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Diagnostic checking in multivariate ARMA models with dependent errors using normalized residual autocorrelations

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Abstract

In this paper we derive the asymptotic distribution of normalized residual empirical autocovariances and autocorrelations under weak assumptions on the noise. We propose new portmanteau statistics for vector autoregressive moving-average (VARMA) models with uncorrelated but non-independent innovations by using a self-normalization approach. We establish the asymptotic distribution of the proposed statistics. This asymptotic distribution is quite different from the usual chi-squared approximation used under the independent and identically distributed assumption on the noise, or the weighted sum of independent chi-squared random variables obtained under nonindependent innovations. A set of Monte Carlo experiments and an application to the daily returns of the CAC40 is presented.

Keywords: Goodness-of-fit test, quasi-maximum likelihood estimation, Box-Pierce and Ljung-Box portmanteau tests, residual autocorrelation, self-normalization, weak (V)ARMA models

1. Introduction

In econometric application, the univariate autoregressive moving-average (ARMA) framework is very restrictive. Consequently the class of vector autoregressive moving-average (VARMA) models are commonly used in time series analysis and econometrics. It describes the possible cross-relationships between the time series and not only the properties of the individual time series (see [Lütkepohl \(2005\)](#)). Moreover, the time series literature shows growing interest in non-linear models. Roughly speaking, these models are those on which the independence assumption on the noise is relaxed. This motivates the framework of our investigations that we further describe below.

We consider a d -dimensional stationary process $X = (X_t)_{t \in \mathbb{Z}}$ that satisfies a VARMA(p, q) representation of the form

$$A_{00}X_t - \sum_{i=1}^p A_{0i}X_{t-i} = B_{00}\epsilon_t - \sum_{i=1}^q B_{0i}\epsilon_{t-i}, \quad \forall t \in \mathbb{Z}. \quad (1)$$

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Without loss of generality, one may assume that A_{0p} and B_{0q} are not both equal to the null matrix. The representation (1) is said to be a VARMA(p, q) representation if the noise process $\epsilon = (\epsilon_t)_{t \in \mathbb{Z}}$ with $\epsilon_t = (\epsilon_{1t}, \dots, \epsilon_{dt})'$ is a multivariate *weak white noise*, that is, if it satisfies

(A0): $\mathbb{E}(\epsilon_t) = 0$, $\text{Var}(\epsilon_t) = \Sigma_0$, and $\text{Cov}(\epsilon_t, \epsilon_{t-h}) = 0$ for all $t \in \mathbb{Z}$ and all $h \neq 0$.

We also assume that Σ_0 is non singular. It is customary to say that $(X_t)_{t \in \mathbb{Z}}$ is a strong VARMA(p, q) model if $(\epsilon_t)_{t \in \mathbb{Z}}$ is an independent and identically distributed (iid for short) sequence of random vectors with zero mean and non singular variance matrix (*i.e. strong white noise*). When $A_{00} = B_{00} = I_d$, where I_d denotes the identity matrix of order d , the VARMA(p, q) representation (1) is said to be in reduced form. Otherwise, it is said to be structural. The structural forms are mainly used in econometrics to introduce instantaneous relationships between economic variables. Of course, constraints are necessary for the identifiability of these representations.

The estimation of ARMA models is however much more difficult in the multivariate case than in univariate case. A first difficulty is that non trivial constraints on the parameters must be imposed for the identifiability of the parameters (see Lütkepohl (2005)). Secondly, the implementation of standard estimation methods (for instance the Gaussian quasi-maximum likelihood estimation) is not obvious because this requires a constrained high-dimensional optimization (see Lütkepohl (2005) for further details). These technical difficulties certainly explain why the vector autoregressive (VAR for short) models are much more used than VARMA in applied context. This is also the reason why the statistical analysis of weak ARMA model is mainly limited to the univariate framework (see Francq and Zakoïan (2005), for a review on weak univariate ARMA models).

After estimating the (V)ARMA process, the next important step in the (V)ARMA modeling consists in checking if the estimated model fits satisfactorily the data. This adequacy checking step validates or invalidates the choice of the orders p and q . In VARMA(p, q) models, the choice of p and q is particularly important as the number of parameters quickly increases with p and q , which entails statistical difficulties.

In 1970, Box and Pierce have proposed (see Box and Pierce (1970)) a goodness-of-fit test, the so-called portmanteau test, for univariate strong ARMA models. A modification of their test has been proposed by Ljung and Box (see Ljung and Box (1978)). It is nowadays one of the most popular diagnostic checking tools in ARMA modeling of time series. Both of these tests are based on the residual empirical autocorrelations $\hat{\rho}(h)$ (see Section 3.1 for a precise notation) and they are defined by

$$Q_m^{\text{BP}} = n \sum_{h=1}^m \hat{\rho}^2(h) \text{ and } Q_m^{\text{LB}} = n(n+2) \sum_{h=1}^m \frac{\hat{\rho}^2(h)}{n-h}, \quad (2)$$

where n is the length of the series and m is a fixed integer. The statistic Q_m^{LB} has the same asymptotic distribution as Q_m^{BP} and has the reputation of doing better for small or medium sized sample (see Ljung and Box (1978)).

Being inspired by the univariate portmanteau statistics defined in (2), Chitturi (see Chitturi (1974)) and Hosking (see Hosking (1980)) have introduced the following multivariate versions of the portmanteau statistics

$$Q_m^{\text{C}} = n \sum_{h=1}^m \text{Tr} \left(\hat{\Gamma}'(h) \hat{\Gamma}^{-1}(0) \hat{\Gamma}(h) \hat{\Gamma}^{-1}(0) \right) \text{ and } Q_m^{\text{H}} = \sum_{h=1}^m \frac{n^2}{(n-h)} \text{Tr} \left(\hat{\Gamma}'(h) \hat{\Gamma}^{-1}(0) \hat{\Gamma}(h) \hat{\Gamma}^{-1}(0) \right), \quad (3)$$

where $\hat{\Gamma}(h)$ is the residual autocovariances matrices function of the multivariate process X , $\text{Tr}(\cdot)$ denotes the trace of a matrix and A' the transpose of a matrix A .

Under the assumption that the noise sequence is iid, the standard test procedure consists in rejecting the null hypothesis of a (V)ARMA(p, q) model if the statistics (2) or (3) are larger than a certain quantile of a chi-squared distribution.

Henceforth, we deal with some models with uncorrelated but dependent noise process ϵ , the so-called weak noise. For such models, the asymptotic distributions of the statistics defined in (2) or (3) are no longer chi-square distributions but a mixture of chi-squared distributions, weighted by eigenvalues of the asymptotic covariance matrix of the vector of autocorrelations (see Francq et al. (2005); Boubacar Mainassara (2011)). Consequently, in order to obtain the asymptotic distribution of the portmanteau statistics under weak assumptions on the noise, one needs a consistent estimator of the asymptotic covariance matrix of the residual autocorrelations vector. In the econometric literature the nonparametric kernel estimator, also called heteroscedastic autocorrelation consistent estimator (see Andrews (1991); Newey and West (1987)), is widely used to estimate covariance matrices. However, this causes serious difficulties regarding the choice of the sequence of weights. An alternative method consists in using a parametric autoregressive estimate of the spectral density of a stationary process. This approach, which has been studied in Berk (1974); den Haan and Levin (1997), is also facing the problem of choosing the truncation parameter. Indeed, this method is based on an infinite autoregressive representation of the stationary process. So the choice of the order of truncation is crucial and difficult. The methodology employed in Francq et al. (2005) (an extension is proposed in Boubacar Mainassara (2011) for VARMA models) presents these difficulties: it supposes to weight appropriately some empirical fourth-order moments by means of a window and a truncation point. Recently, Shao generalized in Shao (2011), the test statistic based on the kernel-based spectral proposed by Hong (1996). However, this test is also confronted to the problem of the choice of the bandwidth parameter. Zhu and Li Zhu and Li (2015) also proposed a bootstrapped spectral test for checking the adequacy of weak ARMA models, in which the limiting distribution depends on the unknown data generating process. They used a block-wise random weighting method to bootstrap their critical values which also need the choice of the block size. When the noise process is observable, Lobato, Nankervis and Savin Lobato et al. (2001, 2002) address the problem of testing the null hypothesis that a time series is uncorrelated up to some fixed order and propose an extension of the Box-Pierce statistic.

In this work, we propose an alternative method where we do not estimate an asymptotic covariance matrix. It is based on a self-normalization based approach to construct a new test-statistic which is asymptotically distribution-free under the null hypothesis. The idea comes from Lobato (see Lobato (2001)) and has been already extended by Kuan and Lee (2006); Shao (010a,b, 2012) to more general frameworks. See also Shao (2015) for a review on some recent developments on the inference of time series data using the self-normalized approach.

In our opinion, there are two major contributions in this work. The first one is to show that the Lobato test statistic can be extended to the residuals of weak ARMA and VARMA models. The second one is to improve the results concerning the statistical analysis of weak ARMA and VARMA models by considering the self-normalization approach for the adequacy problem. Notice that the new tests can replace the standard ones when testing strong ARMA and VARMA models.

We briefly give some details on the test statistic that we introduce in this article. We denote by $A \otimes B$ the Kronecker product of two matrices A and B . Our new test-statistics, for

VARMA models, are defined by

$$Q_m^{\text{SN}} = n \hat{\rho}'_m \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e) \right\} \hat{C}_{md^2}^{-1} \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e) \right\} \hat{\rho}_m, \quad (4)$$

where $\hat{C}_{md^2}^{-1}$ is a normalization matrix (that is defined later in (12)) and \hat{S}_e is a diagonal matrix in which the i -th element is the sample estimate of the variance of the i -th coordinate of the multivariate noise process. The vector of the first m sample autocorrelations is denoted by $\hat{\rho}_m$. We prove in Theorem 2 that the asymptotic distribution of Q_m^{SN} is the distribution of a random variable \mathcal{U}_{md^2} depending on md^2 and is independent of all the parameters of the model. It has an explicit expression by means of Brownian bridges but its law is not explicitly known. Nevertheless it can be easily tabulated by Monte-Carlo experiments (see Table 1 in Lobato (2001)). We emphasize the fact that in Francq et al. (2005); Boubacar Mainassara (2011) the authors have proposed some modified versions of the Box-Pierce and Ljung-Box statistics that are more difficult to implement because their critical values have to be computed from the data. In our case, the critical values are not computed from the data since they are tabulated. In some sense, our method is finally closer to the standard method in which the critical values are simply deduced from a χ^2 -table.

In Monte Carlo experiments, we illustrate that the proposed test statistics have reasonable finite sample performance. Under nonindependent errors, it appears that the standard test statistics are generally non reliable, overrejecting or underrejecting severely, while the proposed tests statistics offer satisfactory levels in most cases. Even for independent errors, they seem preferable to the standard ones, when the number m of autocorrelations is small. Moreover, the error of first kind is well controlled. Concerning the relative powers of the proposed tests, we also show that the proposed tests have similar powers than the standard ones when the critical values are adjusted and when the sample size is large. For all these reasons, we think that the modified versions that we propose in this paper are preferable to the standard ones for diagnosing VARMA models under nonindependent errors.

The article is organised as follows. In the next section, we briefly present the models that we consider here and summarize the results on the quasi-maximum likelihood estimator (QMLE) asymptotic distribution obtained by Boubacar Mainassara and Francq (2011). Our methodology and the main results are given in Section 3. Simulation studies and an illustrative application on real data are presented in Section 4 and Section 5. The numerical tables are gathered in Section 6, after the bibliography. The proofs of the main results are available in the extended online version of this paper.

2. Parametrization and assumptions

The structural VARMA(p, q) representation (1) can be rewritten in a reduced VARMA(p, q) form if the matrices A_{00} and B_{00} are non singular. Indeed, premultiplying (1) by A_{00}^{-1} and introducing the innovation process $e = (e_t)_{t \in \mathbb{Z}}$ with $e_t = A_{00}^{-1} B_{00} \epsilon_t$, with non singular variance $\Sigma_{e0} = A_{00}^{-1} B_{00} \Sigma_0 B_{00}' A_{00}^{-1}$, we obtain the reduced VARMA representation

$$X_t - \sum_{i=1}^p A_{00}^{-1} A_{0i} X_{t-i} = e_t - \sum_{i=1}^q A_{00}^{-1} B_{0i} B_{00}^{-1} A_{00} e_{t-i}. \quad (5)$$

The structural form (1) allows to handle seasonal models, instantaneous economic relationships, VARMA in the so-called echelon form representation, and many other constrained

VARMA representations (see [Lütkepohl \(2005\)](#), chap. 12). The reduced form (5) is more practical from a statistical viewpoint as it gives the forecasts of each component of X according to the past values of the set of the components.

Let $[A_{00} \dots A_{0p} B_{00} \dots B_{0q}]$ be the $d \times (p+q+2)d$ matrix of vector autoregressive and moving average coefficients involved in (1). The matrix Σ_0 is considered as a nuisance parameter. The parameter of interest is denoted ϑ_0 , where ϑ_0 belongs to the interior of the parameter space $\Theta \subset \mathbb{R}^{k_0}$, and k_0 is the number of unknown parameters, which is typically much smaller than $(p+q+2)d^2$. Without loss of generality, we assume that Θ is compact. The matrices $A_{00}, \dots, A_{0p}, B_{00}, \dots, B_{0q}$ involved in (1) and Σ_0 are specified by ϑ_0 . More precisely, we write $A_{0i} = A_i(\vartheta_0)$ and $B_{0j} = B_j(\vartheta_0)$ for $i = 0, \dots, p$ and $j = 0, \dots, q$, and $\Sigma_0 = \Sigma(\vartheta_0)$. The VARMA model (1) can be written more compactly as $A_{\vartheta_0}(L)X_t = B_{\vartheta_0}(L)e_t$ where $A_{\vartheta_0}(L) = A_{00} - \sum_{i=1}^p A_{0i}L^i$ and $B_{\vartheta_0}(L) = B_{00} - \sum_{i=1}^q B_{0i}L^i$, where L stands for the backshift operator. Thus for all $\theta \in \Theta$, we have

$$e_t(\vartheta) = B_{\vartheta}^{-1} A_{\vartheta} X_t = \sum_{i=0}^{\infty} c_i(\vartheta) X_{t-i} \quad (6)$$

with $c_0(\vartheta) = 1$ and $e_t(\vartheta_0) = e_t$.

For the estimation of ARMA and VARMA models, the commonly used estimation method is the quasi-maximum likelihood estimation (QMLE for short), which can be also viewed as a nonlinear least squares estimation (LSE). The asymptotic properties of the QMLE of VARMA models are well-known under the restrictive assumption that the noise ϵ is an iid sequence (see [Dufour and Jouini \(2014\)](#); [Lütkepohl \(2005\)](#)). See also [Mélard et al. \(2006\)](#) where a description of estimation of structured VARMA models allowing unit roots is proposed.

Given a realization X_1, X_2, \dots, X_n satisfying (1), the variable $e_t(\vartheta)$ can be approximated, for $0 < t \leq n$, by $\tilde{e}_t(\vartheta)$ defined recursively by

$$\tilde{e}_t(\vartheta) = X_t - \sum_{i=1}^p A_0^{-1} A_i X_{t-i} + \sum_{i=1}^q A_0^{-1} B_i B_0^{-1} A_0 \tilde{e}_{t-i}(\vartheta), \quad (7)$$

where the unknown initial values are set to zero: $\tilde{e}_0(\vartheta) = \dots = \tilde{e}_{1-q}(\vartheta) = X_0 = \dots = X_{1-p} = 0$. The Gaussian quasi-likelihood is given by

$$\tilde{L}_n(\vartheta, \Sigma_e) = \prod_{t=1}^n \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma_e}} \exp \left\{ -\frac{1}{2} \tilde{e}_t'(\vartheta) \Sigma_e^{-1} \tilde{e}_t(\vartheta) \right\}, \quad \Sigma_e = A_0^{-1} B_0 \Sigma B_0' A_0^{-1}'.$$

A QMLE of (ϑ, Σ_e) is a measurable solution $(\hat{\vartheta}_n, \hat{\Sigma}_e)$ of

$$(\hat{\vartheta}_n, \hat{\Sigma}_e) = \arg \min_{\vartheta, \Sigma_e} \left\{ \log(\det \Sigma_e) + \frac{1}{n} \sum_{t=1}^n \tilde{e}_t(\vartheta) \Sigma_e^{-1} \tilde{e}_t'(\vartheta) \right\}.$$

In the univariate weak case, Francq and Zakoïan [Francq and Zakoïan \(1998\)](#) showed the asymptotic normality of the least squares estimator under mixing assumptions. The asymptotic behavior of the QMLE, of structural VARMA models, has been studied in a much wider context by [Boubacar Mainassara and Francq \(2011\)](#) who proved consistency and asymptotic normality under weak noise process. To ensure the consistency and the asymptotic normality of the QMLE, we assume that the parametrization satisfies the following smoothness conditions as in [Boubacar Mainassara and Francq \(2011\)](#).

(A1): The functions $\vartheta \mapsto A_i(\vartheta)$ $i = 0, \dots, p$, $\vartheta \mapsto B_j(\vartheta)$ $j = 0, \dots, q$ and $\vartheta \mapsto \Sigma(\vartheta)$ admit continuous third order derivatives for all $\vartheta \in \Theta$.

For simplicity, we now write A_i , B_j and Σ instead of $A_i(\vartheta)$, $B_j(\vartheta)$ and $\Sigma(\vartheta)$. Let $A_\vartheta(z) = A_0 - \sum_{i=1}^p A_i z^i$ and $B_\vartheta(z) = B_0 - \sum_{i=1}^q B_i z^i$. We denote by $\det(A)$ the determinant of a matrix A and by $\text{vec}(A)$ the vector obtained by stacking the columns of A . We assume that Θ corresponds to stable and invertible representations, namely

(A2): for all $\vartheta \in \Theta$, we have $\det A_\vartheta(z) \det B_\vartheta(z) \neq 0$ for all $|z| \leq 1$.

In the structural VARMA model (1), the assumption **(A2)** does not guarantee the identifiability of the parameter. Thus, we make the following global assumption for all $\vartheta \in \Theta$.

(A3): For all $\vartheta \in \Theta$ such that $\vartheta \neq \vartheta_0$, either the transfer functions $A_0^{-1} B_0 B_\vartheta^{-1}(z) A_\vartheta(z) \neq A_0^{-1} B_0 B_{\vartheta_0}^{-1}(z) A_{\vartheta_0}(z)$ for some $z \in \mathbb{C}$, or $A_0^{-1} B_0 \Sigma B_0' A_0^{-1'} \neq A_0^{-1} B_0 \Sigma_0 B_0' A_0^{-1'}$.

(A4): The process $e = (e_t)_{t \in \mathbb{Z}}$ is ergodic and strictly stationary. Moreover, e_t has a positive density on some neighborhood of zero.

In the reduced VARMA representation (5), the condition $A_0^{-1} B_0 \Sigma B_0' A_0^{-1'} \neq A_0^{-1} B_0 \Sigma_0 B_0' A_0^{-1'}$ in **(A3)** can be dropped but may be important for structural VARMA forms. In particular, **(A3)** is satisfied when we impose that $A_0 = B_0 = I_d$, that **(A2)** holds, that the common left divisors of $A_\vartheta(L)$ and $B_\vartheta(L)$ are unimodular (*i.e.* with nonzero constant determinant), and finally that the matrix $[A_p : B_q]$ is of full rank. In contrast, the echelon form guarantees uniqueness of the VARMA representation and is the most widely identified VARMA representation employed in the literature. The identifiability of VARMA processes has been studied in Hannan (1976) where the author has given several procedures ensuring identifiability.

For the asymptotic normality of the QMLE, additional assumptions are required. It is necessary to assume that ϑ_0 is not on the boundary of the parameter space Θ .

(A5): We have $\vartheta_0 \in \overset{\circ}{\Theta}$, where $\overset{\circ}{\Theta}$ denotes the interior of Θ .

At last, we use the matrix M_{ϑ_0} of the coefficients of the reduced form (5), where

$$M_{\vartheta_0} = [A_{00}^{-1} A_{01} : \dots : A_{00}^{-1} A_{0p} : A_{00}^{-1} B_{01} B_{00}^{-1} A_{00} : \dots : A_{00}^{-1} B_{0q} B_{00}^{-1} A_{00}].$$

We need a local identifiability assumption which completes **(A3)** and specifies how the matrix M_{ϑ_0} depends on the parameter ϑ_0 .

(A6): The matrix \mathbf{M}_{ϑ_0} , defined as $\partial \text{vec}(M_\vartheta) / \partial \vartheta'$ evaluated at ϑ_0 , is of full rank k_0 .

We recall that the strong mixing coefficients $\alpha_e(h)$ of the stationary process e are defined by

$$\alpha_e(h) = \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+h}^{+\infty}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where $\mathcal{F}_{-\infty}^t = \sigma(e_u, u \leq t)$ and $\mathcal{F}_{t+h}^{+\infty} = \sigma(e_u, u \geq t+h)$. We denote $\|\cdot\|$ the Euclidian norm. We will make an integrability assumption on the moment of the noise and a summability condition on the strong mixing coefficients $(\alpha_e(k))_{k \geq 0}$.

(A7): We have $\mathbb{E}\|e_t\|^{4+2\nu} < \infty$ and $\sum_{k=0}^{\infty} \{\alpha_e(k)\}^{\frac{\nu}{2+\nu}} < \infty$ for some $\nu > 0$. (8)

Under the above assumptions, Boubacar Mainassara and Francq [Boubacar Mainassara and Francq \(2011\)](#) showed that $\hat{\vartheta}_n \rightarrow \vartheta_0$ *a.s.* as $n \rightarrow \infty$ and $\sqrt{n}(\hat{\vartheta}_n - \vartheta_0) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, V := J^{-1} I J^{-1})$ where

$$J := J(\vartheta_0, \Sigma_{e0}) = 2\mathbb{E} \left\{ \frac{\partial e'_t(\vartheta_0)}{\partial \vartheta} \right\} \Sigma_{e0}^{-1} \left\{ \frac{\partial e_t(\vartheta_0)}{\partial \vartheta'} \right\} \text{ and}$$

$$I := I(\vartheta_0, \Sigma_{e0}) = \sum_{h=-\infty}^{+\infty} \text{Cov}(\Upsilon_t, \Upsilon_{t-h}) \text{ with}$$

$$\Upsilon_t := \Upsilon_t(\vartheta_0, \Sigma_{e0}) = \frac{\partial}{\partial \vartheta} \left\{ \log \det \Sigma_{e0} + e'_t(\vartheta_0) \Sigma_{e0}^{-1} e_t(\vartheta_0) \right\} = 2 \frac{\partial e'_t(\vartheta_0)}{\partial \vartheta} \Sigma_{e0}^{-1} e_t(\vartheta_0).$$

3. Main results

In order to state our results, we need further notations that will be used both in univariate and multivariate contexts. In all the sequel, we denote by \xrightarrow{d} , respectively by $\xrightarrow{\mathbb{P}}$ the convergence in distribution, respectively in probability. The symbol $o_{\mathbb{P}}(1)$ is used for a sequence of random variables that converges to zero in probability.

Let $(B_K(r))_{r \geq 0}$ be a K -dimensional Brownian motion starting from 0. For $K \geq 1$, we denote \mathcal{U}_K the random variable defined by

$$\mathcal{U}_K = B'_K(1) V_K^{-1} B_K(1) \tag{9}$$

where

$$V_K = \int_0^1 (B_K(r) - r B_K(1)) (B_K(r) - r B_K(1))' dr. \tag{10}$$

The critical values for \mathcal{U}_K have been tabulated by Lobato in [Lobato \(2001\)](#). The random variable \mathcal{U}_K will intervene in the asymptotic behaviour of the laws of the statistics proposed in (4).

3.1. Multivariate ARMA models

When $p > 0$ or $q > 0$, we may define $\hat{e}_t = \tilde{e}_t(\hat{\vartheta}_n)$ for $t \geq 1$, the quasi-maximum likelihood residuals as

$$\hat{e}_t = X_t - \sum_{i=1}^p A_0^{-1}(\hat{\vartheta}_n) A_i(\hat{\vartheta}_n) \hat{X}_{t-i} + \sum_{i=1}^q A_0^{-1}(\hat{\vartheta}_n) B_i(\hat{\vartheta}_n) B_0^{-1}(\hat{\vartheta}_n) A_0(\hat{\vartheta}_n) \hat{e}_{t-i},$$

for $t = 1, \dots, n$, with $\hat{X}_t = 0$ for $t \leq 0$ and $\hat{X}_t = X_t$ for $t \geq 1$. We denote by

$$\Gamma_e(h) = \frac{1}{n} \sum_{t=h+1}^n e_t e'_{t-h} \quad \text{and} \quad \hat{\Gamma}_e(h) = \frac{1}{n} \sum_{t=h+1}^n \hat{e}_t \hat{e}'_{t-h}$$

the multivariate white noise "empirical" autocovariances and residual autocovariances. For a fixed integer $m \geq 1$, let

$$\Gamma_m = (\{\text{vec} \Gamma_e(1)\}', \dots, \{\text{vec} \Gamma_e(m)\}')', \quad \hat{\Gamma}_m = \left(\{\text{vec} \hat{\Gamma}_e(1)\}', \dots, \{\text{vec} \hat{\Gamma}_e(m)\}' \right)'$$

Denoting the diagonal matrices S_e and \hat{S}_e by

$$S_e = \text{Diag}(\sigma_e(1), \dots, \sigma_e(d)) \quad \text{and} \quad \hat{S}_e = \text{Diag}(\hat{\sigma}_e(1), \dots, \hat{\sigma}_e(d)),$$

where $\sigma_e^2(i)$ is the variance of the i -th coordinate of e_t and $\hat{\sigma}_e^2(i)$ is its sample estimate (*i.e.* $\sigma_e^2(i) = \mathbb{E}e_{it}^2$ and $\hat{\sigma}_e^2(i) = n^{-1} \sum_{t=1}^n \hat{e}_{it}^2$). The theoretical and sample autocorrelations at lag ℓ are respectively defined by $R_e(\ell) = S_e^{-1} \Gamma_e(\ell) S_e^{-1}$ and $\hat{R}_e(\ell) = \hat{S}_e^{-1} \hat{\Gamma}_e(\ell) \hat{S}_e^{-1}$, with $\Gamma_e(\ell) := \mathbb{E}e_t e'_{t-\ell} = 0$ for all $\ell \neq 0$. In the sequel, we will also need the vector of the first m sample autocorrelations

$$\hat{\rho}_m = \left(\left\{ \text{vec} \hat{R}_e(1) \right\}', \dots, \left\{ \text{vec} \hat{R}_e(m) \right\}' \right)'.$$

In [Boubacar Mainassara \(2011\)](#), it is proved that

$$\text{vec}(\hat{S}_e^{-1} \hat{\Gamma}_e(h) \hat{S}_e^{-1}) - \text{vec}(S_e^{-1} \hat{\Gamma}_e(h) S_e^{-1}) = o_{\mathbb{P}}(1). \quad (11)$$

Now using the elementary identities $\text{vec}(ABC) = (C' \otimes A) \text{vec}(B)$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ when A and B are invertible, it follows from (11) that

$$\begin{aligned} \hat{\rho}_m &= \left(\left\{ \text{vec} \hat{R}_e(1) \right\}', \dots, \left\{ \text{vec} \hat{R}_e(m) \right\}' \right)' \\ &= \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e)^{-1} \right\} \hat{\Gamma}_m = \left\{ I_m \otimes (S_e \otimes S_e)^{-1} \right\} \hat{\Gamma}_m + o_{\mathbb{P}}(1). \end{aligned}$$

Based on the residual empirical autocorrelations $\hat{R}_e(h)$, the statistics (3) are usually used to test the following null hypothesis

(H0) : $(X_t)_{t \in \mathbb{Z}}$ satisfies a VARMA(p, q) representation;

against the alternative

(H1) : $(X_t)_{t \in \mathbb{Z}}$ does not admit a VARMA(p, q) representation or $(X_t)_{t \in \mathbb{Z}}$ satisfies a VARMA(p', q') representation with $p' > p$ or $q' > q$.

As mentioned before, our strategy relies on new statistics and we need further notations to explain our purpose. We denote Λ the matrix in $\mathbb{R}^{md^2 \times (k_0 + md^2)}$ defined in block formed by

$$\Lambda = (\Phi_m J^{-1} | I_{md^2}) \quad \text{and} \quad \Phi_m = \mathbb{E} \left\{ \begin{pmatrix} e_{t-1} \\ \vdots \\ e_{t-m} \end{pmatrix} \otimes \frac{\partial e_t}{\partial \vartheta'} \right\}.$$

A Taylor expansion around ϑ_0 gives

$$\sqrt{n} \hat{\Gamma}_m = \frac{1}{\sqrt{n}} \sum_{t=1}^n \Lambda U_t + o_{\mathbb{P}}(1), \quad \text{where} \quad U_t = \left(\left(-2 \frac{\partial e_t'}{\partial \vartheta} \Sigma_{e0}^{-1} e_t \right)', (e_{t-1} \otimes e_t)', \dots, (e_{t-m} \otimes e_t)' \right)'.$$

One refers to the extended online version of this paper for further details about the above expressions. At this stage, we do not rely on the classical method that would consist in estimating the asymptotic covariance matrix of ΛU_t . We rather try to apply Lemma 1 in [Lobato \(2001\)](#). So we need to check that a functional central limit theorem holds for the process $U := (U_t)_{t \geq 1}$.

Without the first entry in U_t , we would be in the context of Lobato and the functional central limit theorem would be clear thanks to the mixing condition on the noise process ϵ . Unfortunately, it is more difficult to deal with the process U itself. In order to prove that $\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \Lambda U_j$ converges on the Skorokhod space to a Brownian motion, we will employ a cutoff argument on the representation of $\partial e_t(\vartheta_0)/\partial \vartheta$ as an infinite sum of the past of the noise. This difficulty is overcome in the proof of Theorem 3 (see the extended online version of this paper for further details).

Finally, we define the normalization matrix $C_{md^2} \in \mathbb{R}^{md^2 \times md^2}$ by

$$C_{md^2} = \frac{1}{n^2} \sum_{t=1}^n S_t S_t' \text{ where } S_t = \sum_{j=1}^t (\Lambda U_j - \Gamma_m).$$

The following theorem states the asymptotic distributions of the sample autocovariances and autocorrelations.

Theorem 1. *We assume that $p > 0$ or $q > 0$. Under Assumptions (A0)–(A7) and under the null hypothesis (H0), we have*

$$n \hat{\Gamma}'_m C_{md^2}^{-1} \hat{\Gamma}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_{md^2}.$$

The sample autocorrelations satisfy

$$n \hat{\rho}'_m \{I_m \otimes (S_e \otimes S_e)\} C_{md^2}^{-1} \{I_m \otimes (S_e \otimes S_e)\} \hat{\rho}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_{md^2}$$

The proof of this result is available in the extended online version of this paper.

Of course, the above theorem is useless for practical purpose, because it does not involve any observable quantities. This gap will be fixed below (see Theorem 2). Right now, we make several important comments that are necessary to compare Theorem 1 with the existing results.

Remark 1. *When $p = q = 0$, we no longer need to estimate the unknown parameter ϑ_0 . Thus a careful reading of the proofs shows that the vector U_t is replaced by*

$$\tilde{U}_t = ((e_{t-1} \otimes e_t)', \dots, (e_{t-m} \otimes e_t)')$$

and Λ is replaced by the identity matrix. Then we obtain the result of Lobato (see Lemma 1 in Lobato (2001)) and we thus generalized his result to the VARMA model.

In practice, one has to replace the matrix C_{md^2} and the variance of the noise Σ_{e0} by their empirical or observable counterparts. For C_{md^2} , the idea is to use $\hat{e}_t = \tilde{e}_t(\hat{\vartheta}_n)$ instead of the unobservable noise e_t . The matrices J and Φ_m can be easily estimated by their empirical counterparts

$$\hat{J} = \frac{2}{n} \sum_{t=1}^n \left\{ \frac{\partial \tilde{e}'_t(\hat{\vartheta}_n)}{\partial \vartheta} \hat{\Sigma}_{e0}^{-1} \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta'} \right\} \text{ and } \hat{\Phi}_m = \frac{1}{n} \sum_{t=1}^n \left\{ \begin{pmatrix} \hat{e}_{t-1} \\ \vdots \\ \hat{e}_{t-m} \end{pmatrix} \otimes \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta'} \right\},$$

where $\hat{\Sigma}_{e0} = \hat{\Gamma}_e(0) = n^{-1} \sum_{t=1}^n \hat{e}_t \hat{e}_t'$. Thus we define

$$\hat{\Lambda} = \left(\hat{\Phi}_m \hat{J}_n^{-1} | I_{md^2} \right) \text{ and } \hat{U}_t = \left(\left(-2 \frac{\partial \tilde{e}'_t(\hat{\vartheta}_n)}{\partial \vartheta} \hat{\Sigma}_{e0}^{-1} \tilde{e}_t(\hat{\vartheta}_n) \right)', (\hat{e}_{t-1} \otimes \hat{e}_t)', \dots, (\hat{e}_{t-m} \otimes \hat{e}_t)' \right)'.$$

Finally we denote the normalization matrix $\hat{C}_{md^2} \in \mathbb{R}^{md^2 \times md^2}$ as

$$\hat{C}_{md^2} = \frac{1}{n^2} \sum_{t=1}^n \hat{S}_t \hat{S}_t' \text{ where } \hat{S}_t = \sum_{j=1}^t (\hat{\Lambda} \hat{U}_j - \hat{\Gamma}_m). \quad (12)$$

The above quantities are all observable and we are able to state our second theorem which is the applicable counterpart of Theorem 1.

Theorem 2. *Under the assumptions of Theorem 1, we have*

$$n \hat{\Gamma}_m' \hat{C}_{md^2}^{-1} \hat{\Gamma}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_{md^2}.$$

The sample autocorrelations satisfy

$$Q_m^{SN} := n \hat{\rho}_m' \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e) \right\} \hat{C}_{md^2}^{-1} \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e) \right\} \hat{\rho}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_{md^2}$$

The proof of this result is available in the extended online version.

Based on the above result, we propose a version of Hosking [Hosking \(1980\)](#) statistic when one uses the following one

$$\tilde{Q}_m^{SN} = n^2 \hat{\rho}_m' \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e) \right\} D_{n,md^2} \hat{C}_{md^2}^{-1} \left\{ I_m \otimes (\hat{S}_e \otimes \hat{S}_e) \right\} \hat{\rho}_m, \quad (13)$$

where the matrix $D_{n,md^2} \in \mathbb{R}^{md^2 \times md^2}$ is diagonal with $(n/(n-1), \dots, n/(n-m))$ as diagonal terms.

3.2. Examples

In order to make our presentation more readable, the results of the previous section are presented in the one dimensional case (*i.e* when $d = 1$) which requires less technical notation. The first example is on an univariate weak ARMA(p, q) model. In the second example, we give an explicit expression of the normalized matrix C_m for one dimensional AR(1) and MA(1) models.

3.2.1. Univariate ARMA models

When $d = 1$, in the representation (5) we have $\mathbb{E}(\epsilon_t^2) = \sigma_{e0}^2$ and $a_{00} = b_{00} = 1$. So (5) takes the following form

$$X_t - \sum_{i=1}^p a_{0i} X_{t-i} = \epsilon_t + \sum_{j=1}^q a_{0j} \epsilon_{t-j}. \quad (14)$$

Then with our notations, the unknown parameter $\vartheta_0 = (a_{01}, \dots, a_{0p}, b_{01}, \dots, b_{0q})'$ is supposed to belong to the interior of the parameter space

$$\Theta := \left\{ \theta = (\vartheta_1, \dots, \vartheta_p, \vartheta_{p+1}, \dots, \vartheta_{p+q})' \in \mathbb{R}^{p+q}, A_\theta(z) = 1 - \sum_{i=1}^p \vartheta_i z^i \text{ and } B_\theta(z) = 1 + \sum_{i=p+1}^{p+q} \vartheta_i z^i \right. \\ \left. \text{have all their zeros outside the unit disk and have also no common zero} \right\}.$$

The theoretical, respectively the sample autocorrelations, at lag ℓ take the simpler forms $R_e(\ell) = \Gamma_e(\ell)/\Gamma_e(0)$, respectively $\hat{R}_e(\ell) = \hat{\Gamma}_e(\ell)/\hat{\Gamma}_e(0)$, with $\Gamma_e(0) := \sigma_{e0}^2$. The vector of the first m sample autocorrelations is now written as $\hat{\rho}_m = (\hat{R}_e(1), \dots, \hat{R}_e(m))'$. The statistics

defined in (2) are based on the residual empirical autocorrelations $\hat{R}_e(h)$, and are used to test the null hypothesis stated above. As mentioned before, we obtain the block matrix $\Lambda \in \mathbb{R}^{m \times (p+q+m)}$ formed by

$$\Lambda = \left(\Phi_m J^{-1} | I_m \right), \text{ where } \Phi_m = \mathbb{E} \left\{ \begin{pmatrix} e_{t-1} \\ \vdots \\ e_{t-m} \end{pmatrix} \frac{\partial e_t(\vartheta_0)}{\partial \vartheta'} \right\} \text{ and } J = \frac{2}{\sigma_{e0}^2} \mathbb{E} \left[\frac{\partial e_t(\vartheta_0)}{\partial \vartheta} \frac{\partial e_t(\vartheta_0)}{\partial \vartheta'} \right]. \quad (15)$$

Then one may write

$$\sqrt{n} \hat{\Gamma}_m = \frac{1}{\sqrt{n}} \sum_{t=1}^n \Lambda U_t + o_{\mathbb{P}}(1), \text{ with } U_t = \left(-2 \frac{\partial e_t(\vartheta_0)}{\partial \vartheta} \frac{1}{\sigma_{e0}^2} e_t, e_t e_{t-1}, \dots, e_t e_{t-m} \right)'.$$

Finally, the normalization matrix $C_m \in \mathbb{R}^{m \times m}$ is now defined by

$$C_m = \frac{1}{n^2} \sum_{t=1}^n S_t S_t' \text{ where } S_t = \sum_{j=1}^t (\Lambda U_j - \Gamma_m).$$

We are now able to state the following theorem, which is clearly the univariate version of Theorem 1.

Theorem 3. *Assume that $p > 0$ or $q > 0$. Under the assumptions of Theorem 1, we have*

$$n \hat{\Gamma}_m' C_m^{-1} \hat{\Gamma}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_m. \quad (16)$$

The sample autocorrelations satisfy

$$n \sigma_{e0}^4 \hat{\rho}_m' C_m^{-1} \hat{\rho}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_m. \quad (17)$$

As mentioned in the VARMA case, Theorem 3 has also to be completed. The matrices J , Φ_m and the scalar σ_{e0}^2 can be easily estimated by their empirical counterparts:

$$\hat{J}_n = \frac{1}{\hat{\sigma}_{e0}^2} \frac{2}{n} \sum_{t=1}^n \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta} \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta'}, \hat{\Phi}_m = \frac{1}{n} \sum_{t=1}^n \left\{ (\hat{e}_{t-1}, \dots, \hat{e}_{t-m})' \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta'} \right\} \text{ and } \hat{\sigma}_{e0}^2 = \frac{1}{n} \sum_{t=1}^n \tilde{e}_t^2(\hat{\vartheta}_n).$$

Thus we define

$$\hat{\Lambda} = \left(\hat{\Phi}_m \hat{J}_n^{-1} | I_m \right) \text{ and } \hat{U}_t = \left(-2 \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta} \frac{1}{\hat{\sigma}_{e0}^2} \hat{e}_t, \hat{e}_t \hat{e}_{t-1}, \dots, \hat{e}_t \hat{e}_{t-m} \right)'.$$

Finally we denote the normalization matrix $\hat{C}_m \in \mathbb{R}^{m \times m}$ by

$$\hat{C}_m = \frac{1}{n^2} \sum_{t=1}^n \hat{S}_t \hat{S}_t' \text{ where } \hat{S}_t = \sum_{j=1}^t \left(\hat{\Lambda} \hat{U}_j - \hat{\Gamma}_m \right).$$

The above quantities are all observable and the following result is the applicable counterpart of Theorem 3.

Theorem 4. Assume that $p > 0$ or $q > 0$. Under Assumptions of Theorem 3, we have

$$n \hat{\Gamma}'_m \hat{C}_m^{-1} \hat{\Gamma}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_m.$$

The sample autocorrelations satisfy

$$Q_m^{SN} = n \hat{\sigma}_{e0}^4 \hat{\rho}'_m \hat{C}_m^{-1} \hat{\rho}_m \xrightarrow[n \rightarrow \infty]{d} \mathcal{U}_m.$$

Based on the above result, we propose a modified version of the Ljung-Box statistic when one uses the statistic

$$\tilde{Q}_m^{SN} = n \hat{\sigma}_{e0}^4 \hat{\rho}'_m D_{n,m}^{1/2} \hat{C}_m^{-1} D_{n,m}^{1/2} \hat{\rho}_m$$

where the matrix $D_{n,m} \in \mathbb{R}^{m \times m}$ is diagonal with $((n+2)/(n-1), \dots, (n+2)/(n-m))$ as diagonal terms.

3.2.2. Explicit form of the matrix C_m in AR(1) or MA(1) case

The following example gives an explicit form of the matrix C_m in the univariate AR(1) or MA(1) model.

For instance, consider the AR(1) case, with $a_{00} = b_{00} = 1$, $a_0 = a_{01}$ and $\sigma^2 = \sigma_{e0}^2$. It is classical that the univariate noise derivatives can be represented as $\partial \epsilon_t / \partial \vartheta = \sum_{i=1}^{\infty} \lambda_i \epsilon_{t-i}$ where $\lambda_i = -a_0^{i-1}$. Then we deduce that $J = 2/(1-a_0^2)$ and the expression $\Phi_m = -\sigma^2(1 \ a_0 \ \dots \ a_0^{m-1})'$. Since $U_t = (-2\sigma^{-2} \sum_{i=1}^{\infty} \lambda_i \epsilon_t \epsilon_{t-i} \ \epsilon_t \epsilon_{t-1} \ \dots \ \epsilon_t \epsilon_{t-m})'$ and $\Lambda = (\Phi_m J^{-1} | I_m)$, we obtain

$$\begin{aligned} \Lambda U_t &= (1 - a_0^2) \sum_{i=1}^{\infty} \lambda_i \epsilon_t \epsilon_{t-i} (1 \ a_0 \ \dots \ a_0^{m-1})' + (\epsilon_t \epsilon_{t-1} \ \dots \ \epsilon_t \epsilon_{t-m})' \quad \text{and} \\ S_t &= \sum_{j=1}^t \left[(1 - a_0^2) \sum_{i=1}^{\infty} \lambda_i \epsilon_j \epsilon_{j-i} (1 \ a_0 \ \dots \ a_0^{m-1})' + (\epsilon_j \epsilon_{j-1} \ \dots \ \epsilon_j \epsilon_{j-m})' - (\Gamma_e(1) \ \dots \ \Gamma_e(m))' \right]. \end{aligned}$$

For simplicity, we take $m = 1$ and we obtain the following simple expression for the normalization matrix

$$C_1 = \frac{1}{n^2} \sum_{t=1}^n S_t^2 = \frac{1}{n^2} \sum_{t=1}^n \left\{ \sum_{j=1}^t \left(-(1 - a_0^2) \sum_{i=1}^{\infty} a_0^{i-1} \epsilon_j \epsilon_{j-i} + \epsilon_j \epsilon_{j-1} - \Gamma_e(1) \right) \right\}^2.$$

When $p = 0$ and $q = 1$ the same result holds with a_0 replaced by $b_0 = b_{01}$.

4. Numerical illustrations

In this section, by means of Monte Carlo experiments, we investigate the finite sample properties of the modified version of the portmanteau tests that we introduced in this work. The numerical illustrations of this section are made with the open source statistical software R (see R Development Core Team, 2015) or (see <http://cran.r-project.org/>). The tables are gathered in Section 6.

4.1. Simulated models

First of all, we introduce the models that we simulate. For illustrative purpose, we also consider the standard portmanteau test and the modified portmanteau test proposed by [Francq et al. \(2005\)](#) (resp. by [Boubacar Mainassara \(2011\)](#)) in the univariate ARMA case (resp. in VARMA case).

We indicate the conventions that we adopt in the discussion and in the tables:

- LB_{FRZ} and BP_{FRZ} refer to LB and BP tests using Q_m^{LB} and Q_m^{BP} as in [Francq et al. \(2005\)](#)
- LB_s and BP_s refer to LB and BP tests using the statistics (2) and (3).
- LB_{SN} and BP_{SN} refer to modified test using the statistics (4) and (13)
- LB_{BM} and BP_{BM} refer to LB and BP tests with Q_m^H and Q_m^C as in [Boubacar Mainassara \(2011\)](#)

Compared to the modified portmanteau test proposed by [Francq et al. \(2005\)](#) (resp. by [Boubacar Mainassara \(2011\)](#)), we use a vector autoregressive (VAR) spectral estimator approach to estimate the asymptotic covariance matrix of a vector autocorrelations residuals. The implementation of this method requires a choice of the VAR order r . In the strong (V)ARMA cases we fixed $r = 1$. By contrast, in the weak (V)ARMA cases the VAR order r is set as $r = 1, \dots, 5$ and is automatically selected by Akaike Information Criterion (AIC) using the function `VARselect()` of the `vars` R package.

The p -values of the modified portmanteau tests, introduced by [Francq et al. \(2005\)](#); [Boubacar Mainassara \(2011\)](#), are computed using the Imhof algorithm (see [Imhof \(1961\)](#)) and by using the function `imhof()` of the R package `CompQuadForm`.

We notice that in the tables, the numerical results using the Ljung-Box tests are very close to those of the Box-Pierce tests. Nevertheless, they are still presented here for the sake of completeness. Other tests could have been considered as well but our aim is not to make an exhaustive comparative study.

4.1.1. Univariate ARMA case

To generate the strong and the weak ARMA models, we consider the following ARMA(1, 1) model

$$X_t = a_0 X_{t-1} + \epsilon_t + b_0 \epsilon_{t-1}, \quad (18)$$

with $\vartheta_0 = (a_0, b_0)' = (0.95, -0.6)'$ and the innovation process ϵ follows a strong or weak white noise. Different weak ARMA(1, 1) models are simulated with various examples of weak white noises.

The generalized autoregressive conditional heteroscedastic (GARCH) models is an important example of weak white noises in the univariate case (see [Francq and Zakoïan \(2010\)](#)). So we first assume that in (18) the innovation process ϵ is the following GARCH(1, 1) model defined by

$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t^2 = 1 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \end{cases} \quad (19)$$

where $(\eta_t)_{t \geq 1}$ is a sequence of iid standard Gaussian random variables.

We propose three other sets of experiments with innovation processes ϵ in (18) defined by

$$\epsilon_t = \eta_t \eta_{t-1}, \quad (20)$$

$$\epsilon_t = \eta_t^2 \eta_{t-1}, \quad (21)$$

$$\epsilon_t = \eta_t (|\eta_{t-1}| + 1)^{-1}. \quad (22)$$

The example (20) was proposed in [Romano and Thombs \(1996\)](#) and (21) and (22) are extensions of other types of noise process in [Romano and Thombs \(1996\)](#). To generate the strong ARMA, we assume that in (18) the innovation process follows (19) with $(\alpha_1, \beta_1) = (0, 0)$. Contrary to (20) and (22), the noise defined in (21) is not a martingale difference sequence for which the limit theory is more classical.

4.1.2. Multivariate ARMA case

This section is a direct extension of the univariate section 4.1.1. Now we repeat the same experiment on different weak and strong VARMA(1,1) models. In order to ensure the uniqueness of a VARMA representation, we consider the following bivariate VARMA(1,1) model considered in [Reinsel \(1997\)](#) (chap. 3, example 3.2 p. 81)

$$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} - \begin{pmatrix} b_{11,1} & b_{12,1} \\ b_{21,1} & b_{22,1} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \end{pmatrix} \quad (23)$$

with $\vartheta_0 = (a_{11,1}, a_{21,1}, a_{12,1}, a_{22,1}, b_{11,1}, b_{21,1}, b_{12,1}, b_{22,1})' = (1.2, 0.6, -0.5, 0.3, -0.6, 0.3, 0.3, 0.6)'$ and $\epsilon_t = (\epsilon_{1,t}, \epsilon_{2,t})'$ that follows a strong or weak white noise. Note that

$$\det \left\{ I_2 - z_1 \begin{pmatrix} 1.2 & -0.5 \\ 0.6 & 0.3 \end{pmatrix} \right\} = 0 \text{ and } \det \left\{ I_2 - z_2 \begin{pmatrix} -0.6 & 0.3 \\ 0.3 & 0.6 \end{pmatrix} \right\} = 0$$

for $z_1 = 1.136 \pm 0.473i$ (and hence $|z_1| = 1.23$) and for $z_2 = \pm(0.45)^{-1/2}$ (and hence $|z_2| = 1.49$). Hence Model (23) can be viewed as a bivariate VARMA(1,1) model in echelon form (ARMA_E(1,1)) considered in [Lütkepohl \(2005\)](#) (chap. 12, definition 12.2 p. 453).

Let $\eta = ((\eta_{1,t}, \eta_{2,t})')_{t \geq 1}$ be an iid sequence of random variables such that

$$\begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \end{pmatrix} \stackrel{\text{law}}{=} \mathcal{N}(0, I_2).$$

We first consider the strong VARMA case by assuming that the innovation process $\epsilon = ((\epsilon_{1,t}, \epsilon_{2,t})')_{t \geq 1}$ in (23) is defined by an iid sequence such that

$$\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} \stackrel{\text{law}}{=} \mathcal{N}(0, I_2). \quad (24)$$

The GARCH models have numerous extensions to the multivariate framework (see [Bauwens et al. \(2006\)](#) for a review). [Jeantheau](#) (see [Jeantheau \(1998\)](#)) has proposed a simple extension of the multivariate GARCH(p, q) with conditional constant correlation. For simplicity we consider the following bivariate ARCH(1) model proposed in [Jeantheau \(1998\)](#) and defined by

$$\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} = \begin{pmatrix} h_{11,t} & 0 \\ 0 & h_{22,t} \end{pmatrix} \begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \end{pmatrix} \quad (25)$$

where

$$\begin{pmatrix} h_{11,t}^2 \\ h_{22,t}^2 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.2 \end{pmatrix} + \begin{pmatrix} 0.45 & 0.00 \\ 0.40 & 0.25 \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-1}^2 \\ \epsilon_{2,t-1}^2 \end{pmatrix}.$$

First we assume that in (23) the innovation process ϵ is an ARCH(1) model defined in (25). In three other sets of experiments, we assume that in (23) the noise process ϵ is defined by

$$\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} = \begin{pmatrix} \eta_{1,t}\eta_{2,t-1}\eta_{1,t-2} \\ \eta_{2,t}\eta_{1,t-1}\eta_{2,t-2} \end{pmatrix}, \quad (26)$$

$$\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} = \begin{pmatrix} \eta_{1,t}^2\eta_{2,t-1}\eta_{1,t-2} \\ \eta_{2,t}^2\eta_{1,t-1}\eta_{2,t-2} \end{pmatrix}, \quad (27)$$

$$\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} = \begin{pmatrix} \eta_{1,t}(|\eta_{1,t-1}| + 1)^{-1} \\ \eta_{2,t}(|\eta_{2,t-1}| + 1)^{-1} \end{pmatrix}. \quad (28)$$

These noises are direct extensions of the weak noises defined in Romano and Thombs (1996) in the univariate case.

4.2. Empirical size

We first simulate $N = 1,000$ independent trajectories of size $n = 10,000$ of models (18) and (23) (the same series of random numbers is used to generate the noises for the different cases). The same series is partitioned as three series of sizes $n = 500$, $n = 2,000$ and $n = 10,000$. For each of these N replications, we use the quasi-maximum likelihood estimation method to estimate the coefficient ϑ_0 and we apply portmanteau tests to the residuals for different values of m , where m is the number of autocorrelations used in the portmanteau test statistic. For instance, $m \in \{1, \dots, 5\}$ in the VARMA case and $m \in \{1, 2, 3, 6, 12\}$ in the univariate ARMA case. The nominal asymptotic level of the tests is $\alpha = 5\%$.

4.2.1. Strong ARMA and VARMA models case

We consider the strong ARMA model (18)–(19), with $(\alpha_1, \beta_1) = (0, 0)$ and of strong VARMA model (23)–(24).

For the standard Box-Pierce test, the model is therefore rejected when the statistic Q_m^{BP} or Q_m^{LB} (resp. Q_m^{C} or Q_m^{H}) is larger than $\chi_{(m-2)}^2(0.95)$ (resp. than $\chi_{(4m-8)}^2(0.95)$) in a univariate ARMA(1, 1) case (resp. in a VARMA(1, 1) case). We know that the asymptotic level of this test is indeed $\alpha = 5\%$ when $\vartheta_0 = (0, 0)'$ (resp. $\vartheta_0 = 0 \in \mathbb{R}^8$) in a univariate ARMA(1, 1) case (resp. in a bivariate VARMA(1, 1) case). Note however that, even when the noise is strong, the asymptotic level is not exactly $\alpha = 5\%$ when $\vartheta_0 \neq (0, 0)'$ (resp. $\vartheta_0 \neq 0 \in \mathbb{R}^8$) in a univariate ARMA case (resp. in a bivariate VARMA case).

For the proposed modified test BP_{SN} or LB_{SN} , the model is rejected when the statistic Q_m^{SN} or \tilde{Q}_m^{SN} is larger than $\mathcal{U}_m(0.95)$ in the univariate ARMA case and than $\mathcal{U}_{4m}(0.95)$ in the bivariate VARMA case, where the critical values $\mathcal{U}_K(0.95)$ (for $K = 1, \dots, 20$) are tabulated in Lobato (see Table 1 in Lobato (2001)).

Table 1 (resp. Table 5) displays the relative rejection frequencies of the null hypothesis H_0 that the data generating process follows an ARMA(1, 1) (resp. a bivariate VARMA(1, 1)), over the N independent replications. When one uses the statistic Q_m^{BP} or Q_m^{LB} , the observed relative rejection frequency of the standard Box-Pierce or Ljung-Box test is very far from the nominal level $\alpha = 5\%$ when the number m of autocorrelations used in the statistic is small.

This observation is in accordance with the results in the literature on the standard (V)ARMA models. The theory that the $\chi^2_{(m-2)}$ (resp. the $\chi^2_{(4m-8)}(0.95)$) approximation is better for larger m is confirmed. In contrast, our modified test better controls the error of first kind in the univariate and multivariate ARMA model, even when m is small. Note that the tests based on LB_{BM} , BP_{BM} , BP_{FRZ} or LB_{FRZ} control well the error of first kind in the univariate ARMA case (see Model I of Table 1) and also in the bivariate VARMA case (see Model I of Table 5). Note that for $m \leq 2$, the empirical size is not available (n.a.) for the standard Box-Pierce or Ljung-Box tests because they are not applicable when $m \leq 2$. For $n = 200$ (the results are not reported here but are presented in the on-line extended version) our test still accurately controls the type I error when $m \leq 5$ in the multivariate case and for $m \leq 12$ in the univariate case.

From these examples we draw the conclusion that the proposed modified version are preferable to the standard ones in these univariate and multivariate strong ARMA models.

4.2.2. Weak ARMA and VARMA models case

We repeat the same experiments on weak (V)ARMA models (see the different models that are proposed in the tables). We first consider the univariate weak ARMA(1,1) models. As expected, Tables 1 and 2 show that the standard LB_s or BP_s test poorly performs in assessing the adequacy of these weak ARMA models (see Models II, III, IV and V in the tables). In view of the observed relative rejection frequencies, the standard test rejects very often the true ARMA(1,1) and all the relative rejection frequencies are very far from the nominal $\alpha = 5\%$. Our modified test and the tests based on BP_{FRZ} or LB_{FRZ} control well the error of first kind for these weak ARMA models except for Model III when $n = 500$ and $m = 12$ (see Table 2).

When $n = 200$ (the results are not reported here, see the on-line extended version for precisions) our test still accurately controls the type I error when $m \leq 12$ for Model V, but the results are less satisfactory for the other cases.

Now, we consider the multivariate weak ARMA(1,1) models. As expected, Tables 5 and 6 show that the standard tests LB_s and BP_s poorly perform in assessing the adequacy of all these weak VARMA models. Table 6 shows that the error of first kind is well controlled by all the tests LB_{SN} , BP_{SN} , BP_{BM} and LB_{BM} in the particular case of Model V. For the models II, III and IV (see Table 5 and 6) the error of first kind is also well controlled by our modified test when $n = 10,000$. When $n \leq 2,000$ the results are less satisfactory except for Model II when $n = 2,000$. When $n = 500$, the results are acceptable for $m = 1$. This is not surprising since the number of parameters in the model is eight and since the dependence structure is complex.

For $n = 200$ (the results are not reported here) our test still accurately controls the type I error for Model V, but the results are not satisfactory in the other weak VARMA cases.

As a conclusion, the tests based on Q_m^{SN} and \tilde{Q}_m^{SN} can be used safely for small and large m for both ARMA and VARMA models.

4.3. Empirical power

In this part we present the results on the univariate ARMA models. The results on the VARMA models are not presented here because the numerical results and their interpretations are very similar to those we will now discuss in the univariate case. Nevertheless, some tables are proposed in the extended version of this work.

In this section we repeat the same experiments as in Section 4.1.1 to examine the power of the tests for the null hypothesis of an ARMA(1, 1) against the following ARMA(2, 1) alternative defined by

$$X_t = X_{t-1} - 0.2X_{t-2} + \epsilon_t + 0.8\epsilon_{t-1}, \quad (29)$$

where the innovation process ϵ follows a strong or weak white noise introduced in Section 4.1.1.

For each of these N replications we fit an ARMA(1, 1) model and perform standard and modified tests based on $m = 1, 2, 3, 6$ and 12 residual autocorrelations.

For these particular strong and weak ARMA models, we have seen that the actual level of the standard version is generally very different from the 5% nominal level (see Tables 1 and 2). In order to compare the powers of the three tests on an equal basis, we use (as in Hong (1996)) the empirical critical values based on $N = 1000$ replications. This power is usually referred as size adjusted power. Tables 3 and 4 display the relative rejection frequencies of over the N independent replications for the univariate ARMA models.

Thanks to these examples, we notice that the standard and modified versions tests have very similar powers when $n \geq 2,000$. In contrast, when $n = 500$ our tests are clearly less powerful than the standard ones. But we repeat that our tests are usually closer from 5% nominal level.

Note that, the empirical critical values strongly depend on the type of weak ARMA which is generated under the null hypothesis. Therefore, this method consisting in adjusting the critical values only works for very specific hypotheses. We also think that this method may be distorted by a too small number of replication that yield to an inaccurate estimation of the empirical critical value. However, in terms of a day-to-day application, we usually wouldn't do this. In that case, what's of interest is the "raw power" of the test, that is to say the rate at which it rejects false null hypotheses when the asymptotic ("wrong") critical value is used. The study of the "raw-power" is presented in the extended version of our work.

5. Illustrative example

We now consider an application to the daily log returns (also simply called the returns) of the CAC40¹. The observations cover the period from March 1, 1990 to July 26, 2010. The length of the series is $n = 5154$. The data can be downloaded from the website Yahoo Finance: <http://fr.finance.yahoo.com/>. Figure 1 plots the closing prices and returns of the CAC40 index from March 1, 1990 to July 26, 2010. It shows that the CAC index series are generally close to a random walk without intercept and that the returns are generally compatible with the second-order stationarity assumption. In financial econometrics, the returns are often assumed to be martingale increments (though they are not generally independent sequences). Moreover it is commonly accepted that the squares of the returns have second-order moments close to those of an ARMA(1,1) (which is compatible with a GARCH(1,1) model for the returns). We will test these hypotheses by fitting weak ARMA models on the returns and on their squares.

First, we apply portmanteau tests for checking the hypothesis that the CAC40 returns constitute a white noise. Table 7 displays the statistics of the standard and modified tests.

¹The CAC 40 index is a linear combination of a selection of 40 shares on the Paris Stock Exchange (CAC stands for «Cotations Assistées en Continu»).

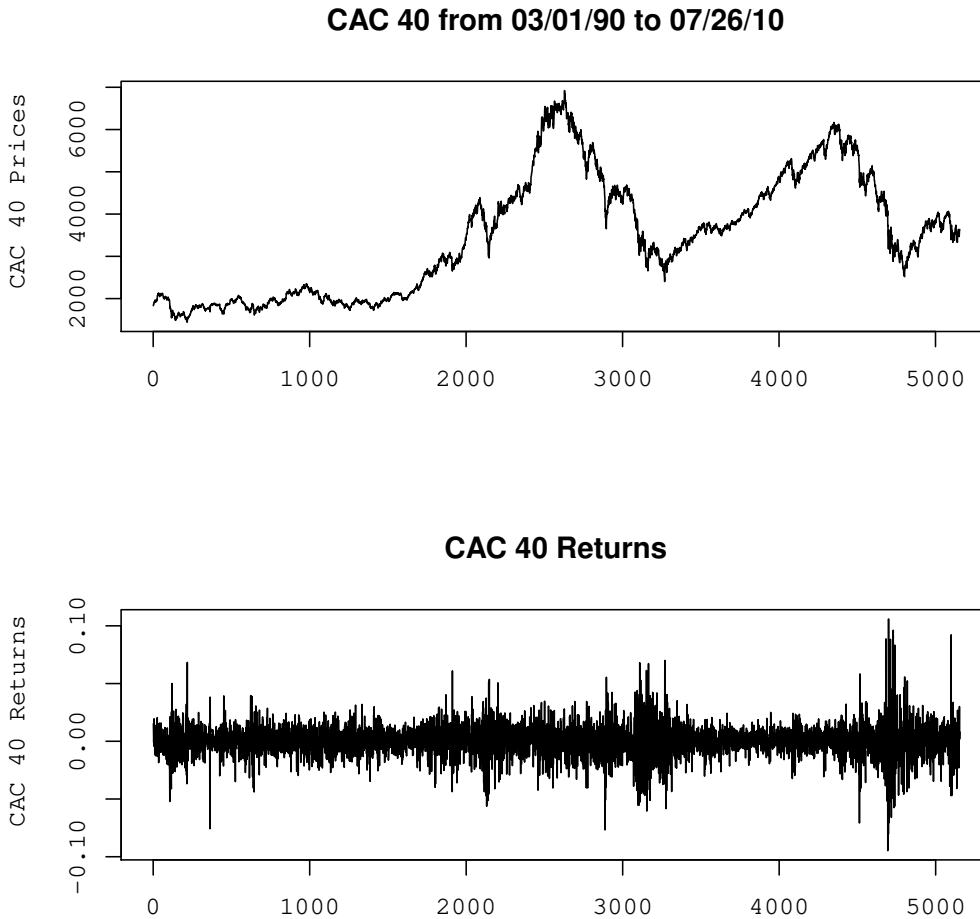


Figure 1: Closing prices and returns of the CAC40 index from March 1, 1990 to July 26, 2010 (5154 observations).

Since the p -values of the standard test are very small, the white noise hypothesis is rejected at the nominal level $\alpha = 1\%$. This is not surprising because the standard tests requires the iid assumption and it is well known that the strong white noise model is not adequate for these series (one can think about the so-called volatility clustering phenomena). In contrast, the white noise hypothesis is not rejected by the modified tests since the statistic is not larger than the critical values (see Table 1 in [Lobato \(2001\)](#)). To summarize, the outputs of Table 7 are in accordance with the common belief that these series are not strong white noises, but could be weak white noises. This is also in accordance with other works devoted to the analysis of stock-market returns [Lobato et al. \(2001\)](#).

Then we focus on the dynamics of the squared returns and we fit an ARMA(1,1) model to the squares of the CAC40 returns. Denoting by $(X_t)_{t \geq 1}$ the mean corrected series of the squared returns, we obtain the following model

$$X_t = 0.97942X_{t-1} + \epsilon_t - 0.89094\epsilon_{t-1}, \text{ where } \text{Var}(\epsilon_t) = 23.5302 \times 10^{-8}.$$

Table 8 displays the statistics of the standard and modified LB and BP tests. From Table 8 we draw the same conclusion on the squares of the previous daily returns: the strong ARMA(1,1) model is rejected, but a weak ARMA(1,1) model is not rejected. Note that the first and second-order structures that we found for the CAC40 returns, namely a weak white noise for the returns and a weak ARMA(1,1) model for the squares of the returns, are compatible with a GARCH(1,1) model.

We emphasize the fact that the assumption of second-order stationarity can be considered for this series. Indeed the estimated autoregressive coefficient is $0.97942 < 1$ and the estimated standard deviation is 0.00569. The procedure described in Proposition 8.1 p. 186 in [Francq and Zakoïan \(2010\)](#) yields a p -value equals to 0,00799 for the test of the second-order stationarity assumption.

We mention that another illustrative example with the Standard & Poor's 500 index is proposed in the extended version of this work.

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6. Tables

Table 1: Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of ARMA(1, 1). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{FRZ}	BP _{FRZ}	LB _S	BP _S
I	$n = 500$	1	4.6	4.5	5.3	5.3	n.a.	n.a.
		2	3.9	3.9	5.4	5.4	n.a.	n.a.
		3	3.6	3.6	4.6	4.4	14.5	14.3
		6	3.9	3.7	4.7	4.6	7.2	7.2
		12	3.4	3.2	4.7	4.1	5.6	5.2
I	$n = 2,000$	1	4.6	4.6	5.1	5.1	n.a.	n.a.
		2	5.1	5.1	5.7	5.7	n.a.	n.a.
		3	4.0	3.9	4.8	4.8	13.8	13.8
		6	3.5	3.5	5.0	4.9	7.2	7.1
		12	5.1	5.0	5.3	5.3	5.9	5.8
I	$n = 10,000$	1	7.0	7.0	5.6	5.6	n.a.	n.a.
		2	5.8	5.8	6.9	6.9	n.a.	n.a.
		3	5.4	5.4	5.9	5.9	16.0	16.0
		6	5.9	5.9	5.6	5.6	8.6	8.6
		12	4.6	4.4	6.2	6.2	7.3	7.3
II	$n = 500$	1	3.2	3.1	4.7	4.7	n.a.	n.a.
		2	3.6	3.6	5.1	5.1	n.a.	n.a.
		3	3.0	3.0	3.6	3.5	25.6	25.3
		6	1.8	1.8	2.4	2.2	17.0	16.5
		12	4.8	4.7	6.4	6.0	17.1	16.0
II	$n = 2,000$	1	4.7	4.6	4.2	4.1	n.a.	n.a.
		2	4.7	4.7	5.0	5.0	n.a.	n.a.
		3	4.4	4.4	4.3	4.3	26.8	26.6
		6	3.2	3.2	4.2	4.1	19.1	19.0
		12	7.3	7.3	4.6	4.6	19.9	19.7
II	$n = 10,000$	1	5.8	5.8	4.1	4.1	n.a.	n.a.
		2	4.6	4.6	5.0	5.0	n.a.	n.a.
		3	5.5	5.5	5.1	5.1	25.0	25.0
		6	4.1	4.1	4.8	4.7	19.5	19.5
		12	6.9	6.9	3.7	3.7	22.5	22.5
III	$n = 500$	1	2.9	2.9	3.9	3.7	n.a.	n.a.
		2	3.6	3.6	3.5	3.4	n.a.	n.a.
		3	3.2	3.2	3.4	3.4	20.9	20.7
		6	2.7	2.7	2.3	2.2	10.8	9.8
		12	1.1	1.1	2.5	1.9	7.4	6.9
III	$n = 2,000$	1	4.9	4.8	4.2	4.2	n.a.	n.a.
		2	4.5	4.5	4.5	4.5	n.a.	n.a.
		3	5.5	5.5	4.3	4.3	21.3	21.0
		6	4.7	4.7	3.8	3.8	11.4	11.3
		12	4.4	4.4	3.2	3.0	8.8	8.7
III	$n = 10,000$	1	4.5	4.5	5.1	5.1	n.a.	n.a.
		2	4.7	4.7	4.9	4.9	n.a.	n.a.
		3	4.4	4.4	4.8	4.8	22.3	22.3
		6	5.0	5.0	5.4	5.4	11.9	11.9

Table 2: Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of ARMA(1, 1). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{FRZ}	BP _{FRZ}	LB _S	BP _S
IV	$n = 500$	1	3.3	3.1	3.2	3.2	n.a.	n.a.
		2	3.5	3.5	4.5	4.2	n.a.	n.a.
		3	3.0	2.7	2.9	2.9	23.5	23.4
		6	0.9	0.8	1.7	1.4	13.5	13.3
		12	1.1	1.0	2.7	2.7	8.8	8.7
IV	$n = 2,000$	1	5.0	5.0	5.7	5.7	n.a.	n.a.
		2	5.2	5.2	5.6	5.6	n.a.	n.a.
		3	4.1	4.1	3.8	3.8	28.1	28.0
		6	3.9	3.9	2.5	2.5	14.1	14.1
		12	1.9	1.8	2.1	2.1	10.5	10.1
IV	$n = 10,000$	1	6.1	6.1	5.5	5.5	n.a.	n.a.
		2	5.9	5.9	5.3	5.3	n.a.	n.a.
		3	5.6	5.6	5.0	5.0	29.2	29.2
		6	6.3	6.3	4.2	4.2	16.8	16.8
		12	3.5	3.5	3.8	3.8	13.4	13.4
V	$n = 500$	1	4.4	4.4	5.4	5.5	n.a.	n.a.
		2	6.3	6.2	5.8	5.8	n.a.	n.a.
		3	5.5	5.1	5.6	5.5	11.6	11.6
		6	4.6	4.5	5.2	5.1	7.6	7.2
		12	4.8	4.6	5.1	4.6	6.1	5.7
V	$n = 2,000$	1	5.6	5.6	4.6	4.6	n.a.	n.a.
		2	5.2	5.2	5.1	5.1	n.a.	n.a.
		3	4.3	4.3	4.8	4.7	12.6	12.6
		6	4.3	4.2	5.2	5.1	7.0	7.0
		12	4.2	4.1	4.5	4.5	5.0	5.0
V	$n = 10,000$	1	6.0	6.0	5.4	5.4	n.a.	n.a.
		2	6.4	6.4	6.1	6.1	n.a.	n.a.
		3	6.2	6.2	5.6	5.6	13.7	13.7
		6	5.0	5.0	5.4	5.4	7.7	7.7
		12	5.1	5.2	4.5	4.5	5.4	5.4

IV: Weak ARMA(1, 1) model (18)-(21).

V: Weak ARMA(1, 1) model (18)-(22).

Table 3: Empirical size adjusted power (in %) of the modified and standard versions of the LB and BP tests at the 5% nominal level in the case of ARMA(2,1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{FRZ}	BP _{FRZ}	LB _S	BP _S
I	$n = 500$	1	80.8	80.8	100.0	100.0	n.a.	n.a.
		2	66.3	66.5	100.0	100.0	n.a.	n.a.
		3	64.9	65.0	100.0	100.0	85.9	85.9
		6	55.3	55.4	100.0	100.0	78.5	78.7
		12	42.3	42.7	100.0	100.0	71.0	71.3
	$n = 2,000$	1	99.7	99.7	100.0	100.0	n.a.	n.a.
		2	97.5	97.5	100.0	100.0	n.a.	n.a.
		3	97.2	97.2	100.0	100.0	100.0	100.0
		6	97.6	97.6	100.0	100.0	100.0	100.0
		12	95.2	95.3	100.0	100.0	100.0	100.0
	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	100.0	100.0	100.0	100.0	100.0	100.0
		6	100.0	100.0	100.0	100.0	100.0	100.0
		12	100.0	100.0	100.0	100.0	100.0	100.0
II	$n = 500$	1	69.5	69.5	100.0	100.0	n.a.	n.a.
		2	54.4	54.4	99.9	99.9	n.a.	n.a.
		3	50.0	50.2	99.9	99.9	71.3	71.3
		6	38.9	38.9	100.0	100.0	61.1	61.1
		12	14.6	14.8	100.0	100.0	50.5	51.0
	$n = 2,000$	1	98.0	98.0	100.0	100.0	n.a.	n.a.
		2	92.3	92.3	100.0	100.0	n.a.	n.a.
		3	92.6	92.6	100.0	100.0	100.0	100.0
		6	92.7	92.6	100.0	100.0	99.8	99.8
		12	50.6	50.8	100.0	100.0	99.8	99.8
	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	99.9	99.9	100.0	100.0	100.0	100.0
		6	100.0	100.0	100.0	100.0	100.0	100.0
		12	100.0	100.0	100.0	100.0	100.0	100.0
III	$n = 500$	1	60.9	60.9	100.0	100.0	n.a.	n.a.
		2	47.8	47.8	99.7	99.7	n.a.	n.a.
		3	40.4	40.5	99.6	99.6	71.1	71.0
		6	30.4	30.3	99.9	99.9	68.4	68.6
		12	20.2	19.3	99.8	99.8	64.8	65.1
	$n = 2,000$	1	95.3	95.3	100.0	100.0	n.a.	n.a.
		2	91.1	91.1	100.0	100.0	n.a.	n.a.
		3	85.0	84.9	100.0	100.0	99.9	99.9
		6	80.5	80.5	100.0	100.0	99.9	99.9
		12	73.0	74.0	100.0	100.0	99.8	99.8
	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	99.6	99.6	100.0	100.0	100.0	100.0
		6	99.7	99.7	100.0	100.0	100.0	100.0

Table 4: Empirical size adjusted power (in %) of the modified and standard versions of the LB and BP tests at the 5% nominal level in the case of ARMA(2,1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{FRZ}	BP _{FRZ}	LB _s	BP _s
IV	$n = 500$	1	50.0	50.0	99.7	99.7	n.a.	n.a.
		2	34.4	34.4	99.8	99.8	n.a.	n.a.
		3	30.6	30.3	99.9	99.9	64.8	64.8
		6	27.9	28.2	99.9	99.9	63.0	63.0
		12	17.1	17.8	100.0	100.0	58.1	58.1
IV	$n = 2,000$	1	84.9	84.9	100.0	100.0	n.a.	n.a.
		2	75.6	75.6	100.0	100.0	n.a.	n.a.
		3	74.2	74.2	100.0	100.0	99.3	99.3
		6	65.3	65.4	100.0	100.0	99.1	99.1
		12	63.1	63.1	100.0	100.0	99.2	99.2
IV	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	99.4	99.4	100.0	100.0	n.a.	n.a.
		3	99.0	99.0	100.0	100.0	100.0	100.0
		6	98.3	98.3	100.0	100.0	100.0	100.0
		12	99.4	99.4	100.0	100.0	100.0	100.0
V	$n = 500$	1	85.6	85.6	100.0	100.0	n.a.	n.a.
		2	72.1	72.1	100.0	100.0	n.a.	n.a.
		3	70.1	70.3	100.0	100.0	88.5	88.5
		6	62.6	62.6	100.0	100.0	80.3	80.5
		12	48.5	49.2	100.0	100.0	71.0	71.1
V	$n = 2,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	99.4	99.4	100.0	100.0	n.a.	n.a.
		3	99.4	99.4	100.0	100.0	100.0	100.0
		6	98.8	98.8	100.0	100.0	100.0	100.0
		12	99.3	99.3	100.0	100.0	100.0	100.0
V	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	100.0	100.0	100.0	100.0	100.0	100.0
		6	100.0	100.0	100.0	100.0	100.0	100.0
		12	100.0	100.0	100.0	100.0	100.0	100.0

IV: Weak ARMA(2,1) model (29)-(21).

V: Weak ARMA(2,1) model (29)-(22).

Table 5: Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of VARMA(1, 1). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
I	$n = 500$	1	4.8	4.9	4.2	4.0	n.a.	n.a.
		2	5.2	5.2	3.4	3.3	n.a.	n.a.
		3	4.0	4.1	4.4	4.2	20.6	19.9
		4	3.5	3.6	4.6	4.3	11.6	11.5
		5	3.5	3.7	3.9	3.8	8.2	7.8
	$n = 2,000$	1	4.6	4.6	2.9	2.9	n.a.	n.a.
		2	4.8	4.8	3.2	3.1	n.a.	n.a.
		3	4.3	4.3	3.9	3.9	19.6	19.4
		4	5.1	5.1	3.5	3.4	10.5	10.4
		5	4.8	4.8	4.3	4.3	8.3	8.1
$n = 10,000$	1	5.7	5.7	3.7	3.7	n.a.	n.a.	
	2	5.4	5.4	4.7	4.7	n.a.	n.a.	
	3	5.3	5.3	4.4	4.4	19.0	19.0	
	4	5.7	5.7	4.5	4.5	10.6	10.6	
	5	6.1	6.1	4.4	4.3	8.5	8.5	
II	$n = 500$	1	3.9	3.9	5.3	5.1	n.a.	n.a.
		2	2.9	2.9	5.5	5.2	n.a.	n.a.
		3	2.1	2.1	2.9	2.7	34.9	34.0
		4	1.1	1.2	2.8	2.7	24.5	23.7
		5	0.9	0.9	2.3	2.0	18.0	17.7
	$n = 2,000$	1	3.9	3.9	6.1	5.9	n.a.	n.a.
		2	5.0	5.0	7.0	6.9	n.a.	n.a.
		3	3.3	3.3	5.4	5.3	43.9	43.8
		4	3.1	3.1	5.5	5.4	32.1	32.0
		5	3.0	3.0	4.9	4.8	26.1	25.9
$n = 10,000$	1	4.4	4.4	5.3	5.3	n.a.	n.a.	
	2	4.9	4.9	6.1	6.1	n.a.	n.a.	
	3	3.9	3.9	5.6	5.6	50.0	49.9	
	4	4.2	4.2	5.4	5.4	37.2	37.1	
	5	3.3	3.3	5.6	5.6	28.4	28.3	
III	$n = 500$	1	3.4	3.4	6.6	6.5	n.a.	n.a.
		2	1.6	1.6	4.6	4.5	n.a.	n.a.
		3	0.3	0.3	2.1	2.1	50.7	50.1
		4	0.1	0.1	1.3	1.3	37.7	36.9
		5	0.0	0.0	1.1	1.1	29.7	28.9
	$n = 2,000$	1	4.8	4.8	6.3	6.3	n.a.	n.a.
		2	3.7	3.7	5.4	5.4	n.a.	n.a.
		3	2.7	2.7	5.0	5.0	56.5	56.4
		4	2.5	2.5	3.6	3.5	41.8	41.3
		5	1.6	1.6	2.4	2.4	34.4	34.3
$n = 10,000$	1	5.1	5.1	6.2	6.2	n.a.	n.a.	
	2	5.2	5.2	6.0	6.0	n.a.	n.a.	
	3	4.0 ²⁶	4.0	5.8	5.8	58.6	58.5	
	4	4.3	4.3	5.6	5.6	44.9	44.9	

Table 6: Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of VARMA(1, 1). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
IV	$n = 500$	1	2.4	2.4	9.5	9.5	n.a.	n.a.
		2	0.7	0.8	6.5	6.5	n.a.	n.a.
		3	0.0	0.0	5.2	5.2	52.0	51.5
		4	0.2	0.2	5.1	5.1	40.5	40.1
		5	0.2	0.2	4.5	4.5	37.3	36.4
IV	$n = 2,000$	1	3.1	3.1	6.0	6.0	n.a.	n.a.
		2	2.5	2.5	4.8	4.8	n.a.	n.a.
		3	2.5	2.5	3.2	3.2	61.8	61.5
		4	2.0	2.0	3.0	3.0	50.2	50.0
		5	0.4	0.4	1.8	1.7	43.9	43.8
IV	$n = 10,000$	1	4.7	4.7	4.1	4.1	n.a.	n.a.
		2	4.4	4.4	5.6	5.6	n.a.	n.a.
		3	3.8	3.8	5.2	5.2	70.4	70.4
		4	3.5	3.5	4.1	4.1	57.3	57.3
		5	3.1	3.1	4.2	4.2	50.3	50.3
V	$n = 500$	1	4.0	4.0	3.6	3.6	n.a.	n.a.
		2	4.5	4.6	3.8	3.8	n.a.	n.a.
		3	3.9	4.1	4.2	4.2	15.1	14.7
		4	3.6	3.8	4.4	4.0	9.6	9.5
		5	3.1	3.2	3.7	3.4	7.8	7.0
V	$n = 2,000$	1	5.5	5.5	3.7	3.7	n.a.	n.a.
		2	5.4	5.5	4.4	4.4	n.a.	n.a.
		3	4.5	4.6	3.8	3.8	16.6	16.4
		4	4.4	4.4	4.0	4.0	9.5	9.3
		5	4.3	4.3	4.3	4.3	7.9	7.8
V	$n = 10,000$	1	4.3	4.3	3.6	3.6	n.a.	n.a.
		2	4.2	4.2	3.2	3.2	n.a.	n.a.
		3	4.7	4.7	3.9	3.9	15.3	15.3
		4	4.4	4.4	4.1	4.1	9.1	9.1
		5	4.8	4.8	3.9	3.9	6.8	6.8

IV: Weak VARMA(1, 1) model (23)-(27).

V: Weak VARMA(1, 1) model (23)-(28).

Table 7: Modified and standard versions of portmanteau tests to check the null hypothesis that the CAC40 returns is a white noise.

Lag m	2	3	4	5	10	18	24
$\hat{\rho}(m)$	-0.02829	-0.05308	0.04064	-0.05296	0.00860	-0.02182	0.00466
LB_{SN}	37.1438	65.4815	141.899	183.391	435.224	669.439	880.159
BP_{SN}	37.1138	65.4176	141.727	183.144	434.646	668.402	878.556
LB_{FRZ}	4.88063	19.4172	27.9413	42.4158	52.7845	61.2431	67.2210
BP_{FRZ}	4.87699	19.3994	27.9134	42.3685	52.7171	61.1467	67.0991
p_{SN}^{LB}	0.23931	0.27453	0.18218	0.22384	0.43726	0.90622	0.98502
p_{SN}^{BP}	0.23951	0.27493	0.18250	0.22440	0.43814	0.90674	0.98519
p_{FRZ}^{LB}	0.29758	0.03480	0.03837	0.00911	0.02085	0.17452	0.24341
p_{FRZ}^{BP}	0.29784	0.03491	0.03850	0.00916	0.02100	0.17544	0.24482
p_S^{LB}	0.08713	0.00022	0.00000	0.00000	0.00000	0.00000	0.00000
p_S^{BP}	0.08729	0.00022	0.00000	0.00000	0.00000	0.00000	0.00000

Table 8: Modified and standard versions of portmanteau tests to check the null hypothesis that the CAC40 squared returns follow an ARMA(1, 1) model.

Lag m	1	2	3	4	5	6	7
$\hat{\rho}(m)$	-0.04724	0.010467	0.01275	-0.00252	0.08579	-0.03532	-0.02772
LB_{SN}	8.96411	17.2907	21.0192	20.9689	21.0344	21.8014	25.0933
BP_{SN}	8.95890	17.2786	21.0026	20.9526	21.0172	21.7755	25.0477
LB_{FRZ}	11.5095	12.0746	12.9140	12.9468	50.9362	57.3777	61.3462
BP_{FRZ}	11.5028	12.0674	12.9061	12.9388	50.8767	57.3081	61.2697
p_{SN}^{LB}	0.30050	0.45977	0.66164	0.84375	0.93811	0.97700	0.98987
p_{SN}^{BP}	0.30061	0.45998	0.66183	0.84391	0.93822	0.97709	0.98991
p_{FRZ}^{LB}	0.11777	0.18293	0.34192	0.48658	0.29325	0.36110	0.38848
p_{FRZ}^{BP}	0.11789	0.18310	0.34218	0.48687	0.29368	0.36159	0.38899
p_S^{LB}	n.a.	n.a.	0.00033	0.00154	0.00000	0.00000	0.00000
p_S^{BP}	n.a.	n.a.	0.00033	0.00155	0.00000	0.00000	0.00000
Lag m	8	9	10	12	18	20	24
$\hat{\rho}(m)$	-0.04099	0.02048	0.04568	0.03584	0.03929	-0.04526	-0.04526
LB_{SN}	27.7828	27.5084	67.8011	93.7494	167.937	155.600	364.860
BP_{SN}	27.7576	27.4791	67.5936	93.1298	169.383	154.868	363.935
LB_{FRZ}	70.0230	72.1898	82.9700	89.6424	117.933	128.546	143.244
BP_{FRZ}	69.9297	72.0919	82.8470	89.5013	117.697	128.265	142.890
p_{SN}^{LB}	0.99599	0.99896	0.99275	0.99673	0.99988	0.99999	0.99995
p_{SN}^{BP}	0.99600	0.99897	0.99278	0.99679	0.99987	0.99999	0.99995
p_{FRZ}^{LB}	0.33766	0.40190	0.33819	0.39101	0.48717	0.45510	0.45367
p_{FRZ}^{BP}	0.33821	0.40250	0.33892	0.39191	0.48829	0.45635	0.45503
p_S^{LB}	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
p_S^{BP}	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Complementary results that are not submitted for publication
They will be put into an on-line extended version

Appendix A. Proofs

We recall that the Skorokhod space $\mathbb{D}^k[0,1]$ is the set of \mathbb{R}^k -valued functions defined on $[0,1]$ which are right continuous and have left limits. It is endowed with the Skorokhod topology and the weak convergence on $\mathbb{D}^k[0,1]$ is mentioned by $\xrightarrow{\mathbb{D}^k}$. We finally denote by $[a]$ the integer part of the real a .

In order to make our presentation more readable, we restrict ourselves to the one dimensional case that requires less technical notation (hence $k_0 = p + q$ in the following). The proofs of Theorems 1 and 2 are the multivariate adaptation of the proofs of Theorems 3 and 4 that are detailed below.

First, we shall need some technical results which are essentially contained in [Francq et al. \(2005\)](#); [Francq and Zakoian \(1998, 2000\)](#). They are necessary to understand the proofs, but are not essential to give the main ideas of the self-normalization approach. This is the reason why these facts are presented here.

Appendix A.1. Reminder on technical issues on quasi likelihood method for ARMA models

We recall that, given a realization X_1, \dots, X_n of length n , the noise $e_t(\vartheta)$ (see Equation (6)) is approximated by $\tilde{e}_t(\vartheta)$ which is defined in (7).

The starting point in the asymptotic analysis is the property that $e_t(\vartheta) - \tilde{e}_t(\vartheta)$ converges uniformly to 0 (almost-surely) as t goes to infinity. Similar properties also hold for the derivatives with respect to ϑ of $e_t(\vartheta) - \tilde{e}_t(\vartheta)$. We sum up the properties that we shall need in the sequel. We refer to the appendix of [Francq and Zakoian \(2000\)](#) (see also [Francq and Zakoian \(1998\)](#)) for a more detailed treatment.

For any $\vartheta \in \Theta \subset \mathbb{R}^{k_0}$ and any $(l, m) \in \{1, \dots, k_0\}^2$, there exists absolutely summable and deterministic sequences $(c_i(\vartheta))_{i \geq 0}$, $(c_{i,l}(\vartheta))_{i \geq 1}$ and $(c_{i,l,m}(\vartheta))_{i \geq 1}$ such that, almost surely,

$$e_t(\vartheta) = \sum_{i=0}^{\infty} c_i(\vartheta) X_{t-i}, \quad \frac{\partial e_t(\vartheta)}{\partial \vartheta_l} = \sum_{i=1}^{\infty} c_{i,l}(\vartheta) X_{t-i} \quad \text{and} \quad \frac{\partial^2 e_t(\vartheta)}{\partial \vartheta_l \partial \vartheta_m} = \sum_{i=2}^{\infty} c_{i,l,m}(\vartheta) X_{t-i} \quad (\text{A.1})$$

$$\tilde{e}_t(\vartheta) = \sum_{i=0}^{t-1} c_i(\vartheta) X_{t-i}, \quad \frac{\partial \tilde{e}_t(\vartheta)}{\partial \vartheta_l} = \sum_{i=1}^{t-1} c_{i,l}(\vartheta) X_{t-i} \quad \text{and} \quad \frac{\partial^2 \tilde{e}_t(\vartheta)}{\partial \vartheta_l \partial \vartheta_m} = \sum_{i=2}^{t-1} c_{i,l,m}(\vartheta) X_{t-i}. \quad (\text{A.2})$$

We strength the fact that $c_0(\vartheta) = 1$ in the above identities. A useful property of the above three sequences that they are asymptotically exponentially small. Indeed there exists $\rho \in]0,1[$ and a positive constant K such that, for all $i \geq 1$,

$$\sup_{\vartheta \in \Theta} \left(|c_i(\vartheta)| + |c_{i,l}(\vartheta)| + |c_{i,l,m}(\vartheta)| \right) \leq K \rho^i.$$

From (14), this implies that there exists some other absolutely summable and deterministic sequences $(d_i(\vartheta))_{i \geq 0}$, $(d_{i,l}(\vartheta))_{i \geq 1}$ and $(d_{i,l,m}(\vartheta))_{i \geq 1}$ such that, almost surely,

$$e_t(\vartheta) = \sum_{i=0}^{\infty} d_i(\vartheta) e_{t-i}, \quad \frac{\partial e_t(\vartheta)}{\partial \vartheta_l} = \sum_{i=1}^{\infty} d_{i,l}(\vartheta) e_{t-i} \quad \text{and} \quad \frac{\partial^2 e_t(\vartheta)}{\partial \vartheta_l \partial \vartheta_m} = \sum_{i=2}^{\infty} d_{i,l,m}(\vartheta) e_{t-i} \quad (\text{A.3})$$

$$\tilde{e}_t(\vartheta) = \sum_{i=0}^{t-1} d_i(\vartheta) \tilde{e}_{t-i}, \quad \frac{\partial \tilde{e}_t(\vartheta)}{\partial \vartheta_l} = \sum_{i=1}^{t-1} d_{i,l}(\vartheta) \tilde{e}_{t-i} \quad \text{and} \quad \frac{\partial^2 \tilde{e}_t(\vartheta)}{\partial \vartheta_l \partial \vartheta_m} = \sum_{i=2}^{t-1} d_{i,l,m}(\vartheta) \tilde{e}_{t-i}.$$

We have $d_0(\vartheta) = 1$ and the three above sequences also satisfy

$$\sup_{\vartheta \in \Theta} \left(|d_i(\vartheta)| + |d_{i,l}(\vartheta)| + |d_{i,l,m}(\vartheta)| \right) \leq K \rho^i . \quad (\text{A.4})$$

Finally, from the above estimates, we are able to deduce that for any $(l, m) \in \{1, \dots, k_0\}^2$

$$\sup_{\vartheta \in \Theta} |e_t(\vartheta) - \tilde{e}_t(\vartheta)| \xrightarrow[t \rightarrow \infty]{\text{a.s.}} 0 , \quad (\text{A.5})$$

$$\rho^t \sup_{\vartheta \in \Theta} |e_t(\vartheta)| \xrightarrow[t \rightarrow \infty]{\text{a.s.}} 0 . \quad (\text{A.6})$$

Analogous estimates to (A.5) and (A.6) are satisfied for first and second order derivatives of e_t and \tilde{e}_t .

Appendix A.2. Proof of Theorem 3

The proof is divided in several steps.

Appendix A.2.1. Taylor's expansion of $\hat{\Gamma}_m$

The aim of this step is to prove that

$$\sqrt{n} \hat{\Gamma}_m = \frac{1}{\sqrt{n}} \sum_{t=1}^n \Lambda U_t + o_{\mathbb{P}}(1). \quad (\text{A.7})$$

Let $\sigma_{e_0}^2$ the common variance of the univariate noise process. In this case, the Gaussian log-quasi-likelihood can be written as

$$\tilde{\ell}_n(\vartheta, \sigma_e^2) = \frac{-2}{n} \log \tilde{L}_n(\vartheta, \sigma_e^2) = \log(2\pi) + \log \sigma_e^2 + \frac{2}{n} \sum_{t=1}^n \frac{\tilde{e}_t^2(\vartheta)}{\sigma_e^2}.$$

The least squares estimator (LSE in short) of ϑ is the value of that minimizes the last quantity as a function of ϑ . Thus the LSE $\hat{\vartheta}_n$ satisfies almost surely

$$Q_n(\hat{\vartheta}_n) = \min_{\vartheta \in \Theta} Q_n(\vartheta), \quad \text{with } Q_n(\vartheta) = \frac{1}{n} \sum_{t=1}^n \tilde{e}_t^2(\vartheta). \quad (\text{A.8})$$

We denote $O_n(\vartheta) = \frac{1}{n} \sum_{t=1}^n e_t^2(\vartheta)$. By (A.5) and (A.6), the sequences $\sqrt{n} \frac{\partial}{\partial \vartheta} Q_n(\vartheta_0)$ and $\sqrt{n} \frac{\partial}{\partial \vartheta} O_n(\vartheta_0)$ have the same asymptotic distribution. More precisely we have

$$\sqrt{n} \left(\frac{\partial}{\partial \vartheta} Q_n(\vartheta_0) - \frac{\partial}{\partial \vartheta} O_n(\vartheta_0) \right) = o_{\mathbb{P}}(1) . \quad (\text{A.9})$$

Then we use the Taylor expansion of the derivative of Q_n which is defined in (A.8). We shall need that $\partial Q_n(\hat{\vartheta}_n) / \partial \vartheta = 0$, which is true because $\hat{\vartheta}_n$ minimizes the function $\vartheta \mapsto Q_n(\vartheta)$. Following the proof of Theorem 2 in [Francq and Zakoïan \(1998\)](#), we obtain that

$$0 = \sqrt{n} \frac{\partial Q_n(\hat{\vartheta}_n)}{\partial \vartheta} = \sqrt{n} \frac{\partial Q_n(\vartheta_0)}{\partial \vartheta} + \frac{\partial^2 Q_n(\vartheta_{n,i,j}^\#)}{\partial \vartheta_i \partial \vartheta_j} \sqrt{n} (\hat{\vartheta}_n - \vartheta_0) ,$$

where the $\vartheta_{n,i,j}^\#$'s are between $\hat{\vartheta}_n$ and ϑ_0 , and consequently

$$0 = \sqrt{n} \frac{\partial O_n(\vartheta_0)}{\partial \vartheta} + \frac{\partial^2 O_n(\vartheta_0)}{\partial \vartheta \partial \vartheta'} \sqrt{n} (\hat{\vartheta}_n - \vartheta_0) + o_{\mathbb{P}}(1). \quad (\text{A.10})$$

With $Y_t = -2e_t(\vartheta_0) \frac{\partial e_t(\vartheta_0)}{\partial \vartheta}$, we deduce from (A.10) that

$$\hat{\vartheta}_n - \vartheta_0 = -J^{-1} \frac{\partial O_n(\vartheta_0)}{\partial \vartheta} + o_{\mathbb{P}}(1) = J^{-1} \frac{1}{n} \sum_{t=1}^n Y_t + o_{\mathbb{P}}(1).$$

Thanks to the results recalled in Section [Appendix A.1](#), we may use the arguments of Lemma A3 in [Francq and Zakoïan \(2000\)](#). We have for $h \in \{1, \dots, m\}$

$$\hat{\Gamma}_e(h) = \Gamma_e(h) + \phi_h (\hat{\vartheta}_n - \vartheta_0) + o_{\mathbb{P}}(1), \quad (\text{A.11})$$

where $\phi_h := \mathbb{E} \left(\frac{\partial e_t(\vartheta_0)}{\partial \vartheta'} e_{t-h}(\vartheta_0) \right) \in \mathbb{R}^{p+q}$. Thus we may rewrite (A.11) as

$$\hat{\Gamma}_e(h) = \frac{1}{n} \sum_{t=h+1}^n e_t e_{t-h} - (\phi_h J^{-1}) \frac{1}{n} \sum_{t=1}^n Y_t + o_{\mathbb{P}}(1).$$

Now, we come back to the vector $\hat{\Gamma}_m = (\hat{\Gamma}_e(1), \dots, \hat{\Gamma}_e(m))'$. We remark that the matrix Φ_m defined by (15) is the matrix in $\mathbb{R}^{m \times (p+q)}$ whose line h is ϕ_h . So we have

$$\sqrt{n} \hat{\Gamma}_m = \frac{1}{\sqrt{n}} \sum_{t=1}^n \Lambda (Y_t', e_t e_{t-1}, \dots, e_t e_{t-m})' + o_{\mathbb{P}}(1). \quad (\text{A.12})$$

Therefore the Taylor expansion (A.7) of $\hat{\Gamma}_m$ is proved. This ends our first step.

Now, it is clear that the asymptotic behaviour of $\hat{\Gamma}_m$ is related to the limit distribution of $U_t = (Y_t', e_t e_{t-1}, \dots, e_t e_{t-m})'$. The next step deals with the asymptotic distribution of ΛU_t .

Appendix A.2.2. Functional central limit theorem for $(\Lambda U_t)_{t \geq 1}$

Our purpose is to prove that there exists a lower triangular matrix Ψ , with nonnegative diagonal entries, such that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \Lambda U_j \xrightarrow[n \rightarrow \infty]{\mathbb{D}^m} \Psi B_m(r) \quad (\text{A.13})$$

where $(B_m(r))_{r \geq 0}$ is a m -dimensional standard Brownian motion.

By (A.3) one rewrites U_t as

$$U_t = \left(-2 \sum_{i=1}^{\infty} d_{i,1}(\vartheta_0) e_t e_{t-i}, \dots, -2 \sum_{i=1}^{\infty} d_{i,k_0}(\vartheta_0) e_t e_{t-i}, e_t e_{t-1}, \dots, e_t e_{t-m} \right)' \quad (\text{A.14})$$

and thus it has zero expectation with values in \mathbb{R}^{k_0+m} . In order to apply the function central limit theorem for strongly mixing process, we need to identify the asymptotic covariance matrix in the classical central limit theorem for the sequence $(U_t)_{t \geq 1}$. It is proved in [Francq and Zakoïan \(1998\)](#) that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n U_t \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{N}(0, \Xi) \quad (\text{A.15})$$

where

$$\Xi := 2\pi f_U(0) = \sum_{h=-\infty}^{+\infty} \text{Cov}(U_t, U_{t-h}) = \sum_{h=-\infty}^{+\infty} \mathbb{E}(U_t U_{t-h}'), \quad (\text{A.16})$$

where $f_U(0)$ is the spectral density of the stationary process $(U_t)_{t \in \mathbb{Z}}$ evaluated at frequency 0 (see for example [Brockwell and Davis \(1991\)](#)). The main issue is the existence of the sum of the right-hand side of (A.16). For that sake, one has to introduce for any integer k , the random variables

$$U_t^k = \left(-2 \sum_{i=1}^k d_{i,1}(\vartheta_0) e_t e_{t-i}, \dots, -2 \sum_{i=1}^k d_{i,k_0}(\vartheta_0) e_t e_{t-i}, e_t e_{t-1}, \dots, e_t e_{t-m} \right)'$$

Since U^k depends on a finite number of values of the noise-process e , it also satisfies a mixing property of the form (8). Based on the Davydov inequality (see [Davydov \(1968\)](#)), the arguments developed in the Lemma A.1 in [Francq et al. \(2005\)](#) (see also [Francq and Zakoïan \(1998\)](#)) imply that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n U_t^k \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{N}(0, \Xi_k) \quad (\text{A.17})$$

where

$$\Xi_k := 2\pi f_{U^k}(0) = \sum_{h=-\infty}^{+\infty} \text{Cov}(U_t^k, U_{t-h}^k) = \sum_{h=-\infty}^{+\infty} \mathbb{E}(U_t^k U_{t-h}^k)'$$

Using this truncation procedure and the Davydov inequality (see [Davydov \(1968\)](#)), the arguments developed in the Lemma A.1 in [Francq et al. \(2005\)](#) (see also [Francq and Zakoïan \(1998\)](#)) imply that (A.15) holds. Moreover we have that $\lim_{k \rightarrow \infty} \Xi_k = \Xi$.

Since the matrix Ξ is positive definite, it can be factorized as $\Xi = \Upsilon \Upsilon'$ where the $(k_0 + m) \times (k_0 + m)$ lower triangular matrix Υ has nonnegative diagonal entries. Therefore, we have

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \Lambda U_t \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{N}(0, \Lambda \Xi \Lambda'),$$

and the new variance matrix can also be factorized as $\Lambda \Xi \Lambda' = (\Lambda \Upsilon)(\Lambda \Upsilon)' := \Psi \Psi'$. Thus, $n^{-1/2} \sum_{t=1}^n \Psi^{-1} \Lambda U_t \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{N}(0, I_m)$ where I_m is the identity matrix of order m . The above arguments also apply to matrix Ξ_k with some matrix Ψ_k which is defined analogously as Ψ . Consequently,

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \Lambda U_t^k \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{N}(0, \Lambda \Xi_k \Lambda'),$$

and we also have $n^{-1/2} \sum_{t=1}^n \Psi_k^{-1} \Lambda U_t^k \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{N}(0, I_m)$

Now we are able to apply the functional central limit theorem (see [Herrndorf \(1984\)](#)). We have for any $r \in (0,1)$,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \Psi_k^{-1} \Lambda U_j^k \xrightarrow[n \rightarrow \infty]{\mathbb{D}^m} B_m(r).$$

We write

$$\Psi^{-1}\Lambda U_j^k = \left(\Psi^{-1} - \Psi_k^{-1}\right)\Lambda U_j^k + \Lambda \Psi_k^{-1}\Lambda U_j^k$$

and we obtain that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \Psi^{-1}\Lambda U_j^k \xrightarrow[n \rightarrow \infty]{\mathbb{D}^m} B_m(r).$$

In order to conclude that (A.13) is true, it remains to observe that, uniformly with respect to n ,

$$Z_n^k(r) := \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \Psi^{-1}\Lambda V_j^k \xrightarrow[k \rightarrow \infty]{\mathbb{D}^m} 0, \quad (\text{A.18})$$

where

$$V_t^k = \left(-2 \sum_{i=k+1}^{\infty} d_{i,1}(\vartheta_0) e_t e_{t-i}, \dots, -2 \sum_{i=k+1}^{\infty} d_{i,k_0}(\vartheta_0) e_t e_{t-i}, e_t e_{t-1}, \dots, e_t e_{t-m} \right)'$$

By Lemma 4 in [Francq and Zakoian \(1998\)](#),

$$\sup_n \text{Var} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^n V_j^k \right) \xrightarrow[k \rightarrow \infty]{} 0$$

and since $\lfloor nr \rfloor \leq n$,

$$\sup_{0 \leq r \leq 1} \sup_n \left(|Z_n^k(r)| \right) \xrightarrow[k \rightarrow \infty]{} 0.$$

Thus (A.18) is true and the proof of (A.13) is achieved.

Appendix A.2.3. Limit theorem

We prove Theorem 3. We follow the arguments developed in Sections 2 and 3 in [Lobato \(2001\)](#). The main difference is that we shall work with the sequence $(\Lambda U_t)_{t \geq 1}$ instead of the sequence $((e_t e_{t-1}, \dots, e_t e_{t-m})'_{t \geq 1})$. The previous step ensures us that Assumption 1 in [Lobato \(2001\)](#) is satisfied for the sequence $(\Lambda U_t)_{t \geq 1}$. Since $C_m = (1/n^2) \sum_{t=1}^n S_t S_t'$, the continuous mapping theorem on the Skorokhod space implies that

$$C_m \xrightarrow[n \rightarrow \infty]{d} \Psi V_m \Psi', \quad (\text{A.19})$$

where the random variable V_m is defined in (10). We assume for the moment that the matrix C_m is invertible (this will be stated and proved in Lemma 5 in Subsection [Appendix A.4](#) at the end of this appendix).

Since by (A.12), $\sqrt{n}\hat{\Gamma}_m = n^{-1/2} \sum_{t=1}^n \Lambda U_t + o_{\mathbb{P}}(1)$, we use (A.13) and (A.19) in order to obtain

$$\begin{aligned} n\hat{\Gamma}'_m C_m^{-1} \hat{\Gamma}_m &= \frac{1}{n} \sum_{t=1}^n ((\Lambda U_t)' C_m^{-1} (\Lambda U_t)) \\ &\xrightarrow[n \rightarrow \infty]{d} (\Psi B_m(1))' (\Psi V_m \Psi')^{-1} (\Psi B_m(1)) = B'_m(1) V_m^{-1} B_m(1), \end{aligned}$$

and we recognize the random variable \mathcal{U}_m defined in (9). Consequently we have proved (16). The property (17) is straightforward since $\hat{\rho}(h) = \hat{\Gamma}_e(h)/\sigma_{e0}^2$ for $h = 1, \dots, m$. The proof of Theorem 3 is then complete.

Appendix A.3. Proof of Theorem 4

We write $\hat{C}_m = C_m + \Upsilon_n$ where $\Upsilon_n = n^{-2} \sum_{t=1}^n (S_t S'_t - \hat{S}_t \hat{S}'_t)$. There are three kinds of entries in the matrix Υ_n . The first one is a sum composed of

$$v_t^{k,k'} = e_t^2(\vartheta_0) e_{t-k}(\vartheta_0) e_{t-k'}(\vartheta_0) - \tilde{e}_t^2(\hat{\vartheta}_n) \tilde{e}_{t-k}(\hat{\vartheta}_n) \tilde{e}_{t-k'}(\hat{\vartheta}_n)$$

for $(k, k') \in \{1, \dots, m\}^2$. By (A.4) and the consistency of $\hat{\vartheta}_n$, we have $v_t^{k,k'} = o_{\mathbb{P}}(1)$ almost surely. The two last kinds of entries of Υ_n come from the following quantities for $i, j \in \{1, \dots, k_0\}$ and $k \in \{1, \dots, m\}$

$$\begin{aligned} \tilde{v}_t^{k,i} &= e_t^2(\vartheta_0) e_{t-k}(\vartheta_0) \frac{\partial e_t(\vartheta_0)}{\partial \vartheta_i} - \tilde{e}_t^2(\hat{\vartheta}_n) \tilde{e}_{t-k}(\hat{\vartheta}_n) \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta_i}, \\ \bar{v}_t^{i,j} &= e_t^2(\vartheta_0) \frac{\partial e_t(\vartheta_0)}{\partial \vartheta_i} \frac{\partial e_t(\vartheta_0)}{\partial \vartheta_j} - \tilde{e}_t^2(\hat{\vartheta}_n) \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta_i} \frac{\partial \tilde{e}_t(\hat{\vartheta}_n)}{\partial \vartheta_j} \end{aligned}$$

and they also satisfy $\tilde{v}_t^{k,i} + \bar{v}_t^{i,j} = o_{\mathbb{P}}(1)$ almost surely. Consequently, $\Upsilon_n = o_{\mathbb{P}}(1)$ almost surely as n goes to infinity. Thus one may find a matrix Υ_n^* , that tends to the null matrix almost surely, such that

$$n \hat{\Gamma}'_m \hat{C}_m^{-1} \hat{\Gamma}_m = n \hat{\Gamma}'_m (C_m + \Upsilon_n)^{-1} \hat{\Gamma}_m = n \hat{\Gamma}'_m C_m^{-1} \hat{\Gamma}_m + n \hat{\Gamma}'_m \Upsilon_n^* \hat{\Gamma}_m.$$

Thanks to the arguments developed in the proof of Theorem 3, $n \hat{\Gamma}'_m \hat{\Gamma}_m$ converges in distribution. So $n \hat{\Gamma}'_m \Upsilon_n^* \hat{\Gamma}_m$ tends to zero in distribution, hence in probability. Then $n \hat{\Gamma}'_m \hat{C}_m^{-1} \hat{\Gamma}_m$ and $n \hat{\Gamma}'_m C_m^{-1} \hat{\Gamma}_m$ have the same limit in distribution and the result is proved.

Appendix A.4. Invertibility of the normalization matrix

We prove in this short subsection why the matrix C_m is invertible.

Lemma 5. *Under the assumption of Theorem 3, the matrix C_m is almost surely non singular.*

Proof. We write the matrix Λ as

$$\Lambda = \left((\lambda_{ij})_{1 \leq i \leq m; 1 \leq j \leq k_0} \mid I_m \right)$$

we have for $t = 1, \dots, n$:

$$S_t = \begin{pmatrix} S_t^1 \\ \vdots \\ S_t^m \end{pmatrix} = \begin{pmatrix} \left(\sum_{j=1}^t (\sum_{l=1}^{k_0} \lambda_{1l} Y_j^l) + e_j e_{j-1} \right) - t \Gamma_e(1) \\ \vdots \\ \left(\sum_{j=1}^t (\sum_{l=1}^{k_0} \lambda_{ml} Y_j^l) + e_j e_{j-m} \right) - t \Gamma_e(m) \end{pmatrix},$$

with $Y_j^l = e_j(\vartheta_0) \frac{\partial e_j(\vartheta_0)}{\partial \vartheta_l}$. We remark that

$$S_{t+1}^i = S_t^i + \left(\sum_{l=1}^{k_0} \lambda_{il} Y_{t+1}^l \right) + e_{t+1} e_{t+1-i} - \Gamma_e(i). \quad (\text{A.20})$$

If the matrix C_m is not invertible, there exists some real constants c_1, \dots, c_m , not all equal to zero, such that for any $t = 1, \dots, n$ we have $\sum_{i=1}^m c_i S_t^i = 0$. By (A.20), it would imply that

$$\sum_{i=1}^m c_i \left(\left(\sum_{l=1}^{k_0} \lambda_{il} Y_t^l \right) + e_t e_{t-i} \right) = \sum_{i=1}^m c_i \Gamma_e(i).$$

By the ergodic Theorem, $\sum_{i=1}^m c_i \Gamma_e(i) \rightarrow 0$ almost-surely as n goes to infinity. Consequently, for any $t \geq 1$, $\sum_{i=1}^m c_i (\sum_{l=1}^{k_0} \lambda_{il} Y_t^l) + c_i e_t e_{t-i} = 0$. Using (A.14) yields that

$$e_t \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^m c_i \sum_{l=1}^{k_0} \lambda_{il} d_{k,l} \right) e_{t-k} + \sum_{k=1}^m c_k e_{t-k} \right) = 0 ,$$

or equivalently

$$e_t \left(\sum_{k=1}^m \left[\sum_{i=1}^m c_i \sum_{l=1}^{k_0} \lambda_{il} d_{k,l} + c_k \right] e_{t-k} + \sum_{k=m+1}^{\infty} \left[\sum_{i=1}^m c_i \sum_{l=1}^{k_0} \lambda_{il} d_{k,l} \right] e_{t-k} \right) = 0 .$$

Therefore, there exists a sequence $(\alpha_k)_{k \geq 1}$, with $\sum_{k \geq 1} |\alpha_k| < \infty$, such that $e_t \sum_{k \geq 1} \alpha_k e_{t-k} = 0$. Thanks to Assumption **(A4)**, e_t has a positive density in some neighborhood of zero and then $e_t \neq 0$ almost-surely. So we would have $\sum_{k \geq 1} \alpha_k e_{t-k} = 0$. Since the variance of the linear innovation process is not equal to zero, all the coefficients α_k vanish. Then

$$0 = \alpha_k = \begin{cases} \sum_{i=1}^m c_i \sum_{l=1}^{k_0} \lambda_{il} d_{k,l} + c_k & \text{if } 1 \leq k \leq m \\ \sum_{i=1}^m c_i \sum_{l=1}^{k_0} \lambda_{il} d_{k,l} & \text{if } k > m . \end{cases}$$

Then we would have $c_1 = \dots = c_m = 0$ which is impossible. Thus we have a contradiction and the matrix C_m is non singular. \square

Remark 2. *The property that e_t has a positive density in some neighborhood of zero is necessary to ensure the invertibility of C_m . Indeed, if we choose an independent noise sequence such that the law of e_t is given by $\frac{1}{2}\delta_0 + \frac{1}{2}\mathcal{N}(0,1)$, then the event $E = \cap_{i=1}^n \{e_i = 0\}$ has a positive probability (actually equals to 2^{-n}). On this event, the matrix C_m is clearly singular because is it equal to the null matrix. It is worth to notice that this counterexample may apply to the work of Lobato (see Lobato (2001)) and it seems that this hypotheses is missing in this work. Of course, the probability of E is very small ($\mathbb{P}(E) < 10^{-60}$ for $n = 200$ which is the lower n chosen in our simulation) and thus it is quite normal that we do obtain any singular matrix C_m in our numerical studies.*

Appendix B. Empirical size for small length series

Appendix C. The Standard & Poor's 500 index as an illustrative example

As we did for the CAC40, we also consider an application to the daily log returns (also simply called the returns) of the S&P500². The observations cover the period from March 1, 1979 to December 31, 2001. The length of the series is $n = 5808$. The data can be downloaded from the website Yahoo Finance: <http://fr.finance.yahoo.com/>. Figure C.2 shows that the S&P500 index series are generally close to a random walk without intercept and that the returns are generally compatible with the second-order stationarity assumption.

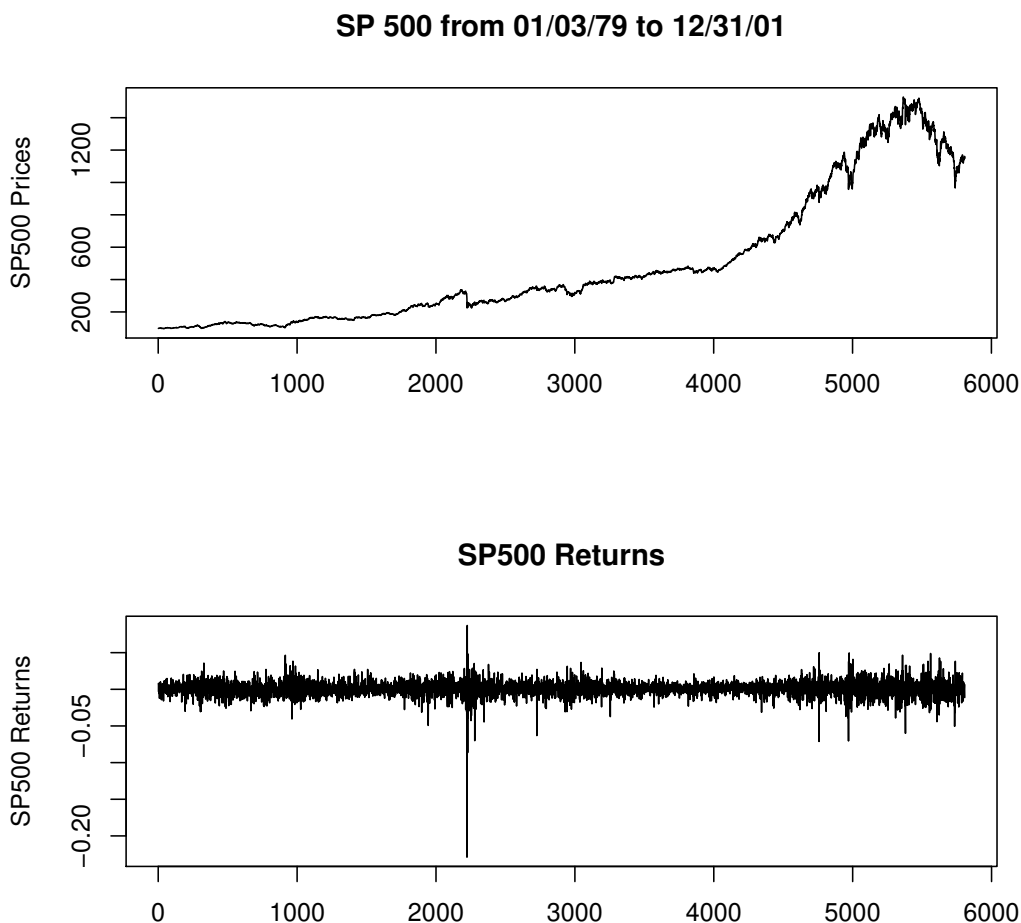


Figure C.2: Closing prices and returns of the S&P 500 index from January 3, 1979 to December 31, 2001 (5807 observations).

First, we apply portmanteau tests for checking the hypothesis that the S&P500 returns constitute a white noise. Table C.11 displays the statistics of the standard and modified BP tests. Since the p -values of the standard test are very small, the white noise hypothesis

²The Standard & Poor's 500 index.

is rejected at the nominal level $\alpha = 1\%$. This is not surprising because the standard tests required the iid assumption and, in particular in view of the so-called volatility clustering, it is well known that the strong white noise model is not adequate for these series. By contrast, the white noise hypothesis is not rejected by the modified tests, since for the modified tests, the statistic is not larger than the critical values (see Table 1 in [Lobato \(2001\)](#)). This is also in accordance with other works devoted to the analysis of stock-market returns [Lobato et al. \(2001\)](#). To summarize, the outputs of Table C.11 are in accordance with the common belief that these series are not strong white noises, but could be weak white noises.

Next, turning to the dynamics of the squared returns, we fit an ARMA(1, 1) model to the squares of the S&P500 returns. Denoting by (X_t) the mean corrected series of the squared returns, we obtain the model

$$X_t = 0.83123X_{t-1} + \epsilon_t - 0.72676\epsilon_{t-1}, \text{ where } \text{Var}(\epsilon_t) = 52.62582 \times 10^{-8}.$$

Table C.12 displays the statistics of the standard and modified BP tests. From Table C.12, we draw the same conclusion, on the squares of the previous daily returns, that the strong ARMA(1, 1) model is rejected, but a weak ARMA(1, 1) model is not rejected. Note that the first and second-order structures we found for the S&P500 returns, namely a weak white noise for the returns and a weak ARMA(1, 1) model for the squares of the returns, are compatible with a GARCH(1, 1) model.

Appendix D. Empirical power using asymptotic critical values

In Section 4.3, we investigate the size adjusted power which is the rate at which the null hypothesis is rejected, when it is false, and when the critical value that is used is the one that will actually ensure that there is no size distortion. This new critical value requires some Monte Carlo simulation. In terms of a day-to-day application, we usually wouldn't do this. In that case, what's of interest is the "raw power" of the test, namely the rate at which it rejects false null hypotheses when the asymptotic critical value is used. This is what we do below in the context described in Section 4.3.

With this notion of power, we remark that the powers our tests become comparable to the standard ones, even when $n = 500$.

Appendix E. Empirical power for VARMA models

Here we use both empirical critical values (leading to the so-called self-adjusted power) and asymptotic critical values (leading to the so-called raw-power) to compare the power of our test when one deals with VARMA models.

For that sake, we simulate $N = 1,000$ independent trajectories of size $n = 10,000$ of the following bivariate VARMA(2, 1) defined by

$$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 1.2 & 0.6 \\ -0.5 & 0.3 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{pmatrix} \begin{pmatrix} X_{1,t-2} \\ X_{2,t-2} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} - \begin{pmatrix} -0.6 & 0.3 \\ 0.3 & 0.6 \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \end{pmatrix}, \quad (\text{E.1})$$

where the multivariate, strong and weak versions of the innovation process ϵ are defined in Section 4.1.2.

For each of these N replications we fit a VARMA(1, 1) model and we perform standard and modified test based on $m = 1, \dots, 5$ residual autocorrelations.

Tables E.15 and E.16 display the relative rejection frequencies of over the N independent replications for the VARMA models when one uses the empirical critical value (size adjusted power of the test).

Tables E.17 and E.18 display the relative rejection frequencies of over the N independent replications for the VARMA models when one uses the asymptotic critical value (raw power of the test).

In these examples, the standard and modified versions of the tests have very similar powers when $n \geq 2000$. When $n = 500$, the empirical size of the tests for the models II, III and IV are very far from the 5% nominal level (see Tables 5 and 6) so the discussion on the power is not relevant. However, the results for the model V (for which the empirical level is close to the nominal) are quite satisfactory whereas it is less the case for the strong model I.

Table B.9: Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of ARMA(1,1). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{FRZ}	BP _{FRZ}	LB _S	BP _S
I	$n = 200$	1	5.9	5.9	4.9	4.5	n.a.	n.a.
		2	4.6	4.4	4.5	4.3	n.a.	n.a.
		3	4.9	4.5	4.3	4.1	11.8	11.3
		4	4.0	3.8	5.1	4.6	9.4	8.9
		5	4.3	4.2	4.5	4.1	8.5	8.0
		6	4.2	3.5	4.9	4.6	7.2	6.5
		8	5.0	4.4	4.4	3.9	6.2	5.2
		10	4.1	3.7	4.8	3.7	6.1	5.1
		12	3.5	2.8	4.2	3.1	5.9	4.7
II	$n = 200$	1	5.6	5.6	5.8	5.6	n.a.	n.a.
		2	4.6	4.2	4.4	4.2	n.a.	n.a.
		3	2.0	2.0	3.6	3.2	19.4	19.0
		4	2.0	2.0	3.4	3.2	15.2	14.6
		5	1.8	1.8	2.4	2.0	16.0	14.8
		6	1.6	1.6	2.8	2.8	15.0	14.2
		8	2.8	2.2	3.0	2.4	12.0	10.4
III	$n = 200$	1	3.4	3.4	3.1	2.9	n.a.	n.a.
		2	3.2	2.9	2.5	2.5	n.a.	n.a.
		3	1.7	1.5	2.2	2.1	18.5	17.6
		4	1.5	1.2	2.4	2.2	13.5	12.6
		5	1.4	1.3	1.9	1.4	10.5	9.4
		6	1.2	1.0	1.6	1.3	8.7	8.0
		8	0.9	0.8	1.5	1.0	7.7	7.2
IV	$n = 200$	1	3.0	3.0	3.3	3.0	n.a.	n.a.
		2	3.7	3.7	2.7	2.7	n.a.	n.a.
		3	1.7	1.3	1.7	1.7	17.7	17.0
		4	0.7	0.6	1.0	1.0	12.0	11.3
		5	0.4	0.3	0.9	0.8	11.4	11.2
		6	0.3	0.3	1.6	1.3	9.2	8.6
V	$n = 200$	1	5.3	5.3	4.3	4.2	n.a.	n.a.
		2	4.0	4.0	5.1	5.0	n.a.	n.a.
		3	3.6	3.3	3.8	3.5	13.3	12.8
		4	4.9	4.7	4.0	4.0	9.0	8.5
		5	4.0	3.8	4.4	4.0	8.1	7.3
		6	4.6	4.4	4.7	4.3	7.7	7.2
		8	3.4	2.9	4.7	4.1	6.7	5.6
		10	4.9	3.9	4.7	3.9	6.5	5.2
		12	4.1	3.2	5.5	3.8	7.2	5.5

I: Strong ARMA(1,1) model (18)-(19) with $(\alpha_1, \beta_1) = (0, 0)$

II: Weak ARMA(1,1) model (18)-(19) with $(\alpha_1, \beta_1) = (0.1, 0.85)$.

III: Weak ARMA(1,1) model (18)-(20).

IV: Weak ARMA(1,1) model (18)-(21).

V: Weak ARMA(1,1) model (18)-(22).

Table B.10: Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of VARMA(1,1). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
I	$n = 200$	1	3.8	3.8	3.6	3.4	n.a.	n.a.
		2	3.7	3.9	3.6	3.5	n.a.	n.a.
		3	3.5	3.7	2.8	2.7	16.7	15.8
		4	3.4	3.6	2.7	2.4	9.3	7.6
		5	3.2	3.5	2.4	2.1	6.9	5.9
II	$n = 200$	1	2.9	2.9	5.9	5.7	n.a.	n.a.
		2	1.9	1.9	4.7	4.6	n.a.	n.a.
		3	0.6	0.7	2.9	2.9	32.5	30.8
		4	0.7	0.9	4.1	4.1	22.3	21.2
		5	1.2	1.2	5.1	5.0	17.4	16.1
III	$n = 200$	1	1.9	1.9	8.3	8.2	n.a.	n.a.
		2	0.4	0.5	12.5	12.2	n.a.	n.a.
		3	0.0	0.0	16.2	15.9	41.6	40.7
		4	0.0	0.0	22.3	22.2	31.6	29.9
		5	0.0	0.0	30.4	29.9	24.5	22.4
IV	$n = 200$	1	0.3	0.3	11.7	11.5	n.a.	n.a.
		2	0.1	0.2	14.8	14.4	n.a.	n.a.
		3	0.2	0.2	18.6	18.5	28.4	27.9
		4	0.4	0.4	23.5	23.2	21.7	21.3
		5	0.5	0.5	33.7	33.3	18.7	18.0
V	$n = 200$	1	3.8	3.8	5.1	4.7	n.a.	n.a.
		2	3.8	4.1	4.3	3.7	n.a.	n.a.
		3	3.4	3.6	3.6	3.3	16.5	15.3
		4	3.4	3.7	0.9	0.6	8.9	8.0
		5	3.1	3.2	0.6	0.5	6.0	5.2

I: Strong VARMA(1,1) model (23)-(24).

II: Weak VARMA(1,1) model (23)-(25).

III: Weak VARMA(1,1) model (23)-(26).

IV: Weak VARMA(1,1) model (23)-(27).

V: Weak VARMA(1,1) model (23)-(28).

Table C.11: Modified and standard versions of portmanteau tests to check the null hypothesis that the S&P 500 returns is a white noise.

Lag m	2	3	4	5	10	18	24
$\hat{\rho}(m)$	-0.03894	-0.03278	-0.01951	0.00802	0.00408	-0.01833	0.01885
LB_{SN}	62.8145	280.930	326.492	529.533	1425.52	1723.81	2323.05
BP_{SN}	62.7759	280.712	326.257	529.140	1424.20	1722.54	2321.68
LB_{FRZ}	16.7123	22.9587	25.1716	25.5460	28.3375	38.4913	46.6186
BP_{FRZ}	16.7021	22.9432	25.1538	25.5278	28.3145	38.4420	46.5352
p_{SN}^{LB}	0.12053	0.01302	0.02585	0.01319	0.01139	0.26024	0.48544
p_{SN}^{BP}	0.12064	0.01306	0.02593	0.01323	0.01142	0.26077	0.48610
p_{FRZ}^{LB}	0.03527	0.01917	0.02594	0.16036	0.25624	0.30714	0.34018
p_{FRZ}^{BP}	0.03533	0.01922	0.02601	0.16054	0.25657	0.30794	0.34149
p_S^{LB}	0.00023	0.00004	0.00004	0.00010	0.00159	0.00333	0.00373
p_S^{BP}	0.00024	0.00004	0.00004	0.00011	0.00160	0.00338	0.00381

Table C.12: Modified and standard versions of portmanteau tests to check the null hypothesis that the S&P 500 squared returns follow an ARMA(1, 1) model.

Lag m	1	2	3	4	5	6	7
$\hat{\rho}(m)$	-0.01978	0.04776	-0.01510	-0.05975	0.08872	-0.02572	-0.03186
LB_{SN}	7.27253	11.5733	94.7095	117.522	120.184	175.809	195.675
BP_{SN}	7.26878	11.5731	94.6567	117.370	120.028	175.496	195.237
LB_{FRZ}	2.27281	15.5306	16.8568	37.6101	83.3760	87.2225	93.1247
BP_{FRZ}	2.27164	15.5203	16.8454	37.5772	83.2880	87.1292	93.0222
p_{SN}^{LB}	0.34481	0.57434	0.16482	0.24529	0.40148	0.38794	0.48841
p_{SN}^{BP}	0.34495	0.57434	0.16495	0.24573	0.40205	0.38879	0.48967
p_{FRZ}^{LB}	0.36613	0.34701	0.32849	0.32611	0.32051	0.31997	0.74110
p_{FRZ}^{BP}	0.36608	0.34719	0.32865	0.32632	0.32076	0.32023	0.74123
p_S^{LB}	n.a.	n.a.	0.00004	0.00000	0.00000	0.00000	0.00000
p_S^{BP}	n.a.	n.a.	0.00004	0.00000	0.00000	0.00000	0.00000
Lag m	8	9	10	12	18	20	24
$\hat{\rho}(m)$	0.01686	0.00422	-0.01346	-0.00686	0.00946	-0.00175	0.01174
LB_{SN}	204.217	271.635	298.802	372.828	1072.04	1403.02	1451.54
BP_{SN}	203.764	270.793	297.731	371.484	1069.14	1397.58	1445.80
LB_{FRZ}	94.7779	94.8815	95.9359	96.3334	97.2783	98.2907	99.4676
BP_{FRZ}	94.6726	94.7760	95.8282	96.2249	97.1667	98.1754	99.3472
p_{SN}^{LB}	0.61390	0.59294	0.66719	0.76105	0.64801	0.60942	0.85991
p_{SN}^{BP}	0.61512	0.59480	0.66914	0.76285	0.65004	0.61268	0.86192
p_{FRZ}^{LB}	0.57299	0.30637	0.51343	0.57000	0.40044	0.44177	0.32510
p_{FRZ}^{BP}	0.57321	0.30664	0.51367	0.57022	0.40072	0.44205	0.32541
p_S^{LB}	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
p_S^{BP}	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table D.14: Empirical power (in %) of the modified and standard versions of the LB and BP tests in the case of ARMA(2, 1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{FRZ}	BP _{FRZ}	LB _S	BP _S
IV	$n = 500$	1	39.8	39.4	62.0	61.4	n.a.	n.a.
		2	29.7	29.6	56.5	56.5	n.a.	n.a.
		3	23.5	23.2	56.5	56.4	86.9	86.8
		6	13.0	12.4	45.2	44.7	75.8	75.7
		12	1.9	1.4	27.9	26.6	65.8	64.8
IV	$n = 2,000$	1	84.9	84.9	98.6	98.6	n.a.	n.a.
		2	76.8	76.7	98.4	98.4	n.a.	n.a.
		3	72.4	72.4	98.4	98.4	100.0	100.0
		6	1.2	61.1	97.8	97.7	99.7	99.7
		12	48.0	47.6	96.4	96.4	99.5	99.5
IV	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	99.5	99.5	100.0	100.0	n.a.	n.a.
		3	99.2	99.2	100.0	100.0	100.0	100.0
		6	98.6	98.6	100.0	100.0	100.0	100.0
		12	99.3	99.3	100.0	100.0	100.0	100.0
V	$n = 500$	1	84.6	84.5	99.0	99.0	n.a.	n.a.
		2	74.8	74.5	94.7	94.6	n.a.	n.a.
		3	70.5	70.3	94.3	94.3	97.0	96.9
		6	60.4	59.8	84.6	84.2	85.3	84.9
		12	46.5	45.9	69.3	68.0	73.7	72.7
V	$n = 2,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	99.4	99.4	100.0	100.0	n.a.	n.a.
		3	99.2	99.2	100.0	100.0	100.0	100.0
		6	98.7	98.7	100.0	100.0	100.0	100.0
		12	99.2	99.1	99.9	99.9	100.0	100.0
V	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	100.0	100.0	100.0	100.0	100.0	100.0
		6	100.0	100.0	100.0	100.0	100.0	100.0
		12	100.0	100.0	100.0	100.0	100.0	100.0

IV: Weak ARMA(2, 1) model (29)-(21).

V: Weak ARMA(2, 1) model (29)-(22).

Table E.15: Empirical size adjusted power (in %) of the modified and standard versions of the LB and BP tests at the 5% nominal level in the case of VARMA(2, 1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
I	$n = 500$	1	29.0	29.0	99.9	99.9	n.a.	n.a.
		2	21.4	21.4	99.8	99.8	n.a.	n.a.
		3	20.7	20.7	99.5	99.5	37.4	37.5
		4	15.6	15.6	99.2	99.2	31.8	31.9
		5	15.5	15.5	99.1	99.1	29.4	29.6
	$n = 2,000$	1	85.0	85.0	100.0	100.0	n.a.	n.a.
		2	78.5	78.5	100.0	100.0	n.a.	n.a.
		3	76.5	76.5	100.0	100.0	98.7	98.7
		4	69.2	69.2	100.0	100.0	97.5	97.5
		5	65.4	65.1	100.0	100.0	96.0	96.0
	$n = 10,000$	1	99.8	99.8	100.0	100.0	n.a.	n.a.
		2	99.9	99.9	100.0	100.0	n.a.	n.a.
		3	99.9	99.9	100.0	100.0	100.0	100.0
		4	100.0	100.0	100.0	100.0	100.0	100.0
		5	100.0	100.0	100.0	100.0	100.0	100.0
II	$n = 500$	1	24.0	24.0	99.1	99.1	n.a.	n.a.
		2	19.8	19.7	98.6	98.7	n.a.	n.a.
		3	18.9	18.7	98.7	98.7	28.8	28.8
		4	14.0	13.7	98.0	98.0	26.0	26.0
		5	11.8	11.6	97.3	97.4	19.7	19.7
	$n = 2,000$	1	63.7	63.7	99.9	99.9	n.a.	n.a.
		2	53.5	53.5	99.8	99.8	n.a.	n.a.
		3	53.1	52.9	99.8	99.8	82.2	82.2
		4	46.8	46.8	99.7	99.7	77.1	77.1
		5	42.2	42.1	99.9	99.9	76.9	76.9
	$n = 10,000$	1	98.8	98.8	100.0	100.0	n.a.	n.a.
		2	98.9	98.9	100.0	100.0	n.a.	n.a.
		3	99.0	99.0	100.0	100.0	100.0	100.0
		4	98.8	98.8	100.0	100.0	100.0	100.0
		5	98.4	98.4	100.0	100.0	100.0	100.0
III	$n = 500$	1	21.0	21.0	98.1	98.1	n.a.	n.a.
		2	18.4	18.4	98.2	98.0	n.a.	n.a.
		3	16.3	16.3	98.3	98.0	22.2	22.2
		4	12.5	12.2	97.0	96.9	20.1	20.1
		5	12.0	12.0	96.5	96.6	20.0	20.2
	$n = 2,000$	1	63.2	63.2	100.0	100.0	n.a.	n.a.
		2	64.2	64.2	100.0	100.0	n.a.	n.a.
		3	59.2	59.2	99.8	99.8	73.6	73.6
		4	53.0	52.9	99.9	99.9	73.1	73.1
		5	51.2	51.0	99.7	99.7	71.1	71.1
	$n = 10,000$	1	99.0	99.0	100.0	100.0	n.a.	n.a.
		2	99.6	99.6	100.0	100.0	n.a.	n.a.
		3	99.4	99.4	100.0	100.0	100.0	100.0
		4	99.7	99.7	100.0	100.0	100.0	100.0

Table E.16: Empirical size adjusted power (in %) of the modified and standard versions of the LB and BP tests at the 5% nominal level in the case of VARMA(2, 1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
IV	$n = 500$	1	28.6	28.6	98.2	98.3	n.a.	n.a.
		2	21.3	21.3	96.9	96.9	n.a.	n.a.
		3	13.6	13.8	97.5	97.5	15.4	15.5
		4	11.9	12.1	97.3	97.5	13.6	13.6
		5	13.6	13.6	97.2	97.1	13.2	13.2
IV	$n = 2,000$	1	65.6	65.6	99.7	99.7	n.a.	n.a.
		2	63.1	63.1	99.5	99.5	n.a.	n.a.
		3	56.7	56.7	98.5	98.6	48.4	48.5
		4	50.9	50.9	99.0	99.0	49.8	49.8
		5	47.7	47.7	98.4	98.4	47.5	47.4
IV	$n = 10,000$	1	97.3	97.3	100.0	100.0	n.a.	n.a.
		2	98.6	98.6	100.0	100.0	n.a.	n.a.
		3	98.4	98.4	100.0	100.0	100.0	100.0
		4	98.9	98.9	100.0	100.0	100.0	100.0
		5	98.8	98.9	100.0	100.0	100.0	100.0
V	$n = 500$	1	38.8	38.8	99.7	99.7	n.a.	n.a.
		2	28.7	28.8	99.6	99.6	n.a.	n.a.
		3	25.9	25.9	99.7	99.7	41.5	41.6
		4	22.9	23.2	99.2	99.2	35.2	35.2
		5	20.3	20.4	99.3	99.3	30.0	30.3
V	$n = 2,000$	1	88.8	88.8	100.0	100.0	n.a.	n.a.
		2	87.1	87.1	100.0	100.0	n.a.	n.a.
		3	85.4	85.4	100.0	100.0	99.2	99.2
		4	80.1	80.1	100.0	100.0	98.3	98.3
		5	79.1	79.1	100.0	100.0	97.0	97.0
V	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	99.9	99.9	100.0	100.0	100.0	100.0
		4	100.0	100.0	100.0	100.0	100.0	100.0
		5	100.0	100.0	100.0	100.0	100.0	100.0

IV: Weak VARMA(2, 1) model (E.1)-(27).

V: Weak VARMA(2, 1) model (E.1)-(28).

Table E.17: Empirical power (in %) of the modified and standard versions of the LB and BP tests in the case of VARMA(2,1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
I	$n = 500$	1	28.4	28.6	41.6	41.4	n.a.	n.a.
		2	21.7	21.8	31.3	30.3	n.a.	n.a.
		3	16.6	16.8	27.4	26.9	66.3	66.0
		4	11.8	11.9	23.8	23.4	48.9	48.3
		5	10.9	11.1	20.7	20.0	40.0	38.8
I	$n = 2,000$	1	83.6	83.7	94.2	94.2	n.a.	n.a.
		2	77.4	77.4	87.4	87.2	n.a.	n.a.
		3	73.5	73.6	88.0	88.0	99.8	99.8
		4	69.2	69.3	86.4	86.4	98.7	98.7
		5	64.5	64.6	85.6	85.6	97.7	97.7
I	$n = 10,000$	1	99.8	99.8	100.0	100.0	n.a.	n.a.
		2	99.9	99.9	100.0	100.0	n.a.	n.a.
		3	99.9	99.9	100.0	100.0	100.0	100.0
		4	100.0	100.0	99.9	99.9	100.0	100.0
		5	100.0	100.0	100.0	100.0	100.0	100.0
II	$n = 500$	1	20.5	20.6	37.4	36.9	n.a.	n.a.
		2	14.4	14.4	27.4	26.7	n.a.	n.a.
		3	9.2	9.2	21.0	20.6	74.6	74.6
		4	6.0	6.4	15.7	15.3	62.6	61.7
		5	3.8	4.0	11.0	10.8	51.5	50.3
II	$n = 2,000$	1	60.8	60.9	89.7	89.7	n.a.	n.a.
		2	53.1	53.3	81.9	81.9	n.a.	n.a.
		3	45.3	45.3	79.4	79.3	99.0	99.0
		4	41.3	41.3	77.2	77.0	97.3	97.3
		5	35.4	35.4	73.4	73.0	96.2	96.2
II	$n = 10,000$	1	98.8	98.8	99.5	99.5	n.a.	n.a.
		2	98.9	98.9	99.4	99.4	n.a.	n.a.
		3	98.8	98.8	99.4	99.4	100.0	100.0
		4	98.6	98.6	99.3	99.3	100.0	100.0
		5	98.1	98.1	99.4	99.4	100.0	100.0
III	$n = 500$	1	17.6	17.6	22.9	22.9	n.a.	n.a.
		2	8.3	8.3	15.2	14.9	n.a.	n.a.
		3	2.4	2.4	7.4	7.3	83.2	82.8
		4	0.9	0.9	3.7	3.5	71.9	71.2
		5	0.3	0.3	3.3	3.2	60.4	59.4
III	$n = 2,000$	1	62.8	62.8	78.8	78.8	n.a.	n.a.
		2	57.9	58.0	63.1	63.1	n.a.	n.a.
		3	49.1	49.1	59.0	58.9	99.7	99.7
		4	40.6	40.7	49.7	49.6	98.5	98.5
		5	35.3	35.2	41.4	41.3	96.5	96.5
III	$n = 10,000$	1	99.0	99.0	100.0	100.0	n.a.	n.a.
		2	99.7	99.7	99.9	99.9	n.a.	n.a.
		3	99.3	99.3	99.9	99.9	100.0	100.0
		4	99.6	99.6	99.9	99.9	100.0	100.0
		5	99.1	99.1	99.9	99.9	100.0	100.0

Table E.18: Empirical power (in %) of the modified and standard versions of the LB and BP tests in the case of VARMA(2, 1) model. The number of replications is $N = 1000$.

Model	Length n	Lag m	LB _{SN}	BP _{SN}	LB _{BM}	BP _{BM}	LB _S	BP _S
IV	$n = 500$	1	15.6	15.7	22.6	22.4	n.a.	n.a.
		2	5.7	6.0	14.5	14.4	n.a.	n.a.
		3	1.7	1.8	9.6	9.6	82.7	82.2
		4	0.4	0.4	10.3	10.1	70.5	70.0
		5	0.8	0.8	8.5	8.5	61.8	61.4
IV	$n = 2,000$	1	59.0	59.2	62.7	62.7	n.a.	n.a.
		2	54.9	54.9	44.5	44.5	n.a.	n.a.
		3	46.1	46.2	36.7	36.6	99.3	99.3
		4	35.5	35.5	29.3	29.3	98.5	98.5
		5	25.5	25.8	23.3	23.2	96.5	96.5
IV	$n = 10,000$	1	97.3	97.3	98.6	98.6	n.a.	n.a.
		2	97.8	97.8	96.9	96.9	n.a.	n.a.
		3	97.9	97.9	96.5	96.5	100.0	100.0
		4	98.4	98.4	96.3	96.3	100.0	100.0
		5	97.4	97.4	96.1	96.1	100.0	100.0
V	$n = 500$	1	34.1	34.1	48.7	48.6	n.a.	n.a.
		2	27.0	27.1	33.2	32.9	n.a.	n.a.
		3	23.2	23.3	27.3	26.5	67.7	67.3
		4	18.6	19.0	23.3	23.0	46.7	45.9
		5	15.0	15.3	20.8	20.2	38.5	37.1
V	$n = 2,000$	1	91.2	91.2	98.9	98.9	n.a.	n.a.
		2	87.4	87.4	94.7	94.7	n.a.	n.a.
		3	84.1	84.2	93.8	93.8	99.9	99.9
		4	79.8	79.8	92.1	92.1	99.2	99.2
		5	76.7	76.8	90.8	90.8	98.3	98.3
V	$n = 10,000$	1	100.0	100.0	100.0	100.0	n.a.	n.a.
		2	100.0	100.0	100.0	100.0	n.a.	n.a.
		3	99.9	99.9	100.0	100.0	100.0	100.0
		4	100.0	100.0	100.0	100.0	100.0	100.0
		5	100.0	100.0	100.0	100.0	100.0	100.0

IV: Weak VARMA(2, 1) model (E.1)-(27).

V: Weak VARMA(2, 1) model (E.1)-(28).