathbb{L}_{\mathcal{P}}^2$ .

*Proof.* Assume  $\rho(\mathbf{X}) < +\infty$ . Recalling that  $\mathbf{I}_n(f_n) = \mathbf{I}_n(\tilde{f}_n)$  and  $\mathcal{I}_n(f_n) = \mathcal{I}_n(\tilde{f}_n)$  where  $\tilde{f}_n$  denotes the symmetrization of  $f_n$ , without loss of generality we assume that all the mappings below are symmetric.

Step 1: Let  $n \in \mathbb{N}^*$  and  $F := \mathcal{I}_n(f_n)$  with  $f_n \in L_s^r(\mathbf{X}^n)$ ,  $r \geq 1$ . Then according to Proposition 12.11 in [?],  $F = \sum_{j=0}^n \mathbf{I}_j(g_j)$  with

$$g_j(\mathbf{x}_1, \dots, \mathbf{x}_j) := \frac{n!}{j!(n-j)!} \int_{\mathbf{X}^{n-j}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \mathbf{x}_1, \dots, \mathbf{x}_j) \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \in L^r(\mathbf{X}^{n-j})$$

as

$$\begin{aligned}
&\| \int_{\mathbf{X}^{n-j}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \cdot) \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \|_{L^r(\mathbf{X}^{n-j})} \\
&\leq \int_{\mathbf{X}^{n-j}} \|f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \cdot)\|_{L^r(\mathbf{X}^{n-j})} \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \\
&= \int_{\mathbf{X}^{n-j}} \left( \int_{\mathbf{X}^j} |f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \mathbf{x}_1, \dots, \mathbf{x}_j)|^r \rho^{\otimes(n-j)}(d\mathbf{x}_1, \dots, d\mathbf{x}_j) \right)^{1/r} \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \\
&\leq (T\pi(\mathbb{X}))^{\frac{r}{r-1}} \|f_n\|_{L^r(\mathbf{X}^n)}.
\end{aligned}$$

Let  $n \in \mathbb{N}^*$  and  $F := \mathbf{I}_n(f_n)$  with  $f_n \in L^r(\mathbf{X}^n)$ ,  $r \geq 1$ . Then

$$F = \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \rho^{\otimes(n-|J|)}(d\mathbf{x}_{J^c})$$



$$\begin{aligned}
&= (-1)^n \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \boldsymbol{\rho}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\
&+ \sum_{J \subset \{1, \dots, n\}; J \neq \emptyset} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \boldsymbol{\rho}^{\otimes(n-|J|)}(d\mathbf{x}_{J^c}) \\
&= (-1)^n \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \boldsymbol{\rho}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\
&+ \sum_{k=1}^n \int_{\mathbf{X}^k} \frac{n!}{k!(n-k)!} (-1)^{n-k} \int_{\mathbf{X}^{n-k}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-k}, \mathbf{x}_1, \dots, \mathbf{x}_k) \boldsymbol{\rho}^{\otimes(n-k)}(d\mathbf{y}) \mathbf{N}^{(k)}(d\mathbf{x}) \\
&= \mathcal{I}_0(g_0) + \sum_{k=1}^n \mathcal{I}_k(g_k)
\end{aligned}$$

with  $g_0 := (-1)^n \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \boldsymbol{\rho}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n)$  and as above

$$g_k := \frac{n!}{k!(n-k)!} (-1)^{n-k} \int_{\mathbf{X}^{n-k}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-k}, \cdot) \boldsymbol{\rho}^{\otimes(n-k)}(d\mathbf{y}) \in L^r(\mathbf{X}^k)$$

Then step 1 together with step 2 implies that

$$\text{Span} \left\{ \mathbf{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \right\} = \text{Span} \left\{ \mathcal{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \right\}.$$

Step 3: Let  $n \in \mathbb{N}^*$  and  $f_n \in L^r_s(\mathbf{X}^n)$ . We have that for any  $(t, x) \in \mathbf{X}$ ,

$$\mathbf{D}_{(t,x)} \mathbf{I}_n(f_n) = n \mathbf{I}_{n-1}(f_n((t, x), \cdot))$$

and thus

$$\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} \mathbf{I}_n(f_n)] = n \mathbf{I}_{n-1} \left( f_n((t, x), \cdot) \mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)} \right),$$

with  $\mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)}((t_2, x_2), \dots, (t_n, x_n)) := \prod_{i=2}^n \mathbf{1}_{\{t_i < t\}}$ . The Clark-Ocone representation (4.1) follows from classical Malliavin's calculus. Representation (4.2) follows from the definition of integrals  $\mathcal{I}$  involving the non-compensated Poisson measure  $\mathbf{N}$ . Thus uniqueness part of the proof gives that (4.2) is in force. The last thing to be proved is that both integrands in (4.1) and (4.2) are predictable.

Step 4: Let  $n \geq 1$  and  $f_n \in L^r(\mathbf{X}^n)$ . Using classical approximations (as  $\boldsymbol{\rho}(\mathbf{X}) < +\infty$ ) there exists a sequence  $(f_n^\ell)_{\ell \geq 1} \subset C_b(\mathbf{X})$  (the set of bounded and continuous functions on  $\mathbf{X}$ ) such that  $f_n^\ell \rightarrow_{\ell \rightarrow +\infty} f_n$  in  $L^r(\mathbf{X}^n)$ . Set  $F^\ell := \mathcal{I}_n(f_n^\ell)$  where by abuse of notation we make use of the same notation of  $f_n^\ell$  for its symmetrization. We have that

$$\mathbb{E} \left[ \|F^\ell - F\| \right] \leq \int_0^T \int_{\mathbb{X}} \mathbb{E} \left[ \left\| n \mathbf{I}_{n-1}(f_n^\ell((t, x), \cdot)) - n \mathbf{I}_{n-1}(f_n((t, x), \cdot)) \right\| \right] \pi(dx) dt \leq \|f_n^\ell - f_n\|_{\mathbb{L}^1_{\mathcal{P}}} \rightarrow 0.$$

Thus,

$$\mathcal{H}_{(t,x)} F^\ell = n \mathbf{I}_{n-1}(f_n^\ell((t, x), \cdot) \mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)})$$

is left continuous in  $t$  for any  $x$ , since  $f_n^\ell$  is a continuous function. Hence we have that  $\mathcal{H}F^\ell$  belongs to  $\mathbb{L}^1_{\mathcal{P}}$  and so is  $\mathcal{H}F$  by Lemma 4.1.

Similarly if  $r = 2$ , thanks to the isometry property of the operator  $\mathbf{I}_n$  (see the comment at the end of Definition 2.8)

$$\begin{aligned} & \int_0^T \int_{\mathbb{X}} \mathbb{E} \left[ \left| \mathbb{E}_{t-}[\mathbf{I}_n(f_n)] - \mathbb{E}_{t-}[\mathbf{I}_n(f_n^\ell)] \right|^2 \right] \pi(dx) dt \\ &= \int_0^T \int_{\mathbb{X}} \mathbb{E} \left[ \left| n \mathbf{I}_{n-1} \left( f_n((t, x), \cdot) \mathbf{1}_{\{[0, t] \times \mathbb{X}\}}^{\otimes(n-1)} \right) - n \mathbf{I}_{n-1} \left( f_n^\ell((t, x), \cdot) \mathbf{1}_{\{[0, t] \times \mathbb{X}\}}^{\otimes(n-1)} \right) \right|^2 \right] \pi(dx) dt \\ &= n^2 \|f_n - f_n^\ell\|_{L^2(\mathbf{X}^n)}^2 \xrightarrow{\ell \rightarrow +\infty} 0. \end{aligned}$$

Once again the continuity of the maps  $f_n^\ell$  gives that  $\mathbb{E}_{t-}[\mathbf{I}_n(f_n)]$  is the predictable projection of  $\mathbf{D}\mathbf{I}_n(f_n)$  and thus that it belongs to  $\mathbb{L}_{\mathcal{P}}^2$ .  $\square$

## Conclusion

This paper extends to any integrable Poisson-functional the Poisson imbedding that provides a representation of any point process (with intensity  $(\lambda_t)_t$ ) as the integral of  $\mathbf{1}_{\{\theta \leq \lambda_t\}}$  with respect to an uncompensated Poisson measure  $\mathbf{N}(dt, d\theta)$ . More precisely, we provide for any  $F \in L^1(\Omega)$  a Pseudo-Clark-Ocone representation formula with respect to the uncompensated Poisson measure and whose integrand is fully characterized as  $\mathcal{H}_{(t,x)}F = \mathbf{D}_{(t,x)}F \circ \tau_t$ . Besides  $\mathcal{H}F$  is well defined in  $\mathbb{L}_{\mathcal{P}}^1$ , contrary to the standard Clark-Ocone formula which requires  $F \in L^2(\Omega)$  and whose integrand  $(\mathbf{D}_{(t,x)}F)^p$  is only defined as a limit in  $L^2(\Omega \times \mathbf{X})$ .

## 5 Appendix

For sake of completeness, we provide below some technical elements on the standard Clark-Ocone formula.

**Lemma 5.1.** *Consider  $X$  a  $\mathcal{F}^{\mathbf{N}} \otimes \mathfrak{X}$ -measurable process with  $\int_{[0, T] \times \mathbb{X}} \mathbb{E}[|X_{(t,x)}|] dt \pi(dx) < +\infty$ . Then there exists a unique element  $X^p \in \mathbb{L}_{\mathcal{P}}^1$  such that for any predictable stopping time  $\tau$*

$$\mathbb{E} [X_{\tau,x} | \mathcal{F}_{\tau-}^{\mathbf{N}}] \mathbf{1}_{\{\tau < \infty\}} = X_{\tau,x}^p \mathbf{1}_{\{\tau < \infty\}}.$$

*Proof.* The proof essentially follows Lemma 3.3 in [?], in which  $X$  is assumed to be bounded. Let  $X$  such that  $\int_{[0, T] \times \mathbb{X}} \mathbb{E}[|X_{(t,x)}|] dt \pi(dx) < +\infty$  and we set  $X^k := -k \vee X \wedge k$  for  $k \in \mathbb{N}^*$ . For any  $k \geq 1$ , there exists a unique  $\mathcal{P}$ -measurable process  $Y^k$  such that for any predictable stopping  $\tau$  and  $x \in \mathbb{X}$

$$\mathbb{E} [X_{\tau,x}^k | \mathcal{F}_{\tau-}^{\mathbf{N}}] \mathbf{1}_{\{\tau < \infty\}} = Y_{\tau,x}^k \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - a.s..$$

By monotone convergence for any  $x \in \mathbb{X}$ ,

$$\mathbb{E} [X_{\tau,x} | \mathcal{F}_{\tau-}^{\mathbf{N}}] \mathbf{1}_{\{\tau < \infty\}} = Y_{\tau,x} \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - a.s.,$$

where  $Y := \lim_{k \rightarrow +\infty} Y^k$  belong to  $\mathbb{L}_{\mathcal{P}}^1$ .  $\square$

We now give a concise proof of the standard Clark-Ocone formula for any  $F \in L^2(\Omega)$  (Theorem 3.2). In particular, following [?, Section 3.2], we make precise the definition of the integrand  $(\mathbf{D}F)^p$  in  $L^2(\Omega \times \mathbf{X})$  in a limiting procedure and based on the continuous feature of the mapping  $F \mapsto (\mathbb{E}_{t-} [\mathbf{D}_{(t,x)}F])_{(t,x)}$  together with the fact that for fixed  $(t, x)$ ,  $\mathbb{E}_{t-} [\mathbf{D}_{(t,x)}F] = (\mathbf{D}_{(t,x)}F)^p \quad \mathbb{P} - a.s..$

*Proof.* The chaotic expansion (Theorem 3.1) entails that  $V := \text{Span} \{ \mathbf{I}_n(f_n); \quad n \in \mathbb{N}, f_n \in L^2(\mathbf{X}) \}$  is dense in  $L^2(\Omega)$ . Let  $F := \mathbf{I}_n(f_n) \in V$  with  $n > 0$  and without loss of generality  $f_n$  is assumed to be symmetric. We have that

$$\mathbf{D}_{(t,x)}F = n\mathbf{I}_{n-1}(f_n((t,x), \cdot))$$

and by Lemma 4.2

$$(\mathbf{D}_{(t,x)}F)^p = \mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F]^p = n\mathbf{I}_{n-1}(f_n((t,x), \cdot)\mathbf{1}_{\{([0,t] \times \mathbb{X})^{\otimes(n-1)}\}}(\cdot))$$

so that

$$F = \int_{[0,T] \times \mathbb{X}} \mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F] \tilde{\mathbf{N}}(dt, dx), \quad (5.1)$$

and

$$\begin{aligned} & \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |\mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F]|^2 dt \pi(dx) \right] \\ &= n^2 \int_{[0,T] \times \mathbb{X}} \mathbb{E} \left[ \left| \mathbf{I}_{n-1}(f_n((t,x), \cdot)\mathbf{1}_{\{([0,t] \times \mathbb{X})^{\otimes(n-1)}\}}(\cdot)) \right|^2 \right] dt \pi(dx) \\ &= n(n-1)! \int_{([0,T] \times \mathbb{X})^n} |f_n|^2 (dt \pi(dx))^{\otimes n} \\ &= \mathbb{E}[|F|^2] < +\infty. \end{aligned}$$

Note that by orthogonality of the operators  $\mathbf{I}_n$  the previous result extends to any element  $F$  on  $V$  with  $\mathbb{E}[F] = 0$ , that is for any  $F \in V$  with  $\mathbb{E}[F] = 0$ ,

$$\mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |\mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F]|^2 dt \pi(dx) \right] = \mathbb{E}[|F|^2] < +\infty.$$

Let the operators  $U$  and  $\tilde{U}$  defined as

$$U : \begin{array}{l} V \rightarrow \mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+ \\ F \mapsto \left( (\mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F])_{(t,x)}; \mathbb{E}[F]^2 \right). \end{array}$$

$$\tilde{U} : \begin{array}{l} V \rightarrow L^2(\Omega) \\ F \mapsto \int_{[0,T] \times \mathbb{X}} \mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F] \tilde{\mathbf{N}}(dt, dx). \end{array}$$

For any  $F \in V$ , we have proved that  $F$  enjoys Representation (5.1) and Relation (2.2) implies that

$$\|U(F)\|_{L^2(\mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+)}^2 = \|F\|_{L^2(\Omega)}^2; \quad \|\tilde{U}(F)\|_{L^2(\Omega)}^2 = \mathbb{E}[F^2] - \mathbb{E}[F]^2 \leq \|F\|_{L^2(\Omega)}^2.$$

Let  $F \in L^2(\Omega)$ . There exists  $(F_n)_n \subset V$  such that  $\lim_{n \rightarrow +\infty} \mathbb{E}[|F_n - F|^2] = 0$ . Using the previous isometries the sequences  $(U(F_n))_n$  and  $(\tilde{U}(F_n))_n$  are Cauchy and thus converging respectively in  $L^2(\mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+)$  and  $L^2(\Omega)$ . Let  $U(F) := (U^1(F), U^2(F))$  and  $\tilde{U}(F)$  their limits where  $U^1$  and  $U^2$  denote the two components of the limits. In addition, the  $L^2(\Omega \times [0, T] \times \mathbb{X})$ -convergence of the process  $U^1(F_n)$  implies that  $U^1(F)$  is  $\mathcal{P}$ -measurable. Hence the process  $t \mapsto \int_{[0,t] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx)$  is a martingale and

$$\mathbb{E} \left[ \left| \int_{[0,T] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] = \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |U^1(F)_{(r,x)}|^2 dr \pi(dx) \right]$$

$$\begin{aligned}
&= \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |U^1(F_n)_{(r,x)}|^2 dr \pi(dx) \right] \\
&= \lim_{n \rightarrow +\infty} \mathbb{E}[F_n^2] - \mathbb{E}[F_n]^2 = \mathbb{E}[F^2] - \mathbb{E}[F]^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E} \left[ \left| F - \mathbb{E}[F] - \int_{[0,T] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] \\
&\leq \lim_{n \rightarrow +\infty} (\mathbb{E} [|F - F_n|^2] + |\mathbb{E}[F] - \mathbb{E}[F_n]|) \\
&+ \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \left| \int_{[0,T] \times \mathbb{X}} U^1(F_n)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) - \int_{[0,T] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] \\
&\leq \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \int_{[0,T] \times \mathbb{X}} |U^1(F_n)_{(r,x)} - U^1(F)|^2 dr \pi(dx) \right] \\
&\leq \lim_{n \rightarrow +\infty} \|U^1(F_n)_{(r,x)} - U^1(F)\|_{L^2(L^2_{\mathbb{P}} \times \mathbb{R}_+)}^2 = 0.
\end{aligned}$$

The proof is complete. □

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