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Sperner's lemma and competitive equilibrium with incomplete financial markets*

Thanh Le[†] Cuong Le Van[‡] Ngoc-Sang Pham[§] Çağrı Sağlam[¶]

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Abstract

We establish the existence of a competitive equilibrium in a two-period stochastic economy with incomplete financial markets by using Sperner's lemma. Our existence result covers (but is not limited to) several results in the literature, including the cases of nominal and numéraire assets. Moreover, there may exist a continuum of equilibrium prices.

Keywords: Sperner's lemma, fixed-point theorem, general equilibrium, incomplete market, indeterminacy.

JEL Classification: C60, C62, D5, G1.

1 Introduction

The issue of the existence of a competitive equilibrium in economies with incomplete financial market is important in economic theory and finance. For instance, [Cass \(2006\)](#) uses the Kakutani fixed point theorem to prove that under mild conditions, there exists an equilibrium whose asset price is a convex combination of the return matrix where the weights are exogenously given.

Given that [Cass \(2006\)](#) only considers the case where all assets are nominal, i.e., the return matrix of assets is constant, we aim to extend [Cass \(2006\)](#) to the case where the payoff matrix of assets depends on commodity prices. We require that the return matrix is a continuous, positive function of commodity prices. Moreover, its rank equals the number of financial assets when all commodity prices are strictly positive. Under these conditions and standard conditions on the utility function, we prove that there exists an equilibrium whose asset price is a convex combination of the return matrix where the weights are exogenously given. This leads to an interesting implication: there may be a continuum of equilibrium prices. Notice that the indeterminacy here is not a simple matter of price normalization but from the structure of our return matrix (whose rank equals the number of financial assets).

It is interesting to highlight that our result applies to nominal and numéraire assets as well. Moreover, we show that in the case of nominal assets, an asset price is an equilibrium price if and only if it is a no-arbitrage price. However, this insight does not hold for the case of numéraire assets.

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Our second contribution concerns the methodology to prove the equilibrium existence result. To establish the equilibrium existence, the Kakutani fixed point theorem (Kakutani, 1941) and its variants have been widely used.¹ We provide a new proof based on Sperner's lemma (which is a combinatorial result on colorings of triangulations) and elementary topology. Note that our approach based on Sperner's lemma is constructive and may be helpful for computing the equilibrium (see Scarf and Hansen (1973), Scarf (1982) for more details) while the standard approach using the Kakutani fixed point theorem is not.²

Let us explain the intuition of our proof. The key point when applying Sperner's lemma is to construct a labeling which is proper (i.e., it satisfies Sperner condition) and, more importantly, will generate a point corresponding to an equilibrium price.³ In a two-period economy with incomplete financial markets, constructing a proper labeling is not easy because the budget sets may have empty interiors when some prices are null.

To overcome this difficulty, we introduce an artificial economy where all agents, except for one, have an additional income ($\epsilon > 0$) in the first period so that their budget sets have a non-empty interior for any prices system in the simplex. For this artificial economy, we use the excess-demand approach to construct a proper labeling and hence prove the existence of an equilibrium which depends on ϵ . Then, we let ϵ go to zero to get an equilibrium for the original economy.

The paper proceeds as follows. In Section 2, we review some basic concepts such as the notions of subsimplex, simplicial subdivision, and Sperner's lemma. In Section 3, we use Sperner's lemma to prove the existence of a competitive equilibrium in an economy with incomplete financial market markets, and discuss the equilibrium properties.

2 Preliminaries and the Sperner lemma

In this section, we present basic notions from combinatorial topology based on which we state Sperner's lemma.

Consider the Euclidean space \mathbb{R}^n . Let $e^1 = (1, 0, 0, \dots, 0)$, $e^2 = (0, 1, 0, \dots, 0)$, \dots , and $e^n = (0, 0, \dots, 0, 1)$ denote the n unit vectors of \mathbb{R}^n . The unit-simplex Δ of \mathbb{R}^n is the convex hull of $\{e^1, e^2, \dots, e^n\}$. A $(m-1)$ -dimensional simplex of \mathbb{R}^n , denoted by $[[x^1, x^2, \dots, x^m]]$, is the convex hull of $\{x^1, x^2, \dots, x^m\}$ where $x^i \in \mathbb{R}^n$ for any $i = 1, \dots, m$, and the vectors $(x^1 - x^2, x^1 - x^3, \dots, x^1 - x^m)$ are linearly independent, or equivalently, the vectors (x^1, x^2, \dots, x^m) are affinely independent (i.e., $\sum_{i=1}^m \lambda_i x_i = 0$ and $\sum_{i=1}^m \lambda_i = 0$ imply that $\lambda_i = 0 \forall i$).

Given a simplex $S = [[x^1, x^2, \dots, x^m]]$, a face of S is the convex hull $[[x^{i_1}, x^{i_2}, \dots, x^{i_h}]]$ with $h < m$, and $\{i_1, i_2, \dots, i_h\} \subset \{1, 2, \dots, m\}$. In particular, a vertex of S is x^i with $i \in \{1, \dots, m\}$.

We now define the notions of simplicial subdivision (or triangulation) and labeling (see Border (1985), Su (1999) or Chapter 23 in Maschler et al. (2013) for a general treatment) before stating Sperner's lemma.

Definition 1. *The set $T = \{\Delta_i : i = 1, \dots, p\}$ of simplices, is a simplicial subdivision of Δ if it satisfies 3 conditions: (i) $\Delta = \cup_{i=1}^p \Delta_i$, (ii) for any $i, j \in \{1, \dots, p\}$, the intersection $\Delta_i \cap \Delta_j$ is either empty or a face of both Δ_i and Δ_j , and (iii) for any $i \in \{1, \dots, p\}$, all of the faces of Δ_i are in T .⁴*

¹See Magill and Quinzii (1996), Florenzano (1999) for excellent treatments of general equilibrium models with incomplete financial markets.

²In a companion paper (Le, Le Van, Pham and Saglam, 2022), we use Sperner's lemma to prove the Gale-Nikaido-Debreu lemma.

³In applying the Kakutani fixed point theorem, a key issue is to construct the correspondence which generates the equilibrium prices. In general, this task is not trivial.

⁴For any positive integer K , there is a simplicial subdivision $T^K = \{\Delta_1^K, \dots, \Delta_{p(K)}^K\}$ of Δ such that

Definition 2. Consider a simplicial subdivision of Δ . Let V denote the set of vertices of all the subsimplices of Δ . A labeling R is a function from V into $\{1, 2, \dots, n\}$. A labeling R is said to be proper if it satisfies the **Sperner condition**:

$$\text{For any } m \leq n, \text{ if } x \in \text{ri}[[e^{i_1}, e^{i_2}, \dots, e^{i_m}]] \text{ then } R(x) \in \{i_1, i_2, \dots, i_m\}.^5$$

In particular, $R(e^i) = i, \forall i$.

Note that the Sperner condition implies that all vertices of the simplex are labeled distinctly. Moreover, the label of any vertex on the edge between the vertices of the original simplex matches with another label of these vertices. With these in mind, we can now state Sperner's lemma.

Lemma 1. (Sperner's lemma) Let $T = \{\Delta_1, \dots, \Delta_p\}$ be a simplicial subdivision of Δ . Let R be a labeling which satisfies the Sperner condition. Then there exists a subsimplex $\Delta_i \in T$ which is completely labeled, i.e., $\Delta_i = [[x^1(i), \dots, x^n(i)]]$ with $R(x^l(i)) = l, \forall l = 1, \dots, n$.

The Sperner lemma guarantees the existence of a completely labeled subsimplex for any simplicially subdivided simplex in accordance with the Sperner condition.⁶

3 Main results

In this section, we use the Sperner lemma to establish the existence of an equilibrium in a two-period stochastic economy with incomplete financial markets. We also discuss implications of our existence result.

3.1 Competitive equilibrium with incomplete financial markets

First, we briefly present here some essential notions. For a full exposition, see [Magill and Quinzii \(1996\)](#) and [Florenzano \(1999\)](#).

Consider an economy with two periods ($t = 0$ and $t = 1$), L consumption goods, J financial assets, and I agents ($I \geq 2$). There is no uncertainty in period 0 while there are S possible states of nature in period 1. In period 0, each agent $i \leq I$ consumes and purchases assets. The consumption prices are denoted by $p_0 \in \mathbb{R}_+^L$ in the first period, $p_s \in \mathbb{R}_+^L$ in the state s of period 1.

Let $p \equiv (p_0, p_1, \dots, p_S) \in \mathbb{R}_+^{L(S+1)}$. Each consumer has endowments of consumption good $\omega_0^i \in \mathbb{R}_+^L$ in period 0 and $\omega_s^i \in \mathbb{R}_+^L$ in state s of period 1. Any agent i has a utility function $U^i(x_0^i, x_1^i, \dots, x_S^i)$ where x_s^i is her consumption in state s . There is a matrix of returns depending on p of financial assets, which is the same for any agent. Typically, if agent $i \leq I$ purchases z^i quantity of assets in period 0, then in period 1, in state s , she/he will obtain an income (positive or negative) $\sum_{j=1}^J R_{s,j}(p)z^j$. The returns $R(p)$ can be represented by a matrix $R = (R_{i,j}(p))_{i=1, \dots, S; j=1, \dots, J}$. We denote by $R_s(p) = (R_{s,1}(p), R_{s,2}(p), \dots, R_{s,J}(p))$ the s^{th} row of $R(p)$.

We make use of the following set of assumptions.

Assumption 1. (i) For any $i = 1, \dots, I$, the consumption set is $X^i = \mathbb{R}_+^{L(S+1)}$, and the assets set is $Z^i = \mathbb{R}^J$.

(ii) For any $i = 1, \dots, I$, $\omega_0^i \in \mathbb{R}_{++}^L$, $\omega_s^i \in \mathbb{R}_{++}^L$ for any state s in period 1.

(iii) For any $i = 1, \dots, I$, U^i is strictly increasing, continuous, and strictly concave.

$\text{Mesh}(T^K) \equiv \max_{i \in \{1, \dots, p(K)\}} \sup_{x, y \in \Delta_i^K} \{\|x - y\| : x, y \in \Delta_i^K\} < 1/K$. For example, we can take equilateral subdivisions or barycentric subdivisions.

⁵Recall that if $\Delta_i = [[x^{i_1}, x^{i_2}, \dots, x^{i_m}]]$, then $\text{ri}(\Delta_i) \equiv \{x \mid x = \sum_{k=1}^m \alpha_k x^k(i); \sum_k \alpha_k = 1; \text{ and } \forall k : \alpha(k) > 0\}$.

⁶A proof of this lemma can be found in several textbooks ([Berge, 1959](#); [Scarf and Hansen, 1973](#); [Border, 1985](#); [Maschler et al., 2013](#)) or papers ([Sperner, 1928](#); [Le Van, 1982](#)).

Assumption 2. The map $p \rightarrow R(p)$ is continuous. $R(p)$ is non-negative for any $p \geq 0$. $R(p)$ has rank J for any $p \gg 0$.⁷

We now introduce the notion of equilibrium in an economy with financial assets.

Definition 3. Consider the economy $\mathcal{E} = ((U^i, X^i, Z^i, \omega^i), R)$. An equilibrium of this economy is a list $((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*))$ where $(x^{i*}, z^{i*})_{i=1}^I \in (X^i)^I \times (Z^i)^I$, $(p^*, q^*) \in \mathbb{R}_{++}^{L(S+1)} \times \mathbb{R}_{++}^J$ such that

- (i) For any i , $(x^{i*}, z^{i*}) \in X^i \times Z^i$, $p_0^* \cdot (x_0^i - \omega_0^i) + q^* \cdot z^i = 0$, $p_s^* \cdot (x_s^i - \omega_s^i) = R_s(p^*) \cdot z^i$ $\forall s = 1, \dots, S$, and x^{i*} solves the problem

$$\max U^i(x_0^i, x_1^i, \dots, x_S^i) \text{ subject to: } x^i \in B^i(p^*, q^*) \quad (1a)$$

where we define

$$B^i(p, q) \equiv \{x^i \in X^i : \exists z^i \in Z^i, p_0 \cdot (x_0^i - \omega_0^i) + q \cdot z^i \leq 0, \\ p_s \cdot (x_s^i - \omega_s^i) \leq R_s(p) \cdot z^i, s = 1, \dots, S\}$$

- (ii) $\sum_{i=1}^I (x_s^{i*} - \omega_s^i) = 0$ for any $s = 0, 1, \dots, S$ and $\sum_{i=1}^I z^{i*} = 0$.

Definition 4. The allocations $((x^i, z^i)_i) \in (X^i)^I \times (Z^i)^I$ are feasible if (i) $\sum_{i=1}^I (x^i - \omega^i) \leq 0$ and (ii) $\sum_{i=1}^I z^i = 0$.

Given $\alpha > 0$ and define the sets $F^\alpha = \{(x^i)_i \in (X^i)^I : \sum_{i=1}^I (x^i - \omega^i) \leq \alpha \mathbf{1}_N\}$, where we denote $N = (S+1)L$ and $\mathbf{1}_N$ is the vector of \mathbb{R}^N , whose coordinates are all equal to 1. Denote the projection of F^α on X^i by \hat{X}^i . Let B^c be a closed ball of \mathbb{R}^L , centered at the origin, which contains all \hat{X}^i in its interior.

An intermediate economy is the economy

$$\tilde{\mathcal{E}} = ((U^i, \tilde{X}^i, Z^i, \omega^i), R),$$

where the consumption set is $\tilde{X}^i = X^c \equiv B^c \cap \mathbb{R}_+^{L(S+1)}$ for any i . An equilibrium in this intermediate economy is defined as in Definition 3.

We can now state our main result.

Theorem 1. Consider the economy \mathcal{E} . Let Assumptions 1 and 2 be satisfied. For any list $(\lambda_0, \lambda_1, \dots, \lambda_S)$ with $\lambda_0 = 1, \lambda_S > 0, s = 1, \dots, S$, there exists an equilibrium $((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*))$ with $p^* \in \Delta$ - the unit-simplex of $\mathbb{R}_+^{L(S+1)}$, and, more importantly,

$$q^* = \sum_{s=1}^S \lambda_s R_s(p^*), \text{ i.e., } q_j^* = \sum_{s=1}^S \lambda_s R_{s,j}(p^*), \forall j = 1, \dots, J. \quad (2)$$

Theorem 1 leads to some interesting consequences.

Corollary 1. Let Assumptions 1 be satisfied. Then the conclusion in Theorem 1 holds for the following cases:

1. (Nominal assets) All assets are nominal: $R(p)$ is a positive constant matrix ($R_{s,j}(p) = R_{s,j} \geq 0 \forall s, \forall j, \forall p \in \Delta$), and $\text{rank}(R) = J$.

⁷Condition " $R(p)$ has rank J for any $p \gg 0$ " is quite similar to Assumptions F0 in (Aouani and Cornet (2011)) which can also cover nominal and numéraire assets.

2. (*Numéraire assets*) All assets are numéraire assets: $R(p) = Q(p) \times G$ where G is a positive constant $S \times J$ -matrix, $\text{rank}(G) = J$, and

$$Q(p) = \begin{bmatrix} p_1 \cdot e & 0 & \dots & 0 \\ 0 & p_2 \cdot e & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_S \cdot e \end{bmatrix},$$

where $e \in \mathbb{R}^L$ and $e \gg 0$ is a numéraire.⁸

Theorem 1 covers the existence of competitive equilibrium in economies with nominal assets as in Cass (2006) and Florenzano (1999). Note that Cass uses the Kakutani fixed point theorem and Florenzano uses the Gale–Mas-Colell lemma (Gale and Mas-Colell, 1975, 1979).⁹ Theorem 1's contribution is two-fold. First, it provides a new proof based on Sperner's lemma. Second, it shows that if the return matrix is continuous, the equilibrium asset price is a linear combination of the return matrix where the weights are given.¹⁰

Our equilibrium existence covers some cases which have not been considered by Cass (2006) and Florenzano (1999). For example, when we consider an economy with 3 states of nature, 2 assets ($S = 3, J = 2$), and the return matrix is given by

$$R = \begin{bmatrix} p_1 \cdot e & 0 \\ 0 & p_2 \cdot e \\ 0 & 1 \end{bmatrix}$$

where $e \in \mathbb{R}^L$ and $e \gg 0$ is a numéraire. This means that the asset delivery is in terms of numéraire asset for the states of nature 1 and 2 but in terms of nominal asset for the state of nature 3.

Before presenting our proof, we point out some implications of Theorem 1:

Remark 1 (Continuum of equilibria). *In the case of nominal assets where the return matrix is constant, we have $q^* = \sum_{s=1}^S \lambda_s R_s$, i.e.,*

$$q_j^* = \sum_{s=1}^S \lambda_s R_{j,s}, \forall j = 1, \dots, J. \quad (3)$$

While the property $q^* = \sum_{s=1}^S \lambda_s R_s$ is well-known in the case of nominal assets, our paper is the first to show a similar property ($q^* = \sum_{s=1}^S \lambda_s R_s(p^*)$) in the case when the return matrix R satisfies Assumption 2.

Remark 2 (Equilibrium price versus no-arbitrage price). *For the nominal assets, an equilibrium always exists, and an asset price is an asset equilibrium price if and only if it is a no arbitrage price. Indeed, take a no-arbitrage price. Using our above result, we obtain an equilibrium. Conversely, for any financial equilibrium, under the assumption that the utility functions are strictly increasing, the first order conditions show that an equilibrium asset price is a no-arbitrage price.*

However, we do not have this equivalence in the numéraire case. Indeed, in this case, the set of no-arbitrage prices is $\{q : q = \sum_{s=1}^S \lambda_s R_s(p), \lambda_s > 0, \forall s \geq 1, p \in \Delta\}$. If q is an equilibrium price, then by the first order conditions, it is a no-arbitrage price. The converse is not always true. Indeed, if $q = \sum_{s=1}^S \lambda_s R_s(p)$ with $\lambda_s > 0, \forall s \geq 1, p \in \Delta$, it is not sure that this q is an equilibrium price (because the return matrix depends on price p).

⁸Notice that if $p \gg 0$, then $p_s \cdot e > 0 \forall s \geq 1$, and hence $\text{rank}(R(p)) = \text{rank}(G)$.

⁹Cass works with utility functions while Florenzano works with preferences.

¹⁰Notice that Assumption 2 may not be satisfied when all assets are real (i.e., the return matrix equals $(p_s \cdot a_s^j)_{s=1, \dots, S; j=1, \dots, J}$, where $a_s^j \in \mathbb{R}^L$). So, Theorem 1 can not be applied for the case of real assets. Recall that when all assets are real, equilibrium may fail to exist (see Hart (1975) and Florenzano (1999) among others).

3.2 Proof of Theorem 1

First, we prove the existence of equilibrium in the intermediate economy $\tilde{\mathcal{E}}$. To do so, we proceed in two steps: (1) we use the Sperner lemma to prove that there exists actually a *Cass equilibrium*, and (2) from this Cass equilibrium, we construct an equilibrium for the intermediate economy.

We now define and prove the existence of a Cass equilibrium.

Definition 5. A Cass equilibrium associated with $(\lambda_0, \lambda_1, \dots, \lambda_S), \lambda_0 = 1, \lambda_s > 0, \forall s \geq 1$ is a list $((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I, (\bar{p}, \bar{q}))$ such that $((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I) \in (X^c)^I \times (\mathbb{R}^J)^{I-1}$, $(\bar{p}, \bar{q}) \in \mathbb{R}_{++}^{L(1+S)} \times \mathbb{R}_{++}^J$ where

- (i) \bar{x}^1 solves the consumer 1 problem under the constraints: $x^1 \in X^c$, $\bar{p}' \cdot (x^1 - \omega^1) \leq 0$, where $\bar{p}' = (\bar{p}_0, \lambda_1 \bar{p}_1, \dots, \lambda_S \bar{p}_S)$.
- (ii) For $i = 2, \dots, I$, we have $\bar{p}_0 \cdot (\bar{x}_0^i - \omega_0^i) + \bar{q} \cdot \bar{z}^i = 0$, $\bar{p}_s \cdot (\bar{x}_s^i - \omega_s^i) = R_s(\bar{p}) \cdot \bar{z}^i$, $\forall s \geq 1$, and \bar{x}^i solves the consumer i 's problem

$$\max U^i(x_0^i, x_1^i, \dots, x_S^i) \text{ subject to: } x^i \in B_{X^c}^i(\bar{p}, \bar{q})$$

where $B_{X^c}^i(\bar{p}, \bar{q}) \equiv \{x^i \in X^c : \exists z^i \in \mathbb{R}^J : \bar{p}_0 \cdot (x_0^i - \omega_0^i) + \bar{q} \cdot z^i \leq 0, \bar{p}_s \cdot (x_s^i - \omega_s^i) \leq R_s(\bar{p}) \cdot z^i, \forall s \geq 1\}$.

$$(iii) \quad \bar{q} = \sum_s \lambda_s R_s(\bar{p}).$$

$$(iv) \quad \sum_{i=1}^I (\bar{x}^i - \omega^i) = 0.$$

Lemma 2. There exists a Cass equilibrium associated with $(\lambda_0, \lambda_1, \dots, \lambda_S), \lambda_0 = 1, \lambda_s > 0, \forall s \geq 1$.

Proof. Step 1. Let $p = (p_0, p_1, \dots, p_S) \in \Delta$ where Δ denotes the unit-simplex of $\mathbb{R}^{L(S+1)}$. Define $p' = (p_0, \lambda_1 p_1, \dots, \lambda_S p_S)$. Let $\bar{\lambda} = \min_s \lambda_s$. Let $\alpha > 0$ and ϵ such that $0 < \epsilon < \frac{\alpha \bar{\lambda}}{(I-1)}$. Define the following ϵ -return matrix $R'(p, \epsilon)$: $R'((p_{sl})_{s,l}, \epsilon) = R((p_{sl} + \epsilon)_{s,l})$. Obviously, $R'(p, 0) = R(p)$ and $R'(p, \epsilon)$ is of rank J for any $\epsilon > 0$.

Consider the problem of agent 1:

$$\max U^1(x^1) \text{ subject to } x^1 \in B_{X^c}^1(p) \equiv \{x^1 \in X^c : p' \cdot (x^1 - \omega^1) \leq 0\}.$$

Any agent i ($i \geq 2$) solves the following problem:

$$\max U^i(x^i) \text{ subject to: } x^i \in B_{X^c}^{i,\epsilon}(p), \quad (\text{Problem } P^{i,\epsilon}(p))$$

$$\text{where } B_{X^c}^{i,\epsilon}(p) \equiv \left\{ x^i \in X^c : \exists z^i \in \mathbb{R}^J : p_0 \cdot (x_0^i - \omega_0^i) + \left(\sum_{s \geq 1} \lambda_s R'_s(p, \epsilon) \right) \cdot z^i \leq \epsilon, \right.$$

$$\left. p_s \cdot (x_s^i - \omega_s^i) \leq R'_s(p, \epsilon) \cdot z^i \quad \forall s \geq 1 \right\}.$$

These optimization problems have continuous, strictly concave objective functions. Clearly, $B_{X^c}^1(p)$ is compact. We can also prove that $B_{X^c}^{i,\epsilon}(p)$ is compact.¹¹ So, these optimization problems have a unique solution.

¹¹First, $B_{X^c}^{i,\epsilon}(p)$ is bounded. Second, we prove that it is closed. Let $x^{i,n}$ be in $B_{X^c}^{i,\epsilon}(p)$ and converge to x^i . We have to prove that x^i is in $B_{X^c}^{i,\epsilon}(p)$. Let $z^{i,n}$ be in \mathbb{R}^J so that two conditions in the definition of $B_{X^c}^{i,\epsilon}(p)$ hold. It suffices to prove that the sequence $(z^{i,n})$ is bounded. Suppose that $\|z^{i,n}\| \rightarrow +\infty$ when $n \rightarrow +\infty$.

We have $\frac{p_0 \cdot (x_0^{i,n} - \omega_0^i)}{\|z^{i,n}\|} + \left(\sum_{s \geq 1} \lambda_s R'_s(p, \epsilon) \right) \cdot \frac{z^{i,n}}{\|z^{i,n}\|} \leq \frac{\epsilon}{\|z^{i,n}\|}$ and $\frac{p_s \cdot (x_s^{i,n} - \omega_s^i)}{\|z^{i,n}\|} \leq R'_s(p, \epsilon) \cdot \frac{z^{i,n}}{\|z^{i,n}\|}$. Let $n \rightarrow +\infty$, since there exists a subsequence $\frac{z^{i,n_k}}{\|z^{i,n_k}\|}$ which converges, we can, without loss of generality, suppose $\frac{z^{i,n}}{\|z^{i,n}\|} \rightarrow \zeta \neq 0$. So, we have $\left(\sum_{s \geq 1} \lambda_s R'_s(p, \epsilon) \right) \cdot \zeta \leq 0$, $0 \leq R'_s(p, \epsilon) \cdot \zeta, \forall s$. By consequence, we have $R'_s(p, \epsilon) \cdot \zeta = 0$, $\forall s = 1, \dots, S$. Since the matrix $R'(p, \epsilon)$ is of rank J , we imply that $S \geq J$ and we get that $\zeta = 0$, which is a contradiction.

We now prove that the optimal values $x^{*1}, x_\epsilon^{*2}, \dots, x_\epsilon^{*I}$ are continuous functions with respect to p .¹² Indeed, consider a sequence of prices $p(n) \in \Delta$ converging to p when n tends to infinity. Let us consider the problem of agent $i \geq 2$ (the proof for agent 1's problem is similar) and denote $x_\epsilon^{*i}(p(n)), x_\epsilon^{*i}(p)$ the solution of the problem $P^{i,\epsilon}(p(n)), P^{i,\epsilon}(p)$ respectively. By using the same argument in Footnote 11, we can assume that there exists sub-sequences $x_\epsilon^{*i}(p(n_k)), z_\epsilon^{*i}(p(n_k))$ satisfying two conditions in the definition of $B_{X^c}^{i,\epsilon}(p(n_k))$ and they converge. Denote $x^{*i} = \lim_{k \rightarrow \infty} x_\epsilon^{*i}(p(n_k))$.

We claim that $x^{*i} = x_\epsilon^{*i}(p)$. Indeed, the budget constraints are satisfied. Observe that the budget sets of the agents i (with $i \geq 2$) have nonempty interiors.¹³ Let x^i be in the interior of $B_{X^c}^{i,\epsilon}(p)$. We have $p_0 \cdot (x_0^i - \omega_0^i) + (\sum_{s \geq 1} \lambda_s R'_s(p, \epsilon)) \cdot z^i < \epsilon$ and $p_s \cdot (x_s^i - \omega_s^i) < R'_s(p, \epsilon) \cdot z^i$.

Since $p(n_k)$ tends to p , we have $p_0(n_k) \cdot (x_0^i - \omega_0^i) + (\sum_{s \geq 1} \lambda_s R'_s(p(n_k), \epsilon)) \cdot z^i < \epsilon$ and $p_s(n_k) \cdot (x_s^i - \omega_s^i) < R'_s(p(n_k), \epsilon) \cdot z^i$ for any n high enough. This implies that $U^i(x_\epsilon^{*i}(p(n_k))) \geq U^i(x^i)$ for any n high enough. Let k tend to infinity, since $x_\epsilon^{*i}(p(n_k))$ converges to x^{*i} , we get that: $U^i(x_\epsilon^{*i}) \geq U^i(x^i)$ for any x^i in the interior of $B_{X^c}^{i,\epsilon}(p)$.

We now take x^i in the interior of $B_{X^c}^{i,\epsilon}(p)$. Define $x^{im} = \frac{1}{m}x^i + (1 - \frac{1}{m})x_\epsilon^{*i}(p)$. Then x^{im} is in the interior of $B_{X^c}^{i,\epsilon}(p)$ for m high enough. By the concavity, we have that, for m high enough,

$$\frac{1}{m}U^i(x^i) + (1 - \frac{1}{m})U^i(x_\epsilon^{*i}(p)) \leq U^i\left(\frac{1}{m}x^i + (1 - \frac{1}{m})x_\epsilon^{*i}(p)\right) = U^i(x^{im}) \leq U^i(x_\epsilon^{*i}),$$

Let m tend to infinity, we get that $U^i(x_\epsilon^{*i}) \geq U^i(x^{*i})$. So, $x_\epsilon^{*i} = x^{*i}(p)$ (since U^i is strictly concave). We have just proved that $x^{*1}, x_\epsilon^{*2}, \dots, x_\epsilon^{*I}$ are continuous functions with respect to p .

For any p , we have

$$p' \cdot \sum_{i=1}^I (x_\epsilon^{*i}(p) - \omega^i) \leq (I-1)\epsilon,$$

where, by convention, we denote $x_\epsilon^{*1} \equiv x^{*1}$.

Define the excess demand mapping ξ by

$$\xi(p) = \sum_{i=1}^I (x_\epsilon^{*i}(p) - \omega^i).$$

It is obvious that $\forall p \in \Delta, p' \cdot \xi(p) \leq (I-1)\epsilon$.

Step 2 (using the Sperner lemma). Denote $N = (S+1)L$. Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of \mathbb{R}^N such that $Mesh(T^K) < 1/K$. We define the following labeling r . For any $p \in \Delta, r(p) = t$, where t is one of the indices satisfying $p_t \neq 0$ and $\xi_t(p) \leq \frac{(I-1)\epsilon}{\lambda}$. Such a labeling is well defined. Indeed, for $p \in \Delta$, we have $(I-1)\epsilon \geq p' \cdot \xi(p) = \sum_{h=0}^S \lambda_h p_h \xi_h(p) \geq \sum_{h=0}^S \lambda_h p_h \xi_h(p)$. So, there must exist $t \in \{1, \dots, N\}$ with $p_t \neq 0$ and $\xi_t(p) \leq \frac{(I-1)\epsilon}{\lambda}$. This labeling satisfies the Sperner condition. Indeed, if $p \in ri[[e^{i_1}, \dots, e^{i_m}]]$ with $m \leq N$, then we have $\{i : p_i \neq 0\} = \{i_1, \dots, i_m\}$. By definition, we see that $r(p) \in \{i : p_i \neq 0\}$. Therefore, $r(p) \in \{i_1, \dots, i_m\}$. It means that the labeling r satisfies the Sperner condition.

According to Sperner's lemma, there exists a completely labeled subsimplex $[[\bar{p}^1(K), \dots, \bar{p}^N(K)]]$, i.e., $r(\bar{p}^i(K)) = i, \forall i$. This implies that $\xi_t(\bar{p}^t(K)) \leq \frac{(I-1)\epsilon}{\lambda}, \forall t = 1, \dots, N$.

Observe that

$$\forall t = 1, \dots, N, \sum_{i=1}^I (x_{\epsilon,t}^{*i}(\bar{p}^t(K)) - \omega^i) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}} \mathbf{1}_N < \alpha \mathbf{1}_N. \quad (4)$$

¹²We can also prove this continuity by applying the Maximum Theorem (Berge, 1959) and noting that $B_{X^c}^{i,\epsilon}(p)$ has a closed graph (since that matrix $R'(p, \epsilon)$ is of rank J).

¹³Indeed, we can take $x_s^i = \omega_s^i, s = 0, 1, \dots, S$ and $z^i > 0$ such that $\sum_{s \geq 1} \lambda_s R'_s(p, \epsilon) z^i < \epsilon$. Then this allocation is in the interior of $B_{X^c}^{i,\epsilon}(p)$.

where recall that $\mathbf{1}_N$ is the vector of \mathbb{R}^N , whose coordinates are all equal to 1.

Let $K \rightarrow +\infty$, there is a subsequence $\bar{p}^t(K_n) \rightarrow p^*(\epsilon) \in \Delta$ for any $t \in \{1, \dots, N\}$, when n tends to infinity. We have $\xi_v(p^*(\epsilon)) \leq \frac{(I-1)\epsilon}{\lambda} < \alpha$, for all $v = 1, \dots, N$.

It follows from (4) that

$$\sum_{i=1}^I \left(x_{\epsilon,t}^{*i}(p^*(\epsilon)) - \omega^i \right) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}} \mathbf{1}_N < \alpha \mathbf{1}_N. \quad (5)$$

This implies that for any i , $x_{\epsilon,t}^{*i}(p^*(\epsilon))$ is uniformly bounded from above when ϵ is small.

Write $p^*(\epsilon) = (p_0^*(\epsilon), p_1^*(\epsilon), \dots, p_S^*(\epsilon))$, $p'^*(\epsilon) = (p_0^*(\epsilon), \lambda_1 p_1^*(\epsilon), \dots, \lambda_S p_S^*(\epsilon))$. Because of (5), the consumptions are in the interior of the ball B^c . Combining with the fact that utility functions are strictly increasing, we get that: the budget constraint must be binding:

$$p'^*(\epsilon) \cdot (x^{*1}(p^*(\epsilon)) - \omega^1) = 0, \quad (6)$$

and $p'^*(\epsilon) \gg 0$, or, equivalently, $p^*(\epsilon) \gg 0$.

For any agent $i \geq 2$, condition (5) also implies that the consumptions are in the ball B^c . By consequence, there exists z_{ϵ}^{*i} such that

$$\begin{aligned} p_0^*(\epsilon) \cdot (x_{\epsilon,0}^{*i}(p^*(\epsilon)) - \omega_s^i) + \left(\sum_s \lambda_s R'_s(p^*(\epsilon), \epsilon) z_{\epsilon}^{*i} \right) &= \epsilon, \\ p_s^*(\epsilon) \cdot (x_{\epsilon,s}^{*i}(p^*(\epsilon)) - \omega_s^i) &= R'_s(p^*(\epsilon), \epsilon) \cdot z_{\epsilon}^{*i}, s = 1, \dots, S. \end{aligned}$$

Since $R'(p^*(\epsilon), \epsilon)$ is of rank J for $\epsilon > 0$, z_{ϵ}^{*i} is unique.

Step 3. Let $\epsilon = \frac{1}{m} \rightarrow 0$ (where $m \in \mathbb{N}$ and $m \rightarrow \infty$), without loss of generality. Since the sequence $(p^*(\frac{1}{m}))_m$ is bounded, there exists a subsequence $p^*(\frac{1}{m_k})_k$ which converges. So, without loss of generality, we can assume that $p^*(\epsilon)$ converges to $\bar{p} \in \Delta$. Since x^{*1} is continuous, we have $x^{*1}(p^*(\epsilon)) \rightarrow \bar{x}^1 \equiv x^{*1}(\bar{p})$. From this, we have $\bar{p} \gg 0$ (because $x_{\epsilon}^{*i}(p^*(\epsilon))$ is uniformly bounded, see (5)).

By using the same argument in Footnote 11 and noting that the matrix $R(\bar{p})$ is of rank J with $\bar{p} \gg 0$, we can prove that there is $\beta > 0$ such that $\|z^{*i}(p^*(\epsilon))\| \leq \beta$ when ϵ is small enough.¹⁴ Hence, without loss of generality, we can assume that $z^{*i}(p^*(\epsilon))$ converges when ϵ goes to zero.

To sum up, when $\epsilon \rightarrow 0$, we can assume that $p^*(\epsilon) \rightarrow \bar{p} \in \Delta$, $x^{*1}(p^*(\epsilon)) \rightarrow \bar{x}^1 \equiv x^{*1}(\bar{p})$, $\bar{p} \gg 0$. For $i \geq 2$, $x_{\epsilon}^{*i}(p^*(\epsilon)) \rightarrow \bar{x}^i$, $z^{*i}(p^*(\epsilon)) \rightarrow \bar{z}^i$.

Let $\bar{p}' = (\bar{p}_0, \lambda_1 \bar{p}_1, \dots, \lambda_S \bar{p}_S)$. Note that from (5) that $\sum_{i=1}^I (\bar{x}^i - \omega^i) \leq 0$ and from (6) that $\bar{p}' \cdot (\sum_{i=1}^I (\bar{x}^i - \omega^i)) = 0 \Rightarrow \bar{p}_p \sum_i (\bar{x}_p^i - \omega_p^i) = 0, p = 1, \dots, N$. Since $\bar{p} \gg 0$, we deduce that $\sum_{i=1}^I (\bar{x}_p^i - \omega_p^i) = 0, \forall p = 1, \dots, N$, or equivalently $\sum_{i=1}^I (\bar{x}^i - \omega^i) = 0$.

The last step: prove the optimality of \bar{x}^i for each $i \geq 2$. To do so, assume that there is $x^i \in B_{X^c}^i(\bar{p}, \bar{q})$ such that $U^i(x^i) > U^i(\bar{x}^i)$. Without loss of generality, we can assume that $x_s^i \gg 0 \forall s$.¹⁵

¹⁴Indeed, suppose that there exists a sequence $(z_{k_n}^{*i})_{n \geq 1} \equiv (z^{*i}(p^*(k_n)))_{n \geq 1}$, with $(k_n)_{n=1,2,\dots}$ being a decreasing sequence converging to zero, and $\|z_{k_n}^{*i}\| \rightarrow +\infty$ when $n \rightarrow +\infty$. We have, for any n , $\forall s = 1, \dots, S$, $p_s^*(k_n) \cdot (x^{*i}(p^*(k_n)) - \omega_s^i) = R'_s(p^*(k_n), k_n) \cdot z_{k_n}^{*i}$. Then,

$$\frac{p_s^*(k_n) \cdot (x_{k_n}^{*i}(p^*(k_n)) - \omega_s^i)}{\|z_{k_n}^{*i}\|} = R'_s(p^*(k_n), k_n) \cdot \frac{z_{k_n}^{*i}}{\|z_{k_n}^{*i}\|} \quad \forall s = 1, \dots, S, \forall n \quad (7)$$

Let $n \rightarrow +\infty$, we can suppose $\frac{z_{k_n}^{*i}}{\|z_{k_n}^{*i}\|} \rightarrow \zeta \neq 0$. Since $\lim_{n \rightarrow \infty} p^*(k_n) = \bar{p}$, we have $\lim_{n \rightarrow \infty} R'_s(p^*(k_n), k_n) = R(\bar{p})$. Therefore, we get that $0 = R_s(\bar{p}) \cdot \zeta = 0 \forall s = 1, \dots, S$. Since $\bar{p} \gg 0$, the matrix $R(\bar{p})$ is of rank J . This implies that $\zeta = 0$, which is a contradiction.

¹⁵Indeed, we can introduce $x^i(\lambda)$ by $x_s^i(\lambda) = (1 - \lambda)x_s^i + \lambda\omega_s$. Then, $x_s^i(\lambda) \gg 0$ because $\omega_s \gg 0, \forall s$. Moreover, we can choose $\lambda > 0$ small enough so that $U^i(x^i(\lambda)) > U^i(\bar{x}^i)$.

Since $x^i \in B_{X^c}^i(\bar{p}, \bar{q})$, we take any $z^i \in \mathbb{R}^J$ such that $\bar{p}_0 \cdot (x_0^i - \omega_0^i) + \bar{q} \cdot z^i \leq 0$, $\bar{p}_s \cdot (x_s^i - \omega_s^i) \leq R_s(\bar{p}) \cdot z^i \forall s \geq 1$. Notice that $x_s^i \gg 0, \forall s$. So, without loss of generality, we can assume that¹⁶

$$\bar{p}_0 \cdot (x_0^i - \omega_0^i) + \bar{q} \cdot z^i < 0, \quad \bar{p}_s \cdot (x_s^i - \omega_s^i) < R_s(\bar{p}) \cdot z^i, \forall s \geq 1.$$

Thus, we can choose $\epsilon > 0$ small enough such that

$$p_0^*(\epsilon) \cdot (x_0^i - \omega_0^i) + \bar{R}'_s(p^*(\epsilon), \epsilon) \cdot z^i < \epsilon, \quad p_s^*(\epsilon) \cdot (x_s^i - \omega_s^i) < R'_s(p^*(\epsilon), \epsilon) \cdot z^i, \forall s \geq 1.$$

By the optimality of $x_\epsilon^{*i}(p^*(\epsilon))$, we have $U^i(x^i) \leq U^i(x_\epsilon^{*i}(p^*(\epsilon)))$. Let $\epsilon \rightarrow 0$, we get that $U^i(x^i) \leq U^i(\bar{x}^i)$, which is a contradiction. We have proved the existence of a *Cass equilibrium*. \square

We now move from Cass equilibrium to an equilibrium in the intermediate economy.

Lemma 3. *Let $\lambda_0 = 1, \lambda_s > 0, \forall s \geq 1$. There exists an equilibrium in the intermediate economy with $\bar{q} = \sum_s \lambda_s R_s(\bar{p})$.*

Proof. Let $((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I, (\bar{p}, \bar{q}))$ be a Cass equilibrium associated with $(\lambda_0, \lambda_1, \dots, \lambda_S)$. Define $\bar{z}^1 = -\sum_{i \geq 2} \bar{z}^i$. We will claim that $((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=1}^I, (\bar{p}, \bar{q}))$ is an equilibrium in the intermediate economy with $\bar{q} = \sum_{s=1}^S \lambda_s R_s(\bar{p})$. Market clearing conditions are obviously satisfied.

Since $\sum_{i=1}^I (\bar{x}_s^i - \omega_s^i) = 0, \forall s \geq 1$, we have $\lambda_s \bar{p}_s \cdot \sum_{i=1}^I (\bar{x}_s^i - \omega_s^i) = 0$, which implies that $\lambda_s \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) = -\lambda_s \bar{p}_s \cdot \sum_{i=2}^I (\bar{x}_s^i - \omega_s^i), \forall s \geq 1$.

We now have $\bar{p}_s \cdot \sum_{i=2}^I (\bar{x}_s^i - \omega_s^i) = R_s(\bar{p}) \cdot \sum_{i \geq 2} \bar{z}^i = -R_s(\bar{p}) \cdot \bar{z}^1$, which implies that

$$\sum_{s \geq 1} \lambda_s \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) = \left(\sum_{s \geq 1} \lambda_s R_s(\bar{p}) \right) \cdot \bar{z}^1 = \bar{q} \cdot \bar{z}^1.$$

By combining this with the fact that $\bar{p}_0 \cdot (\bar{x}_0^1 - \omega_0^1) + \sum_{s \geq 1} \lambda_s \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) = 0$, we get that $\bar{p}_0 \cdot (\bar{x}_0^1 - \omega_0^1) + \bar{q} \cdot \bar{z}^1 = 0$. Therefore, we can easily prove the optimality of \bar{x}^1 . \square

Lemma 4. *An equilibrium in the intermediate economy $\tilde{\mathcal{E}}$ is an equilibrium in the initial economy \mathcal{E} .*

Proof. Let $((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*))$ be an equilibrium in the intermediate economy $\tilde{\mathcal{E}}$. To prove that this is also an equilibrium in the initial economy \mathcal{E} , it suffices to prove, for each i , the optimality of the allocation (x^{i*}, z^{i*}) : $U^i(x^{i*}) \geq U^i(x^i)$ for any $(x^i, z^i) \in B^i(p^*, q^*)$.

Let $(x^i, z^i) \in B^i(p^*, q^*)$. Since x^{i*} belongs to the interior of X^c , there exists $\lambda \in (0, 1)$ such that $\lambda x^i + (1 - \lambda)x^{i*} \in \tilde{X}^i$. We have $\lambda U^i(x^i) + (1 - \lambda)U^i(x^{i*}) \leq U^i(\lambda x^i + (1 - \lambda)x^{i*}) \leq U^i(x^{i*})$, which implies that $U^i(x^i) \leq U^i(x^{i*})$. \square

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¹⁶Indeed, we can define $x^{i'}$ by $x_{s,l}^{i'} = x_{s,l}^i - \tau \forall s = 0, \dots, S, \forall l = 1, \dots, L$ where $\tau > 0$ small enough so that $U^i(x^{i'}) > U^i(\bar{x}^i)$.

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