Portmanteau test for a class of multivariate asymmetric power GARCH model

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Abstract

We establish the asymptotic behaviour of the sum of squared residuals autocovariances and autocorrelations for the class of multivariate power transformed asymmetric models. We then derive a portmanteau test. We establish the asymptotic distribution of the proposed statistics. These asymptotic results are illustrated by Monte Carlo experiments. An application to a bivariate real financial data is also proposed.

Keywords: Multivariate GARCH, asymmetric power GARCH, multivariate portmanteau test.

1. Introduction

In econometric application, the univariate generalized autoregressive conditional heteroscedasticity (GARCH) framework is very restrictive. Consequently the class of multivariate models is commonly used in time series analysis and econometrics. It describes the possible cross-relationships between the time series and not only the properties of the individual time series (see for instance Francq and Zakoïan (2019), Lütkepohl (2005)). There are many extensions of multivariate GARCH models (MGARCH) with many approaches because the specification of the GARCH model does not suggest a natural extension to the multivariate framework. See for instance Bauwens et al. (2006) for a survey on MGARCH models. See also Silvennoinen and Teräsvirta (2009) and Bauwens et al. (2012) for recent surveys on MGARCH processes. The MGARCH model with conditional constant correlation (CCC-GARCH) introduced by Bollerslev (1990) and extended by Jeantheau (1998), seems to be one of the most popular models used to model multivariate financial series. Francq and Zakoïan (2012) proposed an asymmetric CCC-GARCH (CCC-AGARCH) model that includes the CCC-GARCH introduced by Bollerslev (1990) and its generalization by Jeantheau (1998). In all this work, we use the following notation $\underline{u}^{\underline{v}} := (u_1^{v_1}, \ldots, u_d^{v_d})'$ for $\underline{u}, \underline{v} \in \mathbb{R}^d$, and $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$. We consider the asymmetric power GARCH model with constant conditional correlation (CCC-APGARCH(p, q) for short) proposed by Boubacar Maïnassara et al. (2022) and defined by

$$\begin{cases} \underline{\varepsilon}_t = H_{0t}^{1/2}\eta_t, \\ H_{0t} = D_{0t}R_0D_{0t}, \qquad D_{0t} = \operatorname{diag}(\sqrt{h_{1,0t}}, \ldots, \sqrt{h_{d,0t}}), \\ \underline{h}_{0t}^{\delta_0/2} = \underline{\omega}_0 + \displaystyle\sum_{i=1}^{q} \left\{ A_{0i}^+(\underline{\varepsilon}_{t-i}^+)^{\delta_0/2} + A_{0i}^-(\underline{\varepsilon}_{t-i}^-)^{\delta_0/2} \right\} + \sum_{j=1}^{p} B_{0j}\underline{h}_{0t-j}^{\delta_0/2}, \end{cases} \tag{1.1}$$

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where $\underline{\varepsilon}_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{d,t})'$ is a d-dimensional process, $\underline{h}_{0t} = (h_{1,0t}, \ldots, h_{d,0t})'$, $h_{i,0t}$ is the conditional variance of $\varepsilon_{i,t} = h_{i,0t}^{1/2}\eta_{i,t}$ for $i = 1, \ldots, d$,

$$\underline{\varepsilon}_t^+ = \left(\{\varepsilon_{1,t}^+\}^2, \ldots, \{\varepsilon_{d,t}^+\}^2\right)' \qquad \underline{\varepsilon}_t^- = \left(\{\varepsilon_{1,t}^-\}^2, \ldots, \{\varepsilon_{d,t}^-\}^2\right)',$$

$\underline{\omega}_0$ and $\underline{\delta}_0$ are vectors of size $d \times 1$ with strictly positive coefficients, A_{0i}^+, A_{0i}^- and B_{0j} are matrices of size $d \times d$ with positive coefficients and R_0 is a correlation matrix and where the innovation process $(\eta_t)_t$ is an independent and identically distributed (iid for short) sequence of variables on $\mathbb{R}^d$ with identity covariance matrix and $\mathbb{E}[\eta_t] = 0$. The parameters of the model are the coefficients of the vectors $\underline{\omega}_0$, $\underline{\delta}_0$, the coefficients of the matrices $A_{0i}^+, A_{0i}^-, B_{0j}$ and the coefficients in the lower triangular part excluding the diagonal of the matrix R_0.

Model (1.1) includes various GARCH class models: for $\underline{\delta}_0 = (2, \ldots, 2)'$, we obtain the CCC-AGARCH of Francq and Zakoïan (2012); when $d = 1$ and $\underline{\delta}_0 = 2$, we retrieve the threshold GARCH (TGARCH) of Rabemananjara and Zakoïan (1993). The asymptotic properties of the quasi-maximum likelihood (QML) estimation of the model (1.1) are established by Boubacar Maïnassara et al. (2022) when the power $\underline{\delta}_0$ is assumed to be known or unknown.

As mentioned by Francq and Zakoïan (2012), the attractiveness of the CCC-AGARCH models follows from their tractability. They mention three main reasons : the number of unknown coefficients is less than in other specifications and remains tractable in small dimension; the coefficients are easy to interpret; the conditions ensuring positive definiteness of the conditional variance are simple and explicit. There is also an advantage concerning the strict stationarity conditions which are explicit too. In our work, we pass from a constant power CCC-AGARCH to a component-varying power CCC-AGARCH model. In addition to the theoretical contribution, the numerical illustrations proposed in Section 5 highlight the value of this work. To be more specific, our study on real dataset proves that a component-varying power is relevant for the daily exchange rates of the (Dollar,Yen) against the Euro (see Table 1).

In CCC-APGARCH(p, q) models, the choice of p and q is particularly important because the number of parameters quickly increases with p and q, which entails statistical difficulties. After identification and estimation of the MGARCH processes, the next important step in the modeling consists in checking if the estimated model fits the data satisfactorily. This adequacy checking step allows to validate or invalidate the choice of the orders p and q. Thus it is important to check the validity of a MGARCH(p, q) model, for given orders p and q. Based on the residuals empirical autocorrelations, Box and Pierce (1970) derived a goodness-of-fit test, the portmanteau test, for univariate strong autoregressive moving-average (ARMA) models (i.e. under the assumption that the error term is iid). Ljung and Box (1978) proposed a modified portmanteau test which is nowadays one of the most popular diagnostic checking tools in ARMA modeling of time series. Since the articles by Ljung and Box (1978) and McLeod (1978), portmanteau tests have been important tools in time series analysis, in particular for testing the adequacy of an estimated ARMA(p, q) model. See also Li (2004), for a reference book on the portmanteau tests. The standard portmanteau tests consist in rejecting the adequacy of the model for large values of some quadratic form of the residuals autocorrelations. These tests cannot be applied directly to conditional heteroscedasticity or other processes displaying a second order dependence. Indeed such non-linearities may arise for instance when the observed process follows a GARCH representation. Consequently Li and Mak (1994) and Ling and Li (1997b) proposed a portmanteau test based on the autocorrelations of the squared residuals. The intuition behind this portmanteau test is that when the model is correctly specified, the autocorrelations for squared residuals will be close to zero. Other situations where the standard tests do not give satisfactory results can also be found for instance in Relvas and Paula (2016), Cao et al. (2010), Francq et al. (2005), Boubacar Maïnassara and Ilmi Amir (2020), Boubacar Maïnassara and Saussereau (2018), Boubacar Mainassara (2011).

The asymptotic theory on MGARCH model diagnostic checking is mainly limited to the univariate framework. As above-mentioned, Li and Mak (1994) and Ling and Li (1997b) studied a portmanteau

test based on the autocorrelations of the squared residuals. Berkes et al. (2003) developed an asymptotic theory of portmanteau tests in the standard GARCH framework, Leucht et al. (2015) suggested a consistent specification test for GARCH$(1,1)$ model. Recently, Jiménez-Gamero et al. (2020) proposed goodness-of-fit tests for certain parametrizations of conditionally heteroscedastic time series with unobserved components. Francq et al. (2016) proposed a portmanteau test for the Log-GARCH model and the exponential GARCH (EGARCH) model. For the univariate APGARCH model, a portmanteau test based on the autocovariances of the squared residuals is developed by Carbon and Francq (2011) for the APGARCH model when the power $\underline{\delta}_0$ is known and by Boubacar Maïnassara et al. (2021) when the power $\underline{\delta}_0$ is unknown and is jointly estimated with the other parameters. See also Ben and Jiang (2020) who recently extended the work of Carbon and Francq (2011) (when $\underline{\delta}_0$ is known and when some parameters lie on the boundary) to the class of APGARCH with exogenous covariates (APGARCH-X). In the multivariate analysis, there are a few works. Ling and Li (1997a) proposed portmanteau statistic for multivariate conditional heteroscedasticity models (see also Duchesne and Lalancette (2003) and Duchesne and Lalancette (2010)). Duchesne (2004) (see also Duchesne (2010)) introduced the test which is a direct generalization of the portmanteau test of Li and Mak (1994) to the VEC-GARCH model. Wang and Tsay (2013) extend Duchesne's approach to the case of multivariate GARCH models with Student$-t$ innovations. Recently, Ke et al. (2021) provide a residual-based approach to examine the adequacy of multivariate GARCH models. Other tests for multivariate ARCH models include those developed can be found for instance in Kroner and Ng (1998), Tse and Tsui (1999) and Wong and Li (2002).

Contrary to the univariate APGARCH models, there are no validation tests for the class of the model (1.1). In this paper we generalize the results obtained by Carbon and Francq (2011), Boubacar Maïnassara et al. (2021) and Ling and Li (1997a) to the CCC-APGARCH(p,q) models defined in (1.1). This extension raises difficult problems. First, non trivial constraints on the parameters must be imposed for identifiability of the parameters (see Francq and Zakoïan (2019)). Secondly, the implementation of standard estimation methods (for instance the Gaussian quasi-maximum likelihood estimation) is not obvious because this requires a constrained high-dimensional optimization (see also Lütkepohl (2005)). These technical difficulties certainly explain why univariate GARCH models are much more used than MGARCH in applied works.

The paper is organized as follows. In Section 2 we recall the results on the quasi-maximum likelihood estimator (QMLE) and its asymptotic distribution obtained by Boubacar Maïnassara et al. (2022) when the power $\underline{\delta}_0$ is known or unknown. Section 3 presents our main results which give the asymptotic theory of the sum of squared residuals autocovariances and autocorrelations for the wide class of CCC-APGARCH models (1.1) when the power $\underline{\delta}_0$ is known (Section 3.1) and when the power $\underline{\delta}_0$ is unknown and estimated (Section 3.2). In Section 4 we test the null hypothesis of the CCC-APGARCH model for different values of $\underline{\delta}_0$ in both cases. Section 5 illustrates the proposed tests for CCC-APGARCH models applied to a bivariate exchange rates.

2. Quasi-maximum likelihood estimation

When the power $\underline{\delta}_0' = (\delta_{0,1}, ..., \delta_{0,d})$ is known, we write

$$\theta := (\underline{\omega}', \alpha_1^{+\prime}, \ldots, \alpha_q^{+\prime}, \alpha_1^{-\prime}, \ldots, \alpha_q^{-\prime}, \beta_1', \ldots, \beta_p', \rho')',$$

where α_i^+ and α_i^- are defined by $\alpha_i^\pm = \text{vec}(A_i^\pm)$ for $i = 1, \ldots, q$, $\beta_j = \text{vec}(B_j)$ for $j = 1, \ldots, p$, and $\rho = (\rho_{21}, \ldots \rho_{d1}, \rho_{32}, \ldots, \rho_{d2}, \ldots, \rho_{dd-1})'$ such that the ρ_{ij}'s are the components of the matrix R. The parameter θ belongs to the parameter space

$$\Theta \subset]0, +\infty[^d \times [0, \infty[^{d^2(2q+p)} \times]-1, 1[^{d(d-1)/2}.$$

The unknown true parameter value is denoted by

$$\theta_0 := (\underline{\omega}_0', \alpha_{01}^{+\prime}, \ldots, \alpha_{0q}^{+\prime}, \alpha_{01}^{-\prime}, \ldots, \alpha_{0q}^{-\prime}, \beta_{01}{}', \ldots, \beta_{0p}{}', \rho_0')'.$$

Similarly when the power $\underline{\delta} = (\delta_1, ..., \delta_d)'$ is unknown and is jointly estimated with the parameter θ we denote by $\vartheta := (\theta', \underline{\delta}')'$. The parameter ϑ belongs to the parameter space

$$\Delta \subset]0, +\infty[^d \times [0, \infty[^{d^2(2q+p)} \times]-1, 1[^{d(d-1)/2} \times]0, +\infty[^d.$$

The unknown true parameter value is denoted by $\vartheta_0 := (\theta_0', \underline{\delta}_0')'$, where $\underline{\delta}_0 = (\delta_{0,1}, ..., \delta_{0,d})'$.

2.1. Estimation when the power $\underline{\delta}_0$ is known

The goal is to estimate the $s_0 = d + d^2(p + 2q) + d(d-1)/2$ coefficients of the model (1.1). For all $\theta \in \Theta$ we let $H_t = H_t(\theta)$. We assume that H_t is a strictly stationary and non anticipative solution of

$$\begin{cases} H_t = D_t R D_t, \qquad D_t = \operatorname{diag}\left(\sqrt{h_{1,t}}, \ldots, \sqrt{h_{d,t}}\right), \qquad R = R(\theta), \\ \underline{h}_t^{\underline{\delta}_0/2} := \underline{h}_t^{\underline{\delta}_0/2}(\theta) = \underline{\omega} + \sum_{i=1}^{q} \left\{ A_i^+ (\underline{\varepsilon}_{t-i}^+)^{\underline{\delta}_0/2} + A_i^- (\underline{\varepsilon}_{t-i}^-)^{\underline{\delta}_0/2} \right\} + \sum_{j=1}^{p} B_j \underline{h}_{t-j}^{\underline{\delta}_0/2}. \end{cases} \tag{2.1}$$

Given a realization $(\underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_n)$ of length n satisfying the representation (1.1), the variable H_t can be approximated for $t \geq 1$ by $\tilde{H}_t$ defined recursively by

$$\begin{cases} \tilde{H}_t = \tilde{D}_t R \tilde{D}_t, \qquad \tilde{D}_t = \operatorname{diag}\left(\sqrt{\tilde{h}_{1,t}}, \ldots, \sqrt{\tilde{h}_{d,t}}\right) \\ \tilde{\underline{h}}_t^{\underline{\delta}_0/2} := \tilde{\underline{h}}_t^{\underline{\delta}_0/2}(\theta) = \underline{\omega} + \sum_{i=1}^{q} \left\{ A_i^+ (\underline{\varepsilon}_{t-i}^+)^{\underline{\delta}_0/2} + A_i^- (\underline{\varepsilon}_{t-i}^-)^{\underline{\delta}_0/2} \right\} + \sum_{j=1}^{p} B_j \tilde{\underline{h}}_{t-j}^{\underline{\delta}/2}, \end{cases}$$

conditional to the initial values $\underline{\varepsilon}_0, \ldots, \underline{\varepsilon}_{1-q}, \tilde{\underline{h}}_0, \ldots, \tilde{\underline{h}}_{1-p}$. The quasi-maximum likelihood (QML) method is particularly efficient for the MGARCH class models because it provides consistent and asymptotically normal estimator for strictly stationary MGARCH processes under mild regularity conditions (but with no moment assumptions on the observed process). The quasi-maximum likelihood estimator (QMLE) of model (1.1) is obtained by the standard estimation procedure for MGARCH class models. Thus the QMLE of θ_0 of model (1.1) is defined as any measurable solution $\hat{\theta}_n$ of

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \tilde{l}_t(\theta), \qquad \tilde{l}_t(\theta) = \underline{\varepsilon}_t' \tilde{H}_t^{-1} \underline{\varepsilon}_t + \log(\det(\tilde{H}_t)). \tag{2.2}$$

To ensure the asymptotic properties of the QMLE for model (1.1) obtained by Boubacar Maïnassara et al. (2022), we need the following assumptions:

A1: $\theta_0 \in \Theta$ and Θ is compact.

Now, we rewrite the first equation of (1.1) as

$$\underline{\varepsilon}_t = D_t \overline{\eta}_t, \qquad \text{where } \overline{\eta}_t = (\overline{\eta}_{1,t}, \ldots, \overline{\eta}_{d,t})' = R_0^{1/2} \eta_t.$$

Using the third equation of model (1.1), we may write

$$(\underline{\varepsilon}_t^{\pm})^{\underline{\delta}_0/2} = (\Upsilon_t^{\pm,(\underline{\delta}_0)}) \underline{h}_{0t}^{\underline{\delta}_0/2}, \text{ with } \Upsilon_t^{\pm,(\underline{\delta}_0)} = \operatorname{diag}\left((\pm \overline{\eta}_{1,t}^{\pm})^{\delta_{0,1}}, \ldots, (\pm \overline{\eta}_{d,t}^{\pm})^{\delta_{0,d}} \right).$$

A2: $\forall \theta \in \Theta, \det(\mathcal{B}_0(z)) = 0 \Rightarrow |z| > 1$ for $\mathcal{B}_0(z) = I_d - \sum_{j=1}^{p} B_{0j} z^j$ and $\gamma(\mathbf{C}_0) < 0$, where $\gamma(\cdot)$ is the top Lyapunov exponent of the sequence of matrix $C_0 = \{C_{0t}, t \in \mathbb{Z}\}$ with the matrix C_{0t} of size $(p + 2q)d \times (p + 2q)d$ been defined by

$$C_{0t} = \begin{pmatrix} \Upsilon_t^{+,(\underline{\delta}_0)} A_{01:q}^+ & \Upsilon_t^{+,(\underline{\delta}_0)} A_{01:q}^- & \Upsilon_t^{+,(\underline{\delta}_0)} B_{01:p} \\ I_{d(q-1)} & 0_{d(q-1) \times d(p+q+1)} & \\ \Upsilon_t^{-,(\underline{\delta}_0)} A_{01:q}^+ & \Upsilon_t^{-,(\underline{\delta}_0)} A_{01:q}^- & \Upsilon_t^{-,(\underline{\delta}_0)} B_{01:p} \\ 0_{d(q-1) \times dq} & I_{d(q-1)} & 0_{d(q-1) \times d(p+1)} \\ A_{01:q}^+ & A_{01:q}^- & B_{01:p} \\ 0_{d(p-1) \times 2dq} & I_{d(p-1)} & 0_{d(p-1) \times d} \end{pmatrix},$$

where the $d\times qd$ matrices $A^+_{01:q}=(A^+_{01}\dots A^+_{0q})$, $A^-_{01:q}=(A^-_{01}\dots A^-_{0q})$ and $B_{01:q}=(B_{01}\dots B_{0q})\in\mathbb{R}^{d\times pd}$.

A3: For $i=1,\dots,d$ the distribution of $\overline{\eta}_{it}$ is not concentrated on 2 points and $\mathbb{P}(\overline{\eta}_{it}>0)\in(0,1)$.

A4: For $\mathcal{A}^+_0(z)=\sum_{i=1}^q A^+_{0i}z^i$ and $\mathcal{A}^-_0(z)=\sum_{i=1}^q A^-_{0i}z^i$ if $p>0, \mathcal{A}^+_0(1)+\mathcal{A}^-_0(1)\neq 0$, $\mathcal{A}^+_0(z), \mathcal{A}^-_0(z)$ and $\mathcal{B}_0(z)$ are left-coprime and the matrix

$$M(\mathcal{A}^+_0,\mathcal{A}^-_0,\mathcal{B}_0)=\left[a^+_{q_1^+}(1)\dots a^+_{q_d^+}(d)a^-_{q_1^-}(1)\dots a^-_{q_d^-}(d)b_{p_1}(1)\dots b_{p_d}(d)\right]$$

has full rank d, with $q_i^+=q_i^+(\theta_0), q_i^-=q_i^-(\theta_0)$ and $p_i=p_i(\theta_0)$ for any value of i, where $q_i^+(\theta_0)$, $q_i^-(\theta_0)$, and $p_i(\theta_0)$ denote the maximal degrees for any column i of the matrix operators $\mathcal{A}^+_0$, $\mathcal{A}^-_0$ and $\mathcal{B}_0$. We also denote by $a^+_{q_i^+}(i)$ the column vector of the coefficients $L^{q_i^+}$, by $a^-_{q_i^-}(i)$ the column vector of the coefficients $L^{q_i^-}$ in the column i of $\mathcal{A}^+_0$, respectively $\mathcal{A}^-_0$ and by $b_{p_i}(i)$ the column vector of the coefficients L^{p_i} in the column i of $\mathcal{B}_0$.

A5: R is a positive-definite correlation matrix for all $\theta\in\Theta$.

A6: $\theta_0\in\overset{\circ}{\Theta}$, where $\overset{\circ}{\Theta}$ is the interior of Θ.

A7: $\mathbb{E}\|\eta_t\eta_t'\|^2<\infty$.

Then under Assumptions **A1**–**A7**, Boubacar Maïnassara et al. (2022) showed that $\hat\theta_n\to\theta_0$ a.s. when n goes to infinity and $\sqrt{n}(\hat\theta_n-\theta_0)$ is asymptotically normal with mean 0 and covariance matrix $\Omega:=J^{-1}IJ^{-1}$, where J is a positive-definite matrix and I is a positive semi-definite matrix, defined by

$$I:=I(\theta_0)=\mathbb{E}\left[\frac{\partial l_t(\theta_0)}{\partial\theta}\frac{\partial l_t(\theta_0)}{\partial\theta'}\right],\quad J:=J(\theta_0)=\mathbb{E}\left[\frac{\partial^2 l_t(\theta_0)}{\partial\theta\partial\theta'}\right]\quad\text{with } l_t(\theta)=\underline{\varepsilon}_t'H_t^{-1}\underline{\varepsilon}_t+\log(\det(H_t)).$$

2.2. Estimation when the power $\underline{\delta}_0$ is unknown

Similar to the previous section we have $s_0=2d+d^2(p+2q)+d(d-1)/2$ coefficients of model (1.1) to estimate. In order to ensure that parameter $\underline{\delta}_0$ is identified we need the following additional assumption:

A8: η_t has a positive density on some neighbourhood of zero.

For all $\vartheta\in\Delta$ we let $\mathcal{H}_t=\mathcal{H}_t(\vartheta)$. We assume that $\mathcal{H}_t$ is a strictly stationary and non anticipative solution of

$$\begin{cases}\mathcal{H}_t=D_tRD_t, \qquad D_t=\operatorname{diag}\left(\sqrt{h_{1,t}},\dots,\sqrt{h_{d,t}}\right)\\ \underline{h}_t:=\underline{h}_t(\vartheta)=\left(\underline{\omega}+\sum_{i=1}^q A_i^+(\underline{\varepsilon}^+_{t-i})^{\underline{\delta}/2}+A_i^-(\underline{\varepsilon}^-_{t-i})^{\underline{\delta}/2}+\sum_{j=1}^p B_j\underline{h}_{t-j}^{\underline{\delta}/2}\right)^{2/\underline{\delta}}.\end{cases}\tag{2.3}$$

Conditionally to the initial values $\underline{\varepsilon}_0,\dots,\underline{\varepsilon}_{1-q},\underline{\tilde h}_0,\dots,\underline{\tilde h}_{1-p}$, for $t\geq 1$ the variable $\mathcal{H}_t$ can also be approximated recursively by

$$\begin{cases}\tilde{\mathcal{H}}_t=\tilde D_tR\tilde D_t, \qquad \tilde D_t=\operatorname{diag}\left(\sqrt{\tilde h_{1,t}},\dots,\sqrt{\tilde h_{d,t}}\right)\\ \underline{\tilde h}_t:=\underline{\tilde h}_t(\vartheta)=\left(\underline{\omega}+\sum_{i=1}^q A_i^+(\underline{\varepsilon}^+_{t-i})^{\underline{\delta}/2}+A_i^-(\underline{\varepsilon}^-_{t-i})^{\underline{\delta}/2}+\sum_{j=1}^p B_j\underline{\tilde h}_{t-j}^{\underline{\delta}/2}\right)^{2/\underline{\delta}}.\end{cases}$$

The QMLE of ϑ_0 is defined as any measurable solution $\hat\vartheta_n$ of

$$\hat\vartheta_n=\arg\min_{\vartheta\in\Delta}\frac{1}{n}\sum_{t=1}^n\tilde\ell_t(\vartheta),\qquad\tilde\ell_t(\vartheta)=\underline{\varepsilon}_t'\tilde{\mathcal{H}}_t^{-1}\underline{\varepsilon}_t+\log(\det(\tilde{\mathcal{H}}_t)).\tag{2.4}$$

To ensure the asymptotic properties of the QMLE of ϑ_0 for model (1.1) obtained by Boubacar Maïnassara et al. (2022), we need assumptions similar to those we assumed in the case when the power $\underline{\delta}_0$ is known. We will assume Assumptions **A1**–**A6** with parameter θ replaced by ϑ and the space parameter Θ replaced by Δ.

Under Assumptions **A1**–**A8** Boubacar Maïnassara et al. (2022) showed that $\hat{\vartheta}_n \to \vartheta_0$ a.s. when n goes to infinity and $\sqrt{n}(\hat{\vartheta}_n - \vartheta_0)$ is asymptotically normal with mean 0 and covariance matrix $\Omega := \mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}$, where $\mathcal{J}$ is a positive-definite matrix and $\mathcal{I}$ is a positive semi-definite matrix, defined by

$$\mathcal{I} := \mathcal{I}(\vartheta_0) = \mathbb{E}\left[\frac{\partial \ell_t(\vartheta_0)}{\partial \vartheta}\frac{\partial \ell_t(\vartheta_0)}{\partial \vartheta'}\right], \quad \mathcal{J} := \mathcal{J}(\vartheta_0) = \mathbb{E}\left[\frac{\partial^2 \ell_t(\vartheta_0)}{\partial \vartheta \partial \vartheta'}\right] \quad \text{with } \ell_t(\vartheta) = \underline{\varepsilon}_t' \mathcal{H}_t^{-1}\underline{\varepsilon}_t + \log(\det(\mathcal{H}_t)).$$

In all the sequel we denote by $\xrightarrow{\mathrm{d}}$ the convergence in distribution. The symbol $\mathrm{o}_{\mathbb{P}}(1)$ is used for a sequence of random variables that converge to zero in probability.

3. Diagnostic checking with portmanteau tests

To check the adequacy of a given multivariate time series model, for instance for an estimated VARMA(p, q) model, it is common practice to test the significance of the multivariate residuals autocovariances. In the MGARCH framework this approach is not relevant because the process η_t is always a white noise (possibly a martingale difference) even when the volatility is misspecified. For this reason the following portmanteau test is based on the squared residuals autocovariances. The null hypothesis is

$$\boldsymbol{\mathcal{H}}_0 : \text{the process } (\underline{\varepsilon}_t) \text{ satisfies model (1.1).}$$

3.1. Portmanteau test when the power $\underline{\delta}_0$ is known

Let $\hat{\eta}_t = \tilde{\eta}_t(\hat{\theta}_n) = \tilde{H}_t^{-1/2}(\hat{\theta}_n)\underline{\varepsilon}_t = \hat{H}_t^{-1/2}\underline{\varepsilon}_t$ be the QMLE residuals when $p + q > 0$ and where $\tilde{\eta}_t(\theta) = \tilde{H}_t^{-1/2}(\theta)\underline{\varepsilon}_t$.

We define the autocovariances of the sum of squared residuals at lag $h > 0$, for $h < n$, by

$$\begin{aligned}
\hat{r}_h = \tilde{r}_h(\hat{\theta}_n) \qquad \text{where} \quad \tilde{r}_h(\theta) &= \frac{1}{n}\sum_{t=h+1}^{n} [\mathrm{Tr}(\tilde{s}_t(\theta))][\mathrm{Tr}(\tilde{s}_{t-h}(\theta))] \qquad \text{with } \tilde{s}_t(\theta) = \tilde{\eta}_t(\theta)\tilde{\eta}_t'(\theta) - I_d \\
&= \frac{1}{n}\sum_{t=h+1}^{n} [\tilde{\eta}_t'(\theta)\tilde{\eta}_t(\theta) - d][\tilde{\eta}_{t-h}'(\theta)\tilde{\eta}_{t-h}(\theta) - d] \\
&= \frac{1}{n}\sum_{t=h+1}^{n} [\underline{\varepsilon}_t'\tilde{H}_t^{-1}(\theta)\underline{\varepsilon}_t - d][\underline{\varepsilon}_{t-h}'\tilde{H}_{t-h}^{-1}(\theta)\underline{\varepsilon}_{t-h} - d].
\end{aligned}$$

Similarly we define the "empirical" autocovariances of the sum of squared white noise at lag h by

$$r_h = r_h(\theta_0) \qquad \text{where} \quad r_h(\theta) = \frac{1}{n}\sum_{t=h+1}^{n} [\mathrm{Tr}(s_t(\theta))][\mathrm{Tr}(s_{t-h}(\theta))],$$

with $s_t(\theta) = \eta_t(\theta)\eta_t'(\theta) - I_d$ and $\eta_t(\theta) = H_t^{-1/2}(\theta)\underline{\varepsilon}_t$. It should be noted that r_h is not a statistic (unless if $p = q = 0$) because it depends on the unobserved innovations η_t.

For a fixed integer $m \geq 1$ and in the sequel we will also need these following vectors:

$$\hat{\mathbf{r}}_m = (\hat{r}_1, \ldots, \hat{r}_m)' \quad \text{and} \quad \mathbf{r}_m = (r_1, \ldots, r_m)', \text{ such that } 1 \leq m \leq n.$$

To ensure the invertibility of the asymptotic covariance matrix of the vector of the sum of squared residuals autocovariances we need the following technical assumption on the distribution of η_t.

A9: For $d \geq 2$, η_t takes more than $3(d+1)$ positive values and more than $3(d+1)$ negative values.

Let $\mathbb{S}_{t-1:t-m} = (S_{t-1}, \ldots, S_{t-m})'$, where $S_t = \eta_t'\eta_t - d$. The following theorem gives the asymptotic distribution of the vector of the sum of squared residuals autocovariances.

Theorem 3.1. *Under Assumptions **A1**–**A7** and **A9**, if $\underline{\varepsilon}_t$ is the non-anticipative and stationary solution of the CCC-APGARCH(p,q) model defined in* (1.1)*, then we have*

$$\sqrt{n}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, D), \quad \textit{where} \quad D = \left(\mathbb{E}\left[S_t^2\right]\right)^2 I_m + C_m J^{-1} I J^{-1} C_m' + C_m \Sigma_{\hat{\theta}_n, \mathbf{r}_m} + \Sigma_{\hat{\theta}_n, \mathbf{r}_m}' C_m'$$

is a non-singular matrix and where $\Sigma_{\hat{\theta}_n, \mathbf{r}_m} = \mathbb{E}\left[J^{-1}\mathbf{h}_t' vec(\eta_t\eta_t' - I_d) S_t \mathbb{S}_{t-1:t-m}'\right]$ and the matrix C_m is given by (6.10) *in the proof of Theorem 3.1.*

The proof of Theorem 3.1 is postponed to Section 6.

Remark 3.1. *When we assume that: $\mathbb{E}(\eta_{it}^3) = 0$, for $i, = 1, \ldots, d$; for $i, j \in \{1, \ldots, d\}$ and $i \neq j$, η_{it} and η_{jt} are mutually uncorrelated up to the fourth order and η_{it}'s have the same fourth order moment, we have: $\mathbb{E}\left[S_t^2\right] = d\left(\mathbb{E}\left[\eta_{it}^4\right] - 1\right)$ and $\Sigma_{\hat{\theta}_n, \mathbf{r}_m} = -\left(\mathbb{E}\left[\eta_{it}^4\right] - 1\right) J^{-1} C_m'$. Thus we obtain*

$$D = d^2\left(\mathbb{E}\left[\eta_{it}^4\right] - 1\right)^2 I_m + C_m\left(J^{-1} I J^{-1} - 2\left(\mathbb{E}\left[\eta_{it}^4\right] - 1\right) J^{-1}\right) C_m'.$$

Therefore we retrieve the well-known result obtained by Ling and Li (1997a).

The above theorem is useless for practical purpose because it does not involve any observable quantities. In order to state our second result we need to define a consistent estimator of the asymptotic matrix D (see Theorem 3.1).

In view of Boubacar Maïnassara et al. (2022) the matrices I and J can be estimated by their empirical or observable counterparts given by

$$\hat{I}(i,j) = \frac{1}{n}\sum_{t=1}^{n}\left[Tr\left(\left(\hat{H}_t^{-1} - \hat{H}_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}_t'\hat{H}_t^{-1}\right)\frac{\partial \hat{H}_t}{\partial \theta_i}\right) Tr\left(\left(\hat{H}_t^{-1} - \hat{H}_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}_t'\hat{H}_t^{-1}\right)\frac{\partial \hat{H}_t}{\partial \theta_j}\right)\right]$$

$$\text{and} \quad \hat{J}(i,j) = \frac{1}{n}\sum_{t=1}^{n}\left[Tr\left(\hat{H}_t^{-1}\frac{\partial \hat{H}_t}{\partial \theta_j}\hat{H}_t^{-1}\frac{\partial \hat{H}_t}{\partial \theta_i}\right)\right], \quad \text{for} \quad i,j = 1, \ldots, s_0.$$

Let $\hat{\Sigma}_{\hat{\theta}_n, \mathbf{r}_m}$ and $\hat{C}_m$ be weakly consistent estimators of $\Sigma_{\hat{\theta}_n, \mathbf{r}_m}$ and C_m involved in the asymptotic normality of $\sqrt{n}\hat{\mathbf{r}}_m$. Define the matrix $\hat{C}_m$ of size $m \times s_0$ whose $(h,i)-$th element is given by

$$\hat{C}_m(h,i) = -\frac{1}{n}\sum_{t=h+1}^{n}\hat{S}_{t-h}\mathrm{Tr}\left(\hat{H}_t^{-1}\frac{\partial \tilde{H}_t(\hat{\theta}_n)}{\partial \theta_i}\right) \text{ for } 1 \leq h \leq m \text{ and } 1 \leq i \leq s_0,$$

where $\hat{S}_t = \underline{\varepsilon}_t'\hat{H}_t^{-1}\underline{\varepsilon}_t - d$. The matrix $\Sigma_{\hat{\theta}_n, \mathbf{r}_m}$ can be estimated by its empirical or observable counterpart given by

$$\hat{\Sigma}_{\hat{\theta}_n, \mathbf{r}_m} = \frac{1}{n}\sum_{t=1}^{n}\hat{J}^{-1}\hat{\mathbf{h}}_t'\mathrm{vec}(\hat{\eta}_t\hat{\eta}_t' - d)\hat{S}_t\hat{\mathbb{S}}_{t-1:t-m}' \quad \text{where} \quad \hat{\mathbf{h}}_t(i) = \mathrm{vec}\left(\hat{H}_t^{-1/2}\frac{\partial \tilde{H}_t(\hat{\theta}_n)}{\partial \theta_i}\hat{H}_t^{-1/2}\right) \text{ for } 1 \leq i \leq s_0.$$

Let $\hat{D} = \hat{\kappa}^2 I_m + \hat{C}_m\hat{J}^{-1}\hat{I}\hat{J}^{-1}\hat{C}_m' + \hat{C}_m\hat{\Sigma}_{\hat{\theta}_n, \mathbf{r}_m} + \hat{\Sigma}_{\hat{\theta}_n, \mathbf{r}_m}'\hat{C}_m'$ and $\hat{\kappa} = n^{-1}\sum_{t=1}^{n}\hat{S}_t^2$ be weakly consistent estimators of the matrix D and $\mathbb{E}\left[S_t^2\right]$.

The above quantities are now all observable, we are able to state our second theorem which gives the asymptotic distribution for quadratic forms of the autocovariances.

Theorem 3.2. *Under the assumptions of Theorem 3.1 and $\boldsymbol{\mathcal{H}}_0$, we have*

$$n\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathrm{d}} \chi^2_m.$$

The proof of Theorem 3.2 is postponed to Section 6.

Remark 3.2. *If we focuse on the following alternative hypothesis*

$$\boldsymbol{\mathcal{H}}_1 : \textit{the process } (\underline{\varepsilon}_t) \textit{ does not admit the representation } (1.1) \textit{ with parameter } \theta_0,$$

at least one $r_h^0 = \mathbb{E}[(\eta_t'\eta_t - d)(\eta_{t-h}'\eta_{t-h} - d)] \neq 0$ under $\boldsymbol{\mathcal{H}}_1$. One may prove that under $\boldsymbol{\mathcal{H}}_1$

$$\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathbb{P}} {\mathbf{r}_m^0}'D^{-1}\mathbf{r}_m^0$$

where the vector $\mathbf{r}_m^0 = (r_1^0, \ldots, r_m^0)'$. Therefore the test statistic $n\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m$ is consistent in detecting $\boldsymbol{\mathcal{H}}_1$.

The proof of this remark is also postponed to Section 6.

Consider the vector of the first m autocorrelations of the sum of squared residuals

$$\hat{\rho}_m = (\hat{\rho}(1), \ldots, \hat{\rho}(m))' \quad \text{where} \quad \hat{\rho}(h) = \frac{\hat{r}_h}{\hat{r}_0}.$$

Corollary 3.1. *Under the assumptions of Theorem 3.2, we have*

$$\sqrt{n}\hat{\rho}_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, D_{\hat{\rho}}) \quad \textit{where} \quad D_{\hat{\rho}} = \frac{D}{\left(\mathbb{E}\left[\eta_t'\eta_t - d\right]^2\right)^2}, \tag{3.1}$$

$$n\hat{\rho}_m'\hat{D}_{\hat{\rho}}^{-1}\hat{\rho}_m \xrightarrow[n\to\infty]{\mathrm{d}} \chi^2_m \quad \textit{where} \quad \hat{D}_{\hat{\rho}} = \frac{\hat{D}}{\left(\frac{1}{n}\sum_{t=1}^n \left[\hat{\eta}_t'\hat{\eta}_t - d\right]^2\right)^2}. \tag{3.2}$$

The proof of Corollary 3.1 is postponed to Section 6.

Remark 3.3. *Under the assumptions of Remark 3.1, we have: $\mathbb{E}[\underline{\varepsilon}_t'H_t^{-1}\underline{\varepsilon}_t - d]^2 = \left(\mathbb{E}\left[\eta_{it}^4\right] - 1\right)d$, so that $D_{\hat{\rho}} = D/\left(\mathbb{E}\left[\eta_{it}^4\right] - 1\right)^2 d^2$ and $\hat{D}_{\hat{\rho}} = \hat{D}/\left(\hat{\kappa}_i - 1\right)^2 d^2$, for $i = 1, \ldots, d$, where $\hat{\kappa}_i = n^{-1}\sum_{t=1}^n \hat{\eta}_{it}^4$.*

The adequacy of the CCC-APGARCH(p, q) model defined in (1.1) is then rejected at the asymptotic level α when

$$n\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m > \chi^2_m(1-\alpha) \quad \text{or} \quad n\hat{\rho}_m'\hat{D}_{\hat{\rho}}^{-1}\hat{\rho}_m > \chi^2_m(1-\alpha),$$

where $\chi^2_m(1-\alpha)$ represents the $(1-\alpha)$-quantile of the chi-square distribution with m degrees of freedom.

3.2. Portmanteau test when the power is unknown

The results are close to those given in Section 3.1. It consists to adapt the notations in Section 3.1 by replacing θ_0 (resp. $\hat{\theta}_n$) by ϑ_0 (resp. $\hat{\vartheta}_n$) and H_t (resp. $\tilde{H}_t$) by $\mathcal{H}_t$ (resp. $\tilde{\mathcal{H}}_t$).

To establish the asymptotic distribution of the portmanteau test statistic, when $\underline{\delta}_0$ is unknown, Assumption **A9** is replacing by

A9' : For $d \geq 2$, η_t takes more than $11d+1$ positive values and more than $11d+1$ negative values.

Theorem 3.3. *Under Assumptions **A1**–**A8** and **A9'**, if $\underline{\varepsilon}_t$ is the non-anticipative and stationary solution of the CCC-APGARCH(p, q) model defined in* (1.1)*, then we have*

$$\sqrt{n}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, \mathcal{D}), \quad \textit{where} \quad \mathcal{D} = \left(\mathbb{E}\left[S_t^2\right]\right)^2 I_m + \mathcal{C}_m\mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\mathcal{C}_m' + \mathcal{C}_m\Sigma_{\hat{\vartheta}_n,\mathbf{r}_m} + \Sigma'_{\hat{\vartheta}_n,\mathbf{r}_m}\mathcal{C}_m'$$

is a non-singular matrix and where the matrix $\mathcal{C}_m$ is given by (6.29) *in the proof of Theorem 3.3.*

The proof of Theorem 3.3 is postponed to Section 6.

In view of Boubacar Maïnassara et al. (2022) the matrices $\mathcal{I}$ and $\mathcal{J}$ can be estimated by

$$\hat{\mathcal{I}}(i,j) = \frac{1}{n}\sum_{t=1}^{n}\left[Tr\left(\left(\hat{\mathcal{H}}_t^{-1} - \hat{\mathcal{H}}_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}_t'\hat{\mathcal{H}}_t^{-1}\right)\frac{\partial\hat{\mathcal{H}}_t}{\partial\vartheta_i}\right)Tr\left(\left(\hat{\mathcal{H}}_t^{-1} - \hat{\mathcal{H}}_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}_t'\hat{\mathcal{H}}_t^{-1}\right)\frac{\partial\hat{\mathcal{H}}_t}{\partial\vartheta_j}\right)\right]$$

$$\text{and} \quad \hat{\mathcal{J}}(i,j) = \frac{1}{n}\sum_{t=1}^{n}\left[Tr\left(\hat{\mathcal{H}}_t^{-1}\frac{\partial\hat{\mathcal{H}}_t}{\partial\vartheta_j}\hat{\mathcal{H}}_t^{-1}\frac{\partial\hat{\mathcal{H}}_t}{\partial\vartheta_i}\right)\right], \quad \text{for} \quad i,j = 1,\ldots,s_0.$$

Let $\hat{S}_t = \underline{\varepsilon}_t'\hat{\mathcal{H}}_t^{-1}\underline{\varepsilon}_t - d$ and define the matrix $\hat{C}_m$ of size $m \times s_0$ whose $(h,i)-$th element is given by

$$\hat{\mathcal{C}}_m(h,i) = -\frac{1}{n}\sum_{t=h+1}^{n}\hat{S}_{t-h}\mathrm{Tr}\left(\hat{\mathcal{H}}_t^{-1}\frac{\partial\tilde{\mathcal{H}}_t(\hat{\vartheta}_n)}{\partial\vartheta_i}\right) \text{ for } 1 \le h \le m \text{ and } 1 \le i \le s_0.$$

The matrix $\hat{\mathcal{C}}_m$ is a weakly consistent estimator of $\mathcal{C}_m$. The matrix $\Sigma_{\hat{\vartheta}_n,\mathbf{r}_m}$ can also be estimated by

$$\hat{\Sigma}_{\hat{\vartheta}_n,\mathbf{r}_m} = \frac{1}{n}\sum_{t=1}^{n}\hat{\mathcal{J}}^{-1}\hat{\mathbf{h}}_t'\mathrm{vec}(\hat{\eta}_t\hat{\eta}_t' - d)\hat{S}_t\hat{\mathbb{S}}_{t-1:t-m}' \quad \text{where} \quad \hat{\mathbf{h}}_t(i) = \mathrm{vec}\left(\hat{\mathcal{H}}_t^{-1/2}\frac{\partial\tilde{\mathcal{H}}_t(\hat{\vartheta}_n)}{\partial\vartheta_i}\hat{\mathcal{H}}_t^{-1/2}\right) \text{ for } 1 \le i \le s_0.$$

Let $\hat{\mathcal{D}} = \hat{\kappa}^2 I_m + \hat{\mathcal{C}}_m\hat{\mathcal{J}}^{-1}\hat{\mathcal{I}}\hat{\mathcal{J}}^{-1}\hat{\mathcal{C}}_m' + \hat{\mathcal{C}}_m\hat{\Sigma}_{\hat{\vartheta}_n,\mathbf{r}_m} + \hat{\Sigma}_{\hat{\vartheta}_n,\mathbf{r}_m}'\hat{\mathcal{C}}_m'$ be a weakly consistent estimator of $\mathcal{D}$.

We are able to state the second theorem of this section which gives the asymptotic distribution for quadratic forms of the sum of squared residuals autocovariances.

Theorem 3.4. *Under the assumptions of Theorem 3.3 and $\boldsymbol{\mathcal{H}}_0$, we have*

$$n\hat{\mathbf{r}}_m'\hat{\mathcal{D}}^{-1}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathrm{d}} \chi_m^2.$$

The proof of Theorem 3.4 is postponed to Section 6.

Remark 3.4. *If we focuse on the following alternative hypothesis*

$$\boldsymbol{\mathcal{H}}_1 : \textit{the process } (\underline{\varepsilon}_t) \textit{ does not admit the representation } (1.1) \textit{ with parameter } \vartheta_0,$$

at least one $r_h^0 = \mathbb{E}[(\eta_t'\eta_t - d)(\eta_{t-h}'\eta_{t-h} - d)] \neq 0$ under $\boldsymbol{\mathcal{H}}_1$. One may prove that under $\boldsymbol{\mathcal{H}}_1$

$$\hat{\mathbf{r}}_m'\hat{\mathcal{D}}^{-1}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathbb{P}} \mathbf{r}_m^{0\,\prime}\mathcal{D}^{-1}\mathbf{r}_m^0.$$

Therefore the test statistic $n\hat{\mathbf{r}}_m'\hat{\mathcal{D}}^{-1}\hat{\mathbf{r}}_m$ is consistent in detecting $\boldsymbol{\mathcal{H}}_1$.

The proof of this remark is similar to that of Remark 3.2 and is omitted.

Corollary 3.2. *Under the assumptions of Theorem 3.4, we have*

$$\sqrt{n}\hat{\rho}_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0,\mathcal{D}_{\hat{\rho}}) \quad \textit{where} \quad \mathcal{D}_{\hat{\rho}} = \frac{\mathcal{D}}{\left(\mathbb{E}\left[\eta_t'\eta_t - d\right]^2\right)^2}, \tag{3.3}$$

$$n\hat{\rho}_m'\hat{\mathcal{D}}_{\hat{\rho}}^{-1}\hat{\rho}_m \xrightarrow[n\to\infty]{\mathrm{d}} \chi_m^2 \quad \textit{where} \quad \hat{\mathcal{D}}_{\hat{\rho}} = \frac{\hat{\mathcal{D}}}{\left(\frac{1}{n}\sum_{t=1}^{n}\left[\hat{\eta}_t'\hat{\eta}_t - d\right]^2\right)^2}. \tag{3.4}$$

The proof of Corollary 3.2 is postponed to Section 6.

Remark 3.5. *In view of Remark 3.3, we have:* $\mathcal{D}_{\hat{\rho}} = \mathcal{D}/\left(\mathbb{E}\left[\eta_{it}^4\right]-1\right)^2 d^2$ *and* $\hat{\mathcal{D}}_{\hat{\rho}} = \hat{\mathcal{D}}/(\hat{\kappa}_i-1)^2 d^2$, *for* $i=1,\ldots,d$.

The adequacy of the CCC-APGARCH(p,q) model, define in (1.1) is then rejected at the asymptotic level α when

$$n\hat{\mathbf{r}}_m'\hat{\mathcal{D}}^{-1}\hat{\mathbf{r}}_m > \chi^2_m(1-\alpha) \quad \text{or} \quad n\hat{\rho}_m'\hat{\mathcal{D}}_{\hat{\rho}}^{-1}\hat{\rho}_m > \chi^2_m(1-\alpha).$$

In view of Corollary 3.1 (resp. Corollary 3.2) for any $1 \leq h \leq m$, a $100(1-\alpha)\%$ confidence region for $\rho(h)$ is given by

$$-u_\alpha\frac{\hat{D}_{\hat{\rho}}(h,h)}{\sqrt{n}} \leq \hat{\rho}(h) \leq u_\alpha\frac{\hat{D}_{\hat{\rho}}(h,h)}{\sqrt{n}} \quad \left(\text{resp. } -u_\alpha\frac{\hat{\mathcal{D}}_{\hat{\rho}}(h,h)}{\sqrt{n}} \leq \hat{\rho}(h) \leq u_\alpha\frac{\hat{\mathcal{D}}_{\hat{\rho}}(h,h)}{\sqrt{n}}\right)$$

where u_α denotes the quantile of order $1-\alpha$ of the $\mathcal{N}(0,1)$ distribution.

4. Numerical illustration

By means of Monte Carlo experiments we investigate the finite sample properties of the tests introduced in this paper. The numerical illustrations are made with the free statistical software RStudio (see https://www.rstudio.com).

We generate a bivariate CCC-APGARCH$(0,1)$ model (Model (1.1) with $p=0$ and $q=1$)

$$\begin{cases} \underline{\varepsilon}_t = H_{0t}^{1/2}\eta_t, \\ H_{0t} = D_{0t}R_0D_{0t}, \qquad D_{0t} = \operatorname{diag}(\sqrt{h_{1,0t}},\sqrt{h_{2,0t}}), \\ \underline{h}_{0t}^{\underline{\delta}_0/2} = \underline{\omega}_0 + A_{01}^+(\underline{\varepsilon}_{t-1}^+)^{\underline{\delta}_0/2} + A_{01}^-(\underline{\varepsilon}_{t-1}^-)^{\underline{\delta}_0/2}, \end{cases} \tag{4.1}$$

for different values of $\underline{\delta}_0 \in \{(1,1),(0.8,1.5),(2,2),(3,2.5)\}$ and where $\underline{h}_{0t} = (h_{1,0t},h_{2,0t})'$, $\underline{\varepsilon}_t^+ = \left(\{\varepsilon_{1,t}^+\}^2,\{\varepsilon_{2,t}^+\}^2\right)'$ and $\underline{\varepsilon}_t^- = \left(\{\varepsilon_{1,t}^-\}^2,\{\varepsilon_{2,t}^-\}^2\right)'$. The innovation process (η_t) is defined by

$$\begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \end{pmatrix} \sim \text{IID}\,\mathcal{N}(0,I_2).$$

Considering other distributions for η_t does not affect the conclusion.

The coefficients of the data generating process (DGP for short) in (4.1) are chosen such that Assumption **A2** holds. The coefficient $\underline{\omega}_0$ is a vector of size 2×1, A_{01}^+, A_{01}^- and R_0 are matrices of size 2×2 taken as:

$$\underline{\omega}_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix},\ R_0 = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix} \text{ and } \begin{cases} A_{01}^+ = \begin{pmatrix} 0.25 & 0.10 \\ 0.10 & 0.15 \end{pmatrix},\ A_{01}^- = \begin{pmatrix} 0.45 & 0.25 \\ 0.25 & 0.35 \end{pmatrix},\ \text{if } A_{01}^- \neq A_{01}^+ \\ A_{01}^+ = A_{01}^- = \begin{pmatrix} 0.45 & 0.25 \\ 0.25 & 0.35 \end{pmatrix}, \quad \text{if} \quad A_{01}^- = A_{01}^+. \end{cases} \tag{4.2}$$

We simulated $N = 1,000$ independent replications of size $n = 250$, $n = 500$ and $n = 2,000$ of Model (4.1) with coefficients (4.2).

For each of these N replications of model (4.1), we use the QMLE method to estimate the coefficient $\theta_0 \in \mathbb{R}^{11}$ when the power is known (resp. $\vartheta_0 \in \mathbb{R}^{13}$ when the power is unknown). After estimating the model considered we apply portmanteau test to the sum of squared residuals for different values of $m \in \{1,\ldots,12\}$, where m is the number of autocorrelations used in the portmanteau test statistic.

We use in the following tables 3 nominal levels $\alpha = 1\%$, 5% and 10%. For these nominal levels, the empirical relative frequency of rejection size over the N independent replications should vary respectively within the confidence intervals $[0.3\%, 1.7\%]$, $[3.6\%, 6.4\%]$ and $[8.1\%, 11.9\%]$ with probability 95%

and $[0.3\%, 1.9\%]$, $[3.3\%, 6.9\%]$ and $[7.6\%, 12.5\%]$ with probability 99% under the assumption that the true probabilities of rejection are respectively $\alpha = 1\%$, $\alpha = 5\%$ and $\alpha = 10\%$.

We repeat the same experiments to examine the empirical power of the proposed test for the null hypothesis of a bivariate CCC-APGARCH$(0,1)$ model of the form (4.1) against the following bivariate CCC-APGARCH$(1,1)$ alternative defined by

$$\begin{cases} \underline{\varepsilon}_t = H_{0t}^{1/2}\eta_t, \\ H_{0t} = D_{0t}R_0D_{0t}, \qquad D_{0t} = \mathrm{diag}(\sqrt{h_{1,0t}}, \sqrt{h_{2,0t}}), \\ \underline{h}_{0t}^{\delta_0/2} = \underline{\omega}_0 + A_{01}^+(\underline{\varepsilon}_{t-1}^+)^{\delta_0/2} + A_{01}^-(\underline{\varepsilon}_{t-1}^-)^{\delta_0/2} + B_{01}\underline{h}_{0t-1}^{\delta_0/2}, \end{cases} \tag{4.3}$$

where the matrices $\underline{\omega}_0$, A_{01}^+, A_{01}^- and R_0 are given by (4.2) and $B_{01} = \begin{pmatrix} 0.43 & 0.1 \\ 0.1 & 0.42 \end{pmatrix}$.

4.1. When the power is known

Table 2 (resp. Table 3) displays the empirical relative frequencies of rejection over the N independent replications for the 3 nominal levels $\alpha = 1\%$, 5% and 10% when the DGP is the APGARCH$(0,1)$ model (4.1)–(4.2) with $A_{01}^+ = A_{01}^-$ (resp. with $A_{01}^+ \neq A_{01}^-$).

As expected, Tables 2 and 3 show that the percentages of rejection belong to the confidence interval with probability 95% and 99%. Thus the type I error is better controlled.

In term of power performance, we investigate two experiments given in the following tables:

Table 4 displays (in %) the empirical power of the proposed test for the null hypothesis of the CCC-APGARCH$(0,1)$ model defined by (4.1)–(4.2) with $\underline{\delta}_0 = (1,1)$ against the alternative given by (4.1)–(4.2) when $\underline{\delta}_0 \neq (1,1)$.

Table 5 displays also (in %) the empirical power of the proposed test for the null hypothesis of a bivariate CCC-APGARCH$(0,1)$ model of the form (4.1) against the bivariate CCC-APGARCH$(1,1)$ alternative given by (4.3) when $\underline{\delta}_0$ is known.

We draw the conclusion that:

a) in the first experiment given in Table 4, the portmanteau tests are more disappointing since they fail to detect some alternatives of the form $\underline{\delta}_0 \neq (1,1)$ when the null is $\underline{\delta}_0 = (1,1)$, except for $\underline{\delta}_0 \geq (2,2)$ when n increases.

b) Whereas the second experiment given in Table 5 reveals that the portmanteau tests are much more powerful to detect wrong values of the order (p,q) even when n is small.

4.2. When the power is unknown

In this case, the power $\underline{\delta}_0$ is jointly estimated with the parameter θ_0. As in the case where $\underline{\delta}_0$ is known, Table 6 (resp. Table 7) displays the empirical relative frequencies of rejection over the N independent replications for the 3 nominal levels $\alpha = 1\%$, 5% and 10% when the DGP is the APGARCH$(0,1)$ model (4.1)–(4.2) with $A_{01}^+ = A_{01}^-$ (resp. with $A_{01}^+ \neq A_{01}^-$). Even in this case, Tables 6 and 7 show that the percentages of rejection belong to the confidence interval with probability 95% and 99%. Thus the type I error is better controlled. In term of power performance, Table 8 shows that the powers of the test are quite satisfactory even when n is small.

5. Adequacy of CCC-APGARCH models for real datasets

We consider the daily return of two exchange rates EUR/USD (Euros Dollar) and EUR/JPY (Euros Yen). The observations covered the period from January 4, 1999 to March 9, 2021 which correspond to $n = 5,679$ observations. The data were obtained from the website of the European Central Bank

(http://www.ecb.int/stats/exchange/eurofxref/html/index.en.html). On these series, several CCC-APGARCH(p,q) models of the form (1.1) have been estimated by QML. For each estimated model, we apply the portmanteau tests proposed in Section 3 to the sum of squared residuals for different values of $m \in \{1,\ldots,12\}$ to test the adequacy of CCC-APGARCH models.

Table 1 displays the $p-$values for adequacy of the CCC-APGARCH(p,q) models for daily returns of exchange rates based on m squared residuals autocovariances, as well as the true and estimated powers (denoted $\underline{\delta}_0$ and $\hat{\underline{\delta}}$) and the likelihood. When $\underline{\delta}_0$ is known, the two corresponding models with $\underline{\delta}_0 = (1,1)$ and $\underline{\delta}_0 = (2,2)$ have the same number of parameters so it makes sense to prefer the model with the higher likelihood (the likelihood is given in the last column of Table 1). According to this criterion, the Log$-$likelihood of the preferred model is given in bold face (see Table 1).

Table 1 shows that the CCC-APGARCH$(0,q)$ models (for $q = 1,2,3$) are generally rejected whereas the CCC-APGARCH(p,q) models are not generally rejected and seem more appropriate. When $\underline{\delta}_0$ is known, the CCC-APGARCH$(1,1)$ and CCC-APGARCH$(2,1)$ models seem to be relevant for $\underline{\varepsilon}_t = (\mathrm{USD}_t, \mathrm{JPY}_t)'$. In contrast, when $\underline{\delta}_0$ is estimated, the CCC-APGARCH$(2,1)$ and CCC-APGARCH$(2,2)$ models seem to be relevant for $\underline{\varepsilon}_t$.

From the second last column of Table 1, we can also see that the estimated power $\hat{\delta}$ is not necessary equal to 1 or 2 and is different for each model.

The portmanteau test is thus an important tool in the validation process. From the empirical results and the simulation experiments, we draw the conclusion that the proposed portmanteau tests based on the sum of squared residuals of a CCC-APGARCH(p,q) controls well the error of first kind at different asymptotic level α and is efficient to detect a misspecification of the order (p,q).

Table 1: Portmanteau test $p-$values for adequacy of the CCC-APGARCH(p,q) models for daily returns of exchange rates of the (Dollar,Yen), based on m of the sum of squared residuals autocovariances.

Currency	Lag m 1	2	3	4	5	6	7	8	9	10	11	12	$\underline{\delta}_0$ or $\hat{\underline{\delta}}$	Log-lik
Portmanteau tests for adequacy of the CCC-APGARCH$(0,1)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.880	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(1,1)$	**-0.1295**
(USD,JPY)	0.056	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(2,2)$	-0.1291
Portmanteau tests for adequacy of the CCC-APGARCH$(0,2)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.401	0.520	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(1,1)$	-0.1827
(USD,JPY)	0.492	0.045	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(2,2)$	**-0.1844**
Portmanteau tests for adequacy of the CCC-APGARCH$(0,3)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.496	0.683	0.372	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(1,1)$	-0.2002
(USD,JPY)	0.600	0.114	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(2,2)$	**-0.2010**
Portmanteau tests for adequacy of the CCC-APGARCH$(1,1)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.118	0.214	0.362	0.074	0.022	0.027	0.039	0.063	0.064	0.039	0.035	0.042	$(1,1)$	**-0.3410**
(USD,JPY)	0.280	0.479	0.677	0.232	0.110	0.146	0.216	0.254	0.207	0.159	0.159	0.140	$(2,2)$	-0.3406
Portmanteau tests for adequacy of the CCC-APGARCH$(2,1)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.164	0.279	0.322	0.160	0.092	0.067	0.072	0.099	0.047	0.020	0.009	0.014	$(1,1)$	-0.2492
(USD,JPY)	0.337	0.595	0.545	0.260	0.092	0.073	0.114	0.145	0.135	0.124	0.128	0.169	$(2,2)$	**-0.2858**
Portmanteau tests for adequacy of the CCC-APGARCH$(1,2)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.402	0.082	0.151	0.180	0.081	0.064	0.091	0.139	0.086	0.062	0.030	0.046	$(1,1)$	-0.2402
(USD,JPY)	0.610	0.082	0.102	0.025	0.013	0.004	0.007	0.008	0.006	0.009	0.013	0.007	$(2,2)$	**-0.2937**
Portmanteau tests for adequacy of the CCC-APGARCH$(2,2)$ when $\underline{\delta}_0$ is known														
(USD,JPY)	0.166	0.191	0.206	0.102	0.079	0.060	0.083	0.116	0.076	0.045	0.032	0.047	$(1,1)$	-0.2559
(USD,JPY)	0.152	0.205	0.333	0.077	0.040	0.020	0.027	0.038	0.012	0.012	0.012	0.012	$(2,2)$	**-0.3062**
Portmanteau tests for adequacy of the CCC-APGARCH$(0,1)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.518	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(4.595, 1.201)$	-0.1321
Portmanteau tests for adequacy of the CCC-APGARCH$(0,2)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.779	0.102	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(1.743, 1.496)$	-0.1856
Portmanteau tests for adequacy of the CCC-APGARCH$(0,3)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.952	0.361	0.064	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	$(1.245, 1.518)$	-0.2031
Portmanteau tests for adequacy of the CCC-APGARCH$(1,1)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.044	0.084	0.167	0.040	0.031	0.046	0.077	0.111	0.118	0.102	0.129	0.087	$(2.149, 1.568)$	-0.3520
Portmanteau tests for adequacy of the CCC-APGARCH$(2,1)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.115	0.262	0.253	0.134	0.101	0.151	0.207	0.219	0.194	0.234	0.260	0.269	$(2.229, 1.578)$	-0.2756
Portmanteau tests for adequacy of the CCC-APGARCH$(1,2)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.681	0.288	0.336	0.038	0.018	0.027	0.042	0.035	0.028	0.025	0.032	0.036	$(1.618, 4.547)$	-0.2805
Portmanteau tests for adequacy of the CCC-APGARCH$(2,2)$ when $\underline{\delta}_0$ is unknown														
(USD,JPY)	0.237	0.412	0.407	0.188	0.120	0.176	0.213	0.106	0.070	0.081	0.103	0.097	$(0.921, 2.033)$	-0.2783

Table 2: Empirical size of the proposed test: relative frequencies (in %) of rejection of an APGARCH$(0,1)$.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.7	0.7	0.7	1.3	1.1	0.7	0.7	0.5	0.6	0.5	0.4	0.4
$(1,1)$	250	5%	3.6	4.8	5.3	6.3	6.6	5.2	5.5	4.9	4.6	4.6	4.4	4.0
		10%	8.6	9.8	11.4	11.7	11.9	11.2	11.4	10.6	10.0	9.7	9.3	9.7
		1%	0.8	0.7	0.7	0.8	1.2	1.4	0.9	1.1	0.9	0.9	0.8	0.8
$(1,1)$	500	5%	4.4	4.2	4.3	4.3	4.7	4.3	3.8	3.4	3.3	3.9	3.6	3.6
		10%	8.5	9.1	10.3	8.9	9.8	10.2	10.4	9.4	9.3	8.2	8.5	9.0
		1%	1.2	1.6	1.4	0.9	0.9	0.7	1.3	1.2	1.2	1.4	1.4	1.4
$(1,1)$	$2,000$	5%	4.1	4.7	5.3	5.5	6.0	5.6	6.0	5.0	5.1	5.8	5.7	5.3
		10%	8.9	9.4	9.7	10.4	10.8	11.1	12.6	10.9	12.1	11.5	11.4	10.6
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.6	0.9	0.5	1.2	0.9	0.8	0.4	0.3	0.5	0.4	0.2	0.3
$(0.8,1.5)$	250	5%	2.4	3.9	4.8	5.8	5.0	5.1	5.4	4.3	4.5	3.9	4.0	3.8
		10%	8.6	7.7	10.7	10.3	11.1	11.0	10.7	9.6	9.4	9.1	8.4	8.4
		1%	0.6	0.5	0.9	0.7	1.1	1.2	1.1	1.0	0.7	0.7	0.8	0.9
$(0.8,1.5)$	500	5%	4.2	3.9	3.9	3.9	4.4	4.1	4.3	3.9	3.5	4.7	3.7	4.5
		10%	8.4	8.1	9.6	8.3	9.8	9.7	9.7	9.1	9.1	8.5	7.7	8.7
		1%	1.1	1.7	1.3	1.4	0.9	0.9	1.5	1.3	1.2	1.5	1.6	1.2
$(0.8,1.5)$	$2,000$	5%	4.6	4.9	5.4	5.7	5.8	5.6	5.8	5.6	5.5	5.9	6.2	5.2
		10%	9.0	10.0	9.0	9.7	10.6	11.2	11.8	11.5	11.2	11.3	10.7	11.1
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.8	0.9	1.2	1.7	1.2	1.7	1.0	0.9	0.8	0.8	0.5	0.6
$(2,2)$	250	5%	4.2	6.0	6.7	6.9	6.4	6.7	6.4	5.9	5.7	5.1	4.9	4.6
		10%	9.1	11.8	12.3	12.4	12.4	12.2	11.8	11.0	10.5	10.6	10.6	10.0
		1%	0.6	0.8	0.8	1.0	1.3	1.1	1.2	1.2	1.1	0.6	0.7	0.8
$(2,2)$	500	5%	5.1	5.3	4.9	4.4	5.5	4.6	4.5	3.9	3.8	4.3	4.6	4.4
		10%	9.9	10.2	10.2	9.7	9.4	10.2	9.6	9.2	9.2	10.0	8.7	9.7
		1%	1.2	1.7	1.3	1.2	1.2	1.1	1.2	1.2	1.4	1.7	1.5	1.6
$(2,2)$	$2,000$	5%	4.5	5.5	5.7	6.1	6.3	6.5	6.3	5.0	5.9	5.7	5.8	5.4
		10%	8.5	9.4	10.9	10.8	10.5	11.2	12.5	11.7	12.2	11.8	11.4	10.5
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.9	0.9	1.2	1.7	1.4	1.8	1.3	1.3	0.9	0.8	0.7	0.6
$(3,2.5)$	250	5%	4.7	5.7	6.7	6.9	6.2	6.6	5.9	5.7	5.6	4.9	4.7	4.7
		10%	9.6	12.0	11.7	12.6	12.3	12.3	11.2	11.2	9.3	10.5	10.3	8.9
		1%	0.6	0.9	0.9	1.0	1.3	1.0	1.1	1.3	1.2	0.6	0.7	0.7
$(3,2.5)$	500	5%	5.3	5.3	5.6	4.7	5.7	4.6	4.5	4.1	4.2	4.3	4.6	4.6
		10%	11.1	10.4	9.9	10.1	10.0	10.7	9.9	10.1	9.2	9.5	8.9	9.6
		1%	1.2	1.6	1.3	1.3	1.1	1.0	1.0	1.3	1.4	1.7	1.6	1.6
$(3,2.5)$	$2,000$	5%	4.6	4.9	5.8	6.2	6.0	6.4	6.5	5.3	5.5	5.7	5.8	5.2
		10%	9.2	9.2	10.5	10.8	10.2	11.1	12.1	12.4	12.2	11.9	11.3	10.8

Model (4.1)–(4.2) with $A_{01}^{+} = A_{01}^{-}$ when $\underline{\delta}_0$ is known.

Table 3: Empirical size of the proposed test: relative frequencies (in %) of rejection of an APGARCH$(0,1)$.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.8	1.2	0.9	1.6	1.6	0.9	0.8	0.9	0.7	0.5	0.4	0.5
$(1,1)$	250	5%	4.1	4.9	6.1	5.9	6.4	5.7	5.7	5.3	4.7	4.2	4.6	4.4
		10%	8.7	10.7	12.1	12.5	12.4	11.7	11.4	10.5	11.4	10.2	9.3	9.5
		1%	1.1	1.1	1.0	1.0	1.4	1.6	1.1	1.2	1.0	0.8	0.9	1.0
$(1,1)$	500	5%	4.8	5.5	5.1	4.6	5.2	4.6	4.5	3.9	4.0	4.3	4.3	4.2
		10%	9.6	10.6	11.7	10.9	9.6	10.9	10.6	9.6	9.3	8.9	8.5	9.9
		1%	1.2	1.7	1.6	1.4	1.1	1.3	1.5	1.2	1.2	1.5	1.7	1.4
$(1,1)$	$2,000$	5%	4.4	5.2	5.8	5.1	5.6	5.6	6.4	5.0	6.0	6.3	6.1	5.2
		10%	8.7	9.2	10.6	9.7	11.3	11.1	12.2	10.7	11.7	11.9	11.5	11.1
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.5	0.9	1.3	1.3	1.4	1.3	1.0	0.8	0.8	0.5	0.3	0.6
$(0.8,1.5)$	250	5%	3.7	4.2	5.1	5.9	6.1	5.5	5.7	5.1	4.6	4.5	4.0	3.9
		10%	9.3	9.9	11.6	11.3	11.4	11.6	10.3	10.5	10.5	9.9	9.1	9.0
		1%	0.9	0.9	1.2	1.4	1.7	1.5	1.4	1.2	0.9	0.8	0.9	0.9
$(0.8,1.5)$	500	5%	4.4	5.2	5.2	4.3	5.1	4.8	4.6	4.3	4.2	4.7	4.8	4.6
		10%	9.2	10.6	11.0	10.6	10.2	9.8	9.7	8.8	9.0	9.2	8.3	10.3
		1%	1.1	1.5	1.2	1.3	1.7	1.2	1.5	1.3	1.1	1.5	1.8	1.5
$(0.8,1.5)$	$2,000$	5%	4.9	5.2	5.6	5.3	5.6	6.1	6.1	5.3	6.2	6.3	6.0	5.3
		10%	8.5	10.3	10.9	10.1	10.9	10.6	12.1	11.6	11.4	11.7	11.8	10.6
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.2	1.3	1.1	1.7	1.5	1.6	1.2	1.0	0.9	0.7	0.3	0.5
$(2,2)$	250	5%	5.6	6.5	7.2	7.5	6.7	6.6	6.4	6.2	5.6	4.9	4.6	4.6
		10%	10.2	11.8	13.4	12.7	13.4	13.0	12.3	11.3	11.1	11.6	10.0	10.3
		1%	0.7	1.1	0.9	1.2	1.5	1.5	1.3	1.2	1.0	1.0	1.0	1.1
$(2,2)$	500	5%	5.8	5.8	5.8	5.0	5.6	5.2	5.2	4.4	3.9	4.7	4.7	5.0
		10%	10.5	11.5	11.4	11.4	10.7	11.2	10.5	10.2	10.6	10.0	8.2	9.6
		1%	1.3	1.4	1.1	1.2	1.4	1.3	1.3	1.0	1.3	1.6	1.9	1.7
$(2,2)$	$2,000$	5%	4.5	5.1	5.3	5.0	5.9	6.2	5.9	5.2	6.0	6.1	5.5	5.2
		10%	9.3	9.5	10.4	9.9	10.8	10.9	11.6	11.4	11.8	12.1	11.0	10.7
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.4	1.2	1.4	1.7	1.8	1.8	1.6	1.1	1.0	0.8	0.4	0.4
$(3,2.5)$	250	5%	5.0	6.7	7.3	7.7	6.5	7.1	6.7	5.9	5.6	5.1	4.9	4.3
		10%	9.7	11.8	13.6	12.9	13.1	12.8	11.9	11.5	11.4	11.3	10.5	9.2
		1%	1.0	1.1	1.0	1.2	1.4	1.3	1.2	1.2	1.3	1.0	1.0	1.0
$(3,2.5)$	500	5%	5.9	6.0	5.5	4.9	5.3	5.0	4.8	4.2	3.8	4.4	4.3	4.6
		10%	11.0	10.7	10.9	11.2	10.5	11.3	10.9	9.7	10.2	9.4	8.5	9.6
		1%	1.0	1.5	0.9	1.2	1.3	1.2	1.5	1.0	1.3	1.6	1.7	1.8
$(3,2.5)$	$2,000$	5%	5.0	5.0	5.5	5.6	5.9	6.1	6.6	4.8	5.8	5.8	5.3	5.2
		10%	9.9	9.2	10.5	10.2	10.9	10.8	11.2	11.7	12.1	11.8	11.2	10.1

Model (4.1)–(4.2) with $A_{01}^{+} \neq A_{01}^{-}$ when $\underline{\delta}_0$ is known.

Table 4: Empirical power of the proposed test for the null hypothesis of the CCC-APGARCH$(0,1)$ model defined by (4.1) with $\underline{\delta}_0 = (1,1)$ against the alternative given by (4.1) when $\underline{\delta}_0 \neq (1,1)$.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.8	1.2	0.9	1.6	1.6	0.9	0.8	0.9	0.7	0.5	0.4	0.5
$(1,1)$	250	5%	4.1	4.9	6.1	5.9	6.4	5.7	5.7	5.3	4.7	4.2	4.6	4.4
		10%	8.7	10.7	12.1	12.5	12.4	11.7	11.4	10.5	11.4	10.2	9.3	9.5
		1%	1.1	1.1	1.0	1.0	1.4	1.6	1.1	1.2	1.0	0.8	0.9	1.0
$(1,1)$	500	5%	4.8	5.5	5.1	4.6	5.2	4.6	4.5	3.9	4.0	4.3	4.3	4.2
		10%	9.6	10.6	11.7	10.9	9.6	10.9	10.6	9.6	9.3	8.9	8.5	9.9
		1%	1.2	1.7	1.6	1.4	1.1	1.3	1.5	1.2	1.2	1.5	1.7	1.4
$(1,1)$	$2,000$	5%	4.4	5.2	5.8	5.1	5.6	5.6	6.4	5.0	6.0	6.3	6.1	5.2
		10%	8.7	9.2	10.6	9.7	11.3	11.1	12.2	10.7	11.7	11.9	11.5	11.1
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.5	1.2	1.0	1.6	1.4	1.3	0.8	0.9	0.7	0.7	0.5	0.4
$(0.8,1.5)$	250	5%	4.4	5.2	6.3	5.9	5.3	6.4	5.9	5.8	5.3	4.9	4.5	4.7
		10%	9.5	10.7	11.4	12.4	12.7	12.4	11.6	11.2	10.3	10.3	10.1	10.0
		1%	1.0	1.0	1.1	1.0	1.4	1.5	1.3	1.2	1.0	1.0	0.9	1.1
$(0.8,1.5)$	500	5%	5.3	5.2	5.4	5.0	5.3	4.5	4.4	4.1	4.2	4.5	4.5	4.4
		10%	10.3	10.5	11.4	11.0	10.8	10.1	10.0	9.2	9.2	10.0	9.2	9.2
		1%	1.4	1.5	1.2	1.2	1.7	1.3	1.4	1.4	1.4	1.2	1.4	1.3
$(0.8,1.5)$	$2,000$	5%	4.8	5.3	5.7	5.5	5.9	6.4	5.8	5.8	6.7	6.3	6.2	6.0
		10%	9.9	10.1	11.0	10.5	11.6	11.6	12.1	11.6	11.8	12.9	12.2	11.4
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.8	2.4	1.7	1.9	2.0	1.9	1.5	1.1	0.7	0.6	0.5	0.7
$(2,2)$	250	5%	8.4	9.1	9.8	8.6	8.5	7.8	8.0	7.0	5.8	5.8	5.0	5.1
		10%	14.8	16.2	16.1	16.1	15.7	16.0	13.7	12.4	13.2	11.8	11.2	10.7
		1%	3.1	3.4	3.2	2.7	2.3	2.6	2.8	2.1	1.9	1.9	2.0	1.9
$(2,2)$	500	5%	11.1	13.6	11.7	10.8	9.4	9.5	8.1	7.6	7.6	7.6	6.3	7.1
		10%	18.8	19.9	19.8	18.6	17.6	16.2	15.9	15.0	14.7	14.0	13.0	14.0
		1%	10.3	11.0	9.8	8.3	8.1	6.9	6.4	5.7	5.9	5.5	5.6	4.8
$(2,2)$	$2,000$	5%	25.4	29.1	25.1	23.0	21.2	19.3	17.9	18.0	17.4	16.6	15.7	14.7
		10%	35.8	39.5	36.5	34.5	31.9	30.1	29.7	28.5	27.0	26.2	25.9	26.0
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.9	2.4	1.8	1.7	2.1	2.0	1.6	0.9	0.9	0.6	0.4	0.7
$(3,2.5)$	250	5%	8.2	9.7	9.5	9.2	8.2	7.8	7.1	7.1	5.8	5.3	5.3	5.1
		10%	15.2	16.7	16.1	16.7	16.0	15.8	14.4	12.7	12.7	12.4	10.6	11.2
		1%	3.7	3.6	3.7	2.7	2.5	2.6	2.6	2.5	1.8	2.0	2.2	2.2
$(3,2.5)$	500	5%	11.5	13.1	12.3	11.5	9.9	9.8	9.0	8.6	8.4	7.9	7.2	8.1
		10%	19.4	21.3	20.3	19.7	17.9	17.1	17.0	16.2	14.9	14.3	14.4	14.5
		1%	11.0	12.8	10.9	10.0	9.0	8.2	7.5	7.0	6.8	6.4	6.0	5.4
$(3,2.5)$	$2,000$	5%	26.3	31.7	28.4	25.7	22.7	21.2	20.5	20.8	19.4	18.3	17.5	17.5
		10%	38.7	44.3	40.5	38.0	35.4	33.7	32.6	31.0	29.8	29.4	28.6	27.4

Model (4.1)–(4.2) with $A_{01}^{+} \neq A_{01}^{-}$ when $\underline{\delta}_0$ is known.

Table 5: Empirical power of the proposed test for the null hypothesis of a bivariate CCC-APGARCH$(0,1)$ model of the form (4.1) against the bivariate CCC-APGARCH$(1,1)$ alternative given by (4.3) when $\underline{\delta}_0$ is known.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	11.1	12.7	21.2	22.6	20.2	16.5	13.5	10.7	9.4	7.1	6.5	5.5
$(1,1)$	250	5%	25.0	36.0	49.7	50.5	48.0	44.3	40.3	35.6	32.0	28.9	26.8	24.8
		10%	35.1	52.2	66.6	68.9	65.3	61.4	56.7	52.0	48.9	45.9	42.0	39.9
		1%	27.6	42.6	65.1	72.4	70.6	66.8	63.7	57.7	52.3	48.8	44.0	39.6
$(1,1)$	500	5%	46.1	71.7	89.4	92.2	91.1	90.4	86.7	83.3	80.7	77.0	75.7	72.3
		10%	57.8	84.1	94.9	96.3	96.2	95.1	94.0	92.6	89.9	87.6	85.2	83.7
		1%	78.0	97.5	99.3	99.6	99.8	99.8	99.8	99.8	99.8	99.7	99.7	99.8
$(1,1)$	2,000	5%	87.9	99.3	99.7	99.9	99.9	99.9	99.9	99.9	99.8	99.8	99.8	99.8
		10%	92.3	99.6	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	12.5	12.6	18.8	20.4	16.3	13.9	11.6	9.6	7.2	6.2	5.7	5.0
$(0.8,1.5)$	250	5%	24.4	34.2	47.1	48.3	46.3	41.2	37.4	32.5	28.0	25.0	24.6	21.7
		10%	34.2	49.9	64.1	64.7	62.6	58.6	53.6	50.6	46.6	44.3	40.2	37.2
		1%	26.8	37.7	55.9	62.7	61.3	58.3	53.4	47.8	42.4	38.9	34.6	29.1
$(0.8,1.5)$	500	5%	45.3	65.9	80.8	86.6	85.3	83.2	80.8	77.6	73.6	70.9	68.7	62.8
		10%	55.7	79.4	91.1	93.1	93.0	91.8	89.9	87.8	85.6	83.0	79.9	77.8
		1%	71.9	90.9	95.7	97.8	98.2	98.0	98.0	98.1	97.6	97.1	96.6	96.9
$(0.8,1.5)$	2,000	5%	84.4	96.7	98.3	98.8	99.3	99.2	99.2	99.3	99.1	98.7	98.8	98.8
		10%	87.4	98.3	98.7	99.3	99.3	99.3	99.3	99.3	99.3	99.3	99.1	99.3
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	14.6	13.3	21.8	23.6	22.3	19.0	16.3	12.8	10.7	10.4	9.1	7.4
$(2,2)$	250	5%	30.4	38.3	53.0	57.0	53.8	49.5	45.5	39.8	36.1	33.2	30.1	27.8
		10%	41.7	53.3	68.8	73.2	71.5	67.0	63.4	57.7	54.8	51.6	46.9	44.3
		1%	34.9	44.7	66.3	74.0	74.5	72.6	68.2	64.4	59.6	55.9	50.4	45.8
$(2,2)$	500	5%	55.1	70.9	88.6	92.3	91.5	90.8	89.1	87.4	85.4	81.9	79.2	77.5
		10%	64.4	82.7	94.6	96.2	97.0	96.0	95.3	94.3	92.8	91.1	89.6	87.7
		1%	84.6	96.2	98.7	99.4	99.6	99.8	99.8	99.8	99.7	99.7	99.4	99.3
$(2,2)$	2,000	5%	92.9	99.1	99.5	99.8	100.0	99.9	99.8	99.9	99.9	99.9	99.9	99.9
		10%	95.0	99.5	99.8	100.0	100.0	100.0	100.0	100.0	99.9	99.9	100.0	100.0
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	20.1	17.3	23.9	26.3	25.3	22.9	20.2	18.5	16.2	12.7	11.6	9.5
$(3,2.5)$	250	5%	41.7	42.2	54.8	57.6	55.7	52.2	49.0	46.3	42.3	39.0	37.8	34.7
		10%	51.6	59.0	69.6	73.1	73.1	69.5	67.1	62.9	60.1	57.5	54.0	50.6
		1%	45.8	48.3	59.7	66.0	68.8	67.7	66.0	66.0	61.4	58.0	54.7	50.4
$(3,2.5)$	500	5%	67.0	71.7	83.4	86.2	87.8	87.7	86.0	86.2	85.0	82.3	80.3	77.5
		10%	76.5	82.2	91.4	92.2	93.5	93.7	93.3	92.6	91.9	90.0	89.7	87.9
		1%	86.2	91.5	95.4	96.1	96.9	97.7	97.9	97.9	98.1	98.0	97.9	98.2
$(3,2.5)$	2,000	5%	93.2	96.9	98.1	98.3	98.9	98.8	99.0	99.2	99.1	98.8	98.9	98.9
		10%	94.6	97.6	98.7	99.0	99.2	99.4	99.5	99.5	99.5	99.4	99.5	99.4

Model (4.3) with $A_{01}^{+} \neq A_{01}^{-}$ when $\underline{\delta}_0$ is known.

Table 6: Empirical size of the proposed test: relative frequencies (in %) of rejection of an APGARCH$(0,1)$.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.6	1.1	1.0	1.6	1.3	1.0	1.0	1.0	0.6	0.3	0.3	0.5
$(1,1)$	250	5%	5.1	5.4	6.1	7.2	6.4	6.4	6.1	5.6	5.5	5.2	4.8	4.8
		10%	10.4	10.3	11.4	11.4	12.5	12.3	11.8	10.5	10.1	11.1	10.7	10.5
		1%	0.5	0.9	1.1	1.1	1.3	1.0	1.0	1.0	0.9	1.1	0.8	0.7
$(1,1)$	500	5%	4.3	4.9	4.1	4.8	4.3	4.8	4.8	4.3	4.5	4.5	4.3	4.3
		10%	9.1	10.1	10.2	8.9	9.8	9.5	9.8	10.0	8.9	9.0	9.2	9.9
		1%	0.8	1.3	1.1	1.0	1.4	0.8	1.2	1.2	1.4	1.4	1.5	1.1
$(1,1)$	$2,000$	5%	4.2	4.6	5.9	5.4	5.9	5.8	6.0	5.8	5.3	5.3	5.4	4.7
		10%	8.6	9.4	10.7	10.2	10.0	10.6	12.0	11.3	12.2	12.5	11.4	10.5
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.5	0.8	1.3	1.2	0.9	1.0	0.8	0.5	0.6	0.5	0.5	0.5
$(0.8,1.5)$	250	5%	5.0	4.9	6.4	7.1	6.8	5.8	5.9	5.4	5.7	4.7	4.8	5.0
		10%	9.4	10.4	12.0	13.1	13.0	12.9	12.2	10.5	11.0	10.2	10.0	10.7
		1%	0.9	1.1	1.0	1.3	1.5	1.1	1.2	1.2	1.2	0.9	0.8	1.0
$(0.8,1.5)$	500	5%	4.4	4.2	4.4	5.2	4.9	4.9	4.8	4.6	4.3	4.7	4.3	4.8
		10%	8.2	9.3	9.8	9.9	10.3	10.6	10.1	9.8	9.3	9.5	9.8	10.1
		1%	0.9	1.4	1.2	1.2	1.3	0.9	1.3	1.2	1.3	1.6	1.5	1.4
$(0.8,1.5)$	$2,000$	5%	4.4	5.4	5.6	5.6	6.1	5.4	5.9	6.0	6.3	5.7	6.0	4.9
		10%	9.4	9.6	11.2	10.5	10.1	11.7	12.1	11.3	12.0	11.9	11.4	10.8
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.0	1.6	1.6	1.8	2.0	1.6	1.2	1.2	0.8	1.2	1.4	1.0
$(2,2)$	250	5%	6.2	8.2	7.0	6.4	4.8	5.6	5.8	5.6	6.2	5.0	5.4	5.8
		10%	12.0	13.4	14.4	12.4	10.8	10.8	11.0	10.0	9.6	11.8	11.4	10.6
		1%	0.9	0.9	0.9	1.3	1.2	1.1	1.2	1.2	1.0	0.8	0.7	0.9
$(2,2)$	500	5%	5.2	5.2	4.9	5.3	5.6	4.5	4.8	4.6	4.2	4.6	4.8	4.8
		10%	10.5	10.8	11.0	11.0	10.3	10.8	10.3	10.3	9.7	10.1	9.1	10.9
		1%	1.2	1.4	1.1	0.9	1.4	1.1	1.3	1.3	1.4	1.4	1.5	1.6
$(2,2)$	$2,000$	5%	5.4	5.0	5.2	5.7	6.1	6.4	6.8	5.5	5.8	5.8	5.5	5.9
		10%	8.8	10.5	11.1	10.8	10.9	11.2	12.1	11.9	12.8	12.4	11.2	10.8
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.0	2.6	2.0	1.6	1.6	1.8	1.4	1.4	1.2	1.6	1.4	1.0
$(3,2.5)$	250	5%	7.4	7.4	8.8	7.4	6.4	6.0	5.6	5.8	5.4	5.2	5.2	5.2
		10%	11.8	14.8	13.0	14.4	11.4	12.6	11.6	11.4	10.4	11.4	11.6	10.0
		1%	0.8	0.9	0.7	1.1	1.4	1.5	1.4	1.5	1.1	0.9	0.9	1.0
$(3,2.5)$	500	5%	4.9	4.9	5.6	5.2	5.8	4.3	4.1	4.7	4.8	4.8	4.8	5.1
		10%	11.0	11.0	10.5	10.7	10.9	11.0	11.2	10.1	10.2	9.6	9.5	11.1
		1%	1.0	1.4	1.2	1.4	1.0	1.2	1.1	1.4	1.5	1.4	1.6	1.9
$(3,2.5)$	$2,000$	5%	5.2	4.9	5.5	5.9	6.5	6.4	6.9	5.8	5.9	6.0	6.0	5.9
		10%	9.7	10.8	11.4	10.3	10.5	10.7	12.2	12.3	12.9	12.1	11.7	10.8

Model (4.1)–(4.2) with $A_{01}^{+} = A_{01}^{-}$ when $\underline{\delta}_0$ is unknown.

Table 7: Empirical size of the proposed test: relative frequencies (in %) of rejection of an APGARCH$(0,1)$.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.7	0.9	1.4	1.9	1.1	1.2	0.8	0.5	0.7	0.4	0.3	0.6
$(1,1)$	250	5%	4.2	5.3	5.8	6.6	6.2	5.5	5.9	5.6	5.2	4.7	4.8	4.4
		10%	9.1	9.6	12.3	11.3	12.6	13.2	12.0	10.6	11.0	10.5	9.7	9.7
		1%	0.5	0.9	1.1	1.0	1.5	1.4	1.3	1.2	1.1	1.1	0.9	0.8
$(1,1)$	500	5%	4.6	5.0	4.8	4.8	5.5	5.5	4.8	4.0	4.2	4.5	4.5	4.9
		10%	9.7	10.0	10.4	10.4	10.1	10.7	11.0	10.0	9.3	9.6	8.3	9.8
		1%	1.5	1.4	1.4	1.2	1.5	1.3	1.6	1.0	1.3	1.8	1.8	1.2
$(1,1)$	$2,000$	5%	4.5	4.6	5.1	4.9	5.4	5.8	6.1	5.4	5.8	6.1	6.0	5.4
		10%	9.4	9.2	10.6	10.4	11.0	10.6	10.9	11.4	11.2	12.0	10.9	10.5
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.5	1.1	1.3	1.7	1.5	1.4	1.1	0.6	0.7	0.8	0.5	0.5
$(0.8,1.5)$	250	5%	4.5	5.1	6.2	6.7	5.5	5.7	6.0	5.1	5.0	5.1	4.3	4.4
		10%	9.5	10.0	12.1	11.9	12.4	12.7	11.0	10.7	10.0	10.0	9.6	10.0
		1%	1.2	1.2	1.1	1.2	1.6	1.4	1.2	1.0	1.1	1.3	1.2	1.0
$(0.8,1.5)$	500	5%	5.4	5.7	4.8	5.4	5.7	4.9	5.1	4.4	4.6	4.7	4.9	5.3
		10%	10.0	10.3	11.2	10.3	10.8	10.6	11.1	10.0	9.7	9.4	9.1	10.2
		1%	1.4	1.3	1.0	1.2	1.8	1.4	1.4	1.3	1.4	1.7	1.9	1.4
$(0.8,1.5)$	$2,000$	5%	4.5	5.5	5.7	5.8	5.5	5.8	6.9	5.6	6.3	6.0	6.4	5.4
		10%	9.2	10.2	11.3	10.6	10.1	11.2	11.3	11.6	12.1	12.2	12.2	10.7
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	0.8	1.2	1.8	2.0	1.2	1.0	1.0	0.6	0.6	0.6	1.0	0.8
$(2,2)$	250	5%	5.0	6.4	6.4	6.4	4.8	4.8	5.4	5.2	5.8	5.4	5.0	5.4
		10%	10.6	11.2	13.4	12.4	10.6	10.6	10.4	10.2	10.6	11.6	11.4	10.0
		1%	1.0	1.2	1.1	1.2	1.5	1.6	1.8	1.4	1.2	1.3	1.0	0.9
$(2,2)$	500	5%	5.1	6.1	5.7	5.8	5.7	5.8	5.8	4.2	4.7	4.6	4.8	5.1
		10%	9.8	11.9	12.1	12.4	11.6	11.6	11.1	10.8	10.1	10.0	9.0	9.5
		1%	1.1	1.3	1.1	1.2	1.7	1.3	1.7	1.2	1.3	1.8	1.7	1.6
$(2,2)$	$2,000$	5%	4.8	5.0	5.8	5.5	6.1	5.9	6.1	5.4	5.7	5.5	5.5	5.5
		10%	10.2	10.2	10.1	10.6	10.8	11.4	11.3	11.6	12.4	12.3	11.2	10.1
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	1.0	1.4	1.8	2.0	1.0	1.0	0.8	1.0	0.8	0.8	0.8	1.0
$(3,2.5)$	250	5%	6.0	6.4	7.4	6.2	4.2	5.2	6.2	4.2	5.2	5.4	5.6	4.6
		10%	11.6	11.8	13.6	12.2	10.2	10.2	10.4	10.6	9.8	10.4	10.2	10.6
		1%	1.0	1.1	1.2	1.3	1.5	1.4	1.8	1.8	1.3	1.0	1.1	1.0
$(3,2.5)$	500	5%	6.0	6.3	5.4	5.7	5.7	5.3	4.9	4.3	4.0	4.5	4.8	5.1
		10%	11.8	12.9	11.6	12.0	12.2	12.2	11.5	11.0	10.6	9.7	9.5	10.3
		1%	1.2	1.5	1.1	1.1	1.7	1.4	1.8	1.5	1.4	1.9	1.7	1.7
$(3,2.5)$	$2,000$	5%	5.1	4.9	5.9	6.0	5.7	6.3	6.6	5.7	5.6	5.9	5.6	5.5
		10%	10.8	10.1	10.6	11.0	11.3	10.9	11.4	12.0	12.6	12.0	11.0	10.2

Model (4.1)–(4.2) with $A_{01}^{+} \neq A_{01}^{-}$ when $\underline{\delta}_0$ is unknown.

Table 8: Empirical power of the proposed test for the null hypothesis of a bivariate CCC-APGARCH$(0,1)$ model of the form (4.1) against the bivariate CCC-APGARCH$(1,1)$ alternative given by (4.3) when $\underline{\delta}_0$ is unknown.

δ_0	Length n	Level α	Lag m											
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	21.6	19.6	23.8	27.8	23.0	20.4	16.6	13.0	11.2	10.0	9.8	8.2
$(1,1)$	250	5%	37.0	43.6	54.6	55.0	50.0	46.4	43.0	39.0	34.8	31.4	30.8	27.8
		10%	47.2	56.4	67.2	69.8	68.0	62.4	58.4	54.8	52.2	49.6	45.2	42.2
		1%	46.8	53.8	69.2	72.8	71.8	70.2	65.6	60.0	56.2	53.4	46.8	43.4
$(1,1)$	500	5%	68.6	77.0	88.6	91.0	91.0	88.8	86.2	83.4	82.0	78.2	75.4	73.2
		10%	76.6	87.4	95.2	95.4	96.0	95.4	93.6	92.0	89.4	88.0	86.2	85.8
		1%	96.7	99.2	99.6	99.7	99.9	99.9	99.9	99.9	99.9	99.9	99.9	100.0
$(1,1)$	2,000	5%	98.2	99.8	99.9	99.9	99.9	99.9	99.9	99.9	100.0	99.9	100.0	100.0
		10%	99.1	100.0	99.9	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0	100.0
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	20.8	17.4	20.8	20.2	18.2	13.6	11.2	10.2	8.0	6.8	6.0	5.0
$(0.8,1.5)$	250	5%	38.2	40.6	47.8	48.8	46.8	43.0	37.0	33.6	29.4	24.6	24.4	23.2
		10%	49.6	56.2	66.0	66.6	63.2	59.2	53.8	51.0	46.4	43.4	42.0	39.4
		1%	47.5	49.2	61.7	65.6	63.2	60.9	55.4	50.1	46.2	41.0	38.7	32.9
$(0.8,1.5)$	500	5%	67.5	74.5	84.4	87.5	86.0	83.6	82.4	78.7	75.3	71.7	69.1	65.0
		10%	75.5	84.5	92.1	94.0	93.2	92.4	90.4	88.2	86.8	83.4	81.6	79.5
		1%	92.8	96.2	98.1	98.7	98.8	98.7	98.5	98.4	98.2	97.8	97.5	97.8
$(0.8,1.5)$	2,000	5%	96.0	98.3	99.0	99.3	99.4	99.4	99.4	99.3	99.2	98.9	99.1	99.2
		10%	97.2	99.3	99.3	99.6	99.6	99.5	99.5	99.5	99.5	99.5	99.5	99.5
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	14.4	17.2	23.2	23.0	22.0	19.0	17.8	15.0	11.6	11.2	8.2	7.2
$(2,2)$	250	5%	29.0	38.8	56.8	59.0	55.2	50.4	45.2	43.2	38.0	35.0	32.0	29.0
		10%	37.8	57.8	72.2	72.4	71.8	69.2	65.8	59.0	53.8	52.2	48.4	46.4
		1%	40.4	51.8	69.8	76.6	76.6	73.6	70.6	65.8	61.6	58.2	52.8	47.6
$(2,2)$	500	5%	59.6	76.6	90.0	92.0	91.4	91.8	90.2	87.8	86.0	82.6	79.8	77.6
		10%	69.4	86.2	94.8	97.0	95.6	95.8	95.4	94.8	94.2	93.0	91.2	89.4
		1%	91.8	98.2	99.0	99.4	99.7	99.7	99.8	99.8	99.8	99.8	99.6	99.5
$(2,2)$	2,000	5%	96.0	99.4	99.7	100.0	100.0	99.9	99.8	99.9	99.9	99.9	99.9	99.9
		10%	97.0	99.6	99.9	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	99.9
			1	2	3	4	5	6	7	8	9	10	11	12
		1%	24.6	24.8	33.2	36.8	36.8	33.2	29.6	26.6	23.2	21.8	19.2	16.4
$(3.,2.5)$	250	5%	41.0	52.0	62.0	64.4	64.8	62.4	57.0	54.8	50.8	46.8	43.6	43.2
		10%	48.8	64.4	72.4	76.2	75.0	75.4	72.8	69.2	68.4	65.2	62.2	60.4
		1%	52.8	56.6	67.0	71.6	72.4	70.4	67.4	65.8	63.4	59.2	56.6	55.6
$(3.,2.5)$	500	5%	66.6	77.8	85.8	87.2	88.6	89.4	88.0	88.2	87.4	84.6	81.6	80.0
		10%	74.6	84.6	92.0	92.0	93.6	93.6	93.4	92.8	91.4	91.0	91.2	90.4
		1%	90.4	93.7	95.6	96.7	97.7	97.6	97.8	98.0	97.7	98.0	97.8	97.9
$(3.,2.5)$	2,000	5%	94.4	96.9	98.3	98.8	98.9	99.0	99.2	99.3	99.1	98.8	99.0	99.0
		10%	96.4	98.3	98.9	99.3	99.3	99.4	99.5	99.6	99.3	99.3	99.5	99.5

Model (4.3) with $A_{01}^{+} \neq A_{01}^{-}$ when $\underline{\delta}_0$ is unknown.

6. Appendix : Proofs of the mains results

To prove the main results we need some tools from Boubacar Maïnassara et al. (2022) summarized in the following lemma.

6.1. Preliminaries

For all $\theta \in \Theta$, recall that $\tilde{\underline{h}}_t^{\underline{\delta}_0/2}(\theta)$ is the strictly stationary and non-anticipative solution of (2.1).

Lemma 6.1. *(Boubacar Maïnassara et al. (2022))*
Under Assumptions ***A1-A7*** *and for* $s \in]0,1[$*, we have*

$$\mathbb{E}\|\underline{\varepsilon}_t^{\underline{\delta}_0/2}\|^s < \infty, \qquad \mathbb{E}\sup_{\theta\in\Theta}\|\underline{h}_{0t}^{\underline{\delta}_0/2}\|^s < \infty, \qquad \mathbb{E}\sup_{\theta\in\Theta}\|\tilde{\underline{h}}_{0t}^{\underline{\delta}_0/2}\|^s < \infty. \tag{6.1}$$

Moreover, there exists K a random constant that depends on the past values of $\{\varepsilon_t, t \leq 0\}$ and $0 < \rho < 1$ such that

$$\sup_{\theta\in\Theta} \left\| \underline{h}_t^{\delta_0/2}(\theta) - \underline{\tilde{h}}_t^{\delta_0/2}(\theta) \right\| \leq K\rho^t \ . \tag{6.2}$$

Thus, for $i_1 = 1, \ldots, d$, since $\min\left(h_{i_1,t}^{\delta_{0,i_1}/2}(\theta), \tilde{h}_{i_1,t}^{\delta_{0,i_1}/2}(\theta)\right) \geq \omega = \inf\limits_{1\leq i\leq d} \underline{\omega}(i)$, the mean-value theorem implies that

$$\begin{aligned}\sup_{\theta\in\Theta} \left| h_{i_1,t}(\theta) - \tilde{h}_{i_1,t}(\theta) \right| &\leq \frac{2}{\delta_{0,i_1}} \sup_{\theta\in\Theta} \max\left(h_{i_1,t}^{1-\delta_{0,i_1}/2}(\theta), \tilde{h}_{i_1,t}^{1-\delta_{0,i_1}/2}(\theta)\right) \sup_{\theta\in\Theta} \left| h_{i_1,t}^{\delta_{0,i_1}/2}(\theta) - \tilde{h}_{i_1,t}^{\delta_0,i_1/2}(\theta) \right| \\ &\leq \frac{2K}{\delta_{0,i_1}} \left(\sup_{\theta\in\Theta} \frac{1}{\omega}\right) \sup_{\theta\in\Theta} \max\left(h_{i_1,t}(\theta), \tilde{h}_{i_1,t}(\theta)\right) \rho^t \leq K\rho^t \ , \end{aligned} \tag{6.3}$$

and similarly

$$\begin{aligned}\sup_{\theta\in\Theta} \left| h_{i_1,t}^{1/2}(\theta) - \tilde{h}_{i_1,t}^{1/2}(\theta) \right| &\leq \frac{1}{\delta_{0,i_1}} \sup_{\theta\in\Theta} \max\left(h_{i_1,t}^{(1-\delta_{0,i_1})/2}(\theta), \tilde{h}_{i_1,t}^{(1-\delta_{0,i_1})/2}(\theta)\right) \sup_{\theta\in\Theta} \left| h_{i_1,t}^{\delta_{0,i_1}/2}(\theta) - \tilde{h}_{i_1,t}^{\delta_{0,i_1}/2}(\theta) \right| \\ &\leq \frac{K}{\delta_{0,i_1}} \left(\sup_{\theta\in\Theta} \frac{1}{\omega}\right) \sup_{\theta\in\Theta} \max\left(h_{i_1,t}^{1/2}(\theta), \tilde{h}_{i_1,t}^{1/2}(\theta)\right) \rho^t \leq K\rho^t \ . \end{aligned} \tag{6.4}$$

From (6.3) *we can deduce that, almost surely, we have*

$$\sup_{\theta\in\Theta} \left\| H_t(\theta) - \tilde{H}_t(\theta) \right\| \leq K\rho^t, \qquad \forall t. \tag{6.5}$$

Since $\|R^{-1}\|$ is the inverse of the eigenvalue of smaller module of R and $\|\tilde{D}_t^{-1}\| = [\min_i(h_{ii,t}^{1/2})]^{-1}$ for $i = 1, \ldots, d$, we have

$$\sup_{\theta\in\Theta} \|\tilde{H}_t^{-1}(\theta)\| \leq \sup_{\theta\in\Theta} \|\tilde{D}_t^{-1}\|^2 \|R^{-1}\| \leq \sup_{\theta\in\Theta} \left[\min_i(\underline{\omega}(i))\right]^{-1} \|R^{-1}\| \leq K, \tag{6.6}$$

*by using the fact that R is a positive-definite matrix (see Assumption **A5**), the compactness of Θ and the strict positivity of the components of $\underline{\omega}$. Similarly, we have*

$$\sup_{\theta\in\Theta} \|H_t^{-1}(\theta)\| \leq K. \tag{6.7}$$

There exists a neighborhood $V(\theta_0)$ of $\theta_0 \in \overset{\circ}{\Theta}$ such that: for all $r_0 \geq 1$, $i_1 = 1, \ldots, d$ and all $i, j = 1, \ldots, s_1$ we have

$$\mathbb{E} \sup_{\theta\in V(\theta_0)} \left| \frac{1}{h_{i_1,t}^{\delta_{0,i_1}/2}} \frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}}{\partial\theta_i}(\theta) \right|^{r_0} < \infty \qquad \textit{and} \qquad \mathbb{E} \sup_{\theta\in V(\theta_0)} \left| \frac{1}{h_{i_1,t}^{\delta_{0,i_1}/2}} \frac{\partial^2 h_{i_1,t}^{\delta_{0,i_1}/2}}{\partial\theta_i\partial\theta_j}(\theta) \right|^{r_0} < \infty \ . \tag{6.8}$$

In the case where the power is unknown, the vector of parameter becomes ϑ and we replace H_t by $\mathcal{H}_t$ and θ by ϑ. The previous results must be adapted in consequence.

6.2. Proof of Theorem 3.1

We decomposed this proof in following steps.

(i) Asymptotic impact of the unknown initial values on the statistic $\hat{\mathbf{r}}_m$.

(ii) Asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$.

(*iii*) Invertibility of the matrix D.

Recall that

$$\begin{aligned} r_h(\theta) &= \frac{1}{n}\sum_{t=h+1}^{n}[\mathrm{Tr}(s_t(\theta))][\mathrm{Tr}(s_{t-h}(\theta))] \qquad \text{with } s_t(\theta)=\eta_t(\theta)\eta_t'(\theta)-I_d. \\ &= \frac{1}{n}\sum_{t=h+1}^{n}[\eta_t'(\theta)\eta_t(\theta)-d][\eta_{t-h}'(\theta)\eta_{t-h}(\theta)-d] \\ &= \frac{1}{n}\sum_{t=h+1}^{n}[\underline{\varepsilon}_t'H_t^{-1}(\theta)\underline{\varepsilon}_t-d][\underline{\varepsilon}_{t-h}'H_{t-h}^{-1}(\theta)\underline{\varepsilon}_{t-h}-d]. \end{aligned}$$

(*i*) **Asymptotic impact of the unknown initial values on the statistic $\hat{\mathbf{r}}_m$**
Let $S_t(\theta)=\underline{\varepsilon}_t'H_t^{-1}(\theta)\underline{\varepsilon}_t-d$ and $\tilde{S}_t(\theta)=\underline{\varepsilon}_t'\tilde{H}_t^{-1}(\theta)\underline{\varepsilon}_t-d$. We observe that

$$r_h(\theta)-\tilde{r}_h(\theta) = \frac{1}{n}\sum_{t=h+1}^{n}(a_t+b_t),$$

where $a_t=S_{t-h}(\theta)(S_t(\theta)-\tilde{S}_t(\theta))$ and $b_t=(\tilde{S}_{t-h}(\theta)-S_{t-h}(\theta))\tilde{S}_t(\theta)$. We obtain

$$\begin{aligned} |a_t| &= \left|\underline{\varepsilon}_{t-h}'H_{t-h}^{-1}(\theta)\underline{\varepsilon}_{t-h}-d\right|\left|\underline{\varepsilon}_t'H_t^{-1}(\theta)(H_t(\theta)-\tilde{H}_t(\theta))\tilde{H}_t^{-1}(\theta)\underline{\varepsilon}_t\right| \\ &= \left|\mathrm{Tr}(H_{t-h}^{-1}(\theta)\underline{\varepsilon}_{t-h}\underline{\varepsilon}_{t-h}'-I_d)\right|\left|\mathrm{Tr}(H_t^{-1}(\theta)(H_t(\theta)-\tilde{H}_t(\theta))\tilde{H}_t^{-1}(\theta)\underline{\varepsilon}_t\underline{\varepsilon}_t')\right| \\ &\le \sup_{\theta\in\Theta}\left(\|H_t^{-1}(\theta)\|\|\underline{\varepsilon}_{t-h}\underline{\varepsilon}_{t-h}'\|+\|I_d\|\right)\left(\|H_t^{-1}(\theta)\|\|H_t(\theta)-\tilde{H}_t(\theta)\|\|\tilde{H}_t^{-1}(\theta)\|\|\underline{\varepsilon}_t\underline{\varepsilon}_t'\|\right). \end{aligned}$$

Now using (6.5), (6.6) and (6.7), we have

$$|a_t|\le K\rho^t(\underline{\varepsilon}_{t-h}'\underline{\varepsilon}_{t-h}+d)\underline{\varepsilon}_t'\underline{\varepsilon}_t$$

We have the same bound for $|b_t|$. Using the inequality $(a+b)^s\le a^s+b^s$, for $a,b\ge 0$ and $s\in]0,1[$, (6.1) and Hölder's inequality, we have for some $s^*\in]0,1[$ sufficiently small

$$\begin{aligned} \mathbb{E}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sup_{\theta\in\Theta}|a_t|\right|^{s*} &\le \mathbb{E}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sup_{\theta\in\Theta}\|K\rho^t(\underline{\varepsilon}_{t-h}\underline{\varepsilon}_{t-h}'+I_d)\underline{\varepsilon}_t\underline{\varepsilon}_t'\|\right|^{s*} \\ &\le K\left(\frac{1}{\sqrt{n}}\right)^{s*}\sum_{t=1}^{n}\rho^{ts*}\underset{n\to\infty}{\longrightarrow}0 \end{aligned}$$

We deduce

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sup_{\theta\in\Theta}|a_t|=\mathrm{o}_{\mathbb{P}}(1).$$

We have the same convergence for b_t, and for the derivatives of a_t and b_t. Consequently we obtain

$$\sqrt{n}\|\mathbf{r}_m(\theta_0)-\tilde{\mathbf{r}}_m(\theta_0)\|=\mathrm{o}_{\mathbb{P}}(1), \qquad \sup_{\theta\in\Theta}\left\|\frac{\partial\mathbf{r}(\theta)}{\partial\theta'}-\frac{\partial\tilde{\mathbf{r}}(\theta)}{\partial\theta'}\right\|=\mathrm{o}_{\mathbb{P}}(1). \tag{6.9}$$

The unknown initial values have no asymptotic impact on the statistic $\hat{\mathbf{r}}_m$.

(*ii*) **Asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$**
We now show that the asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$ is deduced from the joint distribution of $\sqrt{n}\mathbf{r}_m$ and the QMLE.

Using (6.9) and a Taylor expansion of $\mathbf{r}_m(\cdot)$ around $\hat{\theta}_n$ and θ_0, we obtain

$$\begin{aligned}\sqrt{n}\hat{\mathbf{r}}_m &= \sqrt{n}\tilde{\mathbf{r}}_m(\theta_0) + \frac{\partial\tilde{\mathbf{r}}_m(\theta^*)}{\partial\theta'}\sqrt{n}(\hat{\theta}_n-\theta_0)\\ &= \sqrt{n}\mathbf{r}_m(\theta_0) + \frac{\partial\mathbf{r}_m(\theta^*)}{\partial\theta'}\sqrt{n}(\hat{\theta}_n-\theta_0)+\mathrm{o}_{\mathbb{P}}(1),\end{aligned}$$

for some θ^* between $\hat{\theta}_n$ and θ_0. For $i,j=1,\ldots,s_0$ the first and the second derivatives of $S_t(\theta)$ give

$$\begin{aligned}\frac{\partial S_t(\theta)}{\partial\theta_i} &= -\mathrm{Tr}\left[H_t^{-1}(\theta)\underline{\varepsilon}_t\underline{\varepsilon}_t'H_t^{-1}(\theta)\frac{\partial H_t(\theta)}{\partial\theta_i}\right],\\ \frac{\partial^2 S_t(\theta)}{\partial\theta_i\partial\theta_j} &= \mathrm{Tr}\left[H_t^{-1}(\theta)\frac{\partial H_t(\theta)}{\partial\theta_j}H_t^{-1}(\theta)\underline{\varepsilon}_t\underline{\varepsilon}_t'H_t^{-1}(\theta)\frac{\partial H_t(\theta)}{\partial\theta_i} - H_t^{-1}(\theta)\underline{\varepsilon}_t\underline{\varepsilon}_t'H_t^{-1}(\theta)\frac{\partial^2 H_t(\theta)}{\partial\theta_i\partial\theta_j}\right.\\ &\qquad\left.+H_t^{-1}(\theta)\underline{\varepsilon}_t\underline{\varepsilon}_t'H_t^{-1}(\theta)\frac{\partial H_t(\theta)}{\partial\theta_j}H_t^{-1}(\theta)\frac{\partial H_t(\theta)}{\partial\theta_i}\right].\end{aligned}$$

In view of (6.8), there exists a neighborhood $V(\theta_0)$ of θ_0 such that

$$\mathbb{E}\sup_{\theta\in V(\theta_0)}\left\|\frac{\partial^2 S_{t-h}(\theta)S_t(\theta)}{\partial\theta\partial\theta'}\right\|<\infty.$$

For $i=1,\ldots,s_0$, let $\mathbf{h}_t(i)=\left[\mathrm{vec}\left(H_{0t}^{-1/2}(\partial H_t(\theta_0)/\partial\theta_i)H_{0t}^{-1/2}\right)\right]$ and we define the matrix of size $d^2\times s_0$, $\mathbf{h}_t=(\mathbf{h}_t(1)|\ldots|\mathbf{h}_t(s_0))$. For a fixed r_h, using the previous inequality, Assumption **A7**, the almost sure convergence of θ^* to θ_0, a second Taylor expansion and the ergodic theorem, we obtain

$$\begin{aligned}\frac{\partial r_h(\theta^*)}{\partial\theta_i} = \frac{\partial r_h(\theta_0)}{\partial\theta_i}+\mathrm{o}_{\mathbb{P}}(1)\underset{n\to\infty}{\longrightarrow}C(h,i) &:= \mathbb{E}\left[S_{t-h}\frac{\partial S_t}{\partial\theta_i}\right] = -\mathbb{E}\left[S_{t-h}\mathrm{Tr}\left(H_{0t}^{-1}\frac{\partial H_t(\theta_0)}{\partial\theta_i}\right)\right]\\ &= -\mathbb{E}\left[S_{t-h}\mathbf{h}_t'(i)\mathrm{vec}(I_d)\right],\end{aligned}$$

by the fact $\mathbb{E}[S_t\partial S_{t-h}(\theta_0)/\partial\theta]=0$ and using the property $\mathrm{Tr}(A'B)=(\mathrm{vec}(A))'\mathrm{vec}(B)$. Note that $C(h,i)$ is the (h,i)-th element of the matrix C_m. Consequently we have

$$\frac{\partial\mathbf{r}_m(\theta_0)}{\partial\theta'}\underset{n\to\infty}{\longrightarrow}C_m:=[C(h,i)]_{1\le h\le m,1\le i\le s_0}=-\mathbb{E}\left[(\mathbb{S}_{t-1:t-m})\left(\mathbf{h}_t'\mathrm{vec}(I_d)\right)'\right],\tag{6.10}$$

where $\mathbb{S}_{t-1:t-m}=(S_{t-1},\ldots,S_{t-m})'$. It follows that

$$\sqrt{n}\hat{\mathbf{r}}_m=\sqrt{n}\mathbf{r}_m+C_m\sqrt{n}(\hat{\theta}_n-\theta_0)+\mathrm{o}_{\mathbb{P}}(1).\tag{6.11}$$

From (6.11) it is clear that the asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$ is related to the asymptotic behavior of $\sqrt{n}(\hat{\theta}_n'-\theta_0',\mathbf{r}_m')'$. We note that

$$\sqrt{n}(\hat{\theta}_n-\theta_0)=-J^{-1}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial l_t(\theta_0)}{\partial\theta}\right),$$

with $l_t(\theta)=\underline{\varepsilon}_t'H_t^{-1}(\theta)\underline{\varepsilon}_t+\log(\det(H_t(\theta)))$. The derivatives are recursively calculated with respect to $H_t(\theta)$ for a fixed $i=1,\ldots,s_0$

$$\frac{\partial l_t(\theta_0)}{\partial\theta_i}=\mathrm{Tr}\left[(H_{0t}^{-1}-H_{0t}^{-1}\underline{\varepsilon}_t\underline{\varepsilon}_t'H_{0t}^{-1})\frac{\partial H_t(\theta_0)}{\partial\theta_i}\right]=-\left[\mathrm{vec}\left(H_{0t}^{-1/2}\frac{\partial H_t(\theta_0)}{\partial\theta_i}H_{0t}^{-1/2}\right)\right]'\mathrm{vec}(s_t)=-\mathbf{h}_t'(i)\mathrm{vec}(s_t).$$

We then deduce that

$$\frac{\partial l_t(\theta_0)}{\partial\theta}=\left(\frac{\partial l_t(\theta_0)}{\partial\theta_1},\ldots,\frac{\partial l_t(\theta_0)}{\partial\theta_{s_0}}\right)'=-\mathbf{h}_t'\mathrm{vec}(s_t).$$

Observe that $\sqrt{n}\underline{\mathbf{r}}_m = n^{-1}\sum_{t=1}^{n}\mathbb{S}_{t-1:t-m}S_t$. Now we can obtained the asymptotic distribution of $\sqrt{n}(\hat{\theta}'_n - \theta'_0, \mathbf{r}'_m)'$ by applying the central limit theorem to the multivariate martingale difference

$$\left\{\Upsilon_t = \left(\left\{J^{-1}\mathbf{h}'_t\text{vec}(s_t)\right\}', \mathbb{S}_{t-1:t-m}S_t\right)'; \mathcal{F}^{\eta}_{t-1} := \sigma(\eta_u, u \leq t)\right\}.$$

The expectation of the distribution is given by

$$\mathbb{E}\left[\Upsilon_t|\mathcal{F}^{\eta}_{t-1}\right] = \mathbb{E}\left[\left.\begin{pmatrix} J^{-1}\mathbf{h}'_t\text{vec}(s_t) \\ S_{t-1}S_t \\ \vdots \\ S_{t-m}S_t \end{pmatrix}\right| \mathcal{F}^{\eta}_{t-1}\right] = \begin{pmatrix} J^{-1}\mathbf{h}'_t\mathbb{E}[\text{vec}(s_t)|\mathcal{F}^{\eta}_{t-1}] \\ S_{t-1}\mathbb{E}[S_t|\mathcal{F}^{\eta}_{t-1}] \\ \vdots \\ S_{t-m}\mathbb{E}[S_t|\mathcal{F}^{\eta}_{t-1}] \end{pmatrix} = 0,$$

because $(S_{t-i})_{i\geq 1}$ is measurable with respect to the σ-field $\mathcal{F}^{\eta}_{t-1}$ and $\mathbb{E}[S_t] = \mathbb{E}[\eta'_t\eta_t] - d = 0$ and $\mathbb{E}[\text{vec}(s_t)] = \text{vec}[\mathbb{E}(\eta_t\eta'_t) - I_d] = 0$. For $i = 1, \ldots, d$, the variance is given by

$$\begin{aligned}\Xi := \mathbb{E}\left[\Upsilon_t\Upsilon'_t\right] &= \begin{pmatrix} \Sigma_{\hat{\theta}_n} & \Sigma_{\hat{\theta}_n,\mathbf{r}_m} \\ \Sigma'_{\hat{\theta}_n,\mathbf{r}_m} & \Sigma_{\mathbf{r}_m} \end{pmatrix} \\ &= \begin{pmatrix} J^{-1}IJ^{-1} & \mathbb{E}\left[J^{-1}\mathbf{h}'_t\text{vec}(s_t)S_t\mathbb{S}'_{t-1:t-m}\right] \\ \mathbb{E}\left[\mathbb{S}_{t-1:t-m}\left(\text{vec}(s_t)\right)'S_t\mathbf{h}_tJ^{-1}\right] & \left(\mathbb{E}\left[S_t^2\right]\right)^2 I_m \end{pmatrix}, \end{aligned} \tag{6.12}$$

which leads to

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\Upsilon_t \xrightarrow[n\to\infty]{\text{d}} \mathcal{N}(0, \Xi).$$

Using (6.9) and (6.12), the asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$ gives

$$\sqrt{n}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\text{d}} \mathcal{N}(0, D),$$

where D is a matrix defined as follows

$$\begin{aligned} D := \lim_{n\to\infty}\text{Var}(\sqrt{n}\hat{\mathbf{r}}_m) &= \lim_{n\to\infty}\text{Var}(\sqrt{n}\mathbf{r}_m) + C_m\left[\lim_{n\to\infty}\text{Var}(\sqrt{n}(\hat{\theta}_n - \theta_0))\right]C'_m \\ &\quad + C_m\left[\lim_{n\to\infty}\text{Cov}(\sqrt{n}(\hat{\theta}_n - \theta_0), \sqrt{n}\mathbf{r}_m)\right] \\ &\quad + \left[\lim_{n\to\infty}\text{Cov}(\sqrt{n}(\hat{\theta}_n - \theta_0), \sqrt{n}\mathbf{r}_m)\right]C'_m \\ &= \Sigma_{\mathbf{r}_m} + C_mJ^{-1}IJ^{-1}C'_m + C_m\Sigma_{\hat{\theta}_n,\mathbf{r}_m} + \Sigma'_{\hat{\theta}_n,\mathbf{r}_m}C'_m. \end{aligned}$$

(*iii*) **Invertibility of the matrix** D

Note that by using the relation $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$ we can also rewrite $\mathbf{h}_t(i)$ as follows

$$\mathbf{h}_t(i) = \left[\text{vec}\left(H_{0t}^{-1/2}\frac{\partial H_t(\theta_0)}{\partial\theta_i}H_{0t}^{-1/2}\right)\right] = \left[H_{0t}^{-1/2}\otimes H_{0t}^{-1/2}\right]\text{vec}\left(\frac{\partial H_t(\theta_0)}{\partial\theta_i}\right) = \mathbf{H}_t\mathbf{d}_t(i),$$

where $\mathbf{H}_t = \left[H_{0t}^{-1/2}\otimes H_{0t}^{-1/2}\right]$ and $\mathbf{d}_t(i) = \text{vec}\left(\partial H_t(\theta_0)/\partial\theta_i\right)$. Thus we define the matrix of size $d^2 \times s_0$, $\mathbf{d}_t = (\mathbf{d}_t(1)|\ldots|\mathbf{d}_t(s_0))$ such that $\mathbf{h}_t = \mathbf{H}_t\mathbf{d}_t$.

To study the invertibility of the matrix D we write $V = \mathbb{S}_{t-1:t-m}S_t - C_mJ^{-1}\dfrac{\partial l_t(\theta_0)}{\partial\theta}$ such that

$$\begin{aligned}\mathbb{E}[VV'] &= \mathbb{E}\left[(S_t^2)\mathbb{S}_{t-1:t-m}\mathbb{S}'_{t-1:t-m}\right] - \mathbb{E}\left[(S_t)\mathbb{S}_{t-1:t-m}\frac{\partial l_t(\theta_0)}{\partial\theta}\right]J^{-1}C'_m \\ &\quad - C_mJ^{-1}\mathbb{E}\left[\frac{\partial l_t(\theta_0)}{\partial\theta}(\mathbb{S}'_{t-1:t-m}S_t)\right] + C_mJ^{-1}\mathbb{E}\left[\frac{\partial l_t(\theta_0)}{\partial\theta}\frac{\partial l_t(\theta_0)}{\partial\theta'}\right]J^{-1}C'_m \\ &= \Sigma_{\mathbf{r}_m} + \Sigma'_{\hat{\theta}_n,\mathbf{r}_m}C'_m + C_m\Sigma_{\hat{\theta}_n,\mathbf{r}_m} + C_mJ^{-1}IJ^{-1}C_m. \end{aligned}$$

We can rewrite the vector V as

$$V = \mathbb{S}_{t-1:t-m} S_t + C_m J^{-1} \mathbf{d}_t' \mathbf{H}_t' \text{vec}(s_t).$$

If the matrix $\mathbb{E}[VV']$ is singular, then there exist a vector $\lambda = (\lambda_1, \ldots, \lambda_m)'$ not equal to zero such that

$$\lambda' V = \lambda' \mathbb{S}_{t-1:t-m} S_t + \mu \mathbf{d}_t' \mathbf{H}_t' \text{vec}(s_t) = 0, \quad \text{a.s.}, \tag{6.13}$$

with $\mu = \lambda' C_m J^{-1}$. We have $\mu \neq 0$, else $\lambda' \mathbb{S}_{t-1:t-m} S_t = 0$ almost surely, that implies there exists $j \in \{1, \ldots, m\}$ such that S_{t-j} be mesurable respect to the σ-field $\{S_r, t-1 \leq r \leq t-m\}$ with $r \neq t-j$. That is impossible because the S_t are independent and not degenerated. Consequently (6.13) becomes

$$\begin{aligned}
\mu' \mathbf{d}_t' &= \sum_{i=1}^{s_0} \mu_i \mathbf{d}_t(i) = \sum_{i=1}^{s_0} \mu_i \text{vec}\left(\frac{\partial H_t(\theta_0)}{\partial \theta_i}\right) = \sum_{i=1}^{s_0} \mu_i \frac{\partial}{\partial \theta_i} \left[(D_{0t} \otimes D_{0t}) \text{vec}(R_0)\right] = 0 \quad \text{a.s.}, \\
&= \sum_{i=1}^{s_1} \mu_i \frac{\partial (D_{0t} \otimes D_{0t})}{\partial \theta_i} \text{vec}(R_0) + \sum_{i=s_1+1}^{s_0} \mu_i (D_{0t} \otimes D_{0t}) \frac{\partial \text{vec}(R_0)}{\partial \theta_i} = 0 \quad \text{a.s.}.
\end{aligned} \tag{6.14}$$

Since the vectors $\partial \text{vec}(R_0)/\partial \theta_i$, $i = s_1 + 1, \ldots, s_0$ are linearly independent, the vector $(\mu_{s_1+1}, \ldots, \mu_{s_0})'$ is null and thus Equation (6.14) yields

$$\sum_{i=1}^{s_1} \mu_i \frac{\partial (D_{0t} \otimes D_{0t})}{\partial \theta_i} \text{vec}(R_0) = 0, \quad \text{a.s.}. \tag{6.15}$$

The rows $1, d+1, \ldots, d^2$ of the Equation (6.15) yield

$$\sum_{i=1}^{s_1} \mu_i \frac{\partial \underline{h}_t(\theta_0)}{\partial \theta_i} = 0, \quad \text{a.s.} \tag{6.16}$$

We have for $i_1 = 1, \ldots, d$ and $i = 1, \ldots, s_1$

$$\frac{\partial h_{i_1,t}(\theta_0)}{\partial \theta_i} = \frac{\partial \left(h_{i_1,t}^{\delta_{0,i_1}/2}\right)^{2/\delta_{0,i_1}}}{\partial \theta_i}(\theta_0) = \frac{2}{\delta_{0,i_1}} h_{i_1,0t} \times \frac{1}{h_{i_1,0t}^{\delta_{0,i_1}/2}} \frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}}{\partial \theta_i}(\theta_0), \tag{6.17}$$

where the derivatives involved in (6.17) are defined for all $\theta \in \Theta$ recursively by

$$\frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}(\theta)}{\partial \theta} = c_t(\theta) + \sum_{i_2=1}^{d} \sum_{i=1}^{p} B_i(i_1, i_2) \frac{\partial h_{i_2,t-i}^{\delta_{0,i_2}/2}}{\partial \theta},$$

with

$$\begin{aligned}
c_t(\theta) = \Bigg(& 0, \ldots, 1, 0, \ldots, \left(\varepsilon_{i_1,t-1}^+\right)^{\delta_{0,i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-1}^+\right)^{\delta_{0,i_d}}, 0, \ldots, \left(\varepsilon_{i_1,t-q}^+\right)^{\delta_{0,i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-q}^+\right)^{\delta_{0,i_d}}, \\
& 0, , \ldots, \left(\varepsilon_{i_1,t-1}^-\right)^{\delta_{0,i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-1}^-\right)^{\delta_{0,i_d}}, 0, \ldots, \left(\varepsilon_{i_1,t-q}^-\right)^{\delta_{0,i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-q}^-\right)^{\delta_{0,i_d}}, \\
& 0, , \ldots, h_{i_1,t-1}^{\delta_{0,i_1}/2}, 0, \ldots, h_{i_d,t-1}^{\delta_{0,i_d}/2}, 0, \ldots, h_{i_1,t-p}^{\delta_{0,i_1}/2}, 0, \ldots, h_{i_d,t-p}^{\delta_{0,i_d}/2}, \ldots, 0 \Bigg)'.
\end{aligned} \tag{6.18}$$

The distribution of η_t is non-degenerated, so Equation (6.13) becomes

$$\lambda' V = \lambda' \mathbb{S}_{t-1:t-m} + \mu' \mathbf{d}' \mathbf{H}' \mathbb{1} = 0, \quad \text{a.s.}$$

where $\mathbb{1}$ represents a vector composed by 1 of size $d^2 \times 1$. For $i_1 = 1, \ldots, d$ and in view of (6.17) we can write

$$\lambda' V = \lambda' \mathbb{S}_{t-1:t-m} h_{i_1,t}^{\delta_{0,i_1}/2}(\theta_0) + \sum_{i=1}^{s_1} \mu_i^* \frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}(\theta_0)}{\partial \theta_i} = 0, \quad \text{a.s.} \tag{6.19}$$

where $\mu_i^* = 2\mu_i/\delta_{0,i_1}$.

Denote by R_t a random variable measurable with respect to $\sigma\{\eta_u, u \leq t\}$ whose value will be modified along the proof. Thus we write

$$h_{i_1,t}^{\delta_{0,i_1}/2}(\theta_0) = \sum_{i_2=1}^{d} \left[A_{01}^+(i_1,i_2)(\varepsilon_{i_2,t-1}^+)^{\delta_{0,i_2}} + A_{01}^-(i_1,i_2)(\varepsilon_{i_2,t-1}^-)^{\delta_{0,i_2}} \right] + R_{t-2}.$$

We remind that $\underline{\varepsilon}_t^+ = H_{0t}^{1/2}\eta_t^+$ and $\underline{\varepsilon}_t^- = H_{0t}^{1/2}\eta_t^-$. We decompose Equation (6.19) in two terms. The first one of (6.19) can be rewritten

$$\begin{aligned}
\lambda' \mathbb{S}_{t-1:t-m} h_{i_1,t}^{\delta_{0,i_1}/2}(\theta_0) = & \left\{ \sum_{i_2=1}^{d} \left[A_{01}^+(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right. \right. \\
& \left. \left. + A_{01}^-(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] \right\} R_{t-2} \\
& + \left\{ \sum_{i_2=1}^{d} \left[A_{01}^+(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right. \right. \\
& \left. \left. + A_{01}^-(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] \right\} \\
& \times \left(\lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 \right) + \left(\lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 \right) R_{t-2} + R_{t-2},
\end{aligned} \tag{6.20}$$

by using the fact that

$$\lambda' \mathbb{S}_{t-1:t-m} = \lambda_1 S_{t-1} + R_{t-2} = \lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 + R_{t-2}.$$

The second term of (6.19) can also be rewritten as

$$\begin{aligned}
\mu^{*\prime} \frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}(\theta_0)}{\partial \theta} &= \sum_{i_2=1}^{d} \left[\mu_{i_1+i_2 d}^* (\varepsilon_{i_2,t-1}^+)^{\delta_{0,i_2}} + \mu_{i_1+(i_2+q)d^2}^* (\varepsilon_{i_2,t-1}^-)^{\delta_{0,i_2}} \right] + R_{t-2} \\
&= \sum_{i_2=1}^{d} \left[\mu_{i_1+i_2 d}^* \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right. \\
&\quad \left. + \mu_{i_1+(i_2+q)d^2}^* \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] + R_{t-2},
\end{aligned} \tag{6.21}$$

where the vector $\mu^* = (\mu_1^*, \ldots, \mu_{s_1}^*)'$.

Combining the expressions (6.20) and (6.21), Equation (6.13) comes down almost surely to

$$\begin{aligned}
\lambda' V = &\left\{ \sum_{i_2=1}^{d} \left[A_{01}^{+}(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{+} \right)^{\delta_{0,i_2}} \right.\right. \\
&\left.\left. + A_{01}^{-}(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-} \right)^{\delta_{0,i_2}} \right] \right\} R_{t-2} \\
&+ \left\{ \sum_{i_2=1}^{d} \left[A_{01}^{+}(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{+} \right)^{\delta_{0,i_2}} \right.\right. \\
&\left.\left. + A_{01}^{-}(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-} \right)^{\delta_{0,i_2}} \right] \right\} \left(\lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 \right) + \left(\sum_{i=1}^{d} \eta_{i,t-1}^2 \right) R_{t-2} \\
&+ R_{t-2} \sum_{i_2=1}^{d} \left[\left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{+} \right)^{\delta_{0,i_2}} + \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-} \right)^{\delta_{0,i_2}} \right] + R_{t-2} = 0,
\end{aligned}$$

or equivalent to the two equations

$$\begin{aligned}
&\left[\sum_{i_2=1}^{d} A_{01}^{+}(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{+} \right)^{\delta_{0,i_2}} \right] \left[\lambda_1 \sum_{i_2=1}^{d} \left(\eta_{i_2,t-1}^{+} \right)^2 + R_{t-2} \right] \\
&+ R_{t-2} \sum_{i_2=1}^{d} \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{+} \right)^{\delta_{0,i_2}} + R_{t-2} \sum_{i_2=1}^{d} \left(\eta_{i_2,t-1}^{+} \right)^2 + R_{t-2} = 0 \quad \text{a.s.} \qquad (6.22)
\end{aligned}$$

$$\begin{aligned}
&\left[\sum_{i_2=1}^{d} A_{01}^{-}(i_1,i_2) \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-} \right)^{\delta_{0,i_2}} \right] \left[\lambda_1 \sum_{i_2=1}^{d} \left(\eta_{i_2,t-1}^{-} \right)^2 + R_{t-2} \right] \\
&+ R_{t-2} \sum_{i_2=1}^{d} \left(\sum_{j_1=1}^{d} H_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-} \right)^{\delta_{0,i_2}} + R_{t-2} \sum_{i_2=1}^{d} \left(\eta_{i_2,t-1}^{-} \right)^2 + R_{t-2} = 0 \quad \text{a.s..} \qquad (6.23)
\end{aligned}$$

When $d = 1$, from (6.22) and (6.23) we retrieve an equation of the following form obtained by Carbon and Francq (2011)

$$f(y) = a|y|^{\underline{\delta}_0+2} + b|y|^{\underline{\delta}_0} + cy^2 + d = 0,$$

which cannot have more than 3 positive roots or more than 3 negative roots, except if $a = b = c = d = 0$.

When $d \geq 2$ and also from (6.22) and (6.23), for a fixed component, we obtain an equation of the form

$$f(y) = \sum_{i=1}^{d} a_i |y|^{\delta_{0,i}+2} + \sum_{i=1}^{d} b_i |y|^{\delta_{0,i}+1} + \sum_{i=1}^{d} c_i |y|^{\delta_{0,i}} + ay^2 + b|y| + c = 0.$$

Note that an equation of this form can not have more than $3(d+1)$ non negative roots or more than $3(d+1)$ non positive roots for $d \geq 2$, except if $a_i = b_i = c_i = a = b = c = 0$.

By Assumption **A9**, Equations (6.22) and (6.23) imply that $\lambda_1 \left[\sum_{i_2=1}^{d} A_{01}^{+}(i_1,i_2) + A_{01}^{-}(i_1,i_2)\right] = 0$. But under the assumption **A4**, if $p > 0$, $\mathcal{A}_0(1)^+ + \mathcal{A}_0^- \neq 0$. It is impossible to have $A_{01}^{+}(i_1,i) = A_{01}^{+}(i_1,i) = 0$, for all $i = 1,\ldots,d$. Then, there exists an i_0 such that $A_1(i_1,i_0)^+ + A_1(i_1,i_0)^- \neq 0$ and we then have $\lambda_1 = 0$.

In the general case, Equation (6.14) necessarily leads

$$A_{01}^{+}(i_1,i_0) + A_{01}^{-}(i_1,i_0) = \cdots = A_{0q}^{+}(i_1,i_0) + A_{0q}^{-}(i_1,i_0) = 0, \qquad \forall i_0, i_1 = 1,\ldots,d,$$

that is impossible under the assumption **A4** and $\lambda = 0$. This is in contradiction with $\lambda' V = 0$, almost surely, that leads that the assumption of non invertibility of matrix D is absurd. □

6.3. Proof of Theorem 3.2

The almost sure convergence of $\hat{D}$ to D as n goes to infinity is easy to show using the consistency result. We remind the expression of the matrix D

$$D = \Sigma_{\mathbf{r}_m} + C_m J^{-1} I J^{-1} C_m' + C_m \Sigma_{\hat{\theta}_n,\mathbf{r}_m} + \Sigma_{\hat{\theta}_n,\mathbf{r}_m}' C_m'.$$

The matrix D can be rewritten as

$$D = \Sigma_{\mathbf{r}_m} + A + B + B',$$

where the matrices A and B are given by

$$\begin{aligned}
A &= (C_m - \hat{C}_m) J^{-1} I J^{-1} C_m' + \hat{C}_m (J^{-1} - \hat{J}^{-1}) I J^{-1} C_m' + \hat{C}_m \hat{J}^{-1} (I - \hat{I}) J^{-1} C_m' \\
&\quad + \hat{C}_m \hat{J}^{-1} \hat{I} (J^{-1} - \hat{J}^{-1}) C_m' + \hat{C}_m \hat{J}^{-1} \hat{I} \hat{J}^{-1} (C_m' - \hat{C}_m') + \hat{A}, \\
B &= (C_m - \hat{C}_m) \Sigma_{\hat{\theta}_n,\mathbf{r}_m} + \hat{C}_m (\Sigma_{\hat{\theta}_n,\mathbf{r}_m} - \hat{\Sigma}_{\hat{\theta}_n,\mathbf{r}_m}) + \hat{B},
\end{aligned}$$

with $\hat{A} = \hat{C}_m \hat{J}^{-1} \hat{I} \hat{J}^{-1} \hat{C}_m'$ and $\hat{B} = \hat{C}_m \hat{\Sigma}_{\hat{\theta}_n,\mathbf{r}_m}$ where $\hat{\Sigma}_{\hat{\theta}_n,\mathbf{r}_m} = -(\hat{\kappa}_i - 1)\, \hat{C}_m \hat{J}^{-1}$. Finally we have

$$D - \hat{D} = (\Sigma_{\mathbf{r}_m} - \hat{\Sigma}_{\mathbf{r}_m}) + (A - \hat{A}) + (B - \hat{B}) + (B' - \hat{B}').$$

For any multiplicative norm we have

$$\|D - \hat{D}\| \leq \|\Sigma_{\mathbf{r}_m} - \hat{\Sigma}_{\mathbf{r}_m}\| + \|A - \hat{A}\| + \|B - \hat{B}\| + \|B' - \hat{B}'\|.$$

Observe that

$$\begin{aligned}
\|A - \hat{A}\| &\leq \|C_m - \hat{C}_m\| \|J^{-1}\| \|I\| \|J^{-1}\| \|C_m'\| + \|\hat{C}_m\| \|J^{-1} - \hat{J}^{-1}\| \|I\| \|J^{-1}\| \|C_m'\| \\
&\quad + \|\hat{C}_m\| \|\hat{J}^{-1}\| \|I - \hat{I}\| \|J^{-1}\| \|C_m'\| + \|\hat{C}_m\| \|\hat{J}^{-1}\| \|\hat{I}\| \|J^{-1} - \hat{J}^{-1}\| \|C_m'\| \\
&\quad + \|\hat{C}_m\| \|\hat{J}^{-1}\| \|\hat{I}\| \|\hat{J}^{-1}\| \|C_m - \hat{C}_m\|, \\
&\leq \|C_m - \hat{C}_m\| \|J^{-1}\| \|I\| \|J^{-1}\| \|C_m'\| + \|\hat{C}_m\| \|J^{-1}\| \|\hat{J} - J\| \|\hat{J}\| \|I\| \|J^{-1}\| \|C_m'\| \\
&\quad + \|\hat{C}_m\| \|\hat{J}^{-1}\| \|I - \hat{I}\| \|J^{-1}\| \|C_m'\| + \|\hat{C}_m\| \|\hat{J}^{-1}\| \|\hat{I}\| \|J^{-1}\| \|\hat{J} - J\| \|\hat{J}\| \|C_m'\| \\
&\quad + \|\hat{C}_m\| \|\hat{J}^{-1}\| \|\hat{I}\| \|\hat{J}^{-1}\| \|C_m - \hat{C}_m\|, \qquad (6.24) \\
\|B - \hat{B}\| &\leq \|C_m - \hat{C}_m\| \|\Sigma_{\hat{\theta}_n,\mathbf{r}_m}\| + \|\hat{C}_m\| \|\Sigma_{\hat{\theta}_n,\mathbf{r}_m} - \hat{\Sigma}_{\hat{\theta}_n,\mathbf{r}_m}\|. \qquad (6.25)
\end{aligned}$$

In view of (6.7) and (6.8), we have $\|C_m\| < \infty$. We also have $\|I\| < \infty$. Because the matrix J is nonsingular, we have $\|J^{-1}\| < \infty$ and

$$\|\hat{J}^{-1} - J^{-1}\| \underset{n\to\infty}{\longrightarrow} 0, \quad a.s.$$

by consistency of $\hat{\theta}_n$. Under Assumption **A7**, we have $|\mathbb{E}\left[\eta_t'\eta_t - d\right]^2| \leq K$. Using the previous arguments and also the strong consistency of $\hat{\theta}_n$, we have

$$|\mathbb{E}\left[\eta_t'\eta_t - d\right]^2 - \hat{\kappa}| \underset{n\to\infty}{\longrightarrow} 0, \quad a.s. \text{ and } \|C_m - \hat{C}_m\| \underset{n\to\infty}{\longrightarrow} 0, \quad a.s.$$

We then deduce that Equations (6.24) and (6.25) converge almost surely to 0 when $n \to \infty$ and the conclusion follows. Thus $\hat{D} \underset{n\to\infty}{\longrightarrow} D$ almost surely.

To conclude the proof of Theorem 3.2, it suffices to use Theorem 3.1 and the following result: if $\sqrt{n}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, D)$, with D nonsingular, and if $\hat{D} \underset{n\to\infty}{\longrightarrow} D$ in probability, then $n\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m \xrightarrow[n\to\infty]{\mathrm{d}} \chi^2_m$. □

6.4. Proof of Remark 3.2

We suppose that $\boldsymbol{\mathcal{H}}_1$ holds true. One may rewrite the above arguments in order to prove that there exists a nonsingular matrix D^* such that

$$\sqrt{n}(\hat{\mathbf{r}}_m - \mathbf{r}_m^0) \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0, D^*) \ . \tag{6.26}$$

The matrix D^* is given by $D^* = \Sigma_{\mathbf{r}_m^0} + C_m^* J^{-1} I J^{-1} C_m^{*\prime} + C_m^* \Sigma_{\hat{\theta}_n, \mathbf{r}_m^0} + \Sigma'_{\hat{\theta}_n, \mathbf{r}_m^0} C_m^{*\prime}$, where the matrices $\Sigma_{\mathbf{r}_m^0}$ and $\Sigma_{\hat{\theta}_n, \mathbf{r}_m^0}$ are obtained from the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n' - \theta_0', \mathbf{r}_m' - \mathbf{r}^{0\prime}{}_m)'$. The (h,i)-th element of the matrix C_m^* is geven by

$$C^*(h,i) := \mathbb{E}\left[S_{t-h}\frac{\partial S_t}{\partial \theta_i} + S_t \frac{\partial S_{t-h}}{\partial \theta_i}\right].$$

Now we write

$$\begin{aligned}\sqrt{n}\hat{D}^{-1/2}\hat{\mathbf{r}}_m &= \hat{D}^{-1/2}\sqrt{n}(\hat{\mathbf{r}}_m - \mathbf{r}_m^0) + \hat{D}^{-1/2}\sqrt{n}\mathbf{r}_m^0 \\ &= D^{-1/2}\sqrt{n}(\hat{\mathbf{r}}_m - \mathbf{r}_m^0) + D^{-1/2}\sqrt{n}\mathbf{r}_m^0 + \mathrm{o}_{\mathbb{P}}(1) \ .\end{aligned}$$

Then it holds that

$$\begin{aligned}n\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m &= \left(\sqrt{n}\hat{D}^{-1/2}\hat{\mathbf{r}}_m\right)' \times \left(\sqrt{n}\hat{D}^{-1/2}\hat{\mathbf{r}}_m\right) \\ &= n(\hat{\mathbf{r}}_m - \mathbf{r}_m^0)'D^{-1}(\hat{\mathbf{r}}_m - \mathbf{r}_m^0) + 2n(\hat{\mathbf{r}}_m - \mathbf{r}_m^0)'D^{-1}\mathbf{r}_m^0 + n\mathbf{r}_m^{0\prime}D^{-1}\mathbf{r}_m^0 + \mathrm{o}_{\mathbb{P}}(1). \end{aligned}\tag{6.27}$$

By the ergodic theorem, $(\hat{\mathbf{r}}_m - \mathbf{r}_m^0)'D^{-1}\mathbf{r}_m^0 = \mathrm{o}_{\mathbb{P}}(1)$. By Lemma 17.1 in van der Vaart (1998), the convergence (6.26) implies that

$$(\hat{\mathbf{r}}_m - \mathbf{r}_m^0)'D^{-1}(\hat{\mathbf{r}}_m - \mathbf{r}_m^0) \xrightarrow[n\to\infty]{\mathrm{d}} \sum_{i=1}^m \lambda_i Z_i^2$$

where $(Z_i)_{1\leq i\leq m}$ are i.i.d. with $\mathcal{N}(0,1)$ laws and the λ_i's are the eigenvalues of the matrix $D^{-1/2}D^*D^{-1/2}$. Reporting these convergences in (6.27), we deduce that

$$\begin{aligned}\hat{\mathbf{r}}_m'\hat{D}^{-1}\hat{\mathbf{r}}_m &= (\hat{\mathbf{r}}_m - \mathbf{r}_m^0)'D^{-1}(\hat{\mathbf{r}}_m - \mathbf{r}_m^0) + 2(\hat{\mathbf{r}}_m - \mathbf{r}_m^0)'D^{-1}\mathbf{r}_m^0 + \mathbf{r}_m^{0\prime}D^{-1}\mathbf{r}_m^0 + \mathrm{o}_{\mathbb{P}}(1) \\ &= \mathbf{r}_m^{0\prime}D^{-1}\mathbf{r}_m^0 + \mathrm{o}_{\mathbb{P}}(1)\end{aligned}$$

and the remark is proved. □

6.5. Proof of Corollary 3.1

Note that if the model is correct we have

$$\hat{r}_0 = \frac{1}{n}\sum_{t=h+1}^{n} [\underline{\varepsilon}_t' \tilde{H}_t^{-1}(\hat{\theta}_n)\underline{\varepsilon}_t - d]^2 \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2 = \mathbb{E}\left[\eta_t'\eta_t - d\right]^2.$$

From (6.11) we have $\sqrt{n}(\hat{r}_0 - r_0) = \mathrm{o}_{\mathbb{P}}(1)$. Applying the central limit theorem to the process $([\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2)_{t\in\mathbb{Z}}$, we obtain

$$\sqrt{n}\,(\hat{r}_0 - r_0)) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left([\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2 - \mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2\right) + \mathrm{o}_{\mathbb{P}}(1) \xrightarrow[n\to\infty]{\text{in law}} \mathcal{N}(0,\Phi).$$

So we have $\sqrt{n}(\hat{r}_0 - r_0) = \mathrm{O}_{\mathbb{P}}(1)$ and $\sqrt{n}(r_0 - \mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2) = \mathrm{O}_{\mathbb{P}}(1)$. Now, using (6.11) and the ergodic theorem, we have

$$n\left(\frac{\hat{r}_h}{\hat{r}_0} - \frac{\hat{r}_h}{\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2}\right) = \sqrt{n}\hat{r}_h \frac{\sqrt{n}\left(\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2 - \hat{r}_0\right)}{\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2 \hat{r}_0} = \mathrm{O}_{\mathbb{P}}(1),$$

which means $\sqrt{n}\hat{\rho}(h) = \sqrt{n}\hat{r}_h/\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2 + \mathrm{O}_{\mathbb{P}}(n^{-1/2})$. For $h = 1,\ldots,m$, it follows that

$$\sqrt{n}\hat{\rho}_m = \frac{\sqrt{n}\hat{\mathbf{r}}_m}{\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2} + \mathrm{o}_{\mathbb{P}}(1). \tag{6.28}$$

Thus from (6.28) the asymptotic distribution of the sum of squared residuals autocorrelations $\sqrt{n}\hat{\rho}_m$ depends on the distribution of $\sqrt{n}\hat{\mathbf{r}}_m$. Consequently we have

$$\lim_{n\to\infty} \mathrm{Var}\left(\sqrt{n}\hat{\rho}_m\right) = \lim_{n\to\infty} \mathrm{Var}\left(\frac{\sqrt{n}\hat{\mathbf{r}}_m}{\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2}\right) =: D_{\hat{\rho}} = \frac{D}{\left(\mathbb{E}[\underline{\varepsilon}_t' H_t^{-1}\underline{\varepsilon}_t - d]^2\right)^2}.$$

Thus the first result (3.1) of Corollary 3.1 is proved.

The proof the second result (3.2) of Corollary 3.1 is the same that the one given for Theorem 3.2 and the proof is completed. □

6.6. Proof of Theorem 3.3

We follow the arguments and the different steps that we used in the proof of Theorem 3.1. As in the case where $\underline{\delta}_0$ was known, the proof is decomposed in the following points which will be treated in separate subsections.

- (i) Asymptotic impact of unknown initials values on the statistic $\hat{\mathbf{r}}_m$.
- (ii) Asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$.
- (iii) Invertibility of the matrix $\mathcal{D}$.

There are many similarities with the proof of Theorem 3.1. We only indicates where the fact that the power is estimated has an importance is our reasoning.

(i) **Asymptotic impact of unknown initials values on the statistic $\hat{\mathbf{r}}_m$**

The proof of the asymptotic impact of the initial values on the statistic $\hat{\mathbf{r}}_m$ is the same than the one where $\underline{\delta}_0$ was known. It suffices to adapt this step by replacing θ by ϑ and H_t by $\mathcal{H}_t$.

(ii) **Asymptotic distribution of $\sqrt{n}\hat{\mathbf{r}}_m$**

The asymptotic distribution of $\sqrt{n}\hat{\underline{\mathbf{r}}}_m$ is similar to that the one when the power $\underline{\delta}_0$ is assumed to be known. We adapt this step by replacing again θ by ϑ and H_t by $\mathcal{H}_t$. The only difference resides in the estimations of the derivatives when we differentiate with respect to δ_i, $i = 1,\ldots,d$.

For instance the (h,i)-th element of the matrix $\mathcal{C}_m$ denoted by $\mathcal{C}(h,i)$ is given by

$$\mathcal{C}(h,i) = \mathbb{E}\left[S_{t-h}\frac{\partial S_t}{\partial \vartheta_i}\right] = -\mathbb{E}\left[S_{t-h}\mathrm{Tr}\left(\mathcal{H}_{0t}^{-1}\frac{\partial \mathcal{H}_t(\vartheta_0)}{\partial \vartheta_i}\right)\right] = -\mathbb{E}\left[S_{t-h}\mathbf{h}_t'(i)\mathrm{vec}(I_d)\right].$$

Consequently we have

$$\mathcal{C}_m := [\mathcal{C}(h,i)]_{1\le h\le m, 1\le i\le s_0} = -\mathbb{E}\left[(\mathbb{S}_{t-1:t-m})\left(\mathbf{h}_t'\mathrm{vec}(I_d)\right)'\right]. \tag{6.29}$$

(iii) **Invertibility of the matrix $\mathcal{D}$**

The proof of the invertibility of matrix $\mathcal{D}$ needs to have some modifications compared to the case where the power $\underline{\delta}_0$ is assumed to be known. The start of the proof stay identical, it suffices only to replace H_t by $\mathcal{H}_t$ and θ by ϑ. We rewrite $\mathbf{h}_t(i)$ as follow

$$\mathbf{h}_t(i) = \left[\mathrm{vec}\left(\mathcal{H}_{0t}^{-1/2}\frac{\partial \mathcal{H}_t(\vartheta_0)}{\partial \vartheta_i}\mathcal{H}_{0t}^{-1/2}\right)\right] = \left[\mathcal{H}_{0t}^{-1/2}\otimes \mathcal{H}_{0t}^{-1/2}\right]\mathrm{vec}\left(\frac{\partial \mathcal{H}_t(\vartheta_0)}{\partial \vartheta_i}\right) = \mathbf{H}_t\mathbf{d}_t(i),$$

where $\mathbf{H}_t = \left[\mathcal{H}_{0t}^{-1/2}\otimes \mathcal{H}_{0t}^{-1/2}\right]$ and $\mathbf{d}_t(i) = \mathrm{vec}\left(\partial \mathcal{H}_t(\vartheta_0)/\partial \vartheta_i\right)$. Thus we define the matrix of size $d^2\times s_0$, $\mathbf{d}_t = (\mathbf{d}_t(1)|\dots|\mathbf{d}_t(s_0))$ such that $\mathbf{h}_t = \mathbf{H}_t\mathbf{d}_t$. To study the invertibility of the matrix $\mathcal{D}$ we let $V = \mathbb{S}_{t-1:t-m}S_t - \mathcal{C}_m\mathcal{J}^{-1}\partial l_t(\vartheta_0)/\partial\vartheta$ such that $\mathbb{E}[VV'] = \mathcal{D}$. We can also rewrite the vector V as

$$V = \mathbb{S}_{t-1:t-m}S_t + \mathcal{C}_m\mathcal{J}^{-1}\mathbf{d}_t'\mathbf{H}_t'\mathrm{vec}(s_t).$$

If the matrix $\mathbb{E}[VV']$ is singular, then there exists a vector $\lambda = (\lambda_1,\dots,\lambda_m)'$ not equal to zero such that

$$\lambda'V = \lambda'\mathbb{S}_{t-1:t-m}S_t + \mu\mathbf{d}_t'\mathbf{H}_t'\mathrm{vec}(s_t) = 0, \quad \text{a.s.}, \tag{6.30}$$

with $\mu = \lambda'\mathcal{C}_m\mathcal{J}^{-1}$. We have $\mu\neq 0$, else $\lambda'\mathbb{S}_{t-1:t-m}S_t = 0$ almost surely, that implies there exists $j\in\{1,\dots,m\}$ such that S_{t-j} be mesurable respect to the σ-field $\{S_r, t-1\le r\le t-m\}$ with $r\neq t-j$. That is impossible because the S_t are independent and not degenerated. Consequently (6.30) becomes

$$\mu'\mathbf{d}_t' = \sum_{i=1}^{s_0}\mu_i\mathbf{d}_t(i) = \sum_{i=1}^{s_0}\mu_i\mathrm{vec}\left(\frac{\partial \mathcal{H}_t(\vartheta_0)}{\partial \vartheta_i}\right) = \sum_{i=1}^{s_0}\mu_i\frac{\partial}{\partial \vartheta_i}\left[(D_{0t}\otimes D_{0t})\mathrm{vec}(R_0)\right], \quad \text{a.s.} \tag{6.31}$$

We can then rewrite (6.31) in order to separate the derivatives of the matrix $\mathcal{H}_t$ when we differentiate with respect to the vectors θ and $\underline{\delta}$. It follows that

$$\begin{aligned}\mu'\mathbf{d}_t' &= \sum_{i=1}^{s_2}\mu_i\frac{\partial\left[(D_{0t}\otimes D_{0t})\mathrm{vec}(R_0)\right]}{\partial\theta_i} + \sum_{i=s_2+1}^{s_0}\mu_i\frac{\partial(D_t\otimes D_t)}{\partial\delta_i}\mathrm{vec}(R_0), \quad \text{a.s.}\\ &= \sum_{i=1}^{s_1}\mu_i\frac{\partial(D_{0t}\otimes D_{0t})}{\partial\theta_i}\mathrm{vec}(R_0) + \sum_{i=s_1+1}^{s_2}\mu_i(D_{0t}\otimes D_{0t})\frac{\partial\mathrm{vec}(R_0)}{\partial\theta_i} + \sum_{i=s_2+1}^{s_0}\mu_i\frac{\partial(D_t\otimes D_t)}{\partial\delta_i}\mathrm{vec}(R_0) = 0 \quad \text{a.s.}\end{aligned} \tag{6.32}$$

Since the vectors $\partial\mathrm{vec}(R_0)/\partial\theta_i$, $i = s_1+1,\dots,s_2$ are linearly independent, the vector $(\mu_{s_1+1},\dots,\mu_{s_2})'$ is null and thus Equation (6.32) yields

$$\sum_{i=1}^{s_1}\mu_i\frac{\partial(D_{0t}\otimes D_{0t})}{\partial\theta_i}\mathrm{vec}(R_0) + \sum_{i=s_2+1}^{s_0}\mu_i\frac{\partial(D_t\otimes D_t)}{\partial\delta_i}\mathrm{vec}(R_0) = 0, \quad \text{a.s.} \tag{6.33}$$

The rows $1, d+1,\dots,d^2$ of the Equation (6.33) yield

$$\sum_{i=1}^{s_1}\mu_i\frac{\partial\underline{h}_t(\vartheta_0)}{\partial\theta_i} + \sum_{i=s_2+1}^{s_0}\mu_i\frac{\partial\underline{h}_t(\vartheta_0)}{\partial\delta_i} = 0, \quad \text{a.s.} \tag{6.34}$$

We have for $i_1 = 1, \ldots, d$ and $i = 1, \ldots, s_1$

$$\frac{\partial h_{i_1,t}(\vartheta_0)}{\partial \theta_i} = \frac{\partial \left(h_{i_1,t}^{\delta_{0,i_1}/2}\right)^{2/\delta_{0,i_1}}}{\partial \theta_i}(\vartheta_0) = \frac{2}{\delta_{0,i_1}} h_{i_1,0t} \times \frac{1}{h_{i_1,0t}^{\delta_{0,i_1}/2}} \frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}}{\partial \theta_i}(\vartheta_0), \tag{6.35}$$

where the derivatives involved in (6.35) are defined for all $\vartheta \in \Delta$ recursively by

$$\frac{\partial h_{i_1,t}^{\delta_{i_1}/2}(\vartheta)}{\partial \theta} = c_t(\vartheta) + \sum_{i_2=1}^{d} \sum_{i=1}^{p} B_i(i_1, i_2) \frac{\partial h_{i_2,t-i}^{\delta_{i_2}/2}(\vartheta)}{\partial \theta},$$

with

$$\begin{aligned} c_t(\vartheta) = \Big(&0, \ldots, 1, 0, \ldots, \left(\varepsilon_{i_1,t-1}^{+}\right)^{\delta_{i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-1}^{+}\right)^{\delta_{i_d}}, 0, \ldots, \left(\varepsilon_{i_1,t-q}^{+}\right)^{\delta_{i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-q}^{+}\right)^{\delta_{i_d}}, \\ &0, , \ldots, \left(\varepsilon_{i_1,t-1}^{-}\right)^{\delta_{i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-1}^{-}\right)^{\delta_{i_d}}, 0, \ldots, \left(\varepsilon_{i_1,t-q}^{-}\right)^{\delta_{i_1}}, 0, \ldots, \left(\varepsilon_{i_d,t-q}^{-}\right)^{\delta_{i_d}}, \\ &0, , \ldots, h_{i_1,t-1}^{\delta_{i_1}/2}, 0, \ldots, h_{i_d,t-1}^{\delta_{i_d}/2}, 0, \ldots, h_{i_1,t-p}^{\delta_{i_1}/2}, 0, \ldots, h_{i_d,t-p}^{\delta_{i_d}/2}, \ldots, 0\Big)'. \end{aligned} \tag{6.36}$$

So we can focus on the derivatives with respect to $\underline{\delta}$:

$$\frac{\partial h_{i_1,t}}{\partial \delta_j} = \frac{2}{\delta_{i_1}} h_{i_1,t} \left[-\boldsymbol{\delta}_{j,i_1} \frac{1}{\delta_{i_1}} \log\left(h_{i_1,t}^{\delta_{i_1}/2}\right) + \frac{1}{h_{i_1,t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1,t}^{\delta_{i_1}/2}}{\partial \delta_j} \right], \qquad j = 1, \ldots, d \tag{6.37}$$

with

$$\begin{aligned} \frac{\partial h_{i_1,t}^{\delta_{i_1}/2}}{\partial \delta_j} &= \sum_{i=1}^{q} \left[A_i^{+}(i_1, j) \log(\varepsilon_{j,t-i}^{+}) (\varepsilon_{j,t-i}^{+})^{\delta_j} + A_i^{-}(i_1, j) \log(\varepsilon_{j,t-i}^{-}) (\varepsilon_{j,t-i}^{-})^{\delta_j} \right] + \sum_{i_2=1}^{d} \sum_{i=1}^{p} B_i(i_1, i_2) \frac{\partial h_{i_2,t-i}^{\delta_{i_2}/2}}{\partial \delta_j} \\ &= A_1^{+}(i_1, j) \log(\varepsilon_{j,t-1}^{+}) (\varepsilon_{j,t-1}^{+})^{\delta_j} + A_1^{-}(i_1, j) \log(\varepsilon_{j,t-1}^{-}) (\varepsilon_{j,t-1}^{-})^{\delta_j} + R_{t-2}, \end{aligned} \tag{6.38}$$

where $\boldsymbol{\delta}_{j,i_1}$ denotes the Kronecker symbol. We also remind that

$$h_{i_1,t}^{\delta_{0,i_1}/2}(\vartheta_0) = \sum_{i_2=1}^{d} \left[A_{01}^{+}(i_1, i_2) (\varepsilon_{i_2,t-1}^{+})^{\delta_{0,i_2}} + A_{01}^{-}(i_1, i_2) (\varepsilon_{i_2,t-1}^{-})^{\delta_{0,i_2}} \right] + R_{t-2}. \tag{6.39}$$

The distribution of η_t is non-degenerated, so Equation (6.30) becomes

$$\lambda' V = \lambda' \mathbb{S}_{t-1:t-m} + \mu' \mathbf{d}' \mathbf{H}' \mathbb{1} = 0, \quad \text{a.s.}$$

In view of (6.35) and (6.37), we can finally write

$$\begin{aligned} \lambda' V = \lambda' \mathbb{S}_{t-1:t-m} h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0) + \sum_{i=1}^{s_1} \mu_i^{*} \frac{\partial h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0)}{\partial \theta_i} + \sum_{i=1}^{d} \mu_{i+s_2}^{*} \frac{\partial h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0)}{\partial \delta_i} \\ - \mu_{i_1+s_2}^{*} h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0) \log\left(h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0)\right) = 0, \quad \text{a.s.} \end{aligned} \tag{6.40}$$

where $\mu_i^* = 2\mu_i/\delta_{0,i_1}$, $\mu_{i+s_2}^* = 2\mu_{i+s_2}/\delta_{0,i_1}$ and when $i = i_1$ we have $\mu_{i_1+s_2}^* = 2\mu_{i_1+s_2}/\delta_{0,i_1}^2$. Recall that $\underline{\varepsilon}_t^+ = \mathcal{H}_t^{1/2}\eta_t^+$ and $\underline{\varepsilon}_t^- = \mathcal{H}_t^{1/2}\eta_t^-$ and we decomposed (6.40) in four terms. The first one leads to

$$\begin{aligned}
\lambda'\mathbb{S}_{t-1:t-m} h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0) = &\left\{ \sum_{i_2=1}^{d} \left[A_{01}^+(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right.\right. \\
&\left.\left. + A_{01}^-(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] \right\} R_{t-2} \\
&+ \left\{ \sum_{i_2=1}^{d} \left[A_{01}^+(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right.\right. \\
&\left.\left. + A_{01}^-(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] \right\} \\
&\times \left(\lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 \right) + \left(\lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 \right) R_{t-2} + R_{t-2}, \qquad (6.41)
\end{aligned}$$

by using (6.39) and the fact that

$$\lambda'\mathbb{S}_{t-1:t-m} = \lambda_1 S_{t-1} + R_{t-2} = \lambda_1 \sum_{i=1}^{d} \eta_{i,t-1}^2 + R_{t-2}.$$

Using (6.36), the second term of (6.40) can be rewritten

$$\begin{aligned}
\mu^{*'} \frac{\partial h_{i_1,t}^{\delta_{0,i_1}/2}(\vartheta_0)}{\partial\theta} &= \sum_{i_2=1}^{d} \left[\mu_{i_1+i_2 d}^* (\varepsilon_{i_2,t-1}^+)^{\delta_{0,i_2}} + \mu_{i_1+(i_2+q)d^2}^* (\varepsilon_{i_2,t-1}^-)^{\delta_{0,i_2}} \right] + R_{t-2} \\
&= \sum_{i_2=1}^{d} \left[\mu_{i_1+i_2 d}^* \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right. \\
&\qquad \left. + \mu_{i_1+(i_2+q)d^2}^* \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] + R_{t-2}, \qquad (6.42)
\end{aligned}$$

where the vector $\mu^* = (\mu_1^*, \ldots, \mu_{s_1}^*)'$.

Now using (6.38) the third term of the equation (6.40) can be rewritten as

$$\begin{aligned}
\sum_{i=1}^{d} \mu_{i+s_2}^* \frac{\partial h_{i_1,t}^{\delta_{i_1}/2}(\vartheta_0)}{\partial\delta_i} = \sum_{i_2=1}^{d} \mu_{i_2+s_2}^* &\left[A_{01}^+(i_1,i_2) \log \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right) \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^+ \right)^{\delta_{0,i_2}} \right. \\
&\left. A_1^-(i_1,i_2) \log \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right) \left(\sum_{j_1=1}^{d} \mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^- \right)^{\delta_{0,i_2}} \right] + R_{t-2}. \qquad (6.43)
\end{aligned}$$

Finally by using (6.39), the last term of (6.40) can be rewritten as

$$\mu^*_{i_1+s_2} h^{\delta_{i_1}/2}_{i_1,t}(\vartheta_0) \log\left(h^{\delta_{i_1}/2}_{i_1,t}(\vartheta_0)\right) = \mu^*_{i_1+s_2} \left[\sum_{i_2=1}^{d} A^+_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}} + \sum_{i_2=1}^{d} A^-_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^-_{j_1,t-1}\right)^{\delta_{0,i_2}} + R_{t-2}\right] \times \log\left[R_{t-2} + \sum_{i_2=1}^{d} A^+_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}} + \sum_{i_2=1}^{d} A^-_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^-_{j_1,t-1}\right)^{\delta_{0,i_2}}\right]. \tag{6.44}$$

Combining Equations (6.41), (6.42), (6.43) and (6.44) and by separating the non negative terms and the non positive terms, Equation (6.40) is equivalent to the two equations

$$\begin{aligned}
&\left[\sum_{i_2=1}^{d} A^+_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}}\right] \left[\lambda_1 \sum_{i_2=1}^{d} \left(\eta^+_{i_2,t-1}\right)^2 + R_{t-2}\right] \\
&+ R_{t-2} \sum_{i_2=1}^{d} \left(\eta^+_{i_2,t-1}\right)^2 + R_{t-2} + \sum_{i_2=1}^{d} \mu^*_{i_1+(i_2+q)d} \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}} \\
&+ \sum_{i_2=1}^{d} \mu^*_{i_2+s_2} A^+_{01}(i_1,i_2) \log\left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}} \\
&- \mu^*_{i_1+s_2} \left[\sum_{i_2=1}^{d} A^+_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}} + R_{t-2}\right] \\
&\times \log\left[R_{t-2} + \sum_{i_2=1}^{d} A^+_{01}(i_1,i_2) \left(\sum_{j_1=1}^{d} \mathcal{H}^{1/2}_{0,t-1}(i_2,j_1)\eta^+_{j_1,t-1}\right)^{\delta_{0,i_2}}\right] = 0, \quad \text{a.s.}
\end{aligned} \tag{6.45}$$

$$\left[\sum_{i_2=1}^{d} A_{01}^{-}(i_1,i_2)\left(\sum_{j_1=1}^{d}\mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-}\right)^{\delta_{0,i_2}}\right]\left[\lambda_1\sum_{i_2=1}^{d}\left(\eta_{i_2,t-1}^{-}\right)^2+R_{t-2}\right]$$
$$+R_{t-2}\sum_{i_2=1}^{d}\left(\eta_{i_2,t-1}^{-}\right)^2+R_{t-2}+\sum_{i_2=1}^{d}\mu_{i_1+(i_2+q)d^2}^{*}\left(\sum_{j_1=1}^{d}\mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-}\right)^{\delta_{0,i_2}}$$
$$+\sum_{i_2=1}^{d}\mu_{i_2+s_2}^{*}A_{01}^{-}(i_1,i_2)\log\left(\sum_{j_1=1}^{d}\mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-}\right)\left(\sum_{j_1=1}^{d}\mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-}\right)^{\delta_{0,i_2}}$$
$$-\mu_{i_1+s_2}^{*}\left[\sum_{i_2=1}^{d}A_{01}^{-}(i_1,i_2)\left(\sum_{j_1=1}^{d}\mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-}\right)^{\delta_{0,i_2}}+R_{t-2}\right]$$
$$\times\log\left[R_{t-2}+\sum_{i_2=1}^{d}A_{01}^{-}(i_1,i_2)\left(\sum_{j_1=1}^{d}\mathcal{H}_{0,t-1}^{1/2}(i_2,j_1)\eta_{j_1,t-1}^{-}\right)^{\delta_{i_2}}\right]=0,\quad\text{a.s.}\tag{6.46}$$

When $d=1$, from (6.45) and (6.46) we retrieve an equation of the following form obtained by Boubacar Maïnassara et al. (2021)

$$a|y|^{\underline{\delta}+2}+[b+c(|y|^{\underline{\delta}})]\log[b+c(|y|^{\underline{\delta}})]+[d+e\log(|y|)]|y|^{\underline{\delta}}+fy^2+g=0$$

which cannot have more than 11 positive roots or more than 11 negative roots, except if $a=b=c=d=e=f=g=0$.

When $d\geq 2$ and also from (6.45) and (6.46), for a fixed component, we obtain an equation of the form

$$\sum_{i=1}^{d}a_i|y|^{\delta_i+2}+\sum_{i=1}^{d}b_i|y|^{\delta_i+1}+\sum_{i=1}^{d}c_i|y|^{\delta_i}+\sum_{i=1}^{d}d_i\log(|y|)|y|^{\delta_i}$$
$$+\left(e+\sum_{i=1}^{d}e_i|y|^{\delta_i}\right)\log\left(f+\sum_{i=1}^{d}f_i|y|^{\delta_i}\right)+gy^2+h|y|+k=0.$$

Note that an equation of this form can not have more than $11d+1$ non negative roots or more than $11d+1$ non positive roots for $d\geq 2$, unless $a_i=b_i=c_i=d_i=e_i=f_i=e=f=g=h=k=0$.
By the assumption $\mathbf{A9}'$, Equations (6.45) and (6.46) imply that
$\lambda_1\left[\sum_{i_2=1}^{d}A_{01}^{+}(i_1,i_2)+A_{01}^{-}(i_1,i_2)\right]=0$ and $\mu_{i+s_2}^{*}\left[\sum_{i_2=1}^{d}A_{01}^{+}(i_1,i_2)+A_{01}^{-}(i_1,i_2)\right]=0$ for all $i=1,\ldots,d$. But under the assumption $\mathbf{A4}$, if $p>0$, $\mathcal{A}_0(1)^{+}+\mathcal{A}_0^{-}\neq 0$. It is impossible to have $A_{01}^{+}(i_1,i)=A_{01}^{+}(i_1,i)=0$, for all $i=1,\ldots,d$. Then, there exists an i_0 such that $A_{01}(i_1,i_0)^{+}+A_{01}(i_1,i_0)^{-}\neq 0$ and we then have $\lambda_1=0$ and $\mu_{i_0+s_2}^{*}=0$.

In the general case, Equation (6.33) necessarily leads

$$A_{01}^{+}(i_1,i_0)+A_{01}^{-}(i_1,i_0)=\cdots=A_{0q}^{+}(i_1,i_0)+A_{0q}^{-}(i_1,i_0)=0,\qquad\forall i_0,i_1=1,\ldots,d,$$

that is impossible under Assumption $\mathbf{A4}$ and then $\lambda=0$. This is in contradiction with $\lambda'V=0$, almost surely, that leads that the assumption of non invertibility of matrix $\mathcal{D}$ is absurd. □

6.7. Proof of Theorem 3.4

The proof is the same to that of Theorem 3.2 in the case where the power $\underline{\delta}_0$ is assumed to be known. □

6.8. Proof of Corollary 3.2

The proof is the same to that of Corollary 3.1 in the case where the power $\underline{\delta}_0$ is assumed to be known. □

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Data availability statement

The data used in Section 5 are available in the supporting information of this article. It is openly available from the website of the European Central Bank: http://www.ecb.int/stats/exchange/eurofxref/html/index.en.html.

Supporting information

Additional Supporting Information may be found online in the supporting information tab for this article.

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