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# Weak and strong admissible triplets associated to Collatz like map

Abderrahman Bouhamidi\*

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## Abstract

In this paper, we will study a class of Collatz like problems associated to what we will call weak or strong admissible triplets. This representation can be seen as a simple extension of the classical Collatz mapping, which will give rise to a tool for generating remarkable triplets with convergence to cycles by avoiding the obstacle of difficult Diophantine equations. Many special and interesting families of admissible triplets will be given as well as some general properties. We will establish some formulas for the corresponding map and for the total stopping time. Interesting conjectures which give rise to a general formulation to the classical Collatz conjecture will be given. Bound lengths for eventual non-trivial cycles will be studied and analyzed. Two algorithms will be given for determining automatically the lower bound length for an eventual cycle. Experimental results and test examples will be presented.

## 1 Introduction

The classical Collatz problem also known as Kakutani, Syracuse or  $3n + 1$  problem, concerns the sequence of positive integers generated by the iterations of the Collatz mapping  $C : \mathbb{N} \rightarrow \mathbb{N}$  defined as:

$$C(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n + 1)/2 & \text{if } n \text{ is odd,} \end{cases} \quad (1)$$

where  $\mathbb{N} = \{1, 2, 3, \dots\}$  is the set of integers  $\geq 1$ . For every  $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ , the notation  $C^{(k)}$  stands for the  $k$ -th iterate of the mapping  $C$ . The well known conjecture of the  $3n + 1$  problem asserts that: Starting with any initial integer  $n \in \mathbb{N}$ , there is a positive integer  $k \in \mathbb{N}_0$  such that  $C^{(k)}(n) = 1$ . The simplicity of its formulation makes this conjecture very popular. Although this problem is very simple to formulate, it is well known that finding a rigorous and a complete proof of it remains an open problem challenge in mathematics that require solid arguments [1]. Despite this difficulty, many interesting results are established in the literature and several generalizations of the Collatz mappings have arisen since many years. Relevant literature and extensive surveys as well as historical discussion on this subject may be found in [1, 2]. In an attempt to understand the outlines of a possible proof of such a conjecture, many authors have studied integer decomposition techniques involving the numbers 2 and 3. Others have instead tried to give generalizations of the conjecture

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that will perhaps allow general properties to be derived. In this article, we have chosen instead to follow the path of generalizations in order to understand certain aspects related to the phenomenon of the Collatz conjecture. So, let us start our paper by given a brief and non exhaustive overview on some generalizations of the Collatz problem previously studied in the literature. For more details, we refer the reader to [1, 2, 3]. Before that, let us introduce the following notation, for a positive integer  $d \geq 2$ , the notation  $[n]_d$  denotes the remainder of the integer  $n$  in the Euclidean division of  $n$  by  $d$  with the standard condition  $0 \leq [n]_d < d$ . Now, we recall few generalizations of the Collatz map:

- First, we recall another version introduced by Collatz him self in 1932 (see [1]), given by the following mapping

$$S(n) = \begin{cases} (2n)/3 & \text{if } n \equiv 0 \pmod{3} \\ (4n - 1)/3 & \text{if } n \equiv 1 \pmod{3} \\ (4n + 1)/3 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \quad (2)$$

Let us observe that if one replace the standard system  $\{0, 1, 2\}$  of congruence modulo 3 in (2) by the system  $\{-1, 0, 1\}$ , then the mapping in (2), may be written in the following equivalent form

$$S'(n) = \begin{cases} (2n)/3 & \text{if } n \equiv 0 \pmod{3} \\ (4n - [n]_3)/3 & \text{if } n \not\equiv 0 \pmod{3}. \end{cases} \quad (3)$$

- In 1990, Lagarias [4], proposed the following map

$$L_\beta(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n + \beta)/2 & \text{if } n \text{ is odd,} \end{cases}$$

with  $\beta \geq 1$  is an integer such that  $\gcd(\beta, 6) = 1$ .

- In 1979, Allouche [5] proposed the following generalization, see the resume in [2],

$$A_{d,m}(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ (mn - m[n]_d)/d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases}$$

in which the parameters  $d$  and  $m$  satisfy  $d \geq 2$ ,  $\gcd(m, d) = 1$ . The generalization of Allouche is an extension to the work introduced previously by Hasse. The author noted that it is easy to show that any mapping in Hasse's class with  $1 \leq m < d$  has a finite number of cycles, and for this mapping all orbits eventually enter one of these cycles.

- In 1984, Matthews and Watts [6, 7], see also [8], proposed a complete generalization to the classical Collatz problem which may be seen as an extension of the mapping previously proposed by Hasse. The generalization of Matthews and Watts, may be summarized briefly as following: Let  $d \geq 2$  be a positive integer and  $m_0, \dots, m_{d-1}$  be non-negative integers and let  $r_0, \dots, r_{d-1}$  be relative integers satisfying the conditions  $r_i \equiv im_i \pmod{d}$ , for  $i = 0, \dots, d - 1$ . Thus, the general mapping denoted here by  $M : \mathbb{Z} \rightarrow \mathbb{Z}$  and defined in [6, 7] is given by the formula

$$M(n) = \frac{m_i n - r_i}{d} \text{ if } n \equiv i \pmod{d}. \quad (4)$$

In this paper, we will restrict our attention only to the mappings acting from  $\mathbb{N}$  into  $\mathbb{N}$ . Our first conjecture we will introduce here is the following, and it is quoted in a standard text that everyone (not specifically mathematician ) can understand:

**Conjecture 1.1.** *Starting with a positive integer  $n \geq 1$ :*

- *If  $n$  is a multiple of 10 then remove the zero on the right. Otherwise, multiply it by 6, add 4 times its last digit and divide the result by 5.*
- *Repeat the process infinitely.*

*Then, regardless the starting number, the process eventually reaches 4 after a finite number of iterations.*

More details on this conjecture are given in [9]. The previous conjecture is in fact associated to the operator  $T : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$T(n) = \begin{cases} n/10 & \text{if } n \equiv 0 \pmod{10} \\ (6n + 4[n]_{10})/5 & \text{if } n \not\equiv 0 \pmod{10}, \end{cases} \text{ or } T(n) = \begin{cases} n/10 & \text{if } n \equiv 0 \pmod{10} \\ (12n + 8[n]_{10})/10 & \text{if } n \not\equiv 0 \pmod{10}. \end{cases} \quad (5)$$

We will study special classes of mappings that generalize the classical Collatz mapping together with our presented conjecture. We will give some properties and formulas related to the specific mappings as well as some formulas on the total stopping time. Moreover, we will give some results and theorems concerning the lower bound lengths of eventual non-trivial cycles. We will also formulate few general and interesting conjectures. Examples and experimental tests will be analyzed to illustrate some theoretical given results.

The outline of this paper is as follows. In section 2, we define, what we will call, the admissible triplets. We will give some definitions and properties. In section 3, we describe some special classes of admissible triplets for which we may obtain easily trivial cycles. Some general conjectures will be formulated. In Section 4, we give few iterate formulas. In Section 5, we study the properties concerning bounds for possible non-trivial cycle lengths. Two algorithms will be derived from specific theoretical results. In the last section, we will present experimental examples illustrating the announced conjectures and the theoretical study. As usual, we end our paper by a master conclusion.

## 2 Weak and strong admissible triplets - Definitions

Let  $\alpha > d \geq 2$  be two positive integers and let  $\beta$  be another integer which may be negative. The integers  $d$ ,  $\alpha$  and  $\beta$  are such that:  $\alpha \not\equiv 0 \pmod{d}$  and  $\beta \not\equiv 0 \pmod{d}$ . In the following definition, we allow  $\gcd(\alpha, d)$  or  $\gcd(\beta, d)$  to be great then 1. A natural and obvious extension to the classical Collatz mapping is the mappings  $T : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ (\alpha n + \beta[\kappa_0 n]_d)/d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \quad (6)$$

where the parameter  $\kappa_0 = \pm 1$ . The triplet  $(d, \alpha, \beta)$  associated to  $T$  will be denoted by  $(d, \alpha, \beta)_+$  and  $(d, \alpha, \beta)_-$  for  $\kappa_0 = +1$  and for  $\kappa_0 = -1$ , respectively. We recall that for all positive integer  $n$  we have  $[-n]_d = d - [n]_d$ . The mapping (6) encompasses obviously the classical Collatz mapping (1) together with the mapping (5). We have the following obvious result.

**Theorem 2.1.** *A necessary and sufficient condition that the mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  given by (6) is well defined from  $\mathbb{N}$  into  $\mathbb{N}$  is that the triplet  $(d, \alpha, \beta)_{\pm}$  associate to  $T$  satisfies the following condition*

$$\alpha + \kappa_0\beta > 0 \quad \text{and} \quad \alpha + \kappa_0\beta \equiv 0 \pmod{d}. \quad (7)$$

*Proof.* Let  $n \in \mathbb{N}$  and consider its Euclidean division by  $d$  of the form  $n = q_nd + [n]_d$ . If the remainder  $[n]_d = 0$ , then  $T(n) = q_n \in \mathbb{N}$ . Else,  $[n]_d \neq 0$  and

$$T(n) = (\alpha n + \beta[\kappa_0 n]_d)/d = \begin{cases} \alpha q_n + \frac{\alpha + \beta}{d}[n]_d & \text{if } \kappa_0 = +1 \\ \alpha q_n + \beta + \frac{\alpha - \beta}{d}[n]_d & \text{if } \kappa_0 = -1. \end{cases}$$

It follows that if the condition (7) is satisfied then  $T(n)$  belongs to  $\mathbb{N}$  for all integer  $n \geq 1$ . Conversely, we have

$$T(1) = \begin{cases} \frac{\alpha + \beta}{d} & \text{if } \kappa_0 = +1 \\ \frac{\alpha + \beta(d-1)}{d} & \text{if } \kappa_0 = -1, \end{cases}$$

as  $T(1) \in \mathbb{N}$  it follows obviously that  $\alpha + \kappa_0\beta \equiv 0 \pmod{d}$  for both cases and we have  $\alpha + \beta > 0$  for  $\kappa_0 = 1$ . For  $\kappa_0 = -1$ , we have  $\alpha - \beta > -\beta d$ . Then, if  $\beta < 0$ , it follows that  $\alpha - \beta > 0$  else,  $\beta > 0$ , this implies  $\alpha - \beta > 0$ . Thus, the condition (7) holds.  $\square$

**Remark 2.1.** *In fact, if the triplet  $(d, \alpha, \beta)_{\pm}$  satisfies the condition (7), then the mapping given in (6) is a special case of the general case given in (4). Indeed,*

- for  $\kappa_0 = 1$  and from (7), we have  $\alpha + \beta \equiv 0 \pmod{d}$ , then if we choose in (4),  $m_0 = 1$ ,  $r_0 = 0$  and for  $i = 1, \dots, d-1$ , we set  $m_i = \alpha$  and  $r_i = -i\beta$ , it follows that  $r_i \equiv i\alpha \pmod{d}$  which gives the conditions  $r_i \equiv im_i \pmod{d}$ , for  $i = 1, \dots, d-1$ .
- for  $\kappa_0 = -1$  and from (7), we have  $\alpha - \beta \equiv 0 \pmod{d}$ , then if we choose in (4),  $m_0 = 1$ ,  $r_0 = 0$  and for  $i = 1, \dots, d-1$ , we set  $m_i = \alpha$  and  $r_i = -\beta(d-i)$ , it follows that  $r_i \equiv -(d-i)m_i \pmod{d}$  which gives the conditions  $r_i \equiv im_i \pmod{d}$ , for  $i = 1, \dots, d-1$ .

Although the mapping given by (6) may be seen as a special case of the general case proposed by Matthews and Watts [6, 7], this present representation is a simple extension which give raise, as we will see, to an easy tool for generating remarkable triplets associated to the mapping  $T$  with trivial cycles by avoiding the obstacle of difficult Diophantine equations.

How to choose a triplet  $(d, \alpha, \beta)_{\pm}$  such that the condition (7) holds? We fix  $\kappa_0 = \pm 1$ , then  $\alpha + \kappa_0\beta$  must be in the form  $\alpha + \kappa_0\beta = ed$ , where  $e > 0$  is a positive integer. Let  $\nu_0$  be the largest possible integer exponent such that  $d^{\nu_0}$  divides exactly  $\alpha + \kappa_0\beta$ , then there exist some positive integer  $\lambda_0 \geq 1$  with  $\lambda_0 \not\equiv 0 \pmod{d}$  such that

$$\alpha + \kappa_0\beta = \lambda_0 d^{\nu_0}. \quad (8)$$

For a given  $\alpha > d$ , we choose two integers  $\nu_1 \geq 1$ ,  $\lambda_1 \geq 1$  and we fix a sign  $\kappa_1 = \pm 1$  such that  $\lambda_1 \not\equiv 0 \pmod{d}$  and  $\delta = -\kappa_1(\alpha - \lambda_1 d^{\nu_1}) \geq 1$ . In conclusion, we choose the integers  $d \geq 2$ ,  $\lambda_0 \geq 1$ ,

$\lambda_1 \geq 1$ ,  $\nu_0 \geq 1$ ,  $\nu_1 \geq 1$ ,  $\delta \geq 1$  and  $\kappa_0 = \pm 1$ ,  $\kappa_1 = \pm 1$  with the conditions  $\lambda_0 \not\equiv 0 \pmod{d}$  and  $\lambda_1 \not\equiv 0 \pmod{d}$ , then, the triplet  $(d, \alpha, \beta)_{\pm}$  is obtained such that

$$\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta \quad \text{and} \quad \beta = \kappa_0 (\lambda_0 d^{\nu_0} - \alpha). \quad (9)$$

The decomposition of  $\alpha$  in (9) is not unique. For instance, for  $d = 4$ , if we choose  $\lambda_1 = \nu_1 = 1$ ,  $\delta = 7$  and  $\kappa_1 = -1$  then  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = 11$ . But, also for  $\lambda_1 = 1$ ,  $\nu_1 = 2$ ,  $\delta = 5$ , and  $\kappa_1 = 1$ , we get also  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = 11$ . The two integers  $\nu_0$  and  $\nu_1$ , as we will see later, play an important role in this construction. We have the following definition.

**Definition 2.1.** *Let  $(d, \alpha, \beta)_{\pm}$  be a triplet of integers. We will say that  $(d, \alpha, \beta)_{\pm}$  is an admissible triplet if and only if the triplet satisfies (9).*

As an example, the triplet  $(2, 3, 1)_{+}$  corresponding to the classical Collatz problem is an admissible triplet. The triplet  $(3, 5, 2)_{+}$  is not an admissible triplet and its associated mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$T(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3} \\ (5n + 2[n]_3)/3 & \text{if } n \not\equiv 0 \pmod{3}, \end{cases}$$

is not well defined on  $\mathbb{N}$ . Indeed, for instance  $T(1) = \frac{7}{3} \notin \mathbb{N}$ .

Let us now recall some definitions. Given  $n \in \mathbb{N}$ , the trajectory (or the orbit) of  $n$  is the set  $\Gamma(n)$  of the successive iterates starting from  $n$ :  $\Gamma(n) = \{n, T(n), T^{(2)}(n), \dots\}$ . A cycle (if it exists), having  $k$  elements (or vertices), associated to the mapping  $T$  (or to the triplet  $(d, \alpha, \beta)_{\pm}$ ) is a finite set  $\Omega$  for which  $T^{(k)}(x) = x$  for all  $x \in \Omega$ . The cardinal  $\#\Omega = \text{Card}(\Omega) = k$ , also called the length of the cycle  $\Omega$ , is also the smallest integer  $k$  such that  $T^{(k)}(x) = x$  for all  $x \in \Omega$ . We will denote by  $\Omega(\omega) = \{\omega, T(\omega), T^{(2)}(\omega), \dots, T^{(k-1)}(\omega)\}$  the cycle (if it exists) associated to  $T$  of length  $k$  where  $\omega$  is the smallest element in the cycle. We will also use this abusive following notation

$$(\omega \rightarrow T(\omega) \rightarrow T^{(2)}(\omega) \rightarrow \dots \rightarrow T^{(k-1)}(\omega) \rightarrow \omega),$$

to denote the cycle  $\Omega(\omega)$ . For each choice of the triplet  $(d, \alpha, \beta)_{\pm}$ , the notation  $\text{Cycl}(d, \alpha, \beta)_{\pm}$  stands for the set of all possible cycles associated to  $T$ , it may be empty, finite or infinite set, then the cardinal  $\#\text{Cycl}(d, \alpha, \beta)_{\pm} \in \mathbb{N}_0 \cup \{\infty\}$ , and will be called the order of the triplet  $(d, \alpha, \beta)_{\pm}$ . For  $n \in \mathbb{N}$ , there are two possible situations for a trajectory  $\Gamma(n) = \{T^{(k)}(n)\}_{k \in \mathbb{N}_0}$  starting with  $n$ :

(i) Convergent trajectory to one or other cycle, assuming that such cycles exist.

(ii) Divergent trajectory:  $\lim_{k \rightarrow \infty} T^{(k)}(n) = \infty$ .

Assume that a cycle  $\Omega(\omega)$  exist, we denote by  $G_{\omega}(d, \alpha, \beta)$  and  $G_{\infty}(d, \alpha, \beta)$  the subsets of  $\mathbb{N}$  given by

$$G_{\omega}(d, \alpha, \beta) = \{n \in \mathbb{N} : \exists k \in \mathbb{N}, T^{(k)}(n) \in \Omega(\omega)\}$$

and

$$G_{\infty}(d, \alpha, \beta) = \{n \in \mathbb{N} : \lim_{k \rightarrow \infty} T^{(k)}(n) = \infty\}.$$

The set  $G_\infty(d, \alpha, \beta)$  may be empty or not and the sets  $G_\omega(d, \alpha, \beta)$  are pairwise disjoint: For all  $\omega, \omega' \in \mathbb{N}$  such that the cycles  $\Omega(\omega)$  and  $\Omega(\omega')$  exist, if  $\omega \neq \omega'$ , then  $G_\omega(d, \alpha, \beta) \cap G_{\omega'}(d, \alpha, \beta) = \emptyset$  and  $G_\infty(d, \alpha, \beta) \cap G(\omega) = \emptyset$ .

The admissibility condition (9) does not ensure that the triplet has at least one cycle. For instance the triplets  $(2, 9, 1)_+$ ,  $(5, 31, 4)_+$  and  $(15, 35, 10)_+$ , although they are admissible triplets, it appears that they do not have any cycle. We have the following definition.

**Definition 2.2.** *Let  $(d, \alpha, \beta)_\pm$  be an admissible triplet with its associated mapping  $T$  given by (6). We will say that:*

- *The admissible triplet  $(d, \alpha, \beta)_\pm$  is a weak admissible triplet if and only if it has at least one cycle and at most a finite number of cycles. If applicable, we have  $1 \leq \#\text{Cycl}(d, \alpha, \beta)_\pm < \infty$ .*
- *The admissible triplet  $(d, \alpha, \beta)_\pm$  is a strong admissible triplet if and only if  $(d, \alpha, \beta)_\pm$  is a weak admissible triplet and the set  $G_\infty(d, \alpha, \beta)$  is empty. Namely, the triplet has a finite number of cycles without any divergent trajectory.*

According to the previous definition, it follows that:

- If  $(d, \alpha, \beta)_\pm$  is an admissible weak triplet of order  $q$ , then the mapping  $T$  has a finite number  $q$  of cycles denoted as  $\Omega(\omega_1), \dots, \Omega(\omega_q)$  with  $1 \leq q < +\infty$  and  $\text{Cycl}(d, \alpha, \beta)_\pm = \{\Omega(\omega_1), \dots, \Omega(\omega_q)\} \neq \emptyset$ . In this case, the set  $G_\infty(d, \alpha, \beta)$  may be empty set or not ( $T$  may have divergent trajectories). We have

$$\mathbb{N} = G_{\omega_1}(d, \alpha, \beta) \cup \dots \cup G_{\omega_q}(d, \alpha, \beta) \cup G_\infty(d, \alpha, \beta).$$

In the case where  $G_\infty(d, \alpha, \beta)$  is not empty, the subsets  $G_{\omega_1}(d, \alpha, \beta), \dots, G_{\omega_q}(d, \alpha, \beta)$  and  $G_\infty(d, \alpha, \beta)$  form a partition of  $\mathbb{N}$ .

- If  $(d, \alpha, \beta)_\pm$  is an admissible strong triplet, then the mapping  $T$  has a finite number  $q$  of cycles denoted as  $\Omega(\omega_1), \dots, \Omega(\omega_q)$  with  $1 \leq q < +\infty$  and  $\text{Cycl}(d, \alpha, \beta)_\pm = \{\Omega(\omega_1), \dots, \Omega(\omega_q)\} \neq \emptyset$ . In this case, the set  $G_\infty(d, \alpha, \beta)$  is an empty set and there is no divergent trajectory. The subsets  $G_{\omega_1}(d, \alpha, \beta), \dots,$  and  $G_{\omega_q}(d, \alpha, \beta)$  form a partition of  $\mathbb{N}$ :

$$\mathbb{N} = G_{\omega_1}(d, \alpha, \beta) \cup \dots \cup G_{\omega_q}(d, \alpha, \beta).$$

In this case,  $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}_0$  such that  $T^{(k)}(n) \in \{\omega_1, \dots, \omega_q\}$ .

The classical Collatz conjecture may be now reformulated as following,

**Conjecture 2.1.** *The triplet  $(2, 3, 1)_+$  is an admissible strong triplet of order one. Its unique cycle is the trivial one  $\Omega(1) = (1 \rightarrow 2 \rightarrow 1)$  of length 2.*

Our conjecture 1.1 presented in the introduction may also reformulated as following:

**Conjecture 2.2.** *The triplet  $(10, 12, 8)_+$  is an admissible strong triplet of order one. Its unique cycle is the trivial one  $\Omega(4) = (4 \rightarrow 8 \rightarrow 16 \rightarrow 24 \rightarrow 32 \rightarrow 40 \rightarrow 4)$  of length 6.*

A difficult challenge is to prove that a given admissible triplet is a weak or strong admissible triplet. This question still an open problem for the classical triplet  $(2, 3, 1)_+$ . In this paper, we will give a general formulation conjecture associated to specific admissible triplets.

Let us now give few examples of admissible triplets that at least have one cycle. The question about weak or strong admissible triplet will be an open problem. The following given assertions are based on our analysis, experience and on many computer tests that we have performed. The motivations for our statements will be explained later in the next section. For now, we ask the reader to accept these examples as they are.

**Examples 2.1.** 1. In the first example, we consider the admissible triplet  $(33445533, 33445534, 33445532)_+$  corresponding to a large  $d = 33445533$  and to the parameters:  $\lambda_1 = \delta = \nu_0 = \nu_1 = \kappa_0 = 1$ ,  $\kappa_1 = -1$  and  $\lambda_0 = 2$ , which gives  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = d + 1 = 33445534$  and  $\beta = \kappa_0(\lambda_0 d^{\nu_0} - \alpha) = d - 1 = 33445532$ . The corresponding mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  is given by

$$T(n) = \begin{cases} n/33554433 & \text{if } n \equiv 0 \pmod{33554433} \\ (33554434n + 33554432[n]_d)/33554433 & \text{if } n \not\equiv 0 \pmod{33554433}. \end{cases} \quad (10)$$

It seems that  $(33445533, 33445534, 33445532)_+$  is an admissible strong triplet of order one. The trivial cycle of length 33554433 is

$$\begin{aligned} \Omega(33554432) = & (33554432 \rightarrow 67108864 \rightarrow 100663296 \rightarrow 134217728 \rightarrow 167772160 \rightarrow 201326592 \rightarrow \\ & 234881024 \rightarrow 268435456 \rightarrow 301989888 \rightarrow 335544320 \rightarrow 369098752 \rightarrow 402653184 \rightarrow \dots \rightarrow \\ & 1125899571298304 \rightarrow 1125899604852736 \rightarrow 1125899638407168 \rightarrow 1125899671961600 \rightarrow \\ & 1125899705516032 \rightarrow 1125899739070464 \rightarrow 1125899772624896 \rightarrow 1125899806179328 \rightarrow \\ & 1125899839733760 \rightarrow 1125899873288192 \rightarrow 1125899906842624 \rightarrow 1125899940397056 \rightarrow 33554432). \end{aligned}$$

If the fact that this triplet is a strong admissible triplet is true, then for all  $n \in \mathbb{N}$  there exists  $k \in \mathbb{N}_0$  such that  $T^{(k)}(n) = 33554432$ . We note that the maximum element of the trivial cycle  $\Omega(33554432)$  is 1125899940397056.

2. In this example, we consider the admissible triplet  $(3, 8, 19)_+$ , corresponding to the parameters:  $d = 3$ ,  $\nu_0 = 3$ ,  $\nu_1 = 2$  and  $\kappa_0 = \kappa_1 = \lambda_0 = \lambda_1 = 1$ ,  $\delta = 2$ , which gives  $\alpha = 8$  and  $\beta = 19$ . The corresponding mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  is given by

$$T(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3} \\ (8n + 19[n]_3)/3 & \text{if } n \not\equiv 0 \pmod{3}. \end{cases} \quad (11)$$

We firmly believe that  $(3, 8, 19)_+$  is an admissible weak triplet with order  $q = 4$ . It has at least the following cycles  $\Omega(1) = (1 \rightarrow 9 \rightarrow 3 \rightarrow 1)$ ,  $\Omega(2) = (2 \rightarrow 18 \rightarrow 6 \rightarrow 2)$ ,  $\Omega(19) = (19 \rightarrow 57 \rightarrow 19)$ ,  $\Omega(38) = (38 \rightarrow 114 \rightarrow 38)$ . For this example we may have divergent trajectories.

3. In this example, we consider the admissible triplet  $(5, 6, 4)_+$ , corresponding to the parameters:  $d = 5$ ,  $\nu_0 = \nu_1 = \delta = \lambda_1 = 1$ ,  $\kappa_1 = -1$ ,  $\kappa_0 = 1$  and  $\lambda_0 = 2$ , we get  $\alpha = 6$  and  $\beta = 4$ . The corresponding mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  is given by

$$T(n) = \begin{cases} n/5 & \text{if } n \equiv 0 \pmod{5} \\ (6n + 4[n]_5)/5 & \text{if } n \not\equiv 0 \pmod{5}. \end{cases} \quad (12)$$



It seems that the triplet  $(5, 6, 4)_+$  is an admissible strong triplet of order one. Its trivial is  $\Omega(4) = (4 \rightarrow 8 \rightarrow 12 \rightarrow 16 \rightarrow 20 \rightarrow 4)$  of length 5.

4. In this example, we consider the admissible triplet  $(5, 6, 3089)_+$ , corresponding to the parameters:  $d = 5$ ,  $\nu_0 = \nu_1 = \delta = \lambda_1 = 1$ ,  $\kappa_1 = -1$ ,  $\kappa_0 = 1$  and  $\lambda_0 = 619$ , we get  $\alpha = 6$  and  $\beta = 3089$ . The corresponding mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  is given by

$$T(n) = \begin{cases} n/5 & \text{if } n \equiv 0 \pmod{5} \\ (6n + 3089[n]_5)/5 & \text{if } n \not\equiv 0 \pmod{5}. \end{cases} \quad (13)$$

We strongly believe that  $(5, 6, 3089)_+$  is an admissible strong triplet of order  $q = 33$ . The triplet  $(5, 6, 3089)_+$  has at least the following 33 cycles all of same length 5:

$$\begin{aligned} &\Omega(11), \Omega(16), \Omega(17), \Omega(21), \Omega(22), \Omega(23), \Omega(26), \Omega(27), \Omega(28), \Omega(29), \Omega(31), \Omega(32), \Omega(33), \\ &\Omega(34), \Omega(37), \Omega(38), \Omega(39), \Omega(43), \Omega(44), \Omega(49), \Omega(56), \Omega(62), \Omega(68), \Omega(74), \Omega(81), \Omega(87), \Omega(93), \Omega(99), \\ &\Omega(106), \Omega(112), \Omega(118), \Omega(124), \Omega(3089). \end{aligned}$$

It is not possible to give here the list of all the 33 cycles.

5. In this example, we consider the admissible triplet  $(4, 10, 54)_+$ . The corresponding mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  is given by

$$T(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod{4} \\ (10n + 54[n]_4)/4 & \text{if } n \not\equiv 0 \pmod{4}, \end{cases} \quad \text{or} \quad T(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod{4} \\ (5n + 27[n]_4)/2 & \text{if } n \not\equiv 0 \pmod{4}. \end{cases} \quad (14)$$

It seems that  $(4, 10, 54)_+$  is a strong admissible triplet with order  $q = 34$ . It has at least the following 34 cycles stored in ascending order of length:  $\Omega(9)$ ,  $\Omega(18)$ ,  $\Omega(27)$  all of length 2;  $\Omega(1)$ ,  $\Omega(2)$ ,  $\Omega(1)$  all of length 3;  $\Omega(477)$ ,  $\Omega(549)$ ,  $\Omega(693)$ ,  $\Omega(702)$ ,  $\Omega(774)$ ,  $\Omega(837)$ ,  $\Omega(918)$ ,  $\Omega(927)$ ,  $\Omega(999)$ ,  $\Omega(1062)$ ,  $\Omega(1143)$ ,  $\Omega(1287)$ , all of length 5;  $\Omega(6)$  of length 6;  $\Omega(639)$  of length 10;  $\Omega(7)$ ,  $\Omega(678)$  of length 15;  $\Omega(189)$  of length 20;  $\Omega(342)$  of length 25;  $\Omega(78)$ ,  $\Omega(93)$  of length 27;  $\Omega(198)$ ,  $\Omega(237)$  of length 30;  $\Omega(13)$  of length 36;  $\Omega(5967)$  of length 98;  $\Omega(1518)$  of length 109;  $\Omega(214)$  of length 247;  $\Omega(4174)$ ,  $\Omega(14927)$  of length 682.

### 3 Special classes of admissible triplets

In this section, we will give some classes of admissible triplets  $(d, \alpha, \beta)_\pm$  which have at least one trivial cycle. The particular case for  $d = 2$ , has been studied, by the author, see [10] for more details. We will also formulate some general conjectures derived from our analysis and study as well as from several experimental tests we have performed by computer and not reported in this paper. More investigations are in progress and will be presented in the state-of-arts in a further work. Few experimental examples will be given in Section 6 illustrating the cases given below in the present Section.

#### 3.1 The admissible triplets $(d, \alpha, \beta)_\pm$ with $\lambda_0 = \lambda_1 = 1$ and $1 \leq \delta \leq d - 1$ .

In this subsection, we consider the admissible triplet  $(d, \alpha, \beta)_\pm$  with  $\lambda_0 = \lambda_1 = 1$  and  $1 \leq \delta \leq d - 1$ , see (9). We have the following theorem.

**Theorem 3.1.** Let  $d \geq 2$  be a given integer and consider the admissible triplet  $(d, \alpha, \beta)_{\pm}$  with  $\alpha = d^{\nu_1} - \kappa_1 \delta$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha)$  and  $\nu_0, \nu_1 \geq 1$ ,  $1 \leq \delta \leq d - 1$ ,  $\kappa_0 = \pm 1$  and  $\kappa_1 = \pm 1$ . Then, the triplet  $(d, \alpha, \beta)_{\pm}$  has at least one cycle. More precisely, we have the following cases:

1. For  $\kappa_0 = 1$ , we have  $\#\text{Cycl}(d, \alpha, \beta)_{+} \geq d - 1$ . The  $d - 1$  trivial cycles of length  $\nu_0$  are:

$$\Omega(r) = (r \rightarrow rd^{\nu_0-1} \rightarrow rd^{\nu_0-2} \rightarrow \dots \rightarrow rd \rightarrow r), \quad (15)$$

for all  $r \in \{1, 2, \dots, d - 1\}$ .

2. For  $\kappa_0 = \pm 1$ , if the condition  $\kappa_1 \beta > 0$  holds, then:

- 2-1. If  $\delta = 1$ , then the triplet  $(d, \alpha, \beta)_{\pm}$  has at least the following  $d - 1$  trivial cycles of length  $\nu_1$ :

$$\Omega(r|\beta) = (r|\beta \rightarrow r|\beta|d^{\nu_1-1} \rightarrow r|\beta|d^{\nu_1-2} \rightarrow \dots \rightarrow r|\beta|d \rightarrow r|\beta), \quad (16)$$

for all  $r \in \{1, \dots, d - 1\}$ . It follows that  $\#\text{Cycl}(d, \alpha, \beta)_{-} \geq d - 1$ . Furthermore, if  $\nu_0 \neq \nu_1$ , then  $\#\text{Cycl}(d, \alpha, \beta)_{+} \geq 2(d - 1)$ . But, if  $\nu_0 = \nu_1$ , then the cycles in the forms (15) and (16) coincide and  $\#\text{Cycl}(d, \alpha, \beta)_{+} \geq d - 1$ .

- 2-2. If  $d - 1 \geq \delta > 1$  and  $\nu_0, \nu_1$  are two integers  $\geq 2$  with  $\nu_0 \neq \nu_1$  and  $q_0 = \gcd(d^{\nu_0-1} - d^{\nu_1-1}, \delta) \geq 2$ , then there exist two integers  $\beta_0$  and  $\delta_0$  such that  $\beta = q_0 \beta_0$  and  $\delta = q_0 \delta_0$  and the set  $\text{Cycl}(d, \alpha, \beta)_{\pm}$  contains the following  $\left\lfloor \frac{d-1}{\delta_0} \right\rfloor$  trivial cycles of length  $\nu_1$ :

$$\Omega(r|\beta_0) = (r|\beta_0 \rightarrow r|\beta_0|d^{\nu_1-1} \rightarrow r|\beta_0|d^{\nu_1-2} \rightarrow \dots \rightarrow r|\beta_0|d \rightarrow r|\beta_0), \quad (17)$$

for all  $r = 1, \dots, \left\lfloor \frac{d-1}{\delta_0} \right\rfloor$ . Thus,  $\#\text{Cycl}(d, \alpha, \beta)_{+} \geq (d-1) + \left\lfloor \frac{d-1}{\delta_0} \right\rfloor$  and  $\#\text{Cycl}(d, \alpha, \beta)_{-} \geq \left\lfloor \frac{d-1}{\delta_0} \right\rfloor$ , where  $\lfloor \cdot \rfloor$  denotes the floor function.

*Proof.* 1. For  $\kappa_0 = 1$  and for all  $r \in \{1, \dots, d - 1\}$ , we have  $T(r) = r \frac{\alpha + \beta}{d} = r \frac{d^{\nu_0}}{d} = r d^{\nu_0-1}$ . It follows that  $T^{(\nu_0)}(r) = r$ , which gives the trivial cycles given by (15). Thus  $\#\text{Cycl}(d, \alpha, \beta, +1) \geq d - 1$ .

2. For  $\kappa_0 = \pm 1$ , if the condition  $\kappa_1 \beta > 0$  holds, then:

- 2-1. If  $\delta = 1$ , we have  $\alpha = d^{\nu_1} - \kappa_1$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha)$  with  $\kappa_1 \beta > 0$ , then, for all  $r \in \{1, \dots, d - 1\}$ , we have  $\kappa_1 r \beta > 0$ ,  $\kappa_0 \kappa_1 r \beta = \kappa_1 r (d^{\nu_0} - d^{\nu_1}) + r$  and  $[\kappa_0 \kappa_1 r \beta]_d = r$ . It follows that,

$$T(\kappa_1 r \beta) = \frac{\alpha \kappa_1 r \beta + \beta [\kappa_0 \kappa_1 r \beta]_d}{d} = \beta r \frac{\kappa_1 \alpha + 1}{d}.$$

But,  $\kappa_1 \alpha = \kappa_1 d^{\nu_1} - 1$ , then  $T(\kappa_1 r \beta) = \kappa_1 r \beta \frac{d^{\nu_1}}{d} = \kappa_1 r \beta d^{\nu_1-1}$ . It follows that  $T^{(\nu_1)}(\kappa_1 r \beta) = \kappa_1 r \beta$ . As  $|\beta| = \kappa_1 \beta$ , this gives rise to the trivial cycles of the form (16) and  $\#\text{Cycl}(d, \alpha, \beta)_{-} \geq d - 1$ . But for  $\kappa_0 = 1$ : If  $\nu_0 = \nu_1$ , then  $\kappa_1 \beta = 1$  and the cycles in (15) and (16) coincide. Thus,  $\#\text{Cycl}(d, \alpha, \beta)_{+} \geq d - 1$ , else if  $\nu_0 \neq \nu_1$ , then the cycles in (15) and (16) are different and  $\#\text{Cycl}(d, \alpha, \beta)_{+} \geq (d - 1) + (d - 1) = 2(d - 1)$ .

2-2. If  $d-1 \geq \delta > 1$  and  $\nu_0$  and  $\nu_1$  are  $\geq 2$  with  $\nu_0 \neq \nu_1$ , then  $d^{\nu_0-1} - d^{\nu_1-1} = q_1 q_0$  and  $\delta = q_0 \delta_0$  for some positive integers  $q_1 > 0$  and  $\delta_0 > 0$  where  $q_0 = \gcd(d^{\nu_0-1} - d^{\nu_1-1}, \delta) \geq 2$ . Thus,  $\beta = \kappa_0(d^{\nu_0} - d^{\nu_1}) + \kappa_0 \kappa_1 \delta = q_0 \beta_0$  with  $\beta_0 = \kappa_0(dq_1 + \kappa_1 \delta_0)$ . It follows that, for all integer  $r \in \{1, \dots, d-1\}$ , we have  $\kappa_1 r \beta_0 = \kappa_0(\kappa_1 r q_1 d + r \delta_0) > 0$  since  $\kappa_1 \beta = q_0 \kappa_1 \beta_0 > 0$  and  $q_0 > 0$ . For  $r$  such that  $r \leq \left\lfloor \frac{d-1}{\delta_0} \right\rfloor$ , we have  $0 < r \delta_0 \leq d-1$  and  $[\kappa_0 \kappa_1 r \beta_0]_d = r \delta_0$ . Then

$$T(\kappa_1 r \beta_0) = \frac{\alpha \kappa_1 r \beta_0 + \beta [\kappa_0 \kappa_1 r \beta_0]_d}{d} = \frac{\alpha \kappa_1 r \beta_0 + r \delta_0 \beta}{d} = \frac{\alpha \kappa_1 r \beta_0 + r \delta_0 q_0 \beta_0}{d} = r \beta_0 \frac{\kappa_1 \alpha + \delta}{d}.$$

But,  $\kappa_1 \alpha = \kappa_1 d^{\nu_1} - \delta$ , we get  $T(\kappa_1 r \beta_0) = \kappa_1 r \beta_0 \frac{d^{\nu_1}}{d} = \kappa_1 r \beta_0 d^{\nu_1-1}$ . As  $|\beta_0| = \kappa_1 \beta_0$ , this gives rise to the trivial cycles given by (17) and then  $\#\text{Cycl}(d, \alpha, \beta)_+ \geq (d-1) + \left\lfloor \frac{d-1}{\delta_0} \right\rfloor$  and  $\#\text{Cycl}(d, \alpha, \beta)_- \geq \left\lfloor \frac{d-1}{\delta_0} \right\rfloor$ .

□

**Remark 3.1.** For the classical Collatz triplet  $(1, 2, 3)_+$ , we have  $d = 2$ ,  $\alpha = 3$ ,  $\beta = 1$ ,  $\kappa_0 = +1$  and  $\alpha + \beta = 4 = 2^{\nu_0}$ , with  $\nu_0 = 2$ . In this case, we may write  $\alpha$  and  $\beta$  in the two following forms:

1.  $\alpha = 3 = 2^{\nu_1} - \kappa_1 \delta$  and  $\beta = 1 = 2^{\nu_0} - \alpha$  with  $\nu_1 = \nu_0 = 2$  and  $\kappa_0 = \kappa_1 = \delta = 1$ .
2.  $\alpha = 3 = 2^{\nu_1} - \kappa_1 \delta$  and  $\beta = 1 = 2^{\nu_0} - \alpha$  with  $\nu_1 = 1$ ,  $\nu_0 = 2$ ,  $\delta = \kappa_0 = 1$  and  $\kappa_1 = -1$ . In this case,  $\kappa_1 \beta < 0$ .

According to the previous theorem, for both cases, we should have at least one trivial cycle of length  $\nu_0 = 2$ :

$\Omega(1) = (1 \rightarrow 2 \rightarrow 1)$  for the classical Collatz case.

Let us now give few examples that illustrate the previous Theorem.

**Examples 3.1.** In all the following examples we fix  $\lambda_0 = \lambda_1 = 1$  and  $1 \leq \delta \leq d-1$ .

1. Let us choose for instance the parameters  $d = 3$ ,  $\kappa_0 = \kappa_1 = \delta = 1$ ,  $\nu_0 = 3$  and  $\nu_1 = 2$ . Then  $\alpha = d^{\nu_1} - \kappa_1 \delta = 8$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha) = 19$ . In this case, we have  $\kappa_1 \beta > 0$ . This corresponds to Item 2-1 in the previous Theorem and this is exactly the case of the example Item-2 in Examples 2.1. The admissible triplet  $(3, 8, 19)_+$  has at least two cycles  $\Omega(1)$  and  $\Omega(2)$  of length  $\nu_0 = 3$  and the two others cycles  $\Omega(19)$  and  $\Omega(2 \times 19 = 38)$  of length  $\nu_1 = 2$ , see the example Item-2 in Examples 2.1.
2. In this example, we choose, for instance, the parameters:  $d = 3$ ,  $\kappa_0 = \kappa_1 = \delta = 1$ ,  $\nu_0 = 2$  and  $\nu_1 = 3$ . Then  $\alpha = d^{\nu_1} - \kappa_1 \delta = 26$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha) = -17$ . The corresponding map to the admissible triplet  $(3, 26, -17)_+$  is given as

$$T(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3} \\ (26n - 17[n]_3)/3 & \text{if } n \not\equiv 0 \pmod{3}. \end{cases}$$

As here  $\kappa_1\beta < 0$ , this example corresponds to the item 1, in the previous Theorem. Then, the order of the triplet  $(3, 26, -17)_+$  is  $\#\text{Cycl}(3, 26, -17)_+ \geq d - 1 = 2$ . In this case, we have at least 2 trivial cycles of length  $\nu_0 = 2$  which are:  $\Omega(1) = (1 \rightarrow 3 \rightarrow 1)$  and  $\Omega(2) = (2 \rightarrow 6 \rightarrow 2)$ .

3. In this example, we take the similar parameters as in the previous example, we just change  $\kappa_1 = -1$ . Then  $\alpha = d^{\nu_1} - \kappa_1\delta = 28$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha) = -19$ . The corresponding mapping to the admissible triplet  $(3, 28, -19)_+$  is given as

$$T(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3} \\ (28n - 19[n]_3)/3 & \text{if } n \not\equiv 0 \pmod{3}. \end{cases}$$

As in this case  $\kappa_1\beta > 0$ ,  $\delta = 1$  and  $\nu_0 \neq \nu_1$ , this example corresponds to the item 2-1 in the previous Theorem. The order  $\#\text{Cycl}(3, 28, -19)_+ \geq 2(d - 1) = 4$ . In this case, we have at least 4 trivial cycles. Two of length  $\nu_0 = 2$  which are:  $\Omega(1) = (1 \rightarrow 3 \rightarrow 1)$  and  $\Omega(2) = (2 \rightarrow 6 \rightarrow 2)$  and two others of length  $\nu_1 = 3$ , which are:  $\Omega(|\beta| = 19) = (19 \rightarrow 171 \rightarrow 57 \rightarrow 19)$  and  $\Omega(2|\beta| = 38) = (38 \rightarrow 342 \rightarrow 114 \rightarrow 38)$ .

4. In this example, we consider the admissible triplet  $(12, 134, 1594)_+$  corresponding to the parameters  $d = 12$ ,  $\nu_0 = 3$ ,  $\nu_1 = 2$ ,  $\delta = 10$  and  $\kappa_0 = \kappa_1 = 1$ . Then  $\alpha = d^{\nu_1} - \kappa_1\delta = 134$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha) = 1594$ . The corresponding mapping is:

$$T(n) = \begin{cases} n/12 & \text{if } n \equiv 0 \pmod{12} \\ (134n + 1594[n]_{12})/12 & \text{if } n \not\equiv 0 \pmod{12}. \end{cases}$$

We have  $\kappa_1\beta > 0$  and  $\nu_0 \neq \nu_1$  with  $\delta = 10 > 1$ . This corresponds to Item 2-2 in the previous Theorem. We have  $d - 1 = 11$  trivial cycles of length  $\nu_0 = 3$ , which are:  $\Omega(r) = (r \rightarrow 144r \rightarrow 12r \rightarrow r)$  for  $1 \leq r \leq 11$ . As  $q_0 = \gcd(d^{\nu_0-1} - d^{\nu_1-1}, \delta) = 2$ . Then  $\delta_0 = 5$ ,  $\beta_0 = 797$  and  $\left\lfloor \frac{d-1}{\delta_0} \right\rfloor = 2$ . It follows that the other two cycles of length  $\nu_1 = 2$  are:  $\Omega(797r) = (797r \rightarrow 9564r \rightarrow 797r)$  for  $r = 1, 2$ . Finally, the admissible triplet  $(12, 134, 1594)_+$  has at least 13 cycles.

5. In this example, we consider the triplet  $(3, 8, 5)_-$  corresponding to the parameters:  $d = 3$ ,  $\nu_0 = 1$ ,  $\nu_1 = 2$ ,  $\delta = 1$ ,  $\kappa_0 = -1$  and  $\kappa_1 = 1$ . Then  $\alpha = d^{\nu_1} - \kappa_1\delta = 8$  and  $\beta = \kappa_0(d^{\nu_0} - \alpha) = 5$ . The associated mapping is:

$$T(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3} \\ (8n + 5[-n]_3)/3 & \text{if } n \not\equiv 0 \pmod{3}, \end{cases} \quad \text{more precisely } T(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3} \\ (8n + 10)/3 & \text{if } n \equiv 1 \pmod{3} \\ (8n + 5)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

We have  $\kappa_1\beta > 0$  and  $\delta = 1$ . This corresponds to Item 2-1 in the previous Theorem. We have  $d - 1 = 2$  trivial cycles of length  $\nu_1 = 2$ , which are:  $\Omega(r|\beta|) = (5r \rightarrow 15r \rightarrow 5r)$  for  $1 \leq r \leq 2$ . The admissible triplet  $(3, 8, 5)_-$  has at least 2 cycles:  $\Omega(5) = (5 \rightarrow 15 \rightarrow 5)$  and  $\Omega(10) = (10 \rightarrow 30 \rightarrow 10)$ .

6. Now let us give a general example: For  $d \geq 2$ , we choose  $\lambda_0 = \lambda_1 = \delta = 1$ ,  $\kappa_1 = -1$  and  $\kappa_0 = 1$  with  $\nu_1 > \nu_0 \geq 2$ , so  $\kappa_1\beta = -\beta > 0$ . According to our experience based on a multitude of tests we have performed with computer, it appears that for all  $d \geq 2$ , the triplet  $(d, \alpha, \beta)_+$  with  $\alpha = d^{\nu_1} + 1$  and  $\beta = d^{\nu_0} - \alpha = d^{\nu_0} - d^{\nu_1} - 1$  with  $\nu_1 > \nu_0 \geq 2$  is an admissible weak triplet of order  $\#\text{Cycl}(d, \alpha, \beta)_+ = 2(d-1)$ . Its trivial cycles are:

$$\Omega(r) = (r \rightarrow rd^{\nu_0-1} \rightarrow rd^{\nu_0-2} \rightarrow \dots \rightarrow rd \rightarrow r),$$

and

$$\Omega(r|\beta) = (r|\beta \rightarrow r|\beta|d^{\nu_1-1} \rightarrow r|\beta|d^{\nu_1-2} \rightarrow \dots \rightarrow r|\beta|d \rightarrow r|\beta),$$

of length  $\nu_0$  and  $\nu_1$ , respectively, for all  $r \in \{1, \dots, d-1\}$ .

### 3.2 The special admissible triplets $(d, d+1, -1)_+$ and $(d, d+1, 1)_-$ .

Special cases of the previous class are the triplets  $(d, d+1, -1)_+$  and  $(d, d+1, 1)_-$  corresponding to the parameters  $d \geq 2$ ,  $\lambda_0 = \lambda_1 = \delta = \nu_0 = \nu_1 = 1$ ,  $\kappa_0 = \pm 1$  and  $\kappa_1 = -1$ . Indeed, as the triplet  $(d, \alpha, \beta)_\pm$  is an admissible triplet, we have  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = d+1$  and  $\beta = \kappa_0(\lambda_0 d^{\nu_0} - \alpha) = -\kappa_0$ , with  $d \geq 2$ . The corresponding mapping is given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ \frac{(d+1)n - \kappa_0[\kappa_0 n]_d}{d} & \text{if } n \not\equiv 0 \pmod{d}. \end{cases} \quad (18)$$

We have the following result.

**Theorem 3.2.** *Let  $d \geq 2$ , both the admissible triplets  $(d, d+1, -1)_+$  and  $(d, d+1, 1)_-$  have at least one cycle, more precisely, we have:*

1. *The triplet  $(d, d+1, -1)_+$  has at least the  $d-1$  trivial cycles of length 1:*

$$\Omega(r) = (r \rightarrow r), \quad (19)$$

for  $r \in \{1, \dots, d-1\}$ . It follows that  $\#\text{Cycl}(d, d+1, -1)_+ \geq d-1$ .

2. *The triplet  $(d, d+1, 1)_-$  has at least the trivial cycle of length  $d$ :*

$$\Omega(1) = (1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow d \rightarrow 1). \quad (20)$$

It follows that  $\#\text{Cycl}(d, d+1, 1)_- \geq 1$ .

*Proof.* 1. For  $\kappa_0 = 1$  and for  $r \in \{1, \dots, d-1\}$ ,  $T(r) = \frac{(d+1)r - [r]_d}{d}$ . Then  $T(r) = \frac{(d+1)r - r}{d} = \frac{dr}{d} = r$ , which gives rise to the cycles  $\Omega(r)$  given by (19) for  $r \in \{1, \dots, d-1\}$ . Thus, the cardinality number  $\#\text{Cycl}(d, d+1, -1, 1) \geq d-1$ .

2. For  $\kappa_0 = -1$  and for  $r \in \{1, \dots, d-1\}$ ,  $T(r) = \frac{(d+1)r + [-r]_d}{d}$ . Then,  $T(r) = \frac{(d+1)r + (d-r)}{d} = \frac{d(r+1)}{d} = r+1$ . It follows that  $T^{(d)}(1) = 1$  which gives rise to the cycle  $\Omega(1)$  given by (20). Thus, the order of the triplet is  $\#\text{Cycl}(d, d+1, 1, -1) \geq 1$ .  $\square$

Let us remark that, the classical Collatz case is obtained for  $d = 2$  with  $\kappa_0 = -1$ . The trivial cycle  $\Omega(1)$  is in the form (20):  $\Omega(1) = (1 \rightarrow 2 \rightarrow 1)$ .

it appears that for all  $d \geq 2$ , the triplets  $(d, d+1, -1)_+$  and  $(d, d+1, 1)_-$  are admissible strong triplets and their respective orders satisfy  $d-1 \leq \#\text{Cycl}(d, d+1, -1)_+$  for  $\kappa_0 = 1$  and  $1 \leq \#\text{Cycl}(d, d+1, 1)_-$  for  $\kappa_0 = -1$ . Their trivial cycles are:

- For  $\kappa_0 = 1$ ,  $\Omega(r) = (r \rightarrow r)$  for all  $r \in \{1, \dots, d-1\}$ .
- For  $\kappa_0 = -1$ ,  $\Omega(1) = (1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow d \rightarrow 1)$ .

### 3.3 The admissible triplets $(d, 2d-1, 1)_+$ evoking the Mersene numbers for $d = 2^{p-1}$ , $p \geq 2$ .

The triplet  $(d, 2d-1, 1)_+$  is obtained by choosing  $\lambda_0 = \lambda_1 = 2$  and  $\nu_0 = \nu_1 = \delta = \kappa_0 = \kappa_1 = 1$ , then  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = 2d-1$  and  $\beta = \kappa_0(\lambda_0 d^{\nu_0} - \alpha) = 1$ . The corresponding mapping is given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ \frac{(2d-1)n + [n]_d}{d} & \text{if } n \not\equiv 0 \pmod{d}. \end{cases} \quad (21)$$

An interesting case may be obtained by setting  $d = 2^{p-1}$  where  $p \geq 2$ . This leads to the admissible triplet  $(2^{p-1}, 2^p-1, 1)_+$  with  $\alpha = 2^p-1$  as a Mersenne number, for  $p \geq 2$ . We recover the classical Collatz problem by choosing  $p = 2$ . We have the following result.

**Theorem 3.3.** *Let  $d_p = 2^{p-1}$  for  $p \geq 2$  and let the integers  $\alpha_p = 2d_p - 1 = 2^p - 1$ ,  $\beta = 1$  and  $\kappa_0 = 1$ , then the admissible triplet  $(2^{p-1}, 2^p - 1, 1)_+$  has at least the following trivial cycle of length  $p$ :*

$$\Omega(1) = (1 \rightarrow 2 \rightarrow 2^2 \rightarrow \dots \rightarrow 2^{p-1} \rightarrow 1), \quad (22)$$

It follows that,  $\#\text{Cycl}(2^{p-1}, 2^p - 1, 1)_+ \geq 1$ .

*Proof.* Let  $p \geq 2$ , for any integer  $k$  with  $0 \leq k \leq p-2$ , we have

$$T(2^k) = \frac{(2^p - 1)2^k + [2^k]_{d_p}}{d_p} = \frac{(2^p - 1)2^k + 2^k}{2^{p-1}} = 2^{k+1}.$$

It follows that  $T^{(p)}(1) = 1$  which gives rise to the trivial cycle (22).  $\square$

For this case, we may formulate the following conjecture.

**Conjecture 3.1.** *For all  $p \geq 2$ , let  $d_p = 2^{p-1}$ ,  $\alpha_p = 2d_p - 1 = 2^p - 1$  and  $\beta = 1$ , then the admissible triplet  $(2^{p-1}, 2^p - 1, 1)_+$  is of order one. It is a strong triplet only for  $p = 2$  and it is a weak triplet for all  $p \geq 3$ . Its trivial cycle of length  $p$  is:  $\Omega(1) = (1 \rightarrow 2 \rightarrow 2^2 \rightarrow \dots \rightarrow 2^{p-1} \rightarrow 1)$ .*

If the last conjecture is true, then for  $p \geq 3$ :  $\#\text{Cycl}(2^{p-1}, 2^p - 1, 1)_+ = 1$  and  $\mathbb{N} = G(1) \cup G(\infty)$ . The triplet  $(2^{p-1}, 2^p - 1, 1)_+$  may have divergent trajectories.

### 3.4 The triplet $(d, d+1, d-1)_+$ with $d \geq 2$ an interesting case corresponding to $d = 2^p + 1$ with $p \geq 0$ .

An interesting triplet that we will examine in this subsection is the one obtained by setting  $\lambda_0 = 2$ ,  $\lambda_1 = \nu_0 = \nu_1 = \delta = \kappa_0 = 1$  and  $\kappa_1 = -1$ . Thus,  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = d+1$  and  $\beta = \kappa_0(\lambda_0 d^{\nu_0} - \alpha) = d-1$ , with  $d \geq 2$ . The corresponding mapping is given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ \frac{(d+1)n + (d-1)[n]_d}{d} & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \quad (23)$$

**Theorem 3.4.** *For  $d \geq 2$ , the admissible triplet  $(d, \alpha, \beta)_+ = (d, d+1, d-1)_+$  has at least the following trivial cycle of length  $d$ :*

$$\Omega(\beta) = (\beta \rightarrow 2\beta \rightarrow \dots \rightarrow d\beta \rightarrow \beta). \quad (24)$$

*Proof.* We have obviously, for  $1 \leq k \leq d-1$ ,  $T(k\beta) = \frac{\alpha k\beta + \beta[k\beta]_d}{d}$ . As  $k\beta = k(d-1)$ , then  $[k\beta]_d = [-k]_d = d-k$ . It follows that  $T(k\beta) = \frac{(d+1)k\beta + \beta(d-k)}{d} = (k+1)\beta$  and  $T^{(d-1)}(\beta) = d\beta$  and  $T^{(d)}(\beta) = \beta$ .  $\square$

More interesting, particular case is the one where  $d = 2^p + 1$  with  $p \geq 0$  is a positive integer. Then,

$$d = d_p := 2^p + 1, \quad \alpha = \alpha_p := d_p + 1 = 2^p + 2, \quad \beta = \beta_p := d_p - 1 = 2^p. \quad (25)$$

Thus the corresponding mapping  $T := T_p$ , for  $p \geq 0$ , is given by

$$T_p(n) = \begin{cases} n/(2^p + 1) & \text{if } n \equiv 0 \pmod{2^p + 1} \\ \frac{(2^p + 2)n + 2^p[n]_p}{2^p + 1} & \text{if } n \not\equiv 0 \pmod{2^p + 1}, \end{cases} \quad (26)$$

here  $[ ]_p = [ ]_{d_p}$  stands for the remainder in the Euclidean division by  $d_p = 2^p + 1$ . We observe that the case  $p = 0$  corresponds to the classical Collatz case (1). We now formulate the following main conjecture obtained according to our computational experiences and tests supported by computer. A particular example from this class will be presented in the last section. We think, with conviction, that the following conjecture that generalizes the classical Collatz one is true.

**Conjecture 3.2.** *1. For all integer  $p \geq 0$  with  $p \notin \{1, 3, 4\}$ , the triplet  $(2^p + 1, 2^p + 2, 2^p)_+$  is an admissible strong triplet of order one ( $q = 1$ ). Its unique cycle is the trivial cycle of length  $d_p = 2^p + 1$ :*

$$\Omega(2^p) = (1 \cdot 2^p \rightarrow 2 \cdot 2^p \rightarrow 3 \cdot 2^p \rightarrow \dots \rightarrow (2^p + 1) \cdot 2^p \rightarrow 2^p).$$

*Then, for all integer  $n \geq 1$ , there exists an integer  $k \geq 0$  such that  $T_p^{(k)}(n) = 2^p$ .*

*2. For  $p \in \{1, 3, 4\}$ , the triplet  $(2^p + 1, 2^p + 2, 2^p)_+$  is also an admissible strong triplet of order two ( $q = 2$ ). Its first trivial cycle of length  $2^p + 1$  is:*

$$\Omega(2^p) = (1 \cdot 2^p \rightarrow 2 \cdot 2^p \rightarrow 3 \cdot 2^p \rightarrow \dots \rightarrow (2^p + 1) \cdot 2^p \rightarrow 2^p).$$

Its second trivial cycle in each case, is the cycle  $\Omega(\omega_p)$  of length 9, 21, 49, respectively and starting with  $\omega_1 = 14$ ,  $\omega_3 = 280$  and  $\omega_4 = 1264$ , respectively. The second cycle is given in each case as:

$$\begin{aligned}\Omega(\omega_1) &= (14 \rightarrow 20 \rightarrow 28 \rightarrow 38 \rightarrow 52 \rightarrow 70 \rightarrow 94 \rightarrow 126 \rightarrow 42 \rightarrow 14), \\ \Omega(\omega_3) &= (280 \rightarrow 312 \rightarrow 352 \rightarrow 392 \rightarrow 440 \rightarrow 496 \rightarrow 552 \rightarrow 616 \rightarrow 688 \rightarrow 768 \rightarrow 856 \rightarrow 952 \rightarrow \\ &\quad 1064 \rightarrow 1184 \rightarrow 1320 \rightarrow 1472 \rightarrow 1640 \rightarrow 1824 \rightarrow 2032 \rightarrow 2264 \rightarrow 2520 \rightarrow 280). \\ \Omega(\omega_4) &= (1264 \rightarrow 1344 \rightarrow 1424 \rightarrow 1520 \rightarrow 1616 \rightarrow 1712 \rightarrow 1824 \rightarrow 1936 \rightarrow 2064 \rightarrow 2192 \rightarrow 2336 \rightarrow \\ &\quad 2480 \rightarrow 2640 \rightarrow 2800 \rightarrow 2976 \rightarrow 3152 \rightarrow 3344 \rightarrow 3552 \rightarrow 3776 \rightarrow 4000 \rightarrow 4240 \rightarrow 4496 \rightarrow \\ &\quad 4768 \rightarrow 5056 \rightarrow 5360 \rightarrow 5680 \rightarrow 6016 \rightarrow 6384 \rightarrow 6768 \rightarrow 7168 \rightarrow 7600 \rightarrow 8048 \rightarrow 8528 \rightarrow \\ &\quad 9040 \rightarrow 9584 \rightarrow 10160 \rightarrow 10768 \rightarrow 11408 \rightarrow 12080 \rightarrow 12800 \rightarrow 13568 \rightarrow 14368 \rightarrow 15216 \rightarrow \\ &\quad 16112 \rightarrow 17072 \rightarrow 18080 \rightarrow 19152 \rightarrow 20288 \rightarrow 21488 \rightarrow 1264).\end{aligned}$$

For all integer  $n \geq 1$ , there exists an integer  $k \geq 0$  such that  $T_p^{(k)}(n) \in \{2^p, \omega_p\}$ .

- Examples 3.2.** 1. The classical Collatz conjecture is a special case of the conjecture 3.2, corresponding to  $p = 0$  with the triplet  $(d_0, \alpha_0, \beta_0)_+ = (2, 3, 1)_+$ . If the conjecture is true, then for all  $n \in \mathbb{N}$ ,  $\exists k \in \mathbb{N}_0$  such that  $T_0^{(k)}(n) = 2^0 = 1$  and its unique cycle is the trivial cycle  $(1 \rightarrow 2 \rightarrow 1)$  of length  $d_0 = 2^0 + 1 = 2$ .
2. Let us give the triplet  $(2^p+1, 2^p+2, 2^p)_+$  for  $p = 25$  which is the triplet  $(33445533, 33445534, 33445532)_+$  given in Item 1 of Examples 2.1. If the previous conjecture is true, then  $(33445533, 33445534, 33445532)_+$  is an admissible strong triplet and its unique trivial cycle of length  $d_{25} = 2^{25} + 1 = 33445533$  is given in Item 1 of Examples 2.1. Thus, for all  $n \in \mathbb{N}$ ,  $\exists k \in \mathbb{N}_0$  such that  $T_{25}^{(k)}(n) = 2^{25} = 33445532$ .
3. The case  $p = 2$  gives the triplet  $(5, 6, 4)_+$  and if the previous conjecture is true, then  $(5, 6, 4)_+$  is an admissible strong triplet. It follows that its unique trivial cycle of length  $d_2 = 2^2 + 1 = 5$  is  $\Omega(4) = (4 \rightarrow 8 \rightarrow 12 \rightarrow 16 \rightarrow 20 \rightarrow 4)$  and for all  $n \in \mathbb{N}$ ,  $\exists k \in \mathbb{N}_0$  such that  $T_2^{(k)}(n) = 2^2 = 4$ .

this case is studied in more details in [11].

### 3.5 The triplet $(d, d+2, d-2)_+$ with $d = 2d' \geq 4$ an interesting case corresponding to $d = 2^p + 2$ with $p \geq 1$ .

Another interesting case is the one obtained by setting  $\lambda_0 = 2$ ,  $\lambda_1 = \nu_0 = \nu_1 = \kappa_0 = 1$ ,  $\delta = 2$  and  $\kappa_1 = -1$ . Thus,  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = d + 2$  and  $\beta = \kappa_0 (\lambda_0 d^{\nu_0} - \alpha) = d - 2$ , with  $d = 2d' \geq 4$  an even integer. The corresponding mapping is given by

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d}, \\ \frac{(d+2)n + (d-2)[n]_d}{d} / d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \quad (27)$$

which may be simplified as

$$T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ \frac{(d'+1)n + (d'-1)[n]_d}{d'} / d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \quad (28)$$



**Theorem 3.5.** For  $d = 2d' \geq 4$  with  $d$  even integer, the admissible triplet  $(d, \alpha, \beta)_+ = (d, d+2, d-2)_+$  has at least the following trivial cycle of length  $\frac{1}{2}d + 1$ :

$$\Omega\left(\frac{1}{2}\beta\right) = \left(\frac{1}{2}\beta \rightarrow \beta \rightarrow 2\beta \rightarrow 3\beta \rightarrow \dots \rightarrow d'\beta \rightarrow \frac{1}{2}\beta\right). \quad (29)$$

*Proof.* As  $d = 2d'$ , then  $\alpha = 2(d' + 1)$  and  $\beta = 2(d' - 1)$ . We have obviously,  $T\left(\frac{1}{2}\beta\right) = \frac{1}{2}\beta \frac{\alpha + \beta}{d} = \beta$ . For  $1 \leq k \leq d' - 1$ , we have  $T(k\beta) = \frac{(d+2)k\beta + (d-2)[k\beta]_d}{d}$ . But  $k\beta = k(d-2)$ , then  $[k\beta]_d = [-2k]_d = d - 2k$ . It follows that  $T(k\beta) = \frac{(d+2)k\beta + (d-2)(d-2k)}{d} = (k+1)\beta$ . Which gives  $T^{(d'-1)}(\beta) = d'\beta$ . But  $d'\beta = 2d'(d' - 1) = \frac{1}{2}\beta d$ . It follows that  $T^{(d'+1)}\left(\frac{1}{2}\beta\right) = \frac{1}{2}\beta$ . which gives rise to the trivial cycle (29).  $\square$

**Example 3.1.** For instance the admissible triplet  $(12, 14, 10)_+$  has the trivial cycle  $\Omega(5) = (5 \rightarrow 10 \rightarrow 20 \rightarrow 30 \rightarrow 40 \rightarrow 50 \rightarrow 60 \rightarrow 5)$  of length 7. But it has also two other cycles which are  $\Omega(4) = (4 \rightarrow 8 \rightarrow 16 \rightarrow 22 \rightarrow 4)$  of length 6 and  $\Omega(1305) = (1305 \rightarrow 1530 \rightarrow 1790 \rightarrow 2090 \rightarrow 2440 \rightarrow 2850 \rightarrow 3330 \rightarrow 3890 \rightarrow 4540 \rightarrow 5300 \rightarrow 6190 \rightarrow 7230 \rightarrow 8440 \rightarrow 9850 \rightarrow 11500 \rightarrow 13420 \rightarrow 15660 \rightarrow 1305)$  of length 17. It seems that the admissible triplet  $(12, 14, 10)_+$  has only the three cycles.

More interesting particular case is the one where  $d = 2^p + 2$  with  $p \geq 1$  is a positive integer. Then,

$$d = d_p := 2^p + 2, \quad \alpha = \alpha_p := d_p + 2 = 2^p + 4 \quad \beta = \beta_p := d_p - 2 = 2^p. \quad (30)$$

Thus the corresponding mapping  $T := T_p$ , for  $p \geq 1$ , is given by

$$T_p(n) = \begin{cases} n/(2^p + 2) & \text{if } n \equiv 0 \pmod{(2^p + 2)} \\ \frac{(2^{p-1} + 2)n + 2^{p-1}[n]_p}{2^{p-1} + 1} & \text{if } n \not\equiv 0 \pmod{(2^p + 2)}, \end{cases} \quad (31)$$

here  $[ ]_p = [ ]_{d_p}$  stands for the remainder in the Euclidean division by  $d_p = 2^p + 2$ . We observe that the case  $p = 3$  corresponds to the case  $d_3 = 10$ ,  $\alpha_3 = 12$  and  $\beta_3 = 8$  which is exactly the case of our conjecture 1.1 given in the introduction. We now formulate the following second main conjecture also obtained according to our computational experiences and tests supported by computer. We believe that the following general conjecture is true.

**Conjecture 3.3.** 1. For all integer  $p \geq 1$  with  $p \neq 2$ , the triplet  $(2^p + 2, 2^p + 4, 2^p)_+$  is an admissible strong triplet of order one, its unique cycle is the trivial cycle of length  $2^{p-1} + 2$ :

$$\Omega(2^{p-1}) = \left(2^{p-1} \rightarrow 2^p \rightarrow 2 \cdot 2^p \rightarrow 3 \cdot 2^p \rightarrow \dots \rightarrow (2^{p-1} + 1) \cdot 2^p \rightarrow 2^{p-1}\right).$$

Then, for all integer  $n \geq 1$ , there exists an integer  $k \geq 0$  such that  $T_p^{(k)}(n) = 2^{p-1}$ .

2. If  $p = 2$ , the triplet  $(6, 8, 4)_+$  is an admissible strong triplet of order two ( $q = 2$ ). Its first trivial cycle is  $\Omega(2)$  of length 4 given as:

$$\Omega(2) = \left(2 \rightarrow 4 \rightarrow 8 \rightarrow 12 \rightarrow 2\right),$$

Its second trivial cycle is the cycle  $\Omega(74)$  of length 7 given as:

$$\Omega(74) = (74 \rightarrow 100 \rightarrow 136 \rightarrow 184 \rightarrow 248 \rightarrow 332 \rightarrow 444 \rightarrow 74).$$

Then, for all integer  $n \geq 1$ , there exists an integer  $k \geq 0$  such that  $T_2^{(k)}(n) \in \{2, 74\}$ .

The last Conjecture 3.3 encompass our conjecture 1.1 obtained for  $p = 3$  and associaed to the operator given in (5). But, it does not encompass the classical Collatz conjecture. However, we can derive a modified version of the classical Collaz conjecture for  $p = 1$ , we get  $(d, \alpha, \beta)_+ = (4, 6, 2)_+$ , which gives the map  $T_1$  as following:

$$T_1(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod{4} \\ \frac{3n + [n]_4}{2} & \text{if } n \not\equiv 0 \pmod{4}, \end{cases} \quad \text{or} \quad T_1(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod{4} \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod{4}, \\ \frac{3n+2}{2} & \text{if } n \equiv 2 \pmod{4}, \\ \frac{3n+3}{2} & \text{if } n \equiv 3 \pmod{4}, \end{cases} \quad (32)$$

In this case where the previous conjecture is true, then the triplet  $(4, 6, 2)_+$  is a strong admissible triplet of order one. Its unique trivial cycle is  $(1 \rightarrow 2 \rightarrow 4 \rightarrow 1)$  and for all  $n \geq 1$ , there exists an integer  $k \geq 0$  such that  $T_1^{(k)}(n) = 2^{p-1} = 1$ .

### 3.6 The admissible triplets $(d, \alpha, \beta)_+$ with $\alpha = d^{\nu_1} + 1$ and $\beta = d^{2\mu_0 + \nu_1} - \alpha^2$ for $d \geq 2$ with $\mu_0 \geq 1$ and $\nu_1 \geq 1$ .

Let us return back to the classical Collatz case. The cycles for the triplets  $(2, 3, k)_+$  problem, known as the  $3x + k$  problem, were previously studied in [4]. In [8], it is indicated that in 2002, A. S. Jones pointed out that for the problem  $3x + k$ , if  $k = 2^{2c+1} - 9$ , the numbers  $2^k + 3$  for  $1 \leq k \leq c$ , generate  $c$  different cycles of length  $2c + 1$ . We aim in this subsection, to give an extension to the result pointed out by A. S. Jones. We consider a particular triplet  $(d, \alpha, \beta)_+$  for which the corresponding parameters are as follows: Let  $d \geq 2$  and let us choose  $\lambda_1 = \delta = \kappa_0 = 1$ ,  $\kappa_1 = -1$ . We choose  $\nu_1 = \nu_0 \geq 1$  and we set  $\lambda_0 = d^{2\mu_0} - d^{\nu_1} - 1$  where  $\mu_0 \geq 1$  satisfies  $2\mu_0 > \nu_1$ , then  $\alpha = \lambda_1 d^{\nu_1} - \kappa_1 \delta = d^{\nu_1} + 1$  and  $\beta = \kappa_0 (\lambda_0 d^{\nu_1} - \alpha) = (d^{2\mu_0} - d^{\nu_1} - 1)d^{\nu_1} - (d^{\nu_1} + 1) = d^{2\mu_0 + \nu_1} - \alpha^2$ . Then, we may state the following result.

**Theorem 3.6.** *Let  $d \geq 2$ ,  $\mu_0 \geq 1$  and  $\nu_0 = \nu_1 \geq 1$  such that  $2\mu_0 > \nu_1$  and consider the admissible triplet  $(d, \alpha, \beta)_+$  with  $\alpha = d^{\nu_1} + 1$  and  $\beta = d^{2\mu_0 + \nu_1} - \alpha^2$ . Then,  $(d, \alpha, \beta)_+$  has at least  $q$  cycles  $\Omega(\omega_{k,r})$  with  $q \geq \mu_0(d-1)^2$  generated by the starting numbers  $\omega_{k,r} = r_1(r_2 d^k + r_3 \alpha)$  for  $k$  and  $r = (r_1, r_2, r_3) \in \mathbb{N}^3$  such that*

$$1 \leq k \leq \mu_0, 1 \leq r_1 \leq d-1, 1 \leq r_2 \leq d-1, 1 \leq r_3 \leq d-1, 1 \leq r_1 r_2 \leq d-1, 1 \leq r_1 r_3 \leq d-1, \quad (33)$$

Furthermore, if  $\nu_1 = 1$ , then  $\text{Cycl}(d, \alpha, \beta)_+$  contains also the cycle  $\Omega(\beta)$  of length  $d$ .

*Proof.* For  $r = (r_1, r_2, r_3)$  such that  $1 \leq r_1 \leq d-1$ ,  $1 \leq r_2 \leq d-1$ ,  $1 \leq r_3 \leq d-1$ ,  $1 \leq r_1 r_2 \leq d-1$  and  $1 \leq r_1 r_3 \leq d-1$  and for  $1 \leq k \leq \mu_0$ , we have  $[\omega_{k,r}]_d = [r_1 r_3 \alpha]_d = r_1 r_3$ , then for  $k \geq 1$ :

$$T(\omega_{k,r}) = \frac{\alpha \omega_{k,r} + \beta r_1 r_3}{d} = \frac{\alpha r_1 (r_2 d^k + r_3 \alpha) + \beta r_1 r_3}{d} = \alpha r_1 r_2 d^{k-1} + r_1 r_3 \frac{\beta + \alpha^2}{d}.$$

As  $\alpha^2 + \beta = d^{2\mu_0 + \nu_1}$ , it follows that  $T(\omega_{k,r}) = \alpha r_1 r_2 d^{k-1} + r_1 r_3 d^{2\mu_0 + \nu_1 - 1}$ . Then, by induction for  $1 \leq k \leq \mu_0$ , we assume that at the step  $k$ , we have

$$T^{(k)}(\omega_{k,r}) = \alpha r_1 r_2 + r_1 r_3 d^{2\mu_0 + \nu_1 - k}.$$

As  $[r_1 r_2 \alpha]_d = r_1 r_2$ , it follows that

$$\begin{aligned} T^{(k+1)}(\omega_{k,r}) &= \frac{\alpha r_1 (\alpha r_2 + r_3 d^{2\mu_0 + \nu_1 - k}) + \beta r_1 r_2}{d} = \alpha r_1 r_3 d^{2\mu_0 + \nu_1 - k - 1} + r_1 r_2 \frac{\beta + \alpha^2}{d} \\ &= \alpha r_1 r_3 d^{2\mu_0 + \nu_1 - k - 1} + r_1 r_2 d^{2\mu_0 + \nu_1 - 1}. \end{aligned}$$

Then, for  $1 \leq k' \leq 2\mu_0 + \nu_1 - k$ , we have  $T^{(k+k')}(\omega_{k,r}) = \alpha r_1 r_3 d^{2\mu_0 + \nu_1 - k - k'} + r_1 r_2 d^{2\mu_0 + \nu_1 - k'}$ , which gives for  $k' = 2\mu_0 + \nu_1 - k$  that  $T^{(2\mu_0 + \nu_1)}(\omega_{k,r}) = \omega_{k,r}$ . Then  $\Omega(\omega_{k,r})$  is a cycle of length  $2\mu_0 + \nu_1$  for all  $k$  and  $r = (r_1, r_2, r_3)$  satisfying (33). Furthermore, if  $\nu_1 = 1$ , we have  $\beta = d^{2\mu_0 + 1} - \alpha^2 = d^{2\mu_0 + 1} - (d+1)^2$ , it follows that  $\alpha - 1 = d$ ,  $[\beta]_d = d - 1$  and

$$T(\beta) = \frac{\alpha\beta + \beta[\beta]_d}{d} = \beta \frac{\alpha + d - 1}{d} = 2\beta.$$

By induction, we have for  $1 \leq k \leq d - 1$ ,  $[k\beta]_d = d - k$  and

$$T^{(k)}(\beta) = \frac{k\alpha\beta + \beta[k\beta]_d}{d} = \beta \frac{k\alpha + d - k}{d} = (k+1)\beta.$$

Then,  $T^{(d)}(\beta) = \beta$ , and  $\Omega(\beta)$  is a cycle of length  $d$ . □

**Examples 3.3.** 1. For the first example, we choose  $d = 5$ ,  $\nu_1 = \nu_0 = 1$  and  $\mu_0 = 2$ . It follows that  $\alpha = d^{\nu_1} + 1 = 6$  and  $\beta = d^{2\mu_0 + \nu_1} - \alpha^2 = 3089$ . Then, the admissible triplet  $(5, 6, 3089)_+$  has at least the following 32 cycles of length  $2\mu_0 + \nu_1 = 5$  generated by the starting numbers  $\omega_{k,r}$  with  $k$  and  $r = (r_1, r_2, r_3)$  satisfying the conditions (33). The 32 cycles are:  $\Omega(11)$ ,  $\Omega(16)$ ,  $\Omega(17)$ ,  $\Omega(21)$ ,  $\Omega(22)$ ,  $\Omega(23)$ ,  $\Omega(26)$ ,  $\Omega(27)$ ,  $\Omega(28)$ ,  $\Omega(29)$ ,  $\Omega(31)$ ,  $\Omega(32)$ ,  $\Omega(33)$ ,  $\Omega(34)$ ,  $\Omega(37)$ ,  $\Omega(38)$ ,  $\Omega(39)$ ,  $\Omega(43)$ ,  $\Omega(44)$ ,  $\Omega(49)$ ,  $\Omega(56)$ ,  $\Omega(62)$ ,  $\Omega(68)$ ,  $\Omega(74)$ ,  $\Omega(81)$ ,  $\Omega(87)$ ,  $\Omega(93)$ ,  $\Omega(99)$ ,  $\Omega(106)$ ,  $\Omega(112)$ ,  $\Omega(118)$  and  $\Omega(124)$ . Furthermore, as in this case  $\nu_0 = 1$ , the triplet  $(5, 6, 3089)_+$  has the additional cycle  $\Omega(\beta = 3089)$  of length  $d = 5$  starting by  $\beta = 3089$ . It appears that the triplet  $(5, 6, 3089)_+$  is an admissible strong triplet with exactly the order = 33. This example is the one given in Item 4 of Examples 2.1.

### 3.7 The triplets $(d, \alpha, \beta_0)_\pm$ and $(d, \alpha, \beta)_\pm$ with $\beta = a_0\beta_0$ and $a_0 \equiv 1 \pmod{d}$ .

In this subsection, we will establish an extension to a result given in [4] for the classical Collatz case. More precisely, in [4], Lagarias proposed the map  $L_\beta$ , see Section 1. We may observe that the trivial cycle  $\Omega(\beta) = (\beta \rightarrow 2\beta \rightarrow \beta)$  of the mapping  $L_\beta$  is obtained by multiplying the trivial cycle  $\Omega(1) = (1 \rightarrow 2 \rightarrow 1)$  of  $L_1$  by  $\beta$ . In the following theorem, we will give a general similar result for an admissible triplet  $(d, \alpha, \beta_0)_\pm$ .

**Theorem 3.7.** For all  $a_0$  positive integer such that  $a_0 \equiv 1 \pmod{d}$ , if  $\Omega_0(\omega)$  is a cycle of length  $\ell$  for the admissible triplet  $(d, \alpha, \beta_0)_\pm$  then  $\Omega(a_0\omega)$  is also a cycle of length  $\ell$  for the admissible cycle  $(d, \alpha, \beta)_\pm$  with  $\beta = a_0\beta_0$ .

*Proof.* Let  $T_0$  and  $T$  be the mappings associated to  $(d, \alpha, \beta_0)_\pm$  and  $(d, \alpha, \beta)_\pm$ , respectively, with  $\beta = a_0\beta_0$ , we have

$$T_0(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ (\alpha n + \beta_0[\kappa_0 n]_d)/d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \quad \text{and } T(n) = \begin{cases} n/d & \text{if } n \equiv 0 \pmod{d} \\ (\alpha n + \beta[\kappa_0 n]_d)/d & \text{if } n \not\equiv 0 \pmod{d}. \end{cases}$$

Let  $n$  be any positive integer, as  $a_0 \equiv 1 \pmod{d}$ , then  $[a_0 n]_d = [n]_d$ ,  $[\kappa_0 a_0 n]_d = [\kappa_0 n]_d$  and

$$T(a_0 n) = \begin{cases} (a_0 n)/d & \text{if } n \equiv 0 \pmod{d} \\ (\alpha a_0 n + a_0 \beta_0 [\kappa_0 n]_d)/d & \text{if } n \not\equiv 0 \pmod{d}, \end{cases}$$

It follows that  $T(a_0 n) = a_0 T_0(n)$ , for any positive integer  $n$ . By induction, assume that at a step  $k$ , we have  $T^{(k)}(a_0 n) = a_0 T_0^{(k)}(n)$ . Then,

$$T^{(k+1)}(a_0 n) = T(T^{(k)}(a_0 n)) = T(a_0 T_0^{(k)}(n)) = a_0 T_0(T_0^{(k)}(n)) = a_0 T_0^{(k+1)}(n).$$

Now, assume that  $(d, \alpha, \beta_0)_\pm$  has a cycle  $\Omega_0(\omega)$  of length  $\ell$ .

$$\Omega_0(\omega) = (\omega \rightarrow T_0(\omega) \rightarrow T_0^{(2)}(\omega) \rightarrow \dots \rightarrow T_0^{(\ell-2)}(\omega) \rightarrow T_0^{(\ell-1)}(\omega) \rightarrow \omega).$$

As

$$a_0 \Omega_0(\omega) = (a_0 \omega \rightarrow a_0 T_0(\omega) \rightarrow a_0 T_0^{(2)}(\omega) \rightarrow \dots \rightarrow a_0 T_0^{(\ell-1)}(\omega) \rightarrow a_0 \omega),$$

and  $T^{(\ell)}(a_0 \omega) = a_0 T_0^{(\ell)}(\omega) = a_0 \omega$ , it follows that

$$a_0 \Omega_0(\omega) = (a_0 \omega \rightarrow T(a_0 \omega) \rightarrow T^{(2)}(a_0 \omega) \rightarrow \dots \rightarrow T^{(\ell-1)}(a_0 \omega) \rightarrow a_0 \omega).$$

Which shows that,  $\Omega(a_0 \omega) = a_0 \Omega_0(\omega)$  and  $\Omega(a_0 \omega)$  is also a cycle for  $(d, \alpha, \beta)_\pm$  of length  $\ell$ .  $\square$

**Example 3.2.** In this example, we apply the previous theorem. Consider the triplet  $(d, \alpha, \beta)_+ = (5, 6, 373769)_+$  with its corresponding mapping  $T : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$T(n) = \begin{cases} n/5 & \text{if } n \equiv 0 \pmod{5} \\ (6n + 373769[n]_5)/5 & \text{if } n \not\equiv 0 \pmod{5}. \end{cases}$$

We observe that here  $\beta = 373769 = a_0 \beta_0 = 121 \times 3089$  with  $\beta_0 = 3089$  and  $a_0 = 121 \equiv 1 \pmod{5}$ . The triplet  $(d, \alpha, \beta_0)_+ = (5, 6, 3089)_+$  is an admissible triplet with  $\beta_0 = 3089$  and its corresponding mapping  $T_0 : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$T_0(n) = \begin{cases} n/5 & \text{if } n \equiv 0 \pmod{5} \\ (6n + 3089[n]_5)/5 & \text{if } n \not\equiv 0 \pmod{5}. \end{cases}$$

It has at least the following 33 cycles of length  $\ell = 33$ :

$\Omega(11), \Omega(16), \Omega(17), \Omega(21), \Omega(22), \Omega(23), \Omega(26), \Omega(27), \Omega(28), \Omega(29), \Omega(31), \Omega(32), \Omega(33), \Omega(34), \Omega(37), \Omega(38), \Omega(39), \Omega(43), \Omega(44), \Omega(49), \Omega(56), \Omega(62), \Omega(68), \Omega(74), \Omega(81), \Omega(87), \Omega(93), \Omega(99), \Omega(106), \Omega(112), \Omega(118), \Omega(124), \Omega(3089)$ , see Item 4 of Examples 2.1.

According to the previous theorem, the triplet  $(5, 6, 373769)_+$  is also an admissible triplet and it has at least the following 33 cycles of length also  $\ell = 33$ :

$\Omega(1331), \Omega(1936), \Omega(2057), \Omega(2541), \Omega(2662), \Omega(2783), \Omega(3146), \Omega(3267), \Omega(3388), \Omega(3509), \Omega(3751), \Omega(3872), \Omega(3993), \Omega(4114), \Omega(4477), \Omega(4598), \Omega(4719), \Omega(5203), \Omega(5324), \Omega(5929), \Omega(6776), \Omega(7502), \Omega(8228), \Omega(8954), \Omega(9801), \Omega(10527), \Omega(11253), \Omega(11979), \Omega(12826), \Omega(13552), \Omega(14278), \Omega(15004), \Omega(373769)$ .

## 4 Some iterate and stopping time formulas

For the case of the classical Collatz problem, Terras [12], defined the notion of total stopping time which is interesting to understand the behaviour of the iterations for the Collatz operator. For a general admissible weak triplet  $(d, \alpha, \beta)_{\pm}$ , the total stopping time of a positive integer  $n$  may be also defined by a similar property as for the classical case. So, assume that  $(d, \alpha, \beta)_{\pm}$  is an admissible weak triplet of order  $q$ , it has at least  $q$  cycles:  $\Omega(\omega_1), \dots, \Omega(\omega_q)$ . We recall that an admissible triplet  $(d, \alpha, \beta)_{\pm}$  satisfies (9).

Let  $n$  be a positive integer. Thus, the total stopping time of  $n$ , denoted by  $\sigma_{\infty}(n)$ , is the smallest integer  $k$  such that  $T^{(k)}(n) \in \{\omega_1, \dots, \omega_q\}$ . If no such integer exists the total stopping time is set to be equal to  $\infty$ , i.e  $\sigma_{\infty}(n) = \infty$ .

In this section, we will state basic formulas on forward iteration of the associated mapping  $T$ . Some relations concerning the total stopping time will also be given. Let us first introduce the following functions, for all integers  $n \geq 1$  and  $k \geq 1$ :

$$s(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{d} \\ 1 & \text{if } n \not\equiv 0 \pmod{d}, \end{cases} \quad \text{and} \quad s_k(n) = \sum_{i=0}^{k-1} s(T^{(i)}(n)).$$

The function  $s_k(n)$  is the number of steps that were  $T^{(i)}(n) \not\equiv 0 \pmod{d}$  for  $i = 0, \dots, k$ . We set  $s_0(n) = 0$ .

**Theorem 4.1.** *Let  $d \geq 2$  and let  $(d, \alpha, \beta)_{\pm}$  be an admissible triplet. For all integers  $a \geq 1$  and  $n \geq 1$ , we have*

$$T(ad^k + n) = \alpha^{s(n)} ad^{k-1} + T(n), \quad \forall k \geq 1, \quad (34)$$

and

$$T^{(k)}(ad^k + n) = \alpha^{s_k(n)} a + T^{(k)}(n), \quad \forall k \geq 0. \quad (35)$$

Thus, we have

$$\sigma_{\infty}(ad^k + n) = \sigma_{\infty}(\alpha^{s_k(n)} a + T^{(k)}(n)) + k. \quad (36)$$

*Proof.* The first relation (34) is obtained directly by disjunction of cases. If  $n \equiv 0 \pmod{d}$ , then  $n = dT(n)$ , it follows that  $ad^k + n = d(ad^{k-1} + T(n))$  and  $T(ad^k + n) = ad^{k-1} + T(n)$ . Else, we have

$$T(ad^k + n) = \frac{\alpha(ad^k + n) + \beta[\kappa_0 n]_d}{d} = \alpha ad^{k-1} + \frac{\alpha n + \beta[\kappa_0 n]_d}{d} = \alpha ad^{k-1} + T(n).$$

For the relation (35), we proceed by induction on  $k \geq 0$ . For  $k = 0$ , the relation is trivially true. For  $k = 1$ , the relation is true by (34). We assume that, for all  $a \geq 1$  and for all  $n \geq 1$ , the formula (35) is true until an order  $k$ . As

$$T^{(k+1)}(ad^{k+1} + n) = T\left[T^{(k)}((ad)d^k + n)\right].$$

By the induction hypothesis, we have  $T^{(k)}((ad)d^k + n) = \alpha^{s_k(n)} ad + T^{(k)}(n)$ . As the relation is true for  $k = 1$ , then we have

$$T^{(k+1)}(ad^{k+1} + n) = \alpha^{s(T^{(k)}(n))} \alpha^{s_k(n)} a + T\left[T^{(k)}(n)\right] = \alpha^{s(T^{(k)}(n)) + s_k(n)} a + T^{(k+1)}(n) = \alpha^{s_{k+1}(n)} a + T^{(k+1)}(n),$$

which show that the formula (35) is true at the order  $k + 1$ . The relation (36) follows immediately from (35).  $\square$

Let us remark that the main interest of such theorem is that the relations given by the theorem may be used to accelerate verification algorithms of the conjecture by using sieves on the numbers we have to check, see for instance [13] for the verification of the classical Collatz conjecture.

In the following theorem, we will give some relation for the particular case of the admissible triplet  $(d, \alpha, \beta)_{\pm}$  where the parameter  $\lambda_0 = 1$  given in (9).

**Theorem 4.2.** *Let  $d \geq 2$  and consider an admissible triplet  $(d, \alpha, \beta)_{\pm}$  corresponding to the parameter  $\lambda_0 = 1$ . For all integers  $a \geq 1$ ,  $k \geq 0$  and for all integer  $r \in \{1, \dots, d - 1\}$  we have*

$$T^{(k)}(ad^k + \kappa_0 r) = \begin{cases} a\alpha^{q_0} + \kappa_0 r & \text{if } k \equiv 0 \pmod{\nu_0}, \\ a\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-r_0} & \text{if } k \not\equiv 0 \pmod{\nu_0}, \end{cases} \quad (37)$$

where  $k = q_0\nu_0 + r_0$  with  $0 \leq r_0 < \nu_0$  and  $ad^k + \kappa_0 r \geq 1$ . Thus, we have

$$\sigma_{\infty}(ad^k + \kappa_0 r) = \begin{cases} \sigma_{\infty}(a\alpha^{q_0} + \kappa_0 r) + k & \text{if } k \equiv 0 \pmod{\nu_0}, \\ \sigma_{\infty}(a\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-r_0}) + k & \text{if } k \not\equiv 0 \pmod{\nu_0}. \end{cases} \quad (38)$$

*Proof.* For the relation (37), we proceed by induction on  $k \geq 0$ . For  $k = 0$ , the relation is trivially true. For  $k = 1$ , we have

$$T(ad + \kappa_0 r) = \frac{\alpha(ad + \kappa_0 r) + \beta r}{d} = a\alpha + \kappa_0 r \frac{\alpha + \kappa_0 \beta}{d} = a\alpha + \kappa_0 r d^{\nu_0-1}, \quad (39)$$

which shows that the formula is also true for  $k = 1$ . We assume that, for all  $a \geq 1$  and for all  $r \in \{1, \dots, d - 1\}$ , the formula (37) is true until an order  $k$ . As

$$T^{(k+1)}(ad^{k+1} + \kappa_0 r) = T\left[T^{(k)}((ad)d^k + \kappa_0 r)\right].$$

So, by the recurrence hypothesis at step  $k$ , it follows that:

- If  $r_0 = 0$ , then  $T^{(k)}((ad)d^k + \kappa_0 r) = (ad)\alpha^{q_0} + \kappa_0 r$ . According to the relation (37) at step 1, we get

$$T^{(k+1)}(ad^{k+1} + \kappa_0 r) = T(a\alpha^{q_0} d + \kappa_0 r) = a\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-1}.$$

- If  $1 \leq r_0 \leq \nu_0 - 2$ , then  $T^{(k)}((ad)d^k + \kappa_0 r) = (ad)\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-r_0}$ . Thus

$$T^{(k+1)}(ad^{k+1} + \kappa_0 r) = T((ad)\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-r_0}) = a\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-r_0-1}.$$

- If  $r_0 = \nu_0 - 1$ , then  $k + 1 = (k_0 + 1)\nu_0$  and

$$T^{(k)}((ad)d^k + \kappa_0 r) = ad\alpha^{q_0+1} + \kappa_0 r d^{\nu_0-(\nu_0-1)} = ad\alpha^{q_0+1} + \kappa_0 r d.$$

It follows that  $T^{(k+1)}(ad^{k+1} + \kappa_0 r) = T(ad\alpha^{q_0+1} + \kappa_0 r d) = a\alpha^{q_0+1} + \kappa_0 r$ .

Which show that the formula (37) is true at the order  $k + 1$ . The relation (38) is an immediate consequence of the previous formulas (37).  $\square$

We may also establish the following results.

**Theorem 4.3.** *Let  $d \geq 2$  and consider an admissible triplet  $(d, \alpha, \beta)_\pm$  corresponding to  $\alpha = d + 1$  and  $\beta = -\kappa_0$  with  $\kappa_0 = \pm 1$ . Then, for all integers  $a \geq 1$ , for all integer  $r \in \{1, \dots, d - 1\}$  and for all integer  $k \geq 0$ , we have the following relations*

$$T^{(k)}(ad^k + \kappa_0 r) = a\alpha^k + \kappa_0 r, \quad (40)$$

and for  $k > 2$ , we have

$$T^{(k)}(ad^k + \kappa_0(2d - 1)) = \begin{cases} a3^{k-2} + \kappa_0 & \text{if } d = 2, \\ a\alpha^{k-1} + 2\kappa_0 & \text{if } d > 2, \end{cases} \quad (41)$$

It follows immediately that for  $k \geq 0$ :

$$\sigma_\infty(ad^k + \kappa_0 r) = \sigma_\infty(a\alpha^k + \kappa_0 r) + k, \quad (42)$$

and for  $k > 2$ :

$$\sigma_\infty(ad^k + \kappa_0(2d - 1)) = \begin{cases} \sigma_\infty(a3^{k-2} + \kappa_0) + k & \text{if } d = 2, \\ \sigma_\infty(a\alpha^{k-1} + 2\kappa_0) + k & \text{if } d > 2. \end{cases} \quad (43)$$

*Proof.* • For the relation (40), we again proceed by induction on  $k \geq 0$ . For  $k = 0$ , the relation is trivially true. For  $k = 1$ , we have  $T(ad + \kappa_0 r) = \frac{\alpha(ad + \kappa_0 r) + \beta r}{d} = a\alpha + \kappa_0 r \frac{\alpha + \kappa_0 \beta}{d}$ . As  $\alpha + \kappa_0 \beta = d$ . Then

$$T(ad + \kappa_0 r) = a\alpha + \kappa_0 r, \quad (44)$$

which shows that the formula is also true for  $k = 1$ . We assume that, for all  $a \geq 1$  and for all  $r \in \{1, \dots, d - 1\}$ , the formula (40) is true until an order  $k$ . So, by the recurrence hypothesis at step  $k$ , we get

$$T^{(k+1)}(ad^{k+1} + \kappa_0 r) = T\left[T^{(k)}((ad)d^k + \kappa_0 r)\right] = T\left[(ad)\alpha^k + \kappa_0 r\right].$$

According to the relation (44) at step 1, we obtain  $T^{(k+1)}(ad^{k+1} + \kappa_0 r) = a\alpha^{k+1} + \kappa_0 r$ . Which show that the formula (40) is true at the order  $k + 1$ . The relation (42) is a consequence of the relation (40).

• For the relation (41), for  $k > 2$  we have

$$T^{(k)}(ad^k + \kappa_0(2d - 1)) = T^{(k-1)}[T(ad^k + \kappa_0(2d - 1))].$$

As  $[\kappa_0(ad^k + \kappa_0(2d - 1))]_d = d - 1$  and  $\alpha - 1 = d$ , it follows

$$T(ad^k + \kappa_0(2d - 1)) = \frac{\alpha(ad^k + \kappa_0(2d - 1)) - \kappa_0(d - 1)}{d} = \alpha ad^{k-1} + 2\kappa_0 d.$$

Then, we have

$$T^{(k)}(ad^k + \kappa_0(2d - 1)) = \begin{cases} T^{(k-3)}[(3a)2^{k-3} + \kappa_0] & \text{if } d = 2, \\ T^{(k-2)}[(\alpha a)d^{k-2} + 2\kappa_0] & \text{if } d > 2, \end{cases} \quad (45)$$

Then, using the relation (40), we obtain the relation (41). The relations (42) and (43) are immediate consequences of the relations (40) and (41), respectively.  $\square$

## 5 Lower bound lengths for cycles associated to an admissible triplet

In this section, we consider an admissible triplet  $(d, \alpha, \beta)_{\pm}$  with  $\alpha > d \geq 2$ . We restrict our attention to the case where the numbers  $d$  and  $\alpha$  are supposed to be coprime,  $cgd(d, \alpha) = 1$  and where  $\beta > 0$ . Let  $\xi$  be an irrational real number. The number  $\mu = \mu(\xi)$  is said to be an effective irrationality measure of  $\xi$  (see [14] for more details) if for all  $\varepsilon > 0$ , there exists an integer  $q_0(\varepsilon) > 0$  (effectively computable) such that

$$\forall (p, q) \in \mathbb{Z} \times \mathbb{N}, \quad q > q_0(\varepsilon) \implies \left| \frac{p}{q} - \xi \right| > \frac{1}{q^{\mu+\varepsilon}}. \quad (46)$$

By definition, see [15, 16], two positive rational numbers are said to be multiplicatively independent if the quotient of their logarithms is irrational. In the following, we fix  $\xi$  to be the real number  $\xi = \log_d(\alpha) := \frac{\log(\alpha)}{\log(d)}$ . So, the number  $\xi$  is irrational. Indeed, suppose that  $\xi$  is a rational number, then there exist a pair of integers  $(p, q) \in \mathbb{N}^2$  such that  $\alpha^q = d^p$ , which is impossible since  $\alpha$  and  $d$  are assumed to be coprime numbers. We have the following result.

**Theorem 5.1.** *The number  $\xi = \frac{\log(\alpha)}{\log(d)}$  is a transcendental real number and the effective irrationality measure  $\mu(\xi)$  of  $\xi$  is a finite number. More precisely, there exists a constant  $C$  such that:*

$$2 \leq \mu(\xi) \leq C(\log \alpha)(\log d). \quad (47)$$

*Proof.* As  $\xi$  is irrational real number and  $d^\xi = \alpha$  is an algebraic number, then according to the Gelfond-Schneider theorem, we get that  $\xi$  is transcendental real number. Let  $a_1 = \alpha > a_2 = 1$  and  $b_1 = d > b_2 = 1$ , as  $\xi$  is irrational, then  $a_1/a_2$  and  $b_1/b_2$  are multiplicatively independent. According to Theorem 1.1 given in [15], see also [16], there exists an absolute, effectively computable, constant  $C$  such that

$$\mu\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq C(\log a_1)(\log b_1).$$

It follows that  $\mu(\xi) \leq C(\log \alpha)(\log d)$ . The convergents of the continued fraction of the irrational number  $\xi$  imply that  $\mu(\xi) \geq 2$ . Thus, the effective irrationality measure of  $\xi$  satisfies the inequality (47).  $\square$



According to (46), for  $\varepsilon = 1$ , there exists an integer  $\widehat{q}_0(\xi) > 0$  such that

$$\forall (p, q) \in \mathbb{Z} \times \mathbb{N}, \quad q > \widehat{q}_0(\xi) \implies \left| \frac{p}{q} - \xi \right| > \frac{1}{q^{\mu(\xi)+1}} \implies |p - q\xi| > \frac{1}{q^{\mu(\xi)}}. \quad (48)$$

Let us now recall some properties for the continued fractions which may be found in the literature of the theory of rational approximation, see for instance [17]. As  $\xi$  is an irrational number, there is a unique representation of  $\xi$  as an infinite simple continued fraction:  $\xi = [a_0, a_1, a_2, a_3, \dots]$ . The integer numbers  $a_n$  are called the partial quotients and the rational numbers  $\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n]$  are called the convergents of  $\xi$ , where  $\gcd(p_n, q_n) = 1$ . The numbers  $x_n = [a_n, a_{n+1}, \dots]$  are called the complete quotients. For all  $n \geq 0$ , we have

$$\xi = [a_0, a_1, \dots, a_n, x_{n+1}] = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}.$$

The integers  $p_n, q_n$  are obtained recursively as follows:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \\ q_n &= a_n q_{n-1} + q_{n-2}, \end{aligned}$$

with the initial values  $p_{-1} = 1$ ,  $p_{-2} = 0$  and  $q_{-1} = 0$ ,  $q_{-2} = 1$ . The sequences  $(p_n)$  and  $(q_n)$  are strictly increasing unbounded sequences. It is well known that the fractions  $p_n/q_n$  satisfy the following properties: For any pair of integers  $(p, q) \in \mathbb{N}^2$  and for  $n \geq 0$  with  $1 < q < q_n$  we have:

$$\frac{1}{q_n + q_{n+1}} < |p_n - q_n \xi| < |p - q \xi|. \quad (49)$$

It is also well known that  $(p_{2n}/q_{2n})_n$  is an increasing sequence and that  $(p_{2n+1}/q_{2n+1})_n$  is a decreasing sequence such that

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \dots < \frac{p_{2n}}{q_{2n}} < \frac{p_{2n+2}}{q_{2n+2}} < \dots < \xi < \dots < \frac{p_{2n+1}}{q_{2n+1}} < \frac{p_{2n-1}}{q_{2n-1}} < \dots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}, \quad (50)$$

and  $\lim_{n \rightarrow \infty} \frac{p_n}{q_n} = \xi$ .

Let  $\Omega$  be an eventual cycle for the admissible triplet  $(d, \alpha, \beta)_\pm$  with  $d \geq 2$ ,  $\text{cgd}(\alpha, d) = 1$  and  $\beta > 0$ . Let  $T$  denote its associated operator. We denote by  $\Omega_d := \{n \in \Omega : n \equiv 0 \pmod{d}\}$  the subset of  $\Omega$  consisting of all the elements in  $\Omega$  that are congruent to zero modulo  $d$  and we denote by  $\overline{\Omega}_d := \{n \in \Omega : n \not\equiv 0 \pmod{d}\}$ , the complement of  $\Omega_d$  in  $\Omega$ . Let  $K = \#\Omega$  and  $\overline{K} = \#\overline{\Omega}_d$  denote the cardinal numbers of  $\Omega$  and  $\overline{\Omega}_d$ , respectively. Let  $\max(\Omega)$  and  $\min(\Omega)$  denote the greatest and smallest elements in the cycle  $\Omega$ , respectively. We have the following Lemma which is an extension to the result given in [18] in the case of the classical Collatz problem.

**Lemma 5.1.** *A necessary condition that a cycle  $\Omega$  associated to the admissible triplet  $(d, \alpha, \beta)_\pm$  exists is that the following inequalities hold:*

$$0 < \overline{K} \log_d \left( 1 + \frac{\beta}{\alpha \max(\Omega)} \right) < K - \overline{K} \xi \leq \sum_{n \in \overline{\Omega}_d} \log_d \left( 1 + \frac{\beta (d-1)}{\alpha n} \right) \leq \frac{\beta (d-1)}{\alpha \log(d)} \sum_{n \in \overline{\Omega}_d} \frac{1}{n}, \quad (51)$$

and

$$0 < K - \overline{K} \xi \leq \overline{K} \log_d \left( 1 + \frac{\beta (d-1)}{\alpha \min(\Omega)} \right) \leq \frac{\overline{K} \beta (d-1)}{\alpha \log(d) \min(\Omega)}. \quad (52)$$

*Proof.* Suppose that a cycle  $\Omega$  of length  $K$  exists, then  $\prod_{n \in \Omega} n = \prod_{n \in \Omega} T(n)$ . It follows that

$$1 = \prod_{n \in \Omega} \left( \frac{T(n)}{n} \right) = \left( \prod_{n \in \Omega_d} \frac{T(n)}{n} \right) \times \left( \prod_{n \in \bar{\Omega}_d} \frac{T(n)}{n} \right) = \frac{1}{d^K} \times \prod_{n \in \bar{\Omega}_d} \left( \alpha + \frac{\beta[\kappa_0 n]_d}{n} \right).$$

Then

$$\left( \alpha + \frac{\beta}{\max(\Omega)} \right)^{\bar{K}} \leq \prod_{n \in \bar{\Omega}_d} \left( \alpha + \frac{\beta}{n} \right) \leq d^K = \prod_{n \in \bar{\Omega}_d} \left( \alpha + \frac{\beta[\kappa_0 n]_d}{n} \right) \leq \prod_{n \in \bar{\Omega}_d} \left( \alpha + \frac{\beta(d-1)}{n} \right),$$

Passing to the logarithms, we obtain the following inequality

$$0 < \bar{K} \log_d \left( 1 + \frac{\beta}{\alpha \max(\Omega)} \right) \leq K - \bar{K} \xi \leq \sum_{n \in \bar{\Omega}_d} \log_d \left( 1 + \frac{\beta(d-1)}{\alpha n} \right).$$

Using the fact that  $\log(1+x) < x$  for  $x > 0$ , we get the inequalities (51). Using again in (51) the fact that  $\log(1+x) < x$  for  $x > 0$ , we obtain the required inequality (52).  $\square$

From the Lemma 5.1, we are ready to give the following theorems.

**Theorem 5.2.** *Let  $(p_n/q_n)$  be the sequence of the convergents of  $\xi = \log_d(\alpha)$ . A necessary condition that a cycle  $\Omega$  of length  $K$  associated to the admissible triplet  $(d, \alpha, \beta)_{\pm}$  exists is that there exists an integer  $n_0 > 0$  such that for all  $n \geq n_0$  and for all  $0 < M \leq \min(\Omega)$  the following inequality holds:*

$$\bar{K} \geq \min \left( q_n, \frac{\gamma_0 M}{q_n^{\mu(\xi)}} \right), \quad (53)$$

where  $\mu(\xi)$  is the irrationality measure of  $\xi$  given by (46) and  $\gamma_0 = \frac{\alpha \log(d)}{\beta(d-1)}$ .

*Proof.* Let  $p_n/q_n$  be the convergents to  $\xi$ . Then,  $(q_n)_n$  is strictly increasing unbounded sequence. It follows that there exists an integer  $n_0 > 0$  such that  $\forall n > n_0$ , we have  $q_n > \hat{q}_0(\xi)$ , where  $\hat{q}_0(\xi)$  is the integer defined by (48). It follows that  $|p_n - q_n \xi| > \frac{1}{q_n^{\mu(\xi)}}$ . If  $\bar{K} \geq q_n$ , the required inequality (53) is trivial. Then, suppose that  $1 < \bar{K} < q_n$ . According to (49) and to Lemma 5.1, for  $n \geq n_0$ , we have  $K - \bar{K} \xi = |K - \bar{K} \xi| \geq |p_n - q_n \xi| > \frac{1}{q_n^{\mu(\xi)}}$ . Then, we have  $0 \leq K - \bar{K} \xi \leq \frac{\bar{K} \beta(d-1)}{\alpha \log(d) \min(\Omega)}$ . It follows, that  $\frac{\bar{K} \beta(d-1)}{\alpha \log(d) \min(\Omega)} \geq \frac{1}{q_n^{\mu(\xi)}}$ , which gives the required result.  $\square$

Now, we will use similar arguments to those of the previous theorem to deduce the following result. We point out that similar results have been given by Crandall [19] and Eliahou [18] in the particular case of the classical Collatz triplet  $(2, 3, 1)_+$ . From the proposed result we may easily derive, as will be seen, an algorithm that allow the computation of a lower bound of  $\#\Omega$ , see Algorithm 1 and the following section.

**Theorem 5.3.** Let  $(p_n/q_n)$  be the sequence of the convergents of  $\xi = \log_d(\alpha)$ . A necessary condition that a cycle  $\Omega$  associated to the admissible triplet  $(d, \alpha, \beta)_{\pm}$  exists is that:

$$\bar{K} \geq R_{\infty} := \max_{n \geq 1} (R_n(M)), \quad (54)$$

where  $R_n(M) = \left\lfloor \min\left(q_n, \frac{\gamma_0 M}{q_n + q_{n+1}}\right) \right\rfloor + 1$  with  $M$  is any real number such that  $0 < M \leq \min(\Omega)$ , the constant  $\gamma_0 = \frac{\alpha \log(d)}{\beta(d-1)}$  and  $\lfloor \cdot \rfloor$  is the floor function.

*Proof.* First, we will show that the following inequality holds: For all  $n \geq 1$ ,

$$\bar{K} \geq \min\left(q_n, \frac{\gamma_0 \min(\Omega)}{q_n + q_{n+1}}\right). \quad (55)$$

Let  $n \geq 1$ , if  $\bar{K} \geq q_n$ , then the required inequality (55) is obvious. Assume that  $1 < \bar{K} < q_n$ , it follows from the inequality (49) that

$$K - \bar{K}\xi = |K - \bar{K}\xi| > |p_n - q_n\xi| > \frac{1}{q_n + q_{n+1}}.$$

From Lemma 5.1, we deduct  $\frac{\bar{K} \beta (d-1)}{\alpha \log(d) \min(\Omega)} > \frac{1}{q_n + q_{n+1}}$ . Which prove the required inequality

(55). Now, as the sequence  $(q_n)_n$  is increasing and unbounded and the sequence  $\left(\frac{\gamma_0 \min(\Omega)}{q_n + q_{n+1}}\right)_{n \geq 1}$  is decreasing and converges to zero. It follows that the sequence  $(R_n(M))_n$  is first increasing until it reaches its maximum and after it is decreasing to 1. So, there exists a positive integer  $n_0 \geq 1$  such that  $\max_{n \geq 1} (R_n(M)) = R_{n_0}(M)$ . The required inequality (54) is now an immediate consequence of the inequality (55).  $\square$

Another lower bound of the cardinal  $\#\Omega$  for an eventual trivial cycle, may be obtained by following the idea given by Eliahou in [18] using Farey fractions. Before that, let us briefly recall some definitions and properties related to the Farey fractions, see [20, 21]. A pair of fractions,  $(a/b, c/d)$  (with  $a, b, c$  and  $d$  non-negative integers) is called a Farey pair if  $bc - ad = \pm 1$ . If  $\frac{a}{b} < \frac{c}{d}$ , then  $bc - ad = 1$ , the median of this Farey pair  $(a/b, c/d)$  is defined as  $(a+c)/(b+d)$ . A straightforward calculation shows that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d},$$

and that  $(a/b, (a+c)/(b+d))$  and  $((a+c)/(b+d), c/d)$  form also two Farey pairs. We may continue the process called Farey process to generate squeezed Farey pairs:

$$\begin{aligned} & \frac{a}{b} < \frac{c}{d} \\ & \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \\ & \frac{a}{b} < \frac{2a+c}{2b+d} < \frac{a+c}{b+d} < \frac{a+2c}{b+2d} < \frac{c}{d}, \\ & \frac{a}{b} < \frac{3a+c}{3b+d} < \frac{2a+c}{2b+d} < \frac{3a+2c}{3b+2d} < \frac{a+c}{b+d} < \frac{2a+3c}{2b+3d} < \frac{a+2c}{b+2d} < \frac{a+3c}{b+3d} < \frac{c}{d}, \end{aligned}$$

and so on. Thus, the sequence generated by the Farey process is called a Farey sequence. We observe that, if  $(a/b, c/d)$  is a Farey pair such that  $\frac{a}{b} < \frac{c}{d}$ , then there are infinite number of fractions of the form  $\frac{\alpha a + \beta c}{\alpha b + \beta d}$  such that

$$\frac{a}{b} < \frac{\alpha a + \beta c}{\alpha b + \beta d} < \frac{c}{d},$$

with  $\alpha \geq 1$  and  $\beta \geq 1$  are positive integers. Usually it is sufficient to construct a farey sequence between the Farey pair  $(0/1, 1/1)$ . Then, to obtain a Farey sequence between any other Farey pair  $(a/b, c/d)$ , we use the bijective mapping  $f : \mathbb{Q} \cap [\frac{0}{1}, \frac{1}{1}] \rightarrow \mathbb{Q} \cap [\frac{a}{b}, \frac{c}{d}]$  given by  $f(t) = \frac{(c-a)t + a}{(d-b)t + b}$  which transform one-to-one a Farey sequence between  $(0/1, 1/1)$  to one between  $(a/b, c/d)$ . The inverse mapping of  $f$  is obviously  $f^{-1} : \mathbb{Q} \cap [\frac{a}{b}, \frac{c}{d}] \rightarrow \mathbb{Q} \cap [\frac{0}{1}, \frac{1}{1}]$  given by  $f^{-1}(r) = \frac{br - a}{(c-dr) + (br - a)}$ . The following Lemma gives a reciprocal property. The proof of this Lemma for the Farey pair  $(0/1, 1/1)$  is given in [20] and in [18] for a general Farey pair  $(a/b, c/d)$ . However, for fun of it and for nice readability, we well rewrite this proof given in [18] with slight modifications.

**Lemma 5.2.** *Let  $\frac{a}{b} < \frac{c}{d}$  be a Faray pair. Then, every rational number  $\frac{x}{y}$  in lowest terms, with  $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$ , appears at some stage of the Farey process, namely, there exist two positive integers  $\alpha \geq 1$  and  $\beta \geq 1$  such that  $\frac{x}{y} = \frac{\alpha a + \beta c}{\alpha b + \beta d}$ . In particular,  $x \geq a + c$  and  $y \geq b + d$ .*

*Proof.* Let us consider the bijective mapping  $f$  and its inverse  $f^{-1}$  given above. Suppose that  $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$ , then we have

$$f^{-1}\left(\frac{x}{y}\right) = \frac{b\frac{x}{y} - a}{(c - d\frac{x}{y}) + (b\frac{x}{y} - a)} = \frac{bx - ay}{(cy - dx) + (bx - ay)} = \frac{\beta}{\alpha + \beta},$$

where  $\alpha = cy - dx > 0$  and  $\beta = bx - ay > 0$  are positive integers. Thus  $\frac{x}{y} = f\left(\frac{\beta}{\alpha + \beta}\right) = \frac{\alpha a + \beta c}{\alpha b + \beta d}$ .  $\square$

Let us notice that it is well known that two consecutive convergents  $(p_n/q_n)$  and  $(p_{n+1}/q_{n+1})$  satisfy the relation  $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$ , namely they form a Farey pair.

Let  $(p_n/q_n)$  be the sequence of the convergents of  $\xi = \log_d(\alpha)$ . Assume that there exists a cycle  $\Omega$  with a minimal element  $\min(\Omega) \geq M$ . Then, if we consider the sequence  $(D_n(M))_n$  given by

$$D_n(M) = \xi + \log_d\left(1 + \frac{\beta(d-1)}{\alpha M}\right) - \frac{p_n}{q_n},$$

for  $n \geq 0$ , we may establish the following theorem

**Theorem 5.4.** *Assume the cycle  $\Omega$  has a minimal element  $\min(\Omega) \geq M$ , where  $M$  is a large enough positive integer such that  $D_1(M) < 0$ . Then there exists an integer  $n_0$  such that the following inequalities*

$$\frac{p_{2n_0+2}}{q_{2n_0+2}} < \xi < \frac{p_{2n_0+1}}{q_{2n_0+1}} < \xi + \log_d \left( 1 + \frac{\beta(d-1)}{\alpha M} \right) < \frac{p_{2n_0-1}}{q_{2n_0-1}}. \quad (56)$$

hold and the cardinal of  $\Omega$  satisfies

$$K = \#\Omega \geq p_{2n_0+1}. \quad (57)$$

*Proof.* According to the inequality (52) in Lemma 5.1 and as  $\min(\Omega) \geq M$ , it follows that

$$\xi < \frac{K}{\overline{K}} \leq \xi + \log_d \left( 1 + \frac{\beta(d-1)}{\alpha \min(\Omega)} \right) \leq \xi + \log_d \left( 1 + \frac{\beta(d-1)}{\alpha M} \right). \quad (58)$$

Now, according to (50), for all integer  $n \geq 0$ , we have  $D_{2n}(M) \geq 0$  and the sub-sequence  $(D_{2n}(M))_{n \geq 0}$  is decreasing. As  $M$  is large enough such that  $D_1(M) < 0$  and the sub-sequence  $(D_{2n+1}(M))_{n \geq 0}$  is increasing, then there exists an integer  $n_0 \geq 0$  such that  $D_1(M) < D_3(M) < \dots < D_{2n_0-1}(M) < 0$  and  $D_{2n_0+1}(M) > 0$  which with (50) leads to the inequality (56). Taking into account of inequalities (58), it follows that either  $\frac{p_{2n_0+2}}{q_{2n_0+2}} < \frac{K}{\overline{K}} < \frac{p_{2n_0+1}}{q_{2n_0+1}}$  or  $\frac{K}{\overline{K}} = \frac{p_{2n_0+1}}{q_{2n_0+1}}$  or  $\frac{p_{2n_0+1}}{q_{2n_0+1}} < \frac{K}{\overline{K}} < \frac{p_{2n_0-1}}{q_{2n_0-1}}$ . Then,  $\frac{K}{\overline{K}}$  is an intermediate fraction between Farey pair and according to the Lemma 5.2, it follows that either  $K \geq p_{2n_0+2} + p_{2n_0+1}$  or  $K = p_{2n_0+1}$  or  $K \geq p_{2n_0-1} + p_{2n_0+1}$ . So, in each case, we have  $K \geq p_{2n_0+1}$ .  $\square$

Note that a similar theorem to the previous one has already been given by Eliahou in [18] for the case of the classical triplet  $(2, 3, 1)_+$ . For Theorem 5.4, we have as in [18] also used Farey fractions. Apart from this common technique, the two methods are different. First, our result is given in the case of a general admissible triplet. Furthermore, it allows to derive an algorithm that gives the boundary length directly without any observation or analysis, see the next section for the algorithm. Here, we used the sign of the introduced sequence  $D_{2n+1}$  which is first negative and becomes positive at the step  $2n_0 + 1$  corresponding to the required value  $p_{2n_0}$ , see the algorithm 2 in the following section.

## 6 Experimental tests

In this section, we will make some experimental tests illustrating some theoretical results given in the previous sections. For instance, We will discuss briefly computer verification and validity of the main conjecture 3.2 and conjecture 3.3. A more detailed verification is under investigations and will be presented in further work. We will also give numerical tests to obtain a lower bound length of an eventual cycle by using Theorem 5.3 and Theorem 5.4. Our computations were carried out using different environments: Python, SageMaths, Mathematica and Matlab. We have used an Intel(R) Core i9-CPU@1.80GHz (8 cores) computer with 16 GB of RAM.

### 6.1 First few verifications

We have performed some verifications on the validity of the conjecture 3.2. The numerous tests we have performed indicate the validity of this conjecture for  $p$  going from 0 up to 15 and for  $n$  from 1 up to  $10^4$ . We recall that the case  $p = 0$  corresponds to the classical triplet  $(2, 3, 1)_+$  and it has been

verified experimentally with a computer by many authors. For instance, in [22], the author claims that he verified in 2009 the classical Collatz conjecture up to  $2^{62.3} \simeq 5.67 \times 10^{18}$  and in [13], the author claims that he verified in 2020 the same conjecture up to  $2^{68} \simeq 2.95 \times 10^{20}$ . The case  $p = 2$ , in the conjecture 3.2, corresponds to the triplet  $(5, 6, 4)_+$ . We have conjectured in this case that  $(5, 6, 4)_+$  is an admissible strong triplet with the trivial cycle  $\Omega(4) = (4 \rightarrow 8 \rightarrow 12 \rightarrow 16 \rightarrow 20 \rightarrow 4)$  as a unique cycle without any divergent trajectory, i.e., for any integer  $n \geq 1$ , there exists an integer  $k \geq 0$  such that  $T^{(k)}(n) = 4$ . We have also verified this affirmation in special case up to  $5 \times 10^{11} \simeq 1.82 \times 2^{38}$ . The speed we have reached was about  $4.6 \times 10^6$  numbers per second, by vectorizing computations in `Matlab`. This took about 30 hours. All the trajectories starting with  $n \leq 1.82 \times 2^{38}$  enter the trivial cycle  $\Omega(4)$ . On the other hand, we may assert that the conjecture 3.2 for  $p = 2$  is true up to  $1.82 \times 2^{38}$ . Of course, we have to improve our speed and our used techniques in order to increase the limit of verification. This is our goal for a further work, in which all the techniques and used algorithms will be explained in details. The tests were also coded in Python, SageMaths, Mathematica using 3 computers in parallel.

## 6.2 Tests for Lower bound length

Algorithm 1 is based on Theorem 5.3, it computes the minimal lower bound length of an eventual trivial cycle with a known minimal element. This algorithm is based on Theorem 5.3. Suppose that a cycle  $\Omega$  associated to the admissible triplet  $(d, \alpha, \beta)_\pm$  exists with its minimal element satisfying  $\min(\Omega) \geq M$ . Then, according to Theorem 5.3, the inequality  $\#\Omega \geq R_\infty := \max_{n \geq 1} R_n(M)$ , holds. The sequence  $(R_n(M))_{n \geq 1}$  is a stationary sequence, more precisely, there exists an integer  $n_0$  such that for all  $n \geq n_0$ , we have  $R_n(M) = 1$ .

---

**Algorithm 1** Algorithm based on Theorem 5.3

---

```

1: Input: The integers  $d, \alpha, \beta, M$  (with the minimal element of the cycle  $\min(\Omega) \geq M$  ).
2: Output The integer  $R_\infty$ 
3:  $\gamma = \alpha \log(d) / (\beta(d - 1))$ ;  $\xi = \log(\alpha) / \log(d)$ ;  $x = \xi$ 
4:  $(p_0, q_0, p_1, q_1) = (0, 1, 1, 0)$ 
5:  $(R_0, R) = (0, 1)$ 
6: while  $R_0 \leq R$  do
7:    $R_0 = R$ 
8:    $a = \lfloor x \rfloor$ 
9:    $p = ap_1 + p_0$ ;  $q = aq_1 + q_0$ ;
10:   $x = 1 / (x - a)$ 
11:   $F = p/q$ ;
12:   $Q = \gamma M / (q_1 + q)$ 
13:   $R = \lfloor \min(q, Q) \rfloor + 1$ ;
14:   $(p_0, q_0) = (p_1, q_1)$ ;  $(p_1, q_1) = (p, q)$ ;
15: end while
16: return  $R$ 

```

---

Algorithm 2 is based on Theorem 5.4. It also computes the minimal lower bound of the cardinal

of an eventual trivial cycle with a known minimal element.

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**Algorithm 2** Algorithm based on Theorem 5.4

---

- 1: **Input:** The integers  $d, \alpha, \beta, M$  (with the minimal element of the cycle  $\min(\Omega) \geq M$ ).
  - 2: **Output** The integer  $p_{2n_0+1}$  (given in (57)).
  - 3:  $\xi = \log(\alpha)/\log(d)$ ;  $x = \xi$
  - 4:  $(p_0, q_0, p_1, q_1) = (0, 1, 1, 0)$
  - 5:  $(D_0, D) = (1, -1)$
  - 6: **while**  $D_0 * D \leq 0$  **do**
  - 7:    $D_0 = D$
  - 8:    $a = \lfloor x \rfloor$
  - 9:    $p = ap_1 + p_0$ ;  $q = aq_1 + q_0$ ;
  - 10:    $x = 1/(x-a)$
  - 11:    $F = p/q$ ;
  - 12:    $D_n = \xi + \log_d\left(1 + \frac{\beta(d-1)}{\alpha M}\right) - F$ ;
  - 13:    $(p_0, q_0) = (p_1, q_1)$ ;  $(p_1, q_1) = (p, q)$ ;
  - 14: **end while**
  - 15: **return**  $p$
- 

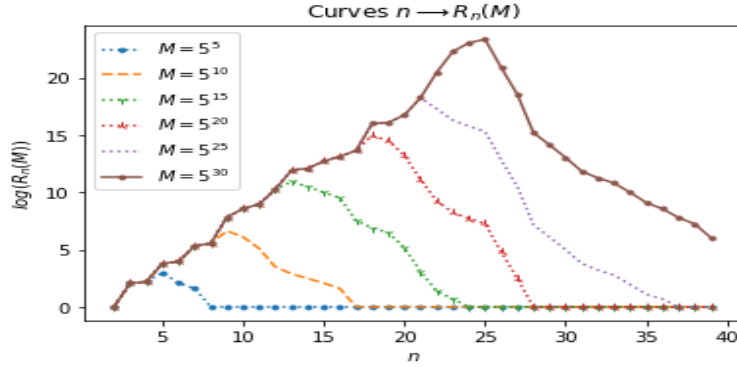


Figure 1: The curves  $n \rightarrow \log(R_n(M))$  for different value of  $M \in \mathbb{M}$  corresponding to the triplet  $(5, 6, 4)_+$ .

We have implemented the Algorithm 1 and Algorithm 2 in Python and tested them in the case of the admissible triplet  $(5, 6, 4)_+$ . In Table 1, we have reported the values of  $R_\infty(M)$  for different values of  $M \in \mathbb{M} = \{5^5, 5^{10}, 5^{15}, 5^{20}, 5^{25}, 5^{30}\}$ , assuming that  $\min(\Omega) \geq M$ , respectively. The integer  $n_0$  corresponds to the step for which the maximum value is reached. In Figure 1, we have plotted the curves  $n \rightarrow \log(R_n(M))$  for the different values of  $M \in \mathbb{M}$ . The curves illustrate the expected behavior of the sequence  $(R_n(M))$ . Actually, we observe that the sequence is first increasing until reaching its maximum  $R_\infty(M)$  and after it decreases to 1. So, for instance, if there exists a non-trivial cycle of minimal element  $\min(\Omega) \geq M = 5^{30}$ , then its length must satisfy  $\#\Omega \geq 15\,032\,816\,36$ .

$M$	$5^5$	$5^{10}$	$5^{15}$	$5^{20}$	$5^{25}$	$5^{30}$
$n_0$	3	7	11	16	20	23
$R_\infty(M)$	36	2134	102678	5905570	208606372	15032816369

Table 1: The results obtained with  $\min(\Omega) \geq M$  with with  $M \in A_2$  for the triplet  $(5, 6, 4)_+$ ,  $\min(\Omega) \geq M$ .

$M$	$5^5$	$5^{10}$	$5^{15}$	$5^{20}$	$5^{25}$	$5^{30}$
$n_0$	2	3	5	8	15	15
$p_{2n_0+1}$	226	2791	167863	10850489	4567472300430581	4567472300430581

Table 2: The results obtained with  $\min(\Omega) \geq M$  with with  $M \in A_2$  for the triplet  $(5, 6, 4)_+$ ,  $\min(\Omega) \geq M$ .

In Table 3, we have given the results obtained by running the Algorithm 1 for  $M = M_1 := 5^{15}$ ,  $M = M_2 := 5^{20}$  and  $M = M_3 := 5^{25}$ . The  $R_\infty(M)$  is the boxed value in the column of the  $R_n(M)$ .

The results from Algorithm 2 are summarized in Table 2 where we give the value of  $p_{2n_0+1}$  for each value of  $m \in \mathbb{M}$ . We have also reported, in Table 4, all the values of the sequence  $D_n(M)$  given by this algorithm for  $M = 5^{15}, 5^{20}, 5^{25}$ . We observe in this table that  $D_{2n+1}$  changes the sign at  $2n_0 + 1$  step, thus we have  $\#\Omega \geq p_{2n_0+1}$ . The  $D_{2n_0+1}$  is the boxed value in the column of the  $D_n(M)$ .

We have observed that numerically the result from Algorithm 2 does not change for  $M \geq 5^{25}$ , then the  $\#\Omega \geq \max(R_\infty, p_{2n_0+1}) = R_\infty$  for  $M \geq 5^{25}$ . However, Algorithm 1 continues to give different results for large values of  $M$ . For instance, for  $\min(\Omega) \geq M = 5^{60}$ , we get the value ( $R_\infty = 869802559919868084225$  at the iteration  $n_0 = 40$ ).

## 7 Conclusion

In this paper, we have introduced and studied a simple and unified extension to the Collatz problem. We have given some general theorems and conjectures. A nice conjecture that generates the classical Collatz one is given. The lower bounds for the cardinal of an eventual cycle are studied. This allow to generate algorithms that compute lower bounds. Examples and verification are given to illustrate the conjectures. More verifications and eventual theoretical results still under investigations by the author. To finish this paper and for fun of it, we recall our new conjecture : Choose a positive integer  $n \geq 1$ . If  $n$  is divided by 10, remove all the zeros on the right. Otherwise, multiply it by 6 and add its last digit 4-times, then divide the result by 5. Repeat this process iteratively, it always reaches 4, regardless the starting number.

## References

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$n$	$p_n$	$q_n$	$R_n(M_1)$	$R_n(M_2)$	$R_n(M_3)$
0	1	1	1	1	1
1	9	8	8	8	8
2	10	9	9	9	9
3	49	44	44	44	44
4	59	53	53	53	53
5	226	203	203	203	203
6	285	256	256	256	256
7	2791	2507	2507	2507	2507
8	5867	5270	5270	5270	5270
9	8658	7777	7777	7777	7777
10	31841	28601	28601	28601	28601
11	167863	150782	102678	150782	150782
12	199704	179383	55786	179383	179383
13	367567	330165	36147	330165	330165
14	567271	509548	21935	509548	509548
15	934838	839713	13651	839713	839713
16	9915651	8906678	1890	5905570	8906678
17	10850489	9746391	988	3085712	9746391
18	20766140	18653069	649	2026729	18653069
19	93915049	84358667	179	558752	84358667
20	866001581	777881072	22	66755	208606372

Table 3: The results obtained with  $\min(\Omega) \geq M_i$  with  $M_1 = 5^{15}$ ,  $M_2 = 5^{20}$ ,  $M_3 = 5^{25}$  for the triplet  $(5, 6, 4)_+$ .

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$n$	$p_n$	$q_n$	$D_n(M_1)$	$D_n(M_2)$	$D_n(M_3)$
0	1	1	1.13e-1	1.13e-1	1.13e-1
1	9	8	-1.17e-2	-1.17e-2	-1.17e-2
2	10	9	2.17e-3	2.17e-3	2.17e-3
3	49	44	-3.54e-4	-3.54e-4	-3.54e-4
4	59	53	7.52e-5	7.52e-5	7.52e-5
5	226	203	-1.77e-5	-1.77e-5	-1.77e-5
6	285	256	1.50e-6	1.50e-6	1.50e-6
7	2791	2507	-5.55e-8	-5.56e-8	-5.56e-8
8	5867	5270	2.02e-8	2.01e-8	2.01e-8
9	8658	7777	-4.23e-9	-4.29e-9	-4.29e-9
10	31841	28601	2.62e-10	2.08e-10	2.08e-10
11	167863	150782	3.05e-11	-2.38e-11	-2.38e-11
12	199704	179383	6.74e-11	1.32e-11	1.32e-11
13	367567	330165	5.06e-11	-3.72e-12	-3.73e-12
14	567271	509548	5.65e-11	2.23e-12	2.21e-12
15	934838	839713	5.42e-11	-1.10e-13	-1.27e-13
16	9915651	8906678	5.43e-11	2.40e-14	6.64e-15
17	10850489	9746391	5.43e-11	1.25e-14	-4.88e-15
18	20766140	18653069	5.43e-11	1.80e-14	6.21e-16
19	93915049	84358667	5.43e-11	1.74e-14	-1.50e-17
20	866001581	777881072	5.43e-11	1.74e-14	2.54e-19
21	5289924535	4751645099	5.43e-11	1.74e-14	-1.65e-20
22	11445850651	10281171270	5.43e-11	1.74e-14	3.93e-21
23	16735775186	15032816369	5.43e-11	1.74e-14	-2.54e-21
24	28181625837	25313987639	5.43e-11	1.74e-14	8.82e-23
25	495823414415	445370606232	5.43e-11	1.74e-14	-5.00e-25
26	4986415769987	4479020049959	5.43e-11	1.74e-14	1.63e-27
27	150088296514025	134815972105002	5.43e-11	1.74e-14	-2.35e-29
28	305163008798037	274110964259963	5.43e-11	1.74e-14	3.50e-30
29	1065577322908136	957148864884891	5.43e-11	1.74e-14	-3.00e-31
30	3501894977522445	3145557558914636	5.43e-11	1.74e-14	1.00e-31
31	4567472300430581	4102706423799527	5.43e-11	1.74e-14	0.00e-29

Table 4: The results obtained with  $\min(\Omega) \geq M_i$  with  $M_1 = 5^{15}$ ,  $M_2 = 5^{20}$ ,  $M_3 = 5^{25}$  for the triplet  $(5, 6, 4)_+$ .

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