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# A COUPLING OF STATIONARY NAVIER-STOKES EQUATIONS FOR TWO FLUIDS WITH NON-LINEAR INTERFACE CONDITION

FRANÇOIS LEGEAIS

ABSTRACT. We study a coupled system of two Navier-Stokes equations with a non-linear friction condition at the interface between the two fluids. In a first part, we prove the existence of weak solutions in the proper functional spaces. To do that, we use a potential vector method to get the pressure as a function of the velocity, which allows us to obtain error estimates and to conclude using a fixed point argument. In a second part we develop a numerical finite elements iterative method which leads us to error estimates and numerical simulations with Freefem++ for different sets of data.

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## 1. INTRODUCTION

We consider a two coupled fluids problem with a rigid lid assumption, driven by two 3D steady Navier-Stokes equations with a Gauckler-Manning condition, also

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called friction law, at the interface [10] [23], and with a Navier condition at the top and bottom boundaries of the domains  $\Omega_1$  and  $\Omega_2$ .

The equations are the following.

$$(1.1) \quad \begin{cases} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i} (\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \end{cases}$$

for  $i = 1, 2$ , where the velocities  $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{x}_h, z_1), \mathbf{u}_2(\mathbf{x}_h, z_2))$  are decomposed as

$$\mathbf{u}_i = (\mathbf{u}_{i,h}, w), \quad \mathbf{u}_{i,h} = (u_{i,x}, u_{i,y}).$$

Moreover,  $\mathbf{x}_h = (x, y) \in \mathbb{T}_2$ , where

$$\mathbb{T}_2 = \frac{[0, L_1] \times [0, L_2]}{\mathbb{Z}^2},$$

is a two dimensional torus, which means that for the sake of the simplicity, we consider horizontal periodic boundary conditions:

$$\forall (n, k, q) \in \mathbf{N} \times \mathbb{Z} \times \mathbb{Z}, \quad D^n \mathbf{u}(x + kL_1, y + qL_2, z) = D^n \mathbf{u}(x, y, z),$$

in the sense of the distributions. The interface  $\Gamma_{Int}$  is given by

$$\Gamma_{Int} = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}.$$

The boundaries  $\Gamma_i$  are given by

$$\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\},$$

which is the top of fluid 1 and

$$\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\},$$

the bottom of the second fluid. For the simplicity we set

$$J_1 = [0, z_1^+], \quad z_2 \in J_2 = [z_2^-, 0],$$

where  $z_1^+ > 0$  and  $z_2^- < 0$ . In other word, the domains  $\Omega_i$  can be defined as

$$\Omega_i = \mathbb{T}_2 \times J_i.$$

Finally, the remaining terms are

- $p_i$  the pressure of the fluid  $i$ ,
- $\mathbf{f}_i$  a source term,
- $\mathbf{V}_i$  a fixed velocity, at the bottom and at the top,
- $\nu_i > 0$  is the cinematic viscosity of the fluid  $i$ ,
- the coefficients  $C_D > 0$  and  $c_{d,i} > 0$  from the Gauckler-Manning's law at the interface and from Navier condition at the top and bottom.

This model suits particularly well to the ocean-atmosphere coupling, especially with the Gauckler-Manning assumptions and Navier assumptions at the top, and bottom, where the fixed Velocities  $\mathbf{V}_i$  can be considered as the speeds of the wind, and of the current flow in higher or lower layers, as depicted in figure 1. The heights  $z_1$  and  $z_2$  of the two layers can be chosen from centimeters to couple of meters.

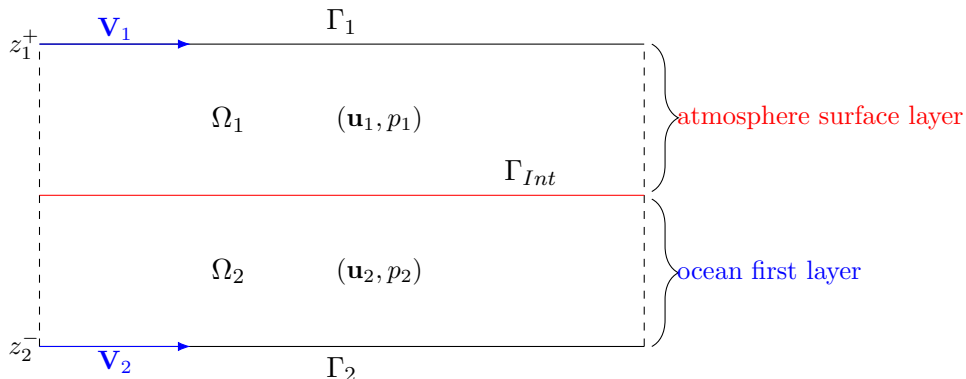


FIGURE 1. Geometry of the domain

The stationary Navier-Stokes problem (1.1) is hard to solve because of the non-linearity coming from transport terms and the Gauckler-Manning condition. The steady Navier-Stokes equations have been studied a lot, but not with such interface conditions. We can find in [6] the study of such an equation for one fluid and without the Navier condition at the top/bottom boundaries.

The aim is to prove the existence of weak solutions of system (1.1) and to give some numerical tools through algorithms and error estimates, to get a better understanding of this model. We recall that we have deliberately chosen this model for ocean-atmosphere interactions. It is less precise than primitive equations, since we don't consider many physical aspects, (see for instance [19], [24], [22]), but also clearly easier to implement. It is a more realistic and complex toy model than the stationary Stokes equations with continuous Navier conditions at the boundary studied in [21].

The paper is organised in two main sections. The first one 2 is dedicated to establish a variational formulation of (1.1) and to prove the existence of weak solutions. The second part 3 is devoted to the numerical analysis of (1.1). Different iterative finite element methods have been developed to obtain numerical simulations of stationary Stokes and Navier-Stokes equations: see for instance [2], [26], [11], [12], [13] or [29]. We follow this steps with a Galerkin finite elements scheme and a linearization algorithm to bypass the non-linearity coming from the convection term and the interface friction condition. We will get theoretical error estimates and Freefem++ numerical simulations for different choices of data.

## 2. PROBLEM FORMULATION

Before we start introducing the functional spaces we need, we give some details of the different steps. Let denote with  $\mathcal{VP}$  the variational formulation of (1.1) given in 2.2. The different operators involved in  $\mathcal{VP}$ , corresponding to transport, diffusion, pressure, friction terms, are analysed in 2.3, especially the compactness properties which will be useful during the whole study. Besides, the subsection 2.4 is dedicated to recall all the useful properties satisfied by bounded sequences in Sobolev spaces (existence of weak subsequential limit, compactness in  $L^q$ , results

on traces, ...).

We will then give in section 2.5 and 2.6 *a priori estimates* for the velocity by standard *energy equality method*, and for the pressure by *potential vector methods* (as developed in [4]). The point is to express the pressure as a function of the velocity. To do this, we will kill two birds with one stone and tackle another concerning issue coming from the non-local free divergence term  $\nabla \cdot \mathbf{u}_i = 0$ . We will approximate this condition by the new system

$$\begin{cases} -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i &= 0 \text{ in } \Omega_i, \\ \frac{\partial p_i}{\partial \mathbf{n}_i} &= 0 \text{ on } \Gamma_i \cup \Gamma_{Int}, \\ \int_{\Omega_i} p_i(\mathbf{x}) d\mathbf{x} &= 0, \end{cases}$$

where  $\varepsilon > 0$  is small. This system admits a unique solution  $p_i = p_i^\varepsilon$  for a given velocity field  $\mathbf{u}_i$  belonging to the suitable functional space. Therefore, we can introduce the system where the free divergence condition is replaced with the approximation introduced above and we denote with  $\mathcal{VP}_\varepsilon$  the corresponding variational formulation, where only  $\mathbf{u}_i = \mathbf{u}_i^\varepsilon$  is involved. We will prove that the weak solutions of  $\mathcal{VP}_\varepsilon$  are converging to the weak solutions of  $\mathcal{VP}$ . Thus it will remain to achieve our goal to show that  $\mathcal{VP}_\varepsilon$  admits weak solutions. This will be done in 2.8.2 by linearization of the non-linear transport and friction terms, and by use of Schauder's fixed point theorem, as summarized in the scheme below:

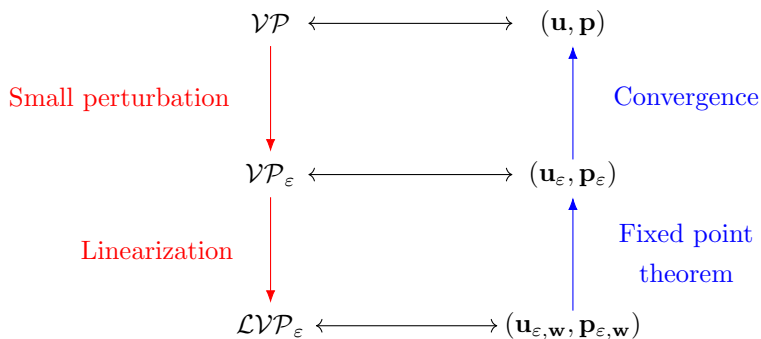


FIGURE 2. Scheme of the proof

**2.1. Functional spaces.** We define the functional space  $\mathcal{W}_i$  by

$$(2.1) \quad \mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i), \mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_{Int} \cap \Gamma_i} = 0\},$$

equipped with

$$(2.2) \quad \|\mathbf{u}\|_{\mathcal{W}_i} = \|\nabla \mathbf{u}\|_{L^2(\mathbb{T}_2 \times J_i)} + \|\text{tru}\|_{L^2(\Gamma_i)},$$

where  $\mathbf{u} \rightarrow \text{tru}$  denotes the trace operator. When there is no ambiguity, we will write  $\mathbf{u}$  instead of  $\text{tru}$ ; this will be the case every time we use the  $L^2$  norms on the boundary.

Let  $W_i$  denotes the completion of  $\mathcal{W}_i$  with respect to this norm. We define the space  $W = W(\Omega)$  as

$$W = W_1 \times W_2,$$

endowed with the norm

$$(2.3) \quad \|\mathbf{u}\|_W^2 = \|\mathbf{u}_1\|_{W_1}^2 + \|\mathbf{u}_2\|_{W_2}^2 = \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \|\mathbf{u}\|_{0,2,\Gamma}^2,$$

where we use the notations

$$\|\nabla \mathbf{u}\|_{0,2,\Omega}^2 := \|\nabla \mathbf{u}_1\|_{0,2,\Omega_1}^2 + \|\nabla \mathbf{u}_2\|_{0,2,\Omega_2}^2$$

and

$$\|\mathbf{u}\|_{0,2,\Gamma}^2 := \|\mathbf{u}_1\|_{0,2,\Gamma_1}^2 + \|\mathbf{u}_2\|_{0,2,\Gamma_2}^2.$$

$W$  is a closed subspace of  $H^1(\Omega_1) \times H^1(\Omega_2)$  so the  $W$ -norm  $\|\cdot\|_W$  is equivalent over  $W$  to the classical  $H^1$ -norm  $\|\cdot\|_{1,2,\Omega}$ .

**Proposition 2.1.** *The  $W$ -norm  $\|\cdot\|_W$  and the  $H^1$  norm  $\|\cdot\|_{1,2}$  are equivalent. Moreover, there exists a constant  $C_{eq} > 0$  such that for every  $\mathbf{v} \in W$ :*

$$(2.4) \quad \frac{1}{C_{eq}} \|\mathbf{v}\|_{1,2} \leq \|\mathbf{v}\|_W \leq C_{eq} \|\mathbf{v}\|_{1,2}.$$

**Remark 2.2.** *We cannot use the semi-norm  $\|\nabla \mathbf{u}\|_{0,2,\Omega}$  because we don't have a Poincaré inequality in our domain.*

We will seek the pressure in the space  $X := X(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$  which is standard for the Navier Stokes equations. Since the pressure is up to a constant, we can even consider the average space  $X_0(\Omega) := L_0^2(\Omega_1) \times L_0^2(\Omega_2)$ , where

$$(2.5) \quad L_0^2(\Omega_i) = \{p_i \in L^2(\Omega_i), \int_{\Omega_i} p_i = 0\}.$$

**2.2. Variational formulation.** We will introduce some notations of the involved operators in order to give the variational formulation of the problem (1.1).

First we recall that the velocity  $\mathbf{u}$  we consider in the following is the couple  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ , where  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are the velocities of the fluids 1 and 2 on the domain  $\Omega_1$  and  $\Omega_2$ . We have in particular

$$\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3.$$

We will also use the notation  $\mathbf{p} = (p_1, p_2)$ , where  $p_1$  and  $p_2$  are the pressure of the fluids on their respective domains. Thus

$$\mathbf{p} = (p_1, p_2) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \times \mathbb{R}.$$

To give a variational formulation of (1.1), we will use different operators:

- a bilinear diffusion operator  $a$ ,
- a trilinear transport operator  $b$ ,
- a bilinear pressure term  $d$ ,
- a trilinear interface term  $g$ .

(1) **Diffusion term**

We introduce the bilinear diffusion operator  $a$  defined by

$$(2.6) \quad a(\mathbf{u}, \mathbf{v}) = a_1(\mathbf{u}_1, \mathbf{v}_1) + a_2(\mathbf{u}_2, \mathbf{v}_2),$$

where  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ , and where the diffusion operators  $a_1$  and  $a_2$  are defined by

$$(2.7) \quad a_i(\mathbf{u}_i, \mathbf{v}_i) := \nu_i \int_{\Omega_i} \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i,$$

for  $i \in \{1, 2\}$ .

We denote by  $A$  the continuous operator  $W \rightarrow W'$  associated with  $a$ , given by:

$$(2.8) \quad \langle A(\mathbf{u}), \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}).$$

(2) **Transport term**

Let  $b$  be the trilinear form defined by:

$$(2.9) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = b_1(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1) + b_2(\mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2),$$

where

$$(2.10) \quad b_i(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = \frac{1}{2} \left( \int_{\Omega_i} (\mathbf{u}_i \cdot \nabla) \mathbf{v}_i \cdot \mathbf{w}_i - \int_{\Omega_i} (\mathbf{u}_i \cdot \nabla) \mathbf{w}_i \cdot \mathbf{v}_i \right).$$

We will see in the next subsection different properties of this operator.

We also will consider  $B : W \times W \rightarrow W'$  which satisfies

$$(2.11) \quad \langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{u}, \mathbf{w}).$$

(3) **Pressure term**

We define the pressure term  $d(\mathbf{u}, \mathbf{p})$  by

$$(2.12) \quad d(\mathbf{u}, \mathbf{p}) = d_1(\mathbf{u}_1, p_1) + d_2(\mathbf{u}_2, p_2),$$

where

$$(2.13) \quad d_i(\mathbf{u}_i, p_i) = \int_{\Omega_i} p_i \nabla \cdot \mathbf{u}_i,$$

for  $i \in \{1, 2\}$ .

We can also consider  $D : X \rightarrow X'$  which verifies

$$(2.14) \quad \langle D(\mathbf{p}), \mathbf{v} \rangle := d(\mathbf{v}, \mathbf{p}),$$

for every  $\mathbf{v} \in W$ .

(4) **Interface term**

We define the non-linear interface term  $g(\mathbf{u}, \mathbf{v}, \mathbf{w})$  by

$$(2.15) \quad g(\mathbf{u}, \mathbf{v}, \mathbf{w}) = C_D \int_{\Gamma_{int}} |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}| (\mathbf{v}_{i,h} - \mathbf{v}_{j,h}) \cdot (\mathbf{w}_{i,h} - \mathbf{w}_{j,h}),$$

and  $G : W \rightarrow W'$  the interface operator by

$$(2.16) \quad \langle G(\mathbf{u}), \mathbf{v} \rangle = g(\mathbf{u}, \mathbf{u}, \mathbf{v}).$$

(5) **Bottom and top terms**

The term we have to consider is

$$(2.17) \quad c_{d,1} \int_{\Gamma_1} (\mathbf{u}_1 - \mathbf{V}_1) \cdot \mathbf{v}_1 + c_{d,2} \int_{\Gamma_2} (\mathbf{u}_2 - \mathbf{V}_2) \cdot \mathbf{v}_2 = \sum_{i=1}^2 c_{d,i} \langle \mathbf{u}_i, \mathbf{v}_i \rangle_{\Gamma_i} - \sum_{i=1}^2 c_{d,i} \langle \mathbf{V}_i, \mathbf{v}_i \rangle_{\Gamma_i},$$

where  $\mathbf{V}_i$  is a given velocity at the bottom and at the top of the domain.

We will split (2.17) into two different terms: the first one

$$c_{d,1} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{u}_2, \mathbf{v}_2 \rangle_{\Gamma_2}$$

will be associated with a bilinear operator and the remaining term

$$c_{d,1} \langle \mathbf{V}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{V}_2, \mathbf{v}_2 \rangle_{\Gamma_2}$$

to a linear one. Let  $h$  be defined by

$$(2.18) \quad h(\mathbf{u}, \mathbf{v}) = h_1(\mathbf{u}_1, \mathbf{v}_1) + h_2(\mathbf{u}_2, \mathbf{v}_2),$$

where

$$(2.19) \quad h_i(\mathbf{u}_i, \mathbf{v}_i) = c_{d,i} \int_{\Gamma_i} \mathbf{u}_i \cdot \mathbf{v}_i,$$

for  $i \in \{1, 2\}$ .

It is a bilinear operator and we can define  $H : W \rightarrow W'$  as

$$(2.20) \quad \langle H(\mathbf{u}), \mathbf{v} \rangle = h(u, v).$$

The remaining term

$$c_{d,1} \langle \mathbf{V}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,1} \langle \mathbf{V}_2, \mathbf{v}_2 \rangle_{\Gamma_2}$$

will be added to the source term in the next paragraph.

**Remark 2.3.** *This term can be seen as a drive speed term at the top and at the bottom. It is classical in lid driven cavity test frameworks for instance.*

### (6) Source term

As explained, we will need to take into consideration the top and bottom given velocities  $\mathbf{V}_i$ , which can be seen as source terms as well. Thus, we define  $(\mathbf{f}, \mathbf{v})_\Omega$  as

$$(2.21) \quad (\mathbf{f}, \mathbf{v})_\Omega = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (\mathbf{f}_2, \mathbf{v}_2)_{\Omega_2} + c_{d,1} \langle \mathbf{V}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{V}_2, \mathbf{v}_2 \rangle_{\Gamma_2},$$

where

$$(2.22) \quad (\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i} := \int_{\Omega_i} \mathbf{f}_i \mathbf{v}_i,$$

for  $i \in \{1, 2\}$ .

Instead of writing  $(\mathbf{f}, \mathbf{v})_\Omega$  we will use  $(\mathbf{f}, \mathbf{v})$  whenever there is no ambiguity.

We are now able to give the variational formulation of the problem (1.1) and the definition of what we mean by weak solutions.

**Definition 2.4.** (*weak solution*)

We say that  $(\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{p} = (p_1, p_2)) \in W \times X$  is a weak solution to problem (1.1) if for every  $(\mathbf{v}, \mathbf{q}) \in W \times X$ ,

$$(\mathcal{VP}) \quad \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - d(\mathbf{v}, \mathbf{p}) + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega, \\ d(\mathbf{u}, \mathbf{q}) = 0. \end{cases}$$

In other words, Problem (1.1) can be written as: find  $(\mathbf{u}, \mathbf{p}) \in W \times X$ , such that

$$(2.23) \quad \begin{cases} A(\mathbf{u}) + B(\mathbf{u}, \mathbf{u}) + G(\mathbf{u}, \mathbf{u}) + H(\mathbf{u}) - D(\mathbf{p}) = \mathbf{f} \in W', \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } X. \end{cases}$$

## 2.3. Properties of the operators.

2.3.1. *Diffusion term.* In the following we will consider  $\nu = \inf(\nu_1, \nu_2)$ .

We have the straightforward inequality we will use a lot in the future estimates:

**Proposition 2.5.**

$$(2.24) \quad \nu \|\nabla \mathbf{u}\|_{0,2,\Omega} \leq a(\mathbf{u}, \mathbf{u}),$$

for every  $\mathbf{u} \in W$ .



2.3.2. *Transport term.* This trilinear form verifies different properties and many different inequalities. In particular the non-linear transport term  $b$  verifies the lemma

**Lemma 2.6.** *Let  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in W^3$ , the trilinear form  $b$  verifies the following properties:*

(1)  *$b$  is continuous on  $W \times W \times W$  and verifies the inequality*

$$(2.25) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C_b \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W,$$

where  $C_b = C_b(\Omega) > 0$  is the best constant satisfying the inequality.

(2)  *$b$  is antisymmetric, i.e*

$$(2.26) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}),$$

in particular  $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$ , and  $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$ .

(3) *When the fluids are incompressible and verify the condition  $\nabla \cdot \mathbf{u}_i = 0$ , then*

$$(2.27) \quad b_i(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = \int_{\Omega_i} (\mathbf{u}_i \cdot \nabla) \mathbf{v}_i \cdot \mathbf{w}_i,$$

which is exactly the transport term in (1.1).

*Proof.* (1) We will mainly use the Cauchy-Schwarz inequality with the Sobolev embedding  $H^1(\Omega_i) \hookrightarrow L^p(\Omega_i)$ ,  $p \leq 6$ . It gives

$$\begin{aligned} |((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i}| &\leq C \|\mathbf{u}\|_{0,4,\Omega_i} \|\nabla \mathbf{v}\|_{0,2,\Omega_i} \|\mathbf{w}\|_{0,4,\Omega_i} \\ &\leq C \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W. \end{aligned}$$

The same estimate can be obtained for the term  $((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{u}_i)_{\Omega_i}$ , hence (2.25).

(2) It follows from the definition of  $b$  and  $b_i$ .

(3) We have

$$((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i} = \sum_{k,j} \int_{\Omega_i} u_i^k(\mathbf{x}) \partial_k v_i^j(\mathbf{x}) w_i^j(\mathbf{x}) d\mathbf{x},$$

where  $u_i^k$  is the  $k$ -th component of  $\mathbf{u}_i$ . We have thanks to the Stokes formula:

$$\begin{aligned} ((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i} &= \sum_{k,j} \left( \int_{\Gamma_i} v_i^j(\mathbf{x}) w_i^j(\mathbf{x}) u_i^k(\mathbf{x}) n^k(\mathbf{x}) d\Gamma(\mathbf{x}) - \int_{\Omega_i} v_i^j(\mathbf{x}) \partial_k (u_i^k w_i^j)(\mathbf{x}) d\mathbf{x} \right) \\ &= (\mathbf{u}_i \cdot \mathbf{n}, \mathbf{v}_i \cdot \mathbf{w}_i)_{\Gamma_i} - (\nabla \cdot \mathbf{u}_i, \mathbf{w}_i \cdot \mathbf{v}_i)_{\Omega_i} - ((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i}, \end{aligned}$$

which holds for any  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in C^1(\overline{\Omega_i})^3$ .

Taking  $\mathbf{u}_i \in \mathcal{W}_i$ , the term  $(\mathbf{u}_i \cdot \mathbf{n}, \mathbf{v}_i \cdot \mathbf{w}_i)_{\Gamma_i}$  vanishes and we get

$$(2.28) \quad ((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i} = -(\nabla \cdot \mathbf{u}_i, \mathbf{w}_i \cdot \mathbf{v}_i)_{\Omega_i} - ((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i}.$$

It remains to show that (2.28) is still valid when  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  belong to  $W$ , without additional regularity. Thanks to the density of  $\mathcal{W}_i$  in  $W_i$  and of  $C^1(\overline{\Omega_i})^3$  in  $H^1(\Omega_i)^3$  and thanks to the continuity estimates

$$\begin{aligned} |(\nabla \cdot \mathbf{u}_i, \mathbf{w}_i \cdot \mathbf{v}_i)_{\Omega_i}| &\leq C_1 \|\mathbf{u}_i\|_{1,2,\Omega_i} \|\mathbf{v}_i\|_{1,2,\Omega_i} \|\mathbf{w}_i\|_{1,2,\Omega_i}, \\ |((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i}| &\leq C_2 \|\mathbf{u}_i\|_{1,2,\Omega_i} \|\mathbf{v}_i\|_{1,2,\Omega_i} \|\mathbf{w}_i\|_{1,2,\Omega_i}, \end{aligned}$$

where  $C_i$  depends only on  $\Omega_i$ , the relation (2.28) holds for any  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in W$ . Finally, the incompressibility condition  $\nabla \cdot \mathbf{u}_i = 0$  gives (2.27).

□

**Remark 2.7.** *The choice of the operator  $b_i$  instead of the more classical  $((\mathbf{u}_i \cdot \nabla)\mathbf{v}_i \cdot \mathbf{w}_i)_{\Omega_i}$  comes from the incompressibility condition. Even without the condition  $\nabla \cdot \mathbf{u}_i = 0$ , the operator  $b_i$  satisfies (2.27), which would have not been the case with  $((\mathbf{u}_i \cdot \nabla)\mathbf{v}_i \cdot \mathbf{w}_i)_{\Omega_i}$ .*

2.3.3. *Interface term.* We recall that the interface terms  $g$  and  $G$  are defined in (2.15) and (2.16) by

$$g(\mathbf{u}, \mathbf{v}, \mathbf{w}) = C_D \int_{\Gamma_{int}} |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}| (\mathbf{v}_{i,h} - \mathbf{v}_{j,h}) \cdot (\mathbf{w}_{i,h} - \mathbf{w}_{j,h}),$$

and

$$\langle G(\mathbf{u}), \mathbf{v} \rangle = g(\mathbf{u}, \mathbf{u}, \mathbf{v}).$$

We will show in this part that the interface terms are well defined in the variational formulation.

**Lemma 2.8.** *For every  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ , the term  $g(\mathbf{u}, \mathbf{v}, \mathbf{w})$  does exist and checks the inequality*

$$(2.29) \quad |g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_g \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W,$$

where the constant  $C_g > 0$  depends only on  $C_D$  and  $\Omega$ .

*Proof.* We have thanks to the Cauchy-Schwarz inequality

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_D \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,4,\Gamma_I} \|\mathbf{v}_1 - \mathbf{v}_2\|_{0,4,\Gamma_I} \|\mathbf{w}_1 - \mathbf{w}_2\|_{0,2,\Gamma_I},$$

and because of the triangular inequality:

$$\begin{aligned} & |g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \\ & \leq C_D (\|\mathbf{u}_1\|_{0,4,\Gamma_I} + \|\mathbf{u}_2\|_{0,4,\Gamma_I}) (\|\mathbf{v}_1\|_{0,4,\Gamma_I} + \|\mathbf{v}_2\|_{0,4,\Gamma_I}) (\|\mathbf{w}_1\|_{0,2,\Gamma_I} + \|\mathbf{w}_2\|_{0,2,\Gamma_I}). \end{aligned}$$

Using the Sobolev trace embedding results  $H^1(\Omega_i) \hookrightarrow L^2(\Gamma_I)$  and  $H^1(\Omega_i) \hookrightarrow L^4(\Gamma_I)$  and the corresponding injection constants  $C_{i,2}$  and  $C_{i,4}$ , we get

$$\begin{aligned} |g(\mathbf{u}, \mathbf{v}, \mathbf{w})| & \leq C_D (C_{1,4} \|\nabla \mathbf{u}_1\|_{0,\Omega_1} + C_{2,4} \|\nabla \mathbf{u}_2\|_{0,\Omega_2}) (C_{1,4} \|\nabla \mathbf{v}_1\|_{0,\Omega_1} + C_{2,4} \|\nabla \mathbf{v}_2\|_{0,\Omega_2}) \\ & \quad (C_{1,2} \|\nabla \mathbf{w}_1\|_{0,\Omega_1} + C_{2,2} \|\nabla \mathbf{w}_2\|_{0,\Omega_2}), \end{aligned}$$

and we obtain

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_t \|\nabla \mathbf{u}\|_{0,2,\Omega} \|\nabla \mathbf{v}\|_{0,2,\Omega} \|\nabla \mathbf{w}\|_{0,2,\Omega},$$

where  $C_t = C_D (C_{1,4} + C_{2,4})^2 (C_{1,2} + C_{2,2})$ , which finally leads to

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_g \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W,$$

using the norms equivalence (2.4), where  $C_g = C_t C_{eq}$ .

□

It is convenient to introduce the function  $\tilde{g} \in W$  defined by

$$(2.30) \quad \tilde{g}(\mathbf{v}) = \mathbf{v} |\mathbf{v}|,$$

which gives in particular

$$g(\mathbf{u}, \mathbf{u}, \mathbf{v}) = C_D \int_{\Gamma_I} \tilde{g}(\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2),$$

for every  $\mathbf{u}, \mathbf{v} \in W$ .

The application  $\tilde{g}$  satisfies the following properties:

**Lemma 2.9.** *There exists a constant  $c$  depending only on  $\Omega$  such that*

(1)  $\tilde{g}$  is positive and satisfies

$$(2.31) \quad 0 \leq \tilde{g}(\mathbf{u}) \cdot \mathbf{u} = |\mathbf{u}|^2,$$

for every  $\mathbf{u} \in \mathbb{R}^3$ .

(2)  $\tilde{g}$  is monotone:

$$(2.32) \quad 0 \leq (\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})) \cdot (\mathbf{u} - \mathbf{v}),$$

and satisfies

$$(2.33) \quad |\nabla \tilde{g}(\mathbf{u})| \leq c|\mathbf{u}|,$$

for every  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ .

(3) Let  $\mathbf{u} \in W$ : then  $\tilde{g}(\mathbf{u}) \in L^2(\Gamma_I)$  and

$$(2.34) \quad \|\tilde{g}(\mathbf{u})\|_{0,2,\Gamma_I} \leq c\|\mathbf{u}\|_W^2.$$

(4) For any  $\mathbf{u}, \mathbf{v} \in W$ ,

$$(2.35) \quad \|\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})\| \leq c(\|\mathbf{u}\|_W + \|\mathbf{v}\|_W)\|\mathbf{u} - \mathbf{v}\|_W.$$

*Proof.* (1) The result is straightforward.

(2) It is straightforward as well.

(3) The proof is similar to the proof of the lemma (2.8) with the use of Sobolev embedding trace theorems.

(4) Doing a Taylor expansion, we obtain for every  $\mathbf{x} \in \Gamma_I$

$$\tilde{g}(\mathbf{u}(\mathbf{x})) - \tilde{g}(\mathbf{v}(\mathbf{x})) = \int_0^1 \nabla \tilde{g}(t\mathbf{u}(\mathbf{x}) + (1-t)\mathbf{v}(\mathbf{x})) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})) dt.$$

Using the control of the gradient (2.33), we have

$$|\tilde{g}(\mathbf{u}(\mathbf{x})) - \tilde{g}(\mathbf{v}(\mathbf{x}))| \leq \frac{c}{2}(|\mathbf{u}(\mathbf{x})| + |\mathbf{v}(\mathbf{x})|)|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})|,$$

which gives by integration and thanks to the Cauchy-Schwarz inequality

$$(2.36) \quad \|\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})\| \leq \frac{c}{2}\|\mathbf{u} + \mathbf{v}\|_{0,4,\Gamma_I}\|\mathbf{u} - \mathbf{v}\|_{0,4,\Gamma_I},$$

and hence to (2.35) using again the injection  $H^1(\Omega_i) \hookrightarrow L^4(\Gamma_I)$ .

□

**Remark 2.10.** When we will look for estimates in the variational formulation, we will take as a test  $\mathbf{v} = \mathbf{u}$  and consider the term  $g(\mathbf{u}, \mathbf{u}, \mathbf{u})$  which verifies

$$(2.37) \quad g(\mathbf{u}, \mathbf{u}, \mathbf{u}) \geq 0,$$

because of the property (2.31) satisfied by  $\tilde{g}$ .

We can now give and prove some properties satisfied by  $G$ :

**Lemma 2.11.** (1) The application  $G : \mathbf{u} \rightarrow G(\mathbf{u})$  maps  $W$  to  $W'$  and verifies

$$(2.38) \quad \|G(\mathbf{u})\|_{W'} \leq C\|\mathbf{u}\|_W^2.$$

(2)  $G$  is positive, i.e.,  $\langle G(\mathbf{u}), \mathbf{u} \rangle \geq 0$  for every  $\mathbf{u} \in W$ .

(3) The friction operator  $G$  satisfies

$$(2.39) \quad \|G(\mathbf{v}) - G(\mathbf{w})\|_{W'} \leq C(1 + \|\mathbf{v}\|_W + \|\mathbf{w}\|_W)\|\mathbf{v} - \mathbf{w}\|_W,$$

(4)  $G$  is monotone.

(5)  $G$  is continuous and compact.

*Proof.* (1) By definition of  $G$  and  $\tilde{g}$ , we have the duality relationship

$$\langle G(\mathbf{u}), \mathbf{v} \rangle = (\tilde{g}(\mathbf{u}), \mathbf{v})_{\Gamma_I},$$

which gives

$$|\langle G(\mathbf{u}), \mathbf{v} \rangle| \leq \|\tilde{g}(\mathbf{u})\|_{0,2,\Gamma_I} \|\mathbf{v}\|_{0,2,\Gamma_I}.$$

Combining it with (2.34) and the Sobolev trace embedding theorem, this gives

$$|\langle G(\mathbf{u}), \mathbf{v} \rangle| \leq c \|\nabla \tilde{g}(\mathbf{u})\|_{0,2,\Omega}^2 \|\nabla \mathbf{v}\|_{0,2,\Omega}.$$

- (2) The results comes from (2.37).  
(3) We have for every  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W$

$$|\langle G(\mathbf{u}) - G(\mathbf{v}), \mathbf{w} \rangle| \leq \|\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})\|_{0,2,\Gamma_I} \|\mathbf{w}\|_{0,2,\Gamma_I},$$

and thanks to the estimate on  $\tilde{g}$  (2.35),

$$|\langle G(\mathbf{u}) - G(\mathbf{v}), \mathbf{w} \rangle| \leq c (\|\nabla \mathbf{u}\|_{0,2,\Omega} + \|\nabla \mathbf{v}\|_{0,2,\Omega}) \|\nabla(\mathbf{u} - \mathbf{v})\|_{0,2,\Omega} \|\nabla \mathbf{w}\|_{0,2,\Omega}.$$

- (4) The monotonicity of  $G$  is derived directly from the monotonicity of  $\tilde{g}$ .  
(5) The continuity of  $G$  comes from the property (2.39), which states that  $G$  is Lipschitz over any bounded set of  $W$ .  
(6) Let  $(\mathbf{u}_n)_n \in W^{\mathbf{N}}$  which weakly converges to  $\mathbf{u} \in W$ . We will show that  $(G(\mathbf{u}_n))_n$  is strongly converging in  $W'$ , up to a subsequence. We will need compact Sobolev embedding to "level up" the weak convergence of  $(\mathbf{u}_n)_n$  into strong convergence of  $(G(\mathbf{u}_n))_n$ . Since  $(\mathbf{u}_n)_n$  is weakly converges to  $\mathbf{u}$  in  $W$ , by definition of the  $W$ -norm, we have the weak convergence of  $(\mathbf{u}_n)_n$  to  $\mathbf{u}$  in the space  $H^{1/2}(\Gamma_I)$ . Moreover, we have the compact embedding  $H^{1/2}(\Gamma_I) \hookrightarrow L^3(\Gamma_I)$ , which means that up to a subsequence, the sequence  $(\|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I})_n$  is converging to zero. We can make appear such  $L^3$ -norms using the Hölder and Cauchy-Schwarz inequality:

$$\begin{aligned} |\langle G(\mathbf{u}_n) - G(\mathbf{u}), \mathbf{v} \rangle| &\leq c (|\mathbf{u}_n| + |\mathbf{u}|) \|\mathbf{u}_n - \mathbf{u}\|_{0,3/2,\Gamma_I} \|\mathbf{v}\|_{0,3,\Gamma_I} \\ &\leq c \|\mathbf{u}_n\|_{0,3,\Gamma_I} + \|\mathbf{u}\|_{0,3,\Gamma_I} \|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I} \|\mathbf{v}\|_{0,3,\Gamma_I}. \end{aligned}$$

The term  $\|\mathbf{u}_n\|_{0,3,\Gamma_I}$  can be bounded by triangular inequality since the sequence  $(\|\mathbf{u}_n\|_W)_n$  is bounded. Still denoting the constant  $c$  which depends on  $\Omega$ ,  $\|\mathbf{u}\|_{0,3,\Gamma_I}$  and on  $\sup_{n \in \mathbf{N}} \|\mathbf{u}_n\|_W$ , we get

$$|\langle G(\mathbf{u}_n) - G(\mathbf{u}), \mathbf{v} \rangle| \leq c \|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I} \|\nabla \mathbf{v}\|_{0,2,\Omega},$$

using one more time the Trace Sobolev embedding  $H^1(\Omega_i) \hookrightarrow L^3(\Gamma_I)$ . Finally

$$\|G(\mathbf{u}_n) - G(\mathbf{u})\|_{W'} \leq c \|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I} \xrightarrow{n \rightarrow \infty} 0,$$

which concludes the proof according to the previous remarks.  $\square$

**Remark 2.12.** *In the proof of the compactness result, we haven't used the estimate of  $\tilde{g}$  with  $L^4(\Gamma_I)$ -norms (2.36) because there is no compact embedding of  $H^{1/2}(\Gamma_I)$  to  $L^4(\Gamma_I)$  for  $n = 3$ . Nevertheless it would have worked for  $n = 2$ .*

**2.4. Compactness tools: VESP and PESP.** In what follows, we will often have to use the same kind of arguments. We will have to consider velocity sequences  $(\mathbf{v}_n)_{n \in \mathbf{N}}$  belonging to  $W$ , which are solutions of an approximate problem, and we would like to pass to the limit in the equations. It will be also the case for pressure sequences  $(\mathbf{p}_n)_{n \in \mathbf{N}}$  belonging to  $X$ .

Lets start with the velocities and introduce the Velocity Extracting Subsequences Principle (VESP).

(1) **VESP.**

Assume that we have an priori estimate obtained from energy inequalities of the form

$$\|\mathbf{v}_n\| \leq C, \quad \forall n \in \mathbf{N},$$

where  $C > 0$  does not depend on  $n$ . Different properties can be retrieved from this inequality. Denote by  $B_C \subset W$  the ball of radius  $C$  centered in zero, we get

- (a) From the Banach Alaoglu theorem: the ball  $B_C$  is relatively compact in  $W$ , wich means there exists a velocity  $\mathbf{v} \in B_C$  and subsequence  $(\mathbf{v}_{n_k})_{k \in \mathbf{N}}$  of  $(\mathbf{v}_n)_{n \in \mathbf{N}}$  such that  $(\mathbf{v}_{n_k})_{k \in \mathbf{N}}$  weakly converges to  $\mathbf{v}$  when  $k \rightarrow \infty$ . Moreover thanks to the Sobolev embedding theorem, the subsequence  $(\mathbf{v}_{n_k})_{k \in \mathbf{N}}$  strongly converges to  $\mathbf{v}$  in  $L^p(\Omega)$ , when  $1 \leq p < 6$  in dimension 3, and when  $1 \leq p$  in dimension 2.
- (b) From the Riesz-Fischer theorem, it also converges a.e in  $\Omega$ , and there exists  $A_p \in L^p(\Omega)$  such that  $\forall k \in \mathbf{N}, |\mathbf{v}_{n_k}| \leq A_p$ , a.e in  $\Omega$ .
- (c) The sequence of the traces  $(\gamma(\mathbf{v}_{n_k}))_{k \in \mathbf{N}}$  weakly converges to the trace  $\gamma(\mathbf{v})$  in  $H^{1/2}(\Gamma)$  and from the trace continuity theorem and the Sobolev embedding results, it also strongly converges in  $L^q(\Gamma)$ , for  $1 \leq q < 4$  in dimension 3 (the dimension of  $\Gamma$  is 2 in that case).
- (d) Again, from the Riesz-Fischer theorem, the sequence  $(\gamma(\mathbf{v}_{n_k}))_{k \in \mathbf{N}}$  converges a.e in  $\Gamma$ , and there exists  $B_q \in L^q(\Gamma)$  such that  $\forall k \in \mathbf{N}, |\gamma(\mathbf{v}_{n_k})| \leq B_q$  a.e in  $\Gamma$ .

**Remark 2.13.** *To simplify, we will always write  $\mathbf{v}_n$  instead of  $\mathbf{v}_{n_k}$  considering that all the convergences are up to a subsequence. To sum up, we have:*

$$\begin{aligned} \mathbf{v}_n &\xrightarrow[n \rightarrow \infty]{W} \mathbf{v}, \\ \mathbf{v}_n &\xrightarrow[n \rightarrow \infty]{L^p(\Omega)} \mathbf{v}, \quad 1 \leq p < 6, \\ \gamma(\mathbf{v}_n) &\xrightarrow[n \rightarrow \infty]{H^{1/2}(\Gamma)} \gamma(\mathbf{v}), \\ \gamma(\mathbf{v}_n) &\xrightarrow[n \rightarrow \infty]{L^q(\Gamma)} \gamma(\mathbf{v}), \quad 1 \leq q < 4. \end{aligned}$$

**Definition 2.14.** *We say that the limit  $\mathbf{v}$  of the subsequence  $(\mathbf{v}_n)_n$  is a VESP-limit of  $(\mathbf{v}_n)_n$  when it satisfies all the previous properties.*

- (2) **PESP.** The Pressure Extracting Subsequence Principle (PESP) is defined as well.
- (3) **Convergence lemma.** We will consider non-linear terms in our equations. It is more difficult to pass to the limit in these terms than in the linear ones.

We give a small lemma to easily pass to the limit in the non linear terms  $b$  and  $g$  whenever we apply the VESP to velocity sequences.

**Lemma 2.15.** *Let  $(\mathbf{v}_n)_{n \in \mathbb{N}}$  and  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be two bounded sequences in  $W$ . Let  $\mathbf{v}$  and  $\mathbf{w}$  be VESP-limits of these sequences. We also assume that  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  strongly converges to  $\mathbf{w}$  in  $W$ . Then*

$$(2.40) \quad \lim_{n \rightarrow \infty} b(\mathbf{v}_n, \mathbf{v}_n, \mathbf{w}_n) = b(\mathbf{v}, \mathbf{v}, \mathbf{w}),$$

$$(2.41) \quad \lim_{n \rightarrow \infty} g(\mathbf{v}_n, \mathbf{v}_n, \mathbf{w}_n) = g(\mathbf{v}, \mathbf{v}, \mathbf{w}).$$

*Proof.* Let give the proof of the first claim on  $b$ . we have by definition

$$b(\mathbf{v}_n, \mathbf{v}_n, \mathbf{w}_n) = \frac{1}{2} \left( \int_{\Omega} \mathbf{v}_n \otimes \mathbf{w}_n : \nabla \mathbf{v}_n - \int_{\Omega} \mathbf{v}_n \otimes \mathbf{v}_n : \nabla \mathbf{w}_n \right),$$

and

$$b(\mathbf{v}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \left( \int_{\Omega} \mathbf{v} \otimes \mathbf{w} : \nabla \mathbf{v} - \int_{\Omega} \mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{w} \right).$$

According to the a-property of the VESP, we have in particular

$$\begin{aligned} \mathbf{v}_n & \xrightarrow[n \rightarrow \infty]{(L^4(\Omega))^3} \mathbf{v}, \\ \mathbf{w}_n & \xrightarrow[n \rightarrow \infty]{(L^4(\Omega))^3} \mathbf{w}. \end{aligned}$$

Thus, we have the strong convergences

$$(2.42) \quad \begin{cases} \mathbf{v}_n \otimes \mathbf{w}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \mathbf{v} \otimes \mathbf{w}, \\ \mathbf{v}_n \otimes \mathbf{v}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \mathbf{v} \otimes \mathbf{v}. \end{cases}$$

Moreover, still from the a-property of the VESP, we have the weak convergences

$$(2.43) \quad \begin{cases} \nabla \mathbf{v}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \nabla \mathbf{v}, \\ \nabla \mathbf{w}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \nabla \mathbf{w} \end{cases}$$

which gives combined with (2.42) the convergence (2.40). The second claim about the convergence of the  $g$ -term is a consequence of the compactness property given in 2.11.  $\square$

## 2.5. A priori estimate for the velocity.

**Proposition 2.16.** *Let  $(\mathbf{u}, \mathbf{p})$  be any weak solution of the problem (1.1). Then  $\mathbf{u}$  satisfies the energy equality*

$$(2.44) \quad a(\mathbf{u}, \mathbf{u}) + g(\mathbf{u}, \mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{u}) = \langle \mathbf{f}, \mathbf{u} \rangle,$$

leading to the estimate

$$(2.45) \quad \mu \|\mathbf{u}\|_W^2 \leq \frac{C_{eq}^2}{\mu} \|\mathbf{f}\|_{0,2}^2 + \frac{(\max(c_{d,1}, c_{d,2}))^2}{\min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2$$

where the constant  $\mu$  is defined by

$$\mu = \min\left(\nu, \frac{1}{2}c_{d,1}, \frac{1}{2}c_{d,2}\right),$$

and where the constant  $C_{eq}$  is given in the norm equivalence result 2.1. In the following, it will be more convenient to use the simplified a priori estimate

$$(2.46) \quad \|\mathbf{u}\|_W^2 \leq C_{vel},$$

where  $C_{vel}$  is given by

$$C_{vel} = \frac{C_{eq}^2}{\mu^2} \|\mathbf{f}\|_{0,2}^2 + \frac{(\max(c_{d,1}, c_{d,2}))^2}{\mu \min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2.$$

*Proof.* We take  $\mathbf{v} = \mathbf{u} \in W$  as a test in  $(\mathcal{VP})$  and we consider each term one after another. Since  $\nabla \cdot \mathbf{u} = 0$ , the pressure term  $\langle \mathbf{p}, \nabla \cdot \mathbf{u} \rangle = 0$ . We also have thanks to antisymmetry property 2.6 of the trilinear form  $b$  that  $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$ . This gives us

$$a(\mathbf{u}, \mathbf{u}) + g(\mathbf{u}, \mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{u}) = \langle \mathbf{f}, \mathbf{u} \rangle,$$

which is the energy equality (2.44). Moreover, the non-linear term  $g$  satisfies  $g(\mathbf{u}, \mathbf{u}, \mathbf{u}) \geq 0$ . Thus:

$$a(\mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{u}) \leq \langle \mathbf{f}, \mathbf{u} \rangle.$$

Recall that the left hand term equals

$$\nu_1 \|\nabla \mathbf{u}_1\|_{0,2,\Omega_1}^2 + \nu_2 \|\nabla \mathbf{u}_2\|_{0,2,\Omega_2}^2 + c_{d,1} \|\mathbf{u}_{1,h}\|_{0,2,\Gamma_1}^2 + c_{d,2} \|\mathbf{u}_{2,h}\|_{0,2,\Gamma_2}^2$$

and the right hand term is

$$\int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{u}_1 + \int_{\Omega_2} \mathbf{f}_2 \cdot \mathbf{u}_2 + c_{d,1} \int_{\Gamma_1} \mathbf{V}_1 \cdot \mathbf{u}_{1,h} + c_{d,2} \int_{\Gamma_2} \mathbf{V}_2 \cdot \mathbf{u}_{2,h},$$

according to (2.21).

Consequently, by definition of the product norms on  $\Omega$ , we get thanks to the Cauchy-Schwarz inequality

$$\nu \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \min(c_{d,1}, c_{d,2}) \|\mathbf{u}\|_{0,2,\Gamma}^2 \leq \|\mathbf{f}\|_{0,2} \|\mathbf{u}\|_{1,2,\Omega} + \max(c_{d,1}, c_{d,2}) \|\mathbf{V}\|_{0,2,\Gamma} \|\mathbf{u}\|_{0,2,\Gamma}.$$

Using the Young's inequality, we obtain

$$\nu \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \frac{1}{2} \min(c_{d,1}, c_{d,2}) \|\mathbf{u}\|_{0,2,\Gamma}^2 \leq \|\mathbf{f}\|_{0,2} \|\mathbf{u}\|_{1,2,\Omega} + \frac{(\max(c_{d,1}, c_{d,2}))^2}{2 \min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2,$$

We use the equivalence result (2.4) between the two norms  $\|\cdot\|_{1,2,\Omega}$  and  $\|\cdot\|_W$  and the Young's inequality to get

$$\mu \|\mathbf{u}\|_W^2 \leq C_{eq} \|\mathbf{f}\|_{0,2} \|\mathbf{u}\|_W + \frac{(\max(c_{d,1}, c_{d,2}))^2}{2 \min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2,$$

where  $\mu = \min(\nu, \frac{1}{2}c_{d,1}, \frac{1}{2}c_{d,2})$ . Finally we get using the Young's inequality again:

$$\mu \|\mathbf{u}\|_W^2 \leq \frac{C_{eq}^2}{\mu} \|\mathbf{f}\|_{0,2}^2 + \frac{(\max(c_{d,1}, c_{d,2}))^2}{\min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2.$$

□

**2.6. A priori estimate for the pressure.** In this section, we will use the potential vector method to express the pressure as a function of the velocities.

To do this, we first need a lemma on the elliptic regularity of a Neumann problem on the Torus.

**Lemma 2.17.** *Let  $u \in H^2(\Omega)$  be the unique solution of the Neumann problem*

$$(2.47) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma \\ \int_{\Omega} u = 0 \end{cases}$$

where  $\Omega = \Pi_2 \times [0, 1]$ . Then we have the estimate:

$$(2.48) \quad \|u\|_{2,2} \leq C \|f\|_{0,2}.$$

*Proof.* See the section 2.9 at the end of this chapter.  $\square$

We will use this lemma to obtain an a priori estimate on the pressure. We recall that

$$(2.49) \quad X = L^2(\Omega_1) \times L^2(\Omega_2)$$

is the pressure space equipped with the norm:

$$\|\mathbf{p}\|_X = \left( \|p_1\|_{L^2(\Omega_1)}^2 + \|p_2\|_{L^2(\Omega_2)}^2 \right)^{1/2},$$

where  $\mathbf{p} = (p_1, p_2) \in X$ .

**Lemma 2.18.** *We can find a test  $\mathbf{w} \in W$  and a constant  $C_{\Omega}$  depending only on  $\Omega$  such that*

$$(2.50) \quad \langle \mathbf{p}, \nabla \cdot \mathbf{w} \rangle = -\|\mathbf{p}\|_X^2,$$

and

$$(2.51) \quad \|\mathbf{w}\|_W \leq C_{\Omega} \|\mathbf{p}\|_X.$$

*Proof.* Let  $v_i \in H^2(\Omega_i)$  be the unique solution of the Neumann problem

$$(2.52) \quad \begin{cases} -\Delta v_i = P_i & \text{in } \Omega_i, \\ \frac{\partial v_i}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_i \cap \Gamma_{int}, \\ \int_{\Omega_i} v_i = 0. \end{cases}$$

for  $i = 1, 2$ . We have

$$\|v_i\|_{H^2(\Omega_i)} \leq C_i \|P_i\|_{L^2(\Omega_i)},$$

where  $C_i$  only depends on  $\Omega_i$ . Let  $\mathbf{w}_i = \nabla v_i$ , which satisfies

$$\mathbf{w}_i \in H^1(\Omega_i)^3, \quad \mathbf{w}_i \cdot \mathbf{n}|_{\Gamma_i \cap \Gamma_{int}} = \frac{\partial v_i}{\partial \mathbf{n}}|_{\Gamma_i \cap \Gamma_{int}},$$

and

$$\|\mathbf{w}_i\|_{H^1(\Omega_i)} \leq C_i \|P_i\|_{L^2(\Omega_i)}.$$

Let  $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ . Then,  $\mathbf{w} \in W$  and there exists a constant  $C_{\Omega}$  such that

$$\|\mathbf{w}\|_W \leq C_{\Omega} \|\mathbf{p}\|_X.$$

Taking  $\mathbf{w}$  as test in (1.1) gives us that

$$\langle \mathbf{p}, \nabla \cdot \mathbf{w} \rangle = -\|\mathbf{p}\|_X^2.$$

$\square$



This will allow us to show the following a priori estimate for the pressure.

**Proposition 2.19.** *Let  $(\mathbf{u}, \mathbf{p})$  be any weak solution of the problem (1.1). Then there exists a positive constant  $C_p$  depending on the domain and on the data such that*

$$(2.53) \quad \|\mathbf{p}\|_X^2 \leq C_p.$$

*Proof.* Let take  $\mathbf{w}$  obtained in the lemma 2.17 as test in (1.1). We have the following equality

$$(2.54) \quad \|\mathbf{p}\|_X^2 = \langle \mathbf{f}, \mathbf{w} \rangle - a(\mathbf{u}, \mathbf{w}) - b(\mathbf{u}, \mathbf{u}, \mathbf{w}) - g(\mathbf{u}, \mathbf{u}, \mathbf{w}) - h(\mathbf{u}, \mathbf{w}).$$

We will give an upper bound of each term of the right hand side. Let  $\rho > 0$  to be fixed later. By using the operator inequalities (2.25) 2.8, Cauchy-Schwarz inequality and Young's inequality, we get

$$(2.55) \quad |a(\mathbf{u}, \mathbf{w})| \leq \frac{\max(\nu_1, \nu_2)^2}{2\rho} \|\mathbf{u}\|_W^2 + \frac{\rho}{2} \|\mathbf{w}\|_W^2,$$

$$(2.56) \quad |b(\mathbf{u}, \mathbf{u}, \mathbf{w})| \leq C_b \|\mathbf{u}\|_W^2 \|\mathbf{w}\|_W \leq \frac{C_b^2}{2\rho} \|\mathbf{u}\|_W^4 + \frac{\rho}{2} \|\mathbf{w}\|_W^2,$$

$$(2.57) \quad |g(\mathbf{u}, \mathbf{u}, \mathbf{w})| \leq \frac{C_g^2}{2\rho} \|\mathbf{u}\|_W^4 + \frac{\rho}{2} \|\mathbf{w}\|_W^2,$$

$$(2.58) \quad |h(\mathbf{u}, \mathbf{w})| \leq \frac{\max(c_{d,1}, c_{d,2})^2}{2\rho} \|\mathbf{u}\|_W^2 + \frac{\rho}{2} \|\mathbf{w}\|_W^2,$$

and

$$(2.59) \quad |\langle \mathbf{f}, \mathbf{w} \rangle| \leq \frac{C_{eq}^2}{2\rho} \|\mathbf{f}\|_{0,2}^2 + \frac{\max(c_{d,1}, c_{d,2})^2}{2\rho} (\|\mathbf{V}_1\|_{0,2,\Gamma_1}^2 + \|\mathbf{V}_2\|_{0,2,\Gamma_2}^2) + \rho \|\mathbf{w}\|_W^2.$$

According to the lemma 2.17,  $\mathbf{w}$  verifies the inequality

$$\|\mathbf{w}\|_W \leq C_\Omega \|\mathbf{p}\|_X.$$

Putting (2.54), (2.55), (2.56), (2.57), (2.58) and (2.59) together, we obtain

$$(2.60) \quad \|\mathbf{p}\|_X^2 \leq +3\rho C_\Omega \|\mathbf{p}\|_X^2 + \frac{1}{2\rho} (2 \max(\nu_1^2, \nu_2^2, c_{d,1}^2, c_{d,2}^2) \|\mathbf{u}\|_W^2 + (C_b + C_g) \|\mathbf{u}\|_W^4 + C_{data}),$$

where  $C_{data}$  is given by

$$C_{data} := C_{eq}^2 \|\mathbf{f}\|_{0,2}^2 + \max(c_{d,1}, c_{d,2})^2 (\|\mathbf{V}_1\|_{0,2,\Gamma_1}^2 + \|\mathbf{V}_2\|_{0,2,\Gamma_2}^2).$$

We can choose

$$\rho = \frac{1}{4C_\Omega^2},$$

and replacing it in (2.60) gives

$$(2.61) \quad \|\mathbf{p}\|_X^2 \leq 8C_\Omega (2 \max(\nu_1^2, \nu_2^2, c_{d,1}^2, c_{d,2}^2) \|\mathbf{u}\|_W^2 + (C_b + C_g) \|\mathbf{u}\|_W^4 + C_{data}).$$

Using the a priori estimate on the velocity (2.46), we finally obtain

$$\|\mathbf{p}\|_X^2 \leq C_p,$$

where  $C_p$  is given by

$$C_p = 8C_\Omega (2 \max(\nu_1^2, \nu_2^2, c_{d,1}^2, c_{d,2}^2) C_{vel} + (C_b + C_g) C_{vel}^2 + C_{data}).$$

□

## 2.7. Uniqueness.

**Theorem 2.20.** *Recall that  $\nu = \min(\nu_1, \nu_2)$ , we also define the constants  $C_{1,\nu}$  and  $C_{2,\nu}$  by*

$$C_{1,\nu} := \frac{C_b}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2},$$

and

$$C_{2,\nu} := \frac{2C}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}.$$

If  $\nu$  verifies the condition

$$(2.62) \quad \nu > C_{1,\nu} + C_{2,\nu},$$

a weak solution  $(\mathbf{u}, \mathbf{p}) \in W \times X_0$  of the problem  $(\mathcal{VP})$  is unique.

*Proof.* Let  $(\mathbf{u}^*, \mathbf{p}^*)$  and  $(\mathbf{u}^{**}, \mathbf{p}^{**})$  be two different solutions of  $(\mathcal{VP})$  and let define  $\mathbf{u} = \mathbf{u}^* - \mathbf{u}^{**}$  and  $\mathbf{p} = \mathbf{p}^* - \mathbf{p}^{**}$ .

We subtract the equations  $(\mathcal{VP})$  corresponding to  $(\mathbf{u}^*, \mathbf{p}^*)$  and  $(\mathbf{u}^{**}, \mathbf{p}^{**})$  and we obtain:

$$(2.63) \quad \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}^*, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}^*, \mathbf{v}) - d(\mathbf{p}, \nabla \cdot \mathbf{v}) + \langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{v} \rangle \\ + \langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{v} \rangle = 0, \\ d(\mathbf{q}, \nabla \cdot \mathbf{v}) = 0, \end{cases}$$

for every  $(\mathbf{v}, \mathbf{q}) \in W \times Q$ .

We take  $\mathbf{v} = \mathbf{u}$  in (2.63), and we study the different terms.

- The transport term verifies  $b(\mathbf{u}^*, \mathbf{u}, \mathbf{u}) = 0$  thanks to 2.6, and we also get

$$(2.64) \quad |b(\mathbf{u}, \mathbf{u}^*, \mathbf{u})| \leq C_b \|\nabla \mathbf{u}\|_{0,2}^2 \|\nabla \mathbf{u}^*\|_{0,2}.$$

Since  $\mathbf{u}^*$  is a solution of  $(\mathcal{VP})$ , we can use the estimate (2.45) in (2.64) which gives: ‘

$$(2.65) \quad |b(\mathbf{u}, \mathbf{u}^*, \mathbf{u})| \leq \frac{C_b}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}.$$

- Since  $\nabla \cdot \mathbf{u} = 0$ , The pressure term  $d(\mathbf{p}, \nabla \cdot \mathbf{u}) = 0$ .
- The friction term  $\langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{u} \rangle$  satisfies the inequality

$$(2.66) \quad \begin{aligned} |\langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{u} \rangle| &\leq C (\|\nabla \mathbf{u}^*\|_{0,2} + \|\nabla \mathbf{u}^{**}\|_{0,2}) \|\nabla \mathbf{u}\|_{0,2}^2 \\ &\leq \frac{2C}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2} \end{aligned}$$

thanks to the continuity of  $G$  given in (2.9).

- Finally the top and bottom boundary term  $\langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{u} \rangle$  verifies

$$\begin{aligned} \langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{u} \rangle &= \int_{\Gamma_1} (\mathbf{u}_1)_\tau^2 + \int_{\Gamma_2} (\mathbf{u}_2)_\tau^2 \\ &= c_{d,1} \|\mathbf{u}_1\|_{0,2,\Gamma_1}^2 + c_{d,2} \|\mathbf{u}_2\|_{0,2,\Gamma_2}^2. \end{aligned}$$

Putting this together, we get by triangular inequality

$$a(\mathbf{u}, \mathbf{u}) + \langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{u} \rangle \leq |b(\mathbf{u}, \mathbf{u}^*, \mathbf{u})| + |\langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{u} \rangle|,$$

hence

$$\begin{aligned} \nu \|\nabla \mathbf{u}\|_{0,2}^2 + \gamma_d \|\mathbf{u}\|_{0,2,\Gamma}^2 &\leq \frac{C_b}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2} \\ &\quad + \frac{2C}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}. \end{aligned}$$

where  $\gamma_d = \min(c_{d,1}, c_{d,2})$ .

We recognize the constants  $C_{1,\nu}$  and  $C_{2,\nu}$ , which gives

$$(\nu - C_{1,\nu} - C_{2,\nu}) \|\nabla \mathbf{u}\|_{0,2}^2 + \gamma_d \|\mathbf{u}\|_{0,2,\Gamma}^2 \leq 0,$$

and finally

$$\min(\nu - C_{1,\nu} - C_{2,\nu}, \gamma_d) \|\mathbf{u}\|_W^2 \leq 0.$$

Using the condition on the data (2.62), we obtain  $\|\mathbf{u}\|_W = 0$ , hence  $\mathbf{u}^* = \mathbf{u}^{**}$ .

Taking as a pressure test  $\mathbf{q} = \mathbf{p}$  gives  $d(\mathbf{p}, \mathbf{w}) = 0$  for every  $\mathbf{w} \in W$ . By using a density result, this remains true for every  $\mathbf{w} \in \mathcal{D}(\Omega)^3$ . Then,  $\mathbf{p} = 0$  a.e in  $\Omega$ , and equal to zero since it belongs to  $X_0$ .  $\square$

**2.8. Approximated Linearized Problem.** The aim of this section is to prove the existence theorem:

**Theorem 2.21.** *If  $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega_1)^3 \times L^2(\Omega_2)^3$  and  $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2) \in L^2(\Gamma_1)^3 \times L^2(\Gamma_2)^3$ , there exists a weak solution  $(\mathbf{u}, \mathbf{p}) \in W \times X_0$  of the initial problem (1.1).*

Since we cannot use "De Rham" arguments to retrieve the pressure from the velocity, we will add a small perturbation on the divergence term. This will allow us to "decouple" the pressure and the velocity, and to obtain  $\varepsilon$ -solutions (approximate solutions).

For a given  $\varepsilon > 0$ , We introduce the following approximate problem

$$(2.67) \quad \left\{ \begin{array}{ll} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i \\ -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{v}_{i,h} - \mathbf{v}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \frac{\partial p_i}{\partial \mathbf{n}_i} = 0, & \text{on } \Gamma_{Int} \cup \Gamma_i \\ \int_{\Omega_i} p_i = 0 & . \end{array} \right.$$

The point is that this new system formally converges to the initial system (1.1) when  $\varepsilon \rightarrow 0$ . We will now consider a pressure problem which will allow us to eliminate the pressure term  $p$  in the system (2.67). Let  $\varepsilon > 0$  and  $\mathbf{u} \in W$  be fixed.

We extract some equations involving the pressure in (2.67) and put them together which gives the Neumann problem:

$$(2.68) \quad \begin{cases} -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i, \\ \frac{\partial p_i}{\partial \mathbf{n}_i} = 0, & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \int_{\Omega_i} p_i = 0 & . \end{cases}$$

This problem is well-posed when we consider the space  $Q_0 = Q_0(\Omega)$  defined by

$$Q_0 := \{q = (q_1, q_2) \in Q = H^1(\Omega_1) \times H^1(\Omega_2), \int_{\Omega_i} q_i = 0, \forall i \in \{1, 2\}\},$$

equipped with the semi-norm  $\|\nabla \mathbf{q}\|_{0,2,\Omega} := (\|\nabla q_1\|_{0,2,\Omega_1}^2 + \|\nabla q_2\|_{0,2,\Omega_2}^2)^{1/2}$  (equivalent to the classical Sobolev norm  $\|\cdot\|_{1,2,\Omega}$  thanks to the Poincaré-Wirtinger inequality). According to the classical Lax-Milgram theorem, the problem (2.68) admits a unique weak solution  $\mathbf{p} \in Q_0 \cup H^2(\Omega)$  which verifies the variational formulation: for every  $\mathbf{q} \in H^1(\Omega)^3$ ,

$$(2.69) \quad \varepsilon \langle \nabla \mathbf{p}, \nabla \mathbf{q} \rangle_{\Omega} + \langle \nabla \cdot \mathbf{u}, \mathbf{q} \rangle_{\Omega} = 0.$$

Since we have existence and uniqueness of the pressure  $\mathbf{p}$  solution of (2.68), we can define the map

$$(2.70) \quad P_{\varepsilon} : \begin{cases} W \rightarrow Q_0, \\ \mathbf{u} \mapsto \mathbf{p}, \text{ the unique weak solution of (2.68).} \end{cases}$$

**Lemma 2.22.** *The map  $P_{\varepsilon}$  is linear and continuous. Moreover, we have the estimate*

$$\|P_{\varepsilon}\| = O\left(\frac{1}{\varepsilon}\right).$$

*Proof.* The linearity is easy to check. Let us prove that  $P_{\varepsilon}$  is continuous. Let  $\mathbf{u} \in W$  and  $\mathbf{p} = P_{\varepsilon}(\mathbf{u})$ . We chose  $\mathbf{q} = \mathbf{p}$  as a test in the variational formulation (2.80). Thanks to the Stokes formula, we get

$$(2.71) \quad \varepsilon \|\nabla \mathbf{p}\|_{0,2,\Omega}^2 + \langle \nabla \cdot \mathbf{u}, \mathbf{p} \rangle_{\Omega} = 0,$$

and using the Neumann condition

$$\frac{\partial p_i}{\partial \mathbf{n}_i} = 0 \text{ on } \Gamma_{Int} \cup \Gamma_i$$

, we get

$$\varepsilon \|\nabla \mathbf{p}\|_{0,2,\Omega}^2 - \langle \mathbf{u}, \nabla \mathbf{p} \rangle_{\Omega} = 0.$$

Thus by Cauchy-Schwarz inequality:

$$\|\nabla \mathbf{p}\|_{0,2,\Omega} \leq \frac{C}{\varepsilon} \|\mathbf{u}\|_W,$$

where  $C > 0$ . □

A direct consequence of the energy equality (2.71) is the following result:

**Lemma 2.23.** *Let  $\mathbf{u} \in W$  and  $\mathbf{p} = P_{\varepsilon}(\mathbf{u})$ . We have the inequality*

$$\langle \nabla \cdot \mathbf{u}, \mathbf{p} \rangle_{\Omega} \leq 0.$$

We can now "transform" the problem (2.67) using the linear map  $P_\varepsilon$ , to "get rid of" the pressure  $\mathbf{p}$ . This problem becomes

$$(2.72) \quad \begin{cases} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i + \frac{1}{2} \mathbf{u}_i (\nabla \cdot \mathbf{u}_i) - \nu_i \Delta \mathbf{u}_i + \nabla P_\varepsilon(\mathbf{u}_i) = \mathbf{f}_i & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{v}_{i,h} - \mathbf{v}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \end{cases}$$

where the only unknown is the velocity field  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ .

The next step is to study the variational formulation associated with (2.72).

2.8.1. *Perturbed variational formulation.* Following our plan, the variational problem  $\mathcal{VP}_\varepsilon$  associated with (2.72) is:

find  $\mathbf{u} \in W$  such that,  $\forall \mathbf{v} \in W$ ,

$$(\mathcal{VP}_\varepsilon) \quad a(\mathbf{u}, \mathbf{v}) + c_d \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \langle P_\varepsilon(\mathbf{u}), \nabla \cdot \mathbf{v} \rangle_\Omega + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + c_d \langle \mathbf{V}, \mathbf{v} \rangle_\Gamma.$$

In the future, we will have to pass to the limit. Therefore, we need some a priori estimates to keep going on.

**Lemma 2.24.** *Let  $\varepsilon > 0$ ,  $\mathbf{u}_\varepsilon$  be any solution of  $\mathcal{VP}_\varepsilon$  and  $\mathbf{p}_\varepsilon = P_\varepsilon(\mathbf{u}_\varepsilon)$ . We have the same kind of estimates as in the previous section, i.e:*

$$(2.73) \quad \|\mathbf{u}_\varepsilon\|_W \leq C_v,$$

and

$$(2.74) \quad \|\mathbf{p}_\varepsilon\|_{0,2,\Omega} \leq C_p,$$

where  $C_v$  and  $C_p$  are the constants obtained in 2.45 and 2.19.

*Proof.* The proof is following the same steps as in 2.16 and 2.19.

We are taking  $\mathbf{u}_\varepsilon$  as a test in  $(\mathcal{VP}_\varepsilon)$  which leads to

$$(2.75) \quad a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + c_d \|\mathbf{u}_\varepsilon\|_{0,2,\Gamma}^2 + b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - \langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{u}_\varepsilon \rangle_\Omega + g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) = \langle \mathbf{f}, \mathbf{u}_\varepsilon \rangle_\Omega + c_d \langle \mathbf{V}, \mathbf{u}_\varepsilon \rangle_\Gamma.$$

The difference between (2.75) and the previous a priori estimates is the pressure term  $-\langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{u}_\varepsilon \rangle_\Omega$  which was vanishing according to free divergence condition in (1.1). Hopefully, we can get rid of this term using the lemma 2.23 stating that  $-\langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{u}_\varepsilon \rangle_\Omega \leq 0$ , which gives

$$(2.76) \quad a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + c_d \|\mathbf{u}_\varepsilon\|_{0,2,\Gamma}^2 + b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \leq \langle \mathbf{f}, \mathbf{u}_\varepsilon \rangle_\Omega + c_d \langle \mathbf{V}, \mathbf{u}_\varepsilon \rangle_\Gamma.$$

The remaining steps of the proof are exactly the same as in subsections 2.5 and 2.6, which yields to the given estimates.  $\square$

The next step is to show that if we are able to have a solution  $\mathbf{u}_\varepsilon$  of the approximated problem  $\mathcal{VP}_\varepsilon$ , we can consider the couple  $(\mathbf{u}_\varepsilon, P_\varepsilon(\mathbf{u}_\varepsilon))$ , pass to the limit and get a solution of the initial problem  $\mathcal{VP}$ . This leads to the proposition:

**Proposition 2.25.** *Let  $\varepsilon > 0$ ,  $\mathbf{u}_\varepsilon$  be a solution of  $\mathcal{VP}_\varepsilon$  and  $\mathbf{p}_\varepsilon = P_\varepsilon(\mathbf{u}_\varepsilon)$ . We consider  $\mathbf{u}$  and  $\mathbf{p}$  any VESP-limit and PESP-limit of the sequences  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  and  $(\mathbf{p}_\varepsilon)_{\varepsilon>0}$ . Then,  $(\mathbf{u}, \mathbf{p})$  is a solution to  $\mathcal{VP}$ .*

*Proof.* By definition of the approximated problem  $\mathcal{VP}_\varepsilon$ , we have for all  $\mathbf{v} \in W$ :

$$(2.77) \quad a(\mathbf{u}_\varepsilon, \mathbf{v}) + c_d \langle \mathbf{u}_\varepsilon, \mathbf{v} \rangle_\Gamma + b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) - \langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{v} \rangle_\Omega + g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{V}, \mathbf{v} \rangle_\Gamma.$$

According to the lemma 2.24,  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$  is bounded in  $W \times Q_0$ . We can then use the VESP and PESP to obtain VESP and PESP limits  $\mathbf{u}$  and  $\mathbf{p}$ . It remains to prove that  $(\mathbf{u}, \mathbf{p})$  is solution of the initial variational problem  $\mathcal{VP}$ . We will pass to the limit in two different problems. First, in the variational formulation (2.77) to retrieve the problem  $\mathcal{VP}$ . Second, in the perturbation pressure problem 2.68.

**First passage to the limit.** Let  $\mathbf{v} \in W$ . The weak convergence of  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  gives

$$(2.78) \quad \begin{cases} a(\mathbf{u}_\varepsilon, \mathbf{v}) & \xrightarrow{\varepsilon \rightarrow 0} a(\mathbf{u}, \mathbf{v}), \\ \langle \mathbf{u}_\varepsilon, \mathbf{v} \rangle_\Gamma & \xrightarrow{\varepsilon \rightarrow 0} \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma, \\ \langle p_\varepsilon, \nabla \cdot \mathbf{v} \rangle_\Omega & \xrightarrow{\varepsilon \rightarrow 0} \langle p, \nabla \cdot \mathbf{v} \rangle_\Omega \end{cases}$$

Moreover, using the convergence properties of lemma 2.15, we get

$$(2.79) \quad \begin{cases} b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) & \xrightarrow{\varepsilon \rightarrow 0} b(\mathbf{u}, \mathbf{u}, \mathbf{v}), \\ g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) & \xrightarrow{\varepsilon \rightarrow 0} g(\mathbf{u}, \mathbf{u}, \mathbf{v}). \end{cases}$$

Since  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  is solution of (2.77) for all  $\mathbf{v} \in W$ , we deduce from (2.78) and (2.79) that  $(\mathbf{u}, p)$  satisfies the variational formulation

$$a(\mathbf{u}, \mathbf{v}) + c_d \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \langle p, \nabla \cdot \mathbf{v} \rangle_\Omega + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{V}, \mathbf{v} \rangle_\Gamma,$$

for all  $\mathbf{v} \in W$ , which is exactly the first equation in  $\mathcal{VP}$ .

**Second passage to the limit.**

We now consider the singular pressure problem 2.68. Let  $\mathbf{q} \in Q$ . The variational formulation (2.80) gives

$$(2.80) \quad -\varepsilon \langle \mathbf{p}_\varepsilon, \Delta \mathbf{q} \rangle_\Omega + \langle \nabla \cdot \mathbf{u}_\varepsilon, \mathbf{q} \rangle_\Omega = 0.$$

Again, using the weak convergences of  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  to  $\mathbf{u}$  in  $W$  and of  $(\mathbf{p}_\varepsilon)_{\varepsilon>0}$  to  $\mathbf{p}$  in  $Q$ , we have

$$(2.81) \quad \begin{cases} \langle \nabla \cdot \mathbf{u}_\varepsilon, \mathbf{q} \rangle_\Omega & \xrightarrow{\varepsilon \rightarrow 0} \langle \nabla \cdot \mathbf{u}, \mathbf{q} \rangle_\Omega, \\ \langle \mathbf{p}_\varepsilon, \Delta \mathbf{q} \rangle_\Omega & \xrightarrow{\varepsilon \rightarrow 0} \langle \mathbf{p}, \Delta \mathbf{q} \rangle_\Omega. \end{cases}$$

This yields  $-\varepsilon \langle \mathbf{p}_\varepsilon, \Delta \mathbf{q} \rangle_\Omega \xrightarrow{\varepsilon \rightarrow 0} 0$  and from (2.80) we obtain

$$(2.82) \quad \langle \nabla \cdot \mathbf{u}, \mathbf{q} \rangle_\Omega = 0,$$

for every  $\mathbf{q} \in Q$ . We recognize the second equation of the variational problem  $\mathcal{VP}$ .

**Conclusion**

We have proved that for a given solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$  of (??), we can retrieve a couple  $(\mathbf{u}, \mathbf{p}) \in W \times Q$ , which satisfies the two equations of the variational formulation  $\mathcal{VP}$ .

It remains to prove that we can find solutions of the perturbed problem.  $\square$

2.8.2. *Linearized problem.* A way to obtain solutions of the problem (??) is to linearize it. The Lax-Milgram theorem will enable us to obtain weak solutions of the linearized problem, and the Schauder fixed-point theorem will allow us to retrieve solutions of the perturbed problem.

. The non-linear terms  $b$  and  $g$  will be linearized as follows: let  $\mathbf{w} \in W$  be a fixed velocity, we replace the terms  $b(\mathbf{u}, \mathbf{u}, \mathbf{v})$  and  $g(\mathbf{u}, \mathbf{u}, \mathbf{v})$  respectively by  $b(\mathbf{w}, \mathbf{u}, \mathbf{v})$  and  $g(\mathbf{w}, \mathbf{u}, \mathbf{v})$ . We then solve the problem

$$(\mathcal{LP}_{\varepsilon, \mathbf{w}}) \quad \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - d(P_\varepsilon(\mathbf{u}), \mathbf{v}) + g(\mathbf{w}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega, \\ \varepsilon a(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{u}) = 0, \end{cases}$$

for every  $(\mathbf{v}, \mathbf{q}) \in W \times Q_0$ . According to the previous construction of  $P_\varepsilon(\mathbf{u})$ , it remains to prove the proposition

**Proposition 2.26.** *Let  $\mathbf{w} \in W$ . The variational formulation: find  $\mathbf{u} \in W$  such that*

$$(2.83) \quad a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - d(P_\varepsilon(\mathbf{u}), \mathbf{v}) + g(\mathbf{w}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega,$$

for every  $\mathbf{v} \in W$ , admits a unique solution  $\mathbf{u}_{\varepsilon, \mathbf{w}} \in W$ .

*Proof.* We will use the Lax-Milgram to prove it. We can rewrite the previous formulation in: find  $\mathbf{u}$  such that

$$\Lambda_\varepsilon(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}),$$

for every  $\mathbf{v} \in W$ , where the bilinear form  $\Lambda_\varepsilon$  is defined by

$$\Lambda_\varepsilon(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - d(P_\varepsilon(\mathbf{u}), \mathbf{v}) + g(\mathbf{w}, \mathbf{u}, \mathbf{v}) + \tilde{h}(\mathbf{u}, \mathbf{v}),$$

and the linear form  $L$  by

$$L(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + h_{\mathbf{v}}(\mathbf{v}).$$

According to the properties of the different operator involved shown in 2.3,  $\Lambda_\varepsilon$  is bilinear continuous. Moreover, since

- (1)  $b(\mathbf{w}, \mathbf{u}, \mathbf{u}) = 0$ , according to lemma 2.6
- (2)  $-d(P_\varepsilon(\mathbf{u}), \mathbf{u}) \geq 0$  according to 2.23,
- (3)  $g(\mathbf{w}, \mathbf{u}, \mathbf{u}) \geq 0$  according to lemma 2.9,

we have

$$\begin{aligned} \Lambda_\varepsilon(\mathbf{u}, \mathbf{u}) &\geq a(\mathbf{u}, \mathbf{u}) + \tilde{h}(\mathbf{u}, \mathbf{u}) \\ &\geq \nu \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \min(c_{d,1}, c_{d,2}) \|\mathbf{u}\|_{0,2,\Gamma}^2 \\ &\geq \min(\nu, c_{d,1}, c_{d,2}) \|\mathbf{u}\|_W^2 \end{aligned}$$

hence  $\Lambda_\varepsilon$  is also verifying the coercivity property. We can use the Lax-Milgram theorem which concludes the proof.  $\square$

We can also give an a priori estimate for the velocity:

**Proposition 2.27.** *Let  $\mathbf{w} \in W$  and  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  the unique solution of  $(\mathcal{LP}_{\varepsilon, \mathbf{w}})$ . Then we have the same estimate as in 2.24, which does not depend nor on  $\varepsilon$ , neither on  $\mathbf{w}$ .*

$$(2.84) \quad \|\mathbf{u}_{\varepsilon, \mathbf{w}}\|_W \leq C_v.$$

We have proved the existence of a weak solution  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  depending on the small perturbation  $\varepsilon$  and on the the given velocity field  $\mathbf{w}$ . The next step is to use a fixed point theorem to get a solution of the problem  $\mathcal{LP}_{\varepsilon}$ .

We introduce the application  $T_{\varepsilon}$  which maps the fixed velocity  $\mathbf{w}$  to the unique solution  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  of previous problem ( $\mathcal{LP}_{\varepsilon, \mathbf{w}}$ ).

$$(2.85) \quad T_{\varepsilon} : \begin{cases} W \rightarrow W, \\ \mathbf{w} \rightarrow \mathbf{u}_{\varepsilon, \mathbf{w}}. \end{cases}$$

The unique solution of ( $\mathcal{LP}_{\varepsilon, \mathbf{w}}$ ) will be denoted  $(\mathbf{u}_{\varepsilon, \mathbf{w}}, \mathbf{p}_{\varepsilon, \mathbf{w}})$ .

We aim to show that  $T_{\varepsilon}$  verifies the assumptions of the Schauder theorem to get a fixed point and a solution of the problem  $\mathcal{LP}_{\varepsilon}$ .

We have to check the following properties:

- (1)  $T_{\varepsilon}$  is continuous over  $W$ ,
- (2) There exists a positive constant  $C$  such that  $T_{\varepsilon}(B_C) \subset B_C$  for  $B_C$  the closed ball of  $W$  of radius  $C$ ,
- (3)  $T_{\varepsilon}$  is compact.

To prove the first point, we will show that  $T_{\varepsilon}$  is weakly continuous, then use it to show that it is continuous.

We need a first weak convergence lemma:

**Lemma 2.28.** *Let  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be weakly convergent to  $\mathbf{w}$  in  $W$ . Then the sequence of the solutions  $(\mathbf{u}_{\varepsilon, \mathbf{w}_n}^n)_n$  of the linear problem  $\mathcal{LP}_{\varepsilon, \mathbf{w}_n}$  associated with  $\mathbf{w}_n$  converges weakly to the solution  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  of the linear problem associated ( $\mathcal{LP}_{\varepsilon, \mathbf{w}}$ ) with  $\mathbf{w}$ .*

To simplify the notation, we will denote by  $\mathbf{u}$  the term  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  and by  $\mathbf{u}^n$  the term  $\mathbf{u}_{\varepsilon, \mathbf{w}_n}$  whenever there is no ambiguity.

*Proof.* We can apply the VESP to  $(\mathbf{w}_n)_{n \in \mathbb{N}}$ : there exists a unique VESP-limit  $\mathbf{w}$ . Let  $(\mathbf{u}^n, \mathbf{p}_n)$  denote the solution of the linear problem  $\mathcal{LP}_{\varepsilon, \mathbf{w}_n}$  associated with  $\mathbf{w}_n$ . We deduce from the estimate (2.84) that  $(\mathbf{u}^n, \mathbf{p}_n)_{n \in \mathbb{N}}$  is bounded in  $W \times X$  uniformly on  $n$ . We can apply the VESP to  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  and the PESP to  $(\mathbf{p}_n)_{n \in \mathbb{N}}$ . Let  $\mathbf{u}$  be a VESP-limit and  $\mathbf{p}$  a PESP-limit.

We want to prove that  $(\mathbf{u}, \mathbf{p})$  is the solution of  $\mathcal{LP}_{\varepsilon, \mathbf{w}}$ . Let  $(\mathbf{v}, \mathbf{q}) \in W \times Q$ . Using the properties (2.40),(2.41) we have the convergences:

$$\begin{aligned} a(\mathbf{u}^n, \mathbf{v}) &\xrightarrow[n \rightarrow \infty]{} a(\mathbf{u}, \mathbf{v}), \\ b(\mathbf{w}_n, \mathbf{u}^n, \mathbf{v}) &\xrightarrow[n \rightarrow \infty]{} b(\mathbf{w}, \mathbf{u}, \mathbf{v}), \\ g(\mathbf{w}_n, \mathbf{u}^n, \mathbf{v}) &\xrightarrow[n \rightarrow \infty]{} g(\mathbf{w}, \mathbf{u}, \mathbf{v}), \\ d(\mathbf{p}_n, \mathbf{v}) &\xrightarrow[n \rightarrow \infty]{} d(\mathbf{p}, \mathbf{v}), \\ d(\mathbf{q}, \mathbf{u}^n) &\xrightarrow[n \rightarrow \infty]{} d(\mathbf{q}, \mathbf{u}^n), \end{aligned}$$

which implies that  $(\mathbf{u}, \mathbf{p})$  is the solution of  $LP_{\varepsilon, \mathbf{w}}$ . □

We can even get a stronger result based on the energy method:

**Lemma 2.29.** *Let  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be weakly convergent to  $\mathbf{w}$  in  $W$ . Then  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  strongly converges in  $W$  to  $\mathbf{u}$ , where  $\mathbf{u}^n = T_{\varepsilon}(\mathbf{w}_n)$ , and  $\mathbf{u} = T_{\varepsilon}(\mathbf{w})$ .*



*Proof.* We know from the previous lemma 2.28 that  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  is weakly converging in  $W$  to  $\mathbf{u}$ . It remains to show that  $(\|\mathbf{u}^n\|_W)_{n \in \mathbb{N}}$  is converging to  $\|\mathbf{u}\|$  and we will have the strong convergence since  $W$  is an Hilbert space. According to the energy equalities, we have

$$\begin{aligned} a(\mathbf{u}^n, \mathbf{u}^n) + g(\mathbf{w}_n, \mathbf{u}^n, \mathbf{u}^n) &= \langle \mathbf{f}, \mathbf{u}^n \rangle, \\ a(\mathbf{u}, \mathbf{u}) + g(\mathbf{w}, \mathbf{u}, \mathbf{u}) &= \langle \mathbf{f}, \mathbf{u} \rangle. \end{aligned}$$

According to the convergence property (2.41) and to the weak convergence, we get

$$\begin{aligned} g(\mathbf{w}_n, \mathbf{u}^n, \mathbf{u}^n) &\xrightarrow[n \rightarrow \infty]{} g(\mathbf{w}, \mathbf{u}, \mathbf{u}), \\ \langle \mathbf{f}, \mathbf{u}^n \rangle &\xrightarrow[n \rightarrow \infty]{} \langle \mathbf{f}, \mathbf{u} \rangle. \end{aligned}$$

We deduce

$$a(\mathbf{u}^n, \mathbf{u}^n) \xrightarrow[n \rightarrow \infty]{} a(\mathbf{u}, \mathbf{u}).$$

The norm induced by the bilinear form  $a$  is equivalent to  $\|\cdot\|_W$  which finally gives

$$\|\mathbf{u}^n\|_W \xrightarrow[n \rightarrow \infty]{} \|\mathbf{u}\|_W,$$

and which combined with the weak convergence of  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  in  $W$  to  $\mathbf{u}$  gives the strong convergence.  $\square$

**Lemma 2.30.** *The application  $T_\varepsilon$  is weakly continuous.*

*Proof.* Let  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be weakly convergent to  $\mathbf{w}$  in  $W$ , and  $\mathbf{u}^n = T(\mathbf{w}_n)$ . We have to prove that  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  weakly converges to  $\mathbf{u} = T(\mathbf{w})$ . We already know from the previous lemma that a subsequence can be extracted from  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  that converges to  $\mathbf{u}$ . But as the solution of the linear problem associated with  $\mathbf{w}$  is unique,  $\mathbf{u}$  is the unique weak limit of  $(\mathbf{u}^n)_{n \in \mathbb{N}}$ .  $\square$

**Lemma 2.31.** *The application  $T_\varepsilon$  is continuous over  $W$ .*

*Proof.* This is as direct consequence of the weak continuity of  $T_\varepsilon$  and of the lemma 2.29.  $\square$

**Proposition 2.32.** *The application  $T_\varepsilon$  has a fixed point. Therefore, the non-linear problem  $(\mathcal{VP}_\varepsilon)$  admits a weak solution  $\mathbf{u}_\varepsilon$ .*

*Proof.* We have proved that  $T_\varepsilon$  is continuous. Let  $C > 0$  the constant given by the a priori estimate  $\|\mathbf{u}\|_W \leq C$  obtained in (2.84). Let  $B_C \subset W$  be the closed ball of radius  $C$  centered in 0. We have by construction  $T(B_C) \subset B_C$ . It remains to prove that  $T(B_C)$  is compact. As  $T(B_C)$  is a closed subspace of the metric space  $W$ , we will show that from any sequence  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  in  $T(B_C)$ , we can extract a subsequence which converges in  $W$ .

Let  $\mathbf{w}_n \in B_C$  be such that  $\mathbf{u}^n = T(\mathbf{w}_n)$ . Since these sequences are bounded, we can extract subsequences that weakly converge respectively to  $\mathbf{w}$  and  $\mathbf{u}$ . We deduce from lemma ?? that  $\mathbf{u} = T(\mathbf{w}) \in B_C$  and from lemma 2.31 that  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  strongly converges to  $\mathbf{u}$ , which proves that  $T(B_C)$  is compact.

We can then use the Schauder's fixed point theorem which concludes the proof.  $\square$

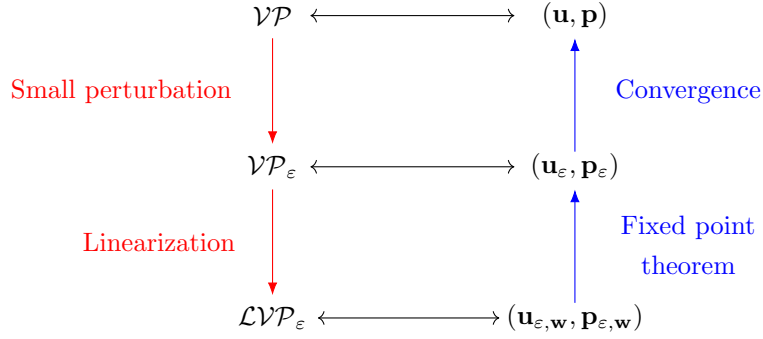


FIGURE 3. Scheme of the proof

2.8.3. *Conclusion.* To summarize what we have done and proved, we recall the scheme of the existence proof in figure 3.

In the previous subsections, we have proved the proposition 2.26, which grants the existence and uniqueness of a weak solution  $(\mathbf{u}_{\varepsilon, \mathbf{w}}, \mathbf{p}_{\varepsilon, \mathbf{w}})$  of the linearized approximate problem  $(\mathcal{LVP}_{\varepsilon, \mathbf{w}})$ , for a given  $\mathbf{w} \in W$ . Then, we deduce from the Schauder's fixed point the existence of a weak solution  $\mathbf{u}_\varepsilon$  of  $(\mathcal{VP}_\varepsilon)$  (proposition 2.32) and consequently the existence of a weak solution  $(\mathbf{u}_E, \mathbf{p}_E)$  to approximate problem (2.67).

Finally, the proposition 2.25 establishes the existence of weak solutions of the initial problem (1.1) by passing to the limit of  $\varepsilon$  towards 0.

Putting this together with the uniqueness theorem 2.20, we have finally proved the existence and uniqueness theorem:

**Theorem 2.33.** *Suppose that  $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega_1)^3 \times L^2(\Omega_2)^3$  and  $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2) \in L^2(\Gamma_1)^3 \times L^2(\Gamma_2)^3$ .*

*There exists a unique weak solution  $(\mathbf{u}, \mathbf{p}) \in W \times X_0$  of the system (1.1), under the condition (2.62). The different given parameters have to ensure the relation:*

$$\nu > C_{1, \nu} + C_{2, \nu},$$

where  $\nu = \min(\nu_1, \nu_2)$  and

$$C_{1, \nu} := \frac{C_b}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0, \Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0, \Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2},$$

and

$$C_{2, \nu} := \frac{2C}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0, \Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0, \Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}.$$

**Remark 2.34.** *This condition comes from the uniqueness theorem 2.20. It is verified whenever the viscosities coefficients  $\nu_1$  and  $\nu_2$  are "big" compared to the other terms involved.*

## 2.9. Technical lemma.

**Lemma 2.35.** *Let  $u \in H^2(\Omega)$  be the unique solution of the Neumann problem*

$$(2.86) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma \\ \int_{\Omega} u = 0 \end{cases}$$

where  $\Omega = \Pi_2 \times [0, 1]$ . Then we have the estimate:

$$(2.87) \quad \|u\|_{2,2} \leq C\|f\|_{0,2}.$$

*Proof.* We consider the Fourier decomposition of  $u$  and  $f$ :

$$u(x, z) = \sum_{k \in \mathbb{Z}} u_k(z) e^{ikx},$$

and

$$f(x, z) = \sum_{k \in \mathbb{Z}} f_k(z) e^{ikx}.$$

Define now the  $H^2$  norm associated with the space  $\Omega = \Pi_2 \times [0, 1]$ :

$$(2.88) \quad \|u\|_{2,2}^2 = \sum_{k \in \mathbb{Z}} \int_0^1 (|u_k''(z)|^2 + k^2 |u_k'(z)|^2 + k^4 |u_k|^2) dz.$$

The Fourier decomposition of (2.86) gives for every  $k \in \mathbb{Z}$ :

$$(2.89) \quad \int_0^1 -u_k''(z) + k^2 u_k(z) dz = \int_0^1 f_k.$$

Taking  $v \in H_0^1$  as a test function and by the Green's formula, we get

$$\int_0^1 u_k' v' + k^2 \int_0^1 u_k v = \int_0^1 f_k v.$$

Taking  $v = u_k$  and thanks to the Young inequality, we obtain the estimate

$$\int_0^1 |u_k'|^2 + k^2 \int_0^1 |u_k|^2 \leq \frac{k^2}{2} \int_0^1 |u_k|^2 + \frac{1}{2k^2} \int_0^1 |f_k|^2,$$

hence

$$\int_0^1 |u_k'|^2 + \frac{k^2}{2} \int_0^1 |u_k|^2 \leq \frac{1}{2k^2} \int_0^1 |f_k|^2,$$

and

$$(2.90) \quad \int_0^1 k^4 |u_k|^2 \leq \int_0^1 |f_k|^2 - 2 \int_0^1 k^2 |u_k'|^2.$$

According to (2.89), we have

$$\begin{aligned} \int_0^1 |u_k''|^2 + k^2 |u_k'|^2 + k^4 |u_k|^2 &= \int_0^1 |k^2 u_k - f_k|^2 + k^2 |u_k'|^2 + k^4 |u_k|^2 \\ &= \int_0^1 2k^4 |u_k|^2 + |f_k|^2 + k^2 |u_k'|^2 \\ &\leq 3 \int_0^1 |f_k|^2 - 3 \int_0^1 k^2 |u_k'|^2 \\ &\leq 3 \int_0^1 |f_k|^2 \end{aligned}$$

using (2.90). Summing this inequality for  $z \in \mathbb{Z}$  gives the estimate we want.  $\square$

### 3. ITERATIVE ELEMENT METHOD FOR STEADY NAVIER-STOKES EQUATIONS WITH FRICTION CONDITIONS

**3.1. Introduction.** We will consider in the following a problem slightly different from the study we have done previously. Instead of considering the spaces  $\Omega_1$  and  $\Omega_2$  as  $\Pi_2 \times [0, z_i]$ , we will assume that they are both bounded domains in  $\mathbf{R}^d$ , where the boundary is at least Lipschitz. Therefore we will also replace the periodic boundary conditions "on the sides" by the Dirichlet condition  $\mathbf{u}_i = 0$  on  $\Gamma_{i,s}$ ,  $1 \leq i \leq 2$ , where  $\Gamma_{i,s}$  denotes the "left" and "right" sides of the boundary of  $\Omega_i$  (see figure 4).

$$(3.1) \quad \begin{cases} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \mathbf{u}_i = 0 & \text{on } \Gamma_{i,s}, \end{cases},$$

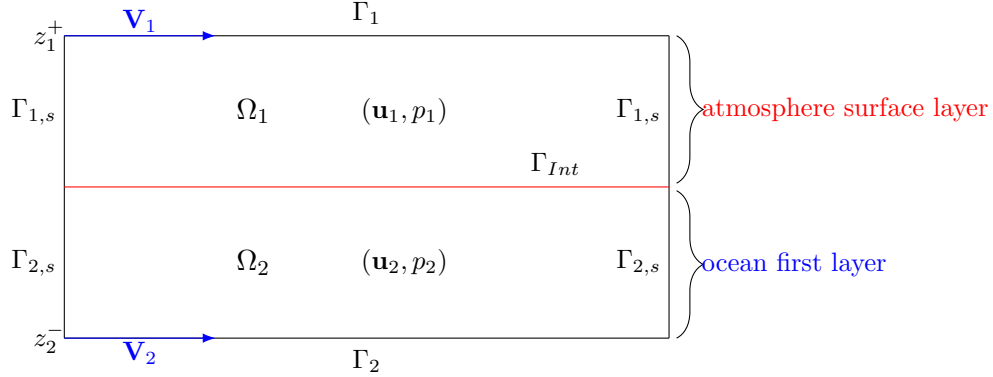


FIGURE 4. Geometry of the domain

We will give the proper functional setting of this problem in the section 3.2 and introduce a finite element Galerkin approximation in section 3.3.

Thanks to the Oseen iterative method which consists in a linearization of the trilinear transport term, we will be able to do some numerical simulations in Freefem++. We establish in section 3.4 the stability and error estimates relatively to the iterative method we have chosen. The plan of what we will do can be summarized in the scheme 5.

**3.2. Functional setting.** The benefit of considering this new problem relies on the Poincaré inequality, which will enable us to change the functional spaces we have introduced in the section 2, and to consider "classical" Sobolev spaces associated with the norm coming from the scalar product.

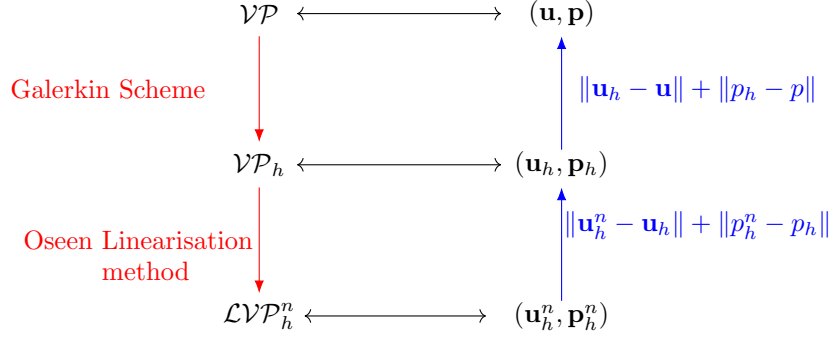


FIGURE 5. Scheme of the error estimates

**Definition 3.1.** We introduce the Sobolev space  $H_{0,s}^1(\Omega_i)^d$  by

$$H_{0,s}^1(\Omega_i)^d := \{\mathbf{u}_i \in H^1(\Omega_i)^d, \text{tr}(\mathbf{u}_i) = 0 \text{ on } \Gamma_{i,s}\}.$$

To simplify the notations, we will denote by  $W_i$  the space  $H_{0,s}^1(\Omega_i)^d$  and we finally introduce the Hilbert product space

$$W = W_1 \times W_2.$$

Thanks to the Poincaré inequality, the spaces  $W_i$  are equipped with the usual scalar product  $\langle \nabla \mathbf{u}_i, \nabla \mathbf{v}_i \rangle$  and with the equivalent norm  $\|\mathbf{u}\|_{W_i} = \|\nabla \mathbf{u}\|_0$ , for  $\mathbf{u}_i, \mathbf{v}_i \in W_i$ .

We denote by  $Y$  the space

$$Y := L^2(\Omega_1)^2 \times L^2(\Omega_2)^2,$$

and by  $X$  the space

$$X := L_0^2(\Omega_1)^2 \times L_0^2(\Omega_2)^2.$$

It remains to take into consideration the free divergence condition. We can introduce two other spaces: we denote by  $V$  the closed subset of  $W$  given by

$$V = \{\mathbf{u} \in W, \nabla \cdot \mathbf{u}_i = 0, \forall 1 \leq i \leq 2\},$$

and we denote by  $H$  the closed subset of  $Y$  defined by

$$H = \{\mathbf{u} \in Y, \nabla \cdot \mathbf{u}_i = 0, \mathbf{u}_i \cdot \mathbf{n}_i = 0 \text{ on } \Gamma_{Int} \cup \Gamma_i, \forall 1 \leq i \leq 2\}.$$

More details on the spaces can be found in [26] [11]. We denote the Stokes operator by  $A = -P\Delta$ , where  $P$  is the  $L^2$ -orthogonal projection of  $Y$  onto  $H$ . Following the notations of the section 2, the variational formulation of problem ... reads: find  $(\mathbf{u}, p) \in W \times X$  such that for all  $(\mathbf{v}, q) \in W \times X$ ,

$$(3.2) \quad a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + d(\mathbf{u}, q) - d(\mathbf{v}, p) + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle.$$

The following existence and uniqueness of solution of (3.2) can be easily adapted from the results of section 2 and from [11], [26].

**Theorem 3.2.** Assume that  $\Omega_1$  and  $\Omega_2$  are smooth enough (if they are convex polygons/polyedra in dimension 2 and 3 for instance). Given  $\mathbf{f}$  in  $W'$ , there exists at least a solution  $(\mathbf{u}, p) \in W \times X$  wich satisfies (3.2) and

$$(3.3) \quad \|\nabla \mathbf{u}\|_0 \leq \frac{c}{\nu} \|\mathbf{f}\|_{-1}, \quad \|\mathbf{f}\|_{-1} = \sup_{\mathbf{v} \in W} \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|_0},$$

where  $\nu = \min(\nu_1, \nu_2)$ . Moreover, if  $\nu$  and  $\mathbf{f}$  satisfy the condition of theorem 2.20, the solution pair is unique.

We have another regularity result we can adapt from He and Wang [14].

**Theorem 3.3.** *Assume that  $\mathbf{f} \in Y$ . The solution pair  $(\mathbf{u}, p)$  of (3.2) satisfies*

$$\nu \|\mathbf{A}\mathbf{u}\|_0 + \|\nabla p\|_0 \leq c \|\mathbf{f}\|_0.$$

**3.3. Finite element Galerkin approximation.** Let  $h > 0$  be a real positive parameter, which will be the length of the mesh triangulation edge. We introduce  $W_h \times X_h$ , the finite element subspace of  $W \times X$  characterized by  $J_h$ , a partitioning of  $\Omega_1 \times \Omega_2$  into triangles. Examples of such finite elements subspaces and partitioning can be found in Girault and Raviart [11] or in Ciarlet [7]. We can also define the subspace  $V_h$  of  $W_h$ , given by

$$(3.4) \quad V_h = \{\mathbf{v}_h \in W_h, d(\mathbf{v}_h, \mathbf{Q}_h) = 0, \forall \mathbf{Q}_h \in X_h\},$$

and the  $L^2$  orthogonal Leray projector  $P_h$  on  $V_h$ .

We assume that the couple  $W_h \times X_h$  satisfies the approximation properties:

- (1) For each  $\mathbf{v} \in D(A)$  and  $\mathbf{Q} \in H^1(\Omega_1) \times H^1(\Omega_2)$ , there exists a velocity approximation  $\pi_h \mathbf{v} \in W_h$  and a pressure approximation  $\rho_h \mathbf{Q} \in X_h$  such that

$$(3.5) \quad \|\nabla(\mathbf{v} - \pi_h \mathbf{v})\|_0 \leq ch \|\mathbf{A}\mathbf{v}\|_0, \quad \|\mathbf{Q} - \rho_h \mathbf{Q}\|_0 \leq ch \|\mathbf{Q}\|_1,$$

- (2) we have the inverse inequality

$$(3.6) \quad \|\nabla \mathbf{v}_h\|_0 \leq ch^{-1} \|\mathbf{v}_h\|_0, \quad \forall \mathbf{v}_h \in W_h$$

- (3) and we have the inf-sup inequality: for each  $\mathbf{Q}_h \in X_h$ , there exists  $\mathbf{v}_h \in W_h$  (different from zero) such that

$$(3.7) \quad d(\mathbf{v}_h, \mathbf{Q}_h) \geq \beta \|\mathbf{Q}_h\|_0 \|\nabla \mathbf{v}_h\|_0,$$

where  $\beta$  is a positive constant depending on  $\Omega$ .

This kind of condition is classical; see [1] for instance.

Examples of spaces  $W_h$  and  $X_h$  such that the approximation hypothesis is verified can be found in Girault Raviart [11] or in Bercovier-Pironneau [2].

Let  $(\mathbf{v}, \mathbf{Q}) \in W \times X$  be given, we will denote by  $(R_h(\mathbf{v}, \mathbf{Q}), \mathbf{Q}_h(\mathbf{v}, \mathbf{Q})) \in W_h \times X_h$  the Stokes projection defined by

$$(3.8) \quad (\nabla R_h, \nabla \mathbf{v}_h) - d(\mathbf{v}_h, \mathbf{Q}_h) + d(R_h, q_h) = (\nabla \mathbf{v}, \nabla \mathbf{v}_h) - d(\mathbf{v}_h, q) + d(\mathbf{v}, q_h),$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ . We have the following inequalities (proved in [26] or [11]):

$$(3.9) \quad \|R_h - \mathbf{v}\|_0 + h \|\nabla(R_h - \mathbf{v})\|_0 \leq ch (\|\nabla \mathbf{v}\|_0 + \|q\|_0),$$

for all  $(\mathbf{v}, q) \in W \times X$  and

$$(3.10) \quad \|R_h - \mathbf{v}\|_0 + h (\|\nabla(R_h - \mathbf{v})\|_0 + \|\mathbf{Q}_h - q\|_0) \leq ch^2 (\|\mathbf{A}\mathbf{v}\|_0 + \|q\|_1),$$

for all  $(\mathbf{v}, q) \in D(A) \times (H^1(\Omega_1) \times H^1(\Omega_2)) \cup X$ . Finally we can define a discrete version of the Stokes operator  $A$  by  $A_h = -P_h \Delta_h$ , given by  $(-\Delta_h \mathbf{u}_h, \mathbf{v}_h) =$

$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)$ , for all  $(\mathbf{u}_h, \mathbf{v}_h) \in W_h^2$ . We make the additional assumption that the projected laplacian operator  $A_h = -P_h \Delta_h$  verifies that

$$(3.11) \quad \langle G(\mathbf{u}_h), A_h \mathbf{u}_h \rangle = 0, \quad \langle H(\mathbf{u}_h), A_h \mathbf{u}_h \rangle = 0$$

We get the gradient control inequalities

$$(3.12) \quad \gamma_0 \|\mathbf{v}_h\|_0^2 \leq \|\nabla \mathbf{v}_h\|_0^2, \quad \gamma_0 \|\nabla \mathbf{v}_h\|_0^2 \leq \|A_h \mathbf{v}_h\|_0^2,$$

for all  $\mathbf{v}_h \in W_h$ .

We will use some inequalities verified by the trilinear form  $b$  ([12]):

**Lemma 3.4.** *The trilinear form  $b$  satisfies the estimates*

$$(3.13) \quad \begin{aligned} & |b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w})| + |b(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w})| + |b(\mathbf{w}, \mathbf{u}_h, \mathbf{v}_h)| \\ & \leq \frac{1}{2} c_0 \|A_h \mathbf{v}_h\|_0^{1/2} \|\nabla \mathbf{v}_h\|_0^{1/2} \|\mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|\mathbf{w}\|_0 \\ & \quad + \frac{1}{2} c_0 \|A_h \mathbf{v}_h\|_0^{1/2} \|\mathbf{v}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|\mathbf{w}\|_0, \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} & |b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w})| + |b(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w})| + |b(\mathbf{w}, \mathbf{u}_h, \mathbf{v}_h)| \\ & \leq c_0 \|\nabla \mathbf{v}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|\mathbf{w}\|_0, \end{aligned}$$

for all  $\mathbf{u}_h, \mathbf{v}_h \in V_h, \mathbf{w} \in W$ , where  $c_0 > 0$  depends only on  $\Omega$ .

The Galerkin approximation corresponding to the problem (3.1) reads as follows: find  $(\mathbf{u}_h, p_h) \in W_h \times X_h$  such that for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ ,

$$(3.15) \quad \begin{aligned} & a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + d(\mathbf{u}_h, q_h) - d(\mathbf{v}_h, p_h) + h(\mathbf{u}_h, \mathbf{v}_h) + g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ & = (\mathbf{f}, \mathbf{v}_h). \end{aligned}$$

We can now prove the following stability and convergence results of the approximation.

**Theorem 3.5.** *Under the assumptions (3.5), (3.6) and (3.7),  $(\mathbf{u}_h, p_h)$  satisfies the stability and error estimates:*

$$(3.16) \quad \|\nabla \mathbf{u}_h\|_0 \leq c_1,$$

$$(3.17) \quad \|A_h \mathbf{u}_h\|_0 \leq \frac{2c}{\nu} \|\mathbf{f}\|_0,$$

$$(3.18) \quad \|\mathbf{u} - \mathbf{u}_h\|_0 + h(c \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0) \leq ch^2,$$

where  $c_1$  depends on the constants  $c_{d,1}, c_{d,2}$ , on the viscosities  $\nu_1$  and  $\nu_2$ , and on the functions  $\|\mathbf{f}\|_{-1}, \|\mathbf{V}_1\|_{0,\Gamma_1}, \|\mathbf{V}_2\|_{0,\Gamma_2}$ .

*Proof.* By taking  $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h) \in W_h \times X_h$  in (3.15), we have the same inequality as in (2.45), which gives  $\|\nabla \mathbf{u}_h\|_0 \leq c_1$ .

Then, we take  $\mathbf{v}_h = A_h \mathbf{u}_h \in W_h$  and  $q_h = 0$  in (3.15) and we use (3.12), (3.13), (3.16), the hypothesis on  $A_h$  (3.11) and the uniqueness condition (2.63) to get:

$$\begin{aligned} \nu \|A_h \mathbf{u}_h\|_0^2 &\leq \frac{1}{4} c_0 \left( \|A_h \mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|\mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|A_h \mathbf{u}_h\|_0 \right. \\ &\quad \left. + \|A_h \mathbf{u}_h\|_0^{1/2} \|\mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0 \|A_h \mathbf{u}_h\|_0 \right) + \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h\|_0 \\ &\leq \frac{1}{2} c_0 \gamma_0^{-1/4} \|A_h \mathbf{u}_h\|_0^{3/2} \|\nabla \mathbf{u}_h\|_0^{3/2} + \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h\|_0, \end{aligned}$$

which gives thanks to the Young inequality and (3.16):

$$\begin{aligned} \nu \|A_h \mathbf{u}_h\|_0 &\leq \frac{\nu}{2} \|A_h \mathbf{u}_h\|_0 + \frac{1}{4} c_0^2 \gamma_0^{-1/2} \|\nabla \mathbf{u}_h\|_0^3 + \|\mathbf{f}\|_0 \\ &\leq \frac{\nu}{2} \|A_h \mathbf{u}_h\|_0 + \frac{1}{4} c_0^2 \gamma_0^{-1/2} c_1^3 + \|\mathbf{f}\|_0 \\ &\leq \frac{\nu}{2} \|A_h \mathbf{u}_h\|_0 + c \|\mathbf{f}\|_0. \end{aligned}$$

It remains to prove the last inequality. We use the Stokes projection in  $(\mathcal{VP})$ , which gives

$$\begin{aligned} &\nu_1 (\nabla R_h^1, \nabla \mathbf{v}_h)_{\Omega_1} + \nu_2 (\nabla R_h^2, \nabla \mathbf{v}_h)_{\Omega_2} - d(\mathbf{v}_h, Q_h^1)_{\Omega_1} - d(\mathbf{v}_h, Q_h^2)_{\Omega_2} \\ (3.19) \quad &+ d(\nu_1 R_h^1, q_h)_{\Omega_1} + d(\nu_2 R_h^2, q_h)_{\Omega_2} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) + \langle G(\mathbf{u}), \mathbf{v}_h \rangle + \langle H(\mathbf{u}), \mathbf{v}_h \rangle \\ &= (\mathbf{f}, \mathbf{v}_h), \end{aligned}$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ , where  $(R_h^i, Q_h^i) = \left( R_h^i(\nu_i \mathbf{u}, p), Q_h^i(\nu_i \mathbf{u}, p) \right)$ ,  $i \in \{1, 2\}$ , and  $(R_h, Q_h) = \left( (R_h^1, R_h^2), (Q_h^1, Q_h^2) \right)$ . Subtracting (3.15) from (3.19) and setting  $(\mathbf{e}_h, \eta_h) = (R_h - \mathbf{u}_h, Q_h - p_h)$ , we get

$$\begin{aligned} (3.20) \quad &a(\mathbf{e}_h, \mathbf{v}_h) - d(\mathbf{v}_h, \eta_h) + d(\nu_1 \mathbf{e}_h, q_h)_{\Omega_1} + d(\nu_2 \mathbf{e}_h, q_h)_{\Omega_2} + b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}, \mathbf{v}_h) \\ &+ b(\mathbf{u}_h, \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{v}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{v}_h \rangle = 0 \end{aligned}$$

We take  $\mathbf{v}_h = \mathbf{e}_h$  and  $q_h = (\nu_1^{-1} \eta_h^1, \nu_2^{-1} \eta_h^2)$  in (3.20) which gives, by calculations and property of the trilinear form  $b$ :

$$\begin{aligned} (3.21) \quad &a(\mathbf{e}_h, \mathbf{e}_h) + b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h) \\ &+ \langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle = 0. \end{aligned}$$

We will use the following estimates (more details can be found in [12]):

$$\begin{aligned} (3.22) \quad &\|\mathbf{e}_h\|_{L^6} \leq c \|\nabla \mathbf{e}_h\|_0, \\ &\|\nabla \mathbf{u}_h\|_{L^3} + \|\mathbf{u}_h\|_{L^\infty} \leq c \|A_h \mathbf{u}_h\|_0, \\ &\|\nabla \mathbf{u}\|_{L^3} + \|\mathbf{u}\|_{L^\infty} \leq c \|A_h \mathbf{u}\|_0. \end{aligned}$$

- The three different transport terms satisfy the inequalities:

$$\begin{aligned} |b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h)| &\leq C_b \|\nabla \mathbf{u}\|_0 \|\nabla \mathbf{e}_h\|_0^2 \\ &\leq C_b c_1 \|\nabla \mathbf{e}_h\|_0^2, \end{aligned}$$

thanks to (3.16).



Besides,

$$\begin{aligned} |b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h)| &\leq C_b \left( \|\nabla \mathbf{u}\|_{L^3} \|\mathbf{e}_h\|_{L^6} + \|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{e}_h\|_0 \right) \|\mathbf{u} - R_h\|_0 \\ &\leq C_b c \|A\mathbf{u}\|_0 \|\nabla \mathbf{e}_h\|_0 \|\mathbf{u} - R_h\|_0 \\ &\leq ch^2 \|\nabla \mathbf{e}_h\|_0, \end{aligned}$$

according to the Stokes projection result (3.10). Last but not least:

$$\begin{aligned} |b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h)| &\leq C_b \left( \|\nabla \mathbf{u}_h\|_{L^3} \|\mathbf{e}_h\|_{L^6} + \|\mathbf{u}_h\|_{L^\infty} \|\nabla \mathbf{e}_h\|_0 \right) \|\mathbf{u} - R_h\|_0 \\ &\leq C_b c \|A_h \mathbf{u}_h\|_0 \|\nabla \mathbf{e}_h\|_0 \|\mathbf{u} - R_h\|_0 \\ &\leq ch^2 \|\nabla \mathbf{e}_h\|_0, \end{aligned}$$

thanks again to (3.10) and to (3.17).

- The  $G$ -term verifies

$$\begin{aligned} |\langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle| &\leq C \left( \|\nabla \mathbf{u}\|_0^2 + \|\nabla \mathbf{u}_h\|_0^2 \right) \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 \|\nabla \mathbf{e}_h\|_0 \\ &\leq 2Cc_1^2 \|\nabla \mathbf{e}_h\|^2 + 2Cc_1^2 \|\nabla(\mathbf{u} - R_h)\|_0 \|\nabla \mathbf{e}_h\|_0 \\ &\leq 2Cc_1^2 \|\nabla \mathbf{e}_h\|^2 + ch \|\nabla \mathbf{e}_h\|_0, \end{aligned}$$

still thanks to (3.10).

- Finally, the  $H$ -term satisfies

$$\begin{aligned} |\langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle| &\leq c \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 \|\nabla \mathbf{e}_h\|_0 \\ &\leq c \|\nabla \mathbf{e}_h\|_0^2 \|\nabla(\mathbf{u} - R_h)\|_0 \|\nabla \mathbf{e}_h\|_0 + c \|\nabla \mathbf{e}_h\|_0^2 \\ &\leq ch \|\nabla \mathbf{e}_h\|_0 + c \|\nabla \mathbf{e}_h\|_0^2. \end{aligned}$$

Putting these five previous inequalities in (3.21), we obtain

$$(\nu - C_b c_1 - 2Cc_1^2 - C) \|\nabla \mathbf{e}_h\|_0 \leq ch + 2ch^2.$$

i.e.,

$$(3.23) \quad \|\nabla \mathbf{e}_h\|_0 \leq ch + O(h^2).$$

Getting back to the variational formulation (3.20) and taking  $\mathbf{v}_h = \mathbf{e}_h, q_h = 0$ , we obtain

$$(3.24) \quad \begin{aligned} &a(\mathbf{e}_h, \mathbf{e}_h) - d(\mathbf{e}_h, \eta_h) + b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h) \\ &+ \langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle = 0. \end{aligned}$$

Using the inf-sup condition (3.7) in (3.24), we get

$$\begin{aligned} \beta \|\eta_h\|_0 \|\nabla \mathbf{e}_h\|_0 &\leq \nu \|\nabla \mathbf{e}_h\|_0^2 + |b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h)| \\ &\quad + |\langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle|, \end{aligned}$$

Hence from the previous inequalities:

$$(3.25) \quad \beta \|\eta_h\|_0 \|\nabla \mathbf{e}_h\|_0 \leq c \|\nabla \mathbf{e}_h\|_0^2 + ch \|\nabla \mathbf{e}_h\|_0 + ch^2 \|\nabla \mathbf{e}_h\|_0^2.$$

We can finally prove the last inequality. Thanks to the triangular inequality:

$$(3.26) \quad \begin{aligned} \nu \|\mathbf{u} - \mathbf{u}_h\|_0 + h(\nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0) &\leq \nu \|\mathbf{e}_h\|_0 + h(\nu \|\nabla(\mathbf{e}_h)\|_0 + \|\eta_h\|_0) \\ &+ \nu \|\mathbf{u} - R_h\|_0 + h(\nu \|\nabla(\mathbf{u} - R_h)\|_0 + \|p - Q_h\|_0), \end{aligned}$$

where

$$\nu \|\mathbf{e}_h\|_0 + h(\nu \|\nabla(\mathbf{e}_h)\|_0 + \|\eta_h\|_0) \leq ch + O(h^2)$$

comes from the inequalities (3.23) and (3.25).

The other term verifies (3.10), thus

$$\nu \|\mathbf{u} - R_h\|_0 + h(\nu \|\nabla(\mathbf{u} - R_h)\|_0 + \|p - Q_h\|_0) \leq ch^2.$$

Putting these two estimates in (3.26), we finally obtain

$$\nu \|\mathbf{u} - \mathbf{u}_h\|_0 + h(\nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0) \leq ch + O(h^2).$$

□

**Remark 3.6.** *We only have a first order estimate in  $h$ , compared to second order in "more classic" Navier Stokes problems (see for instance [16],[13]).*

We can add another control inequality

**Theorem 3.7.**

$$(3.27) \quad \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0 \leq ch + O(h^2)$$

which comes straightforwardly from the previous inequations.

**3.4. Stability of the method and error analysis.** We define the linearized numerical scheme corresponding to the problem (3.1) with:

$$(3.28) \quad \left\{ \begin{array}{ll} (\mathbf{u}_i^n \cdot \nabla) \mathbf{u}_i^{n-1} - \nu_i \Delta \mathbf{u}_i^n + \nabla p_i^n = \mathbf{f}_i & \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i^n = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}^n}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n) |\mathbf{u}_{i,h}^{n-1} - \mathbf{u}_{j,h}^{n-1}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}^n}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h}^n - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i^n \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \mathbf{u}_i^n = 0 & \text{on } \Gamma_{i,s}, \end{array} \right. ,$$

for  $n \geq 1$ .

The weak formulation associated with (3.28) can be seen as an Oseen iterative method (see [13]):

We start from the couple  $(\mathbf{u}_h^0, p_h^0) \in W_h \times X_h$  defined by the discrete Stokes problem on  $\Omega = \Omega_1 \times \Omega_2$

$$(3.29) \quad a(\mathbf{u}_h^0, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^0) + d(\mathbf{u}_h^0, q_h) = (\mathbf{f}, \mathbf{v}_h),$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ .

**Remark 3.8.** *The two Stokes problems for the two different fluids can be solved separately on  $\Omega_1$  and  $\Omega_2$  since there is no boundary condition involving the two fluids at this point.*

The iterative method at order  $n$  reads:

$$(3.30) \quad \begin{aligned} a(\mathbf{u}_h^n, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^n) + d(\mathbf{u}_h^n, q_h) + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) + \langle \tilde{G}(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}), \mathbf{v}_h \rangle \\ + \langle H(\mathbf{u}_h^n), \mathbf{v}_h \rangle = (\mathbf{f}, \mathbf{v}_h), \end{aligned}$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ , where the map  $\tilde{G}$  is the discrete version of  $G$  and verifies

$$(3.31) \quad \langle \tilde{G}(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}), \mathbf{v}_h \rangle = \int_{\Gamma_I} (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n) |\mathbf{u}_{1,h}^{n-1} - \mathbf{u}_{2,h}^{n-1}| (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}).$$

This leads to the the stability result:

**Theorem 3.9.** *The iterative method is unconditionally stable, meaning that the couple  $(\mathbf{u}_h^n, p_h^n)$  defined by the scheme (3.30) satisfies*

$$(3.32) \quad \nu \|\nabla \mathbf{u}_h^n\|_0 \leq \|\mathbf{f}\|_{-1},$$

and

$$(3.33) \quad \|A_h \mathbf{u}_h^n\|_0 \leq c_2,$$

for all  $n \geq 0$ , where  $c_2$  depends on  $\nu_1, \nu_2, \|\mathbf{f}\|_0$  (and on  $\Omega$  through different Sobolev embedding theorems).

*Proof.* The result is verified for  $n = 0$ , by definition of  $\mathbf{u}_h^0$  as a solution of a standard Stokes problem. We assume that the first inequality is true at the step  $n$  and we will prove it for the step  $n + 1$ . Taking  $\mathbf{v}_h = \mathbf{u}_h^{n+1}$ ,  $q_h = p_h^{n+1} \in W_h \times X_h$  in (3.30) at the step  $n + 1$ , we get

$$a(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + \langle \tilde{G}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^n), \mathbf{u}_h^{n+1} \rangle + \langle H(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1} \rangle = (\mathbf{f}, \mathbf{u}_h^{n+1}).$$

Besides, both  $\tilde{G}$  and  $H$ -terms satisfy

$$\langle \tilde{G}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^n), \mathbf{u}_h^{n+1} \rangle \geq 0,$$

and

$$\langle H(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1} \rangle \geq 0.$$

Hence, we get the estimate

$$\nu \|\nabla \mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{f}\|_{-1} \|\nabla \mathbf{u}_h^{n+1}\|_0,$$

and the first claim is proved.

The second claim will be proved without iteration argument. Taking  $\mathbf{v}_h = A_h \mathbf{u}_h^{n+1}$  and  $q_h = 0$  in (3.30) at the step  $n + 1$ , recalling that  $A_h$  satisfy the inequality (3.17) and cancel the  $G$  and  $H$ -terms, we get by Cauchy-Schwarz and transport trilinear term (3.13) inequalities:

$$\nu \|A_h \mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h^{n+1}\|_0 + \frac{1}{2} c_0 \gamma_0^{-1/4} \|\nabla \mathbf{u}_h^n\|_0 \|\nabla \mathbf{u}_h^{n+1}\|_0^{1/2} \|A_h \mathbf{u}_h^{n+1}\|_0^{3/2},$$

and from the Young inequality:

$$\nu \|A_h \mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h^{n+1}\|_0 + \frac{1}{4\nu} c_0^2 \gamma_0^{-1/2} \|\nabla \mathbf{u}_h^n\|_0^2 \|\nabla \mathbf{u}_h^{n+1}\|_0 \|A_h \mathbf{u}_h^{n+1}\|_0 + \frac{\nu}{2} \|A_h \mathbf{u}_h^{n+1}\|_0^2.$$

Thanks to (3.32), we obtain

$$\frac{\nu}{2} \|A_h \mathbf{u}_h^{n+1}\|_0 \leq \|\mathbf{f}\|_0 + \frac{1}{4\nu^4} c_0^2 \gamma_0^{-1/2} \|\mathbf{f}\|_{-1}^3,$$

for all  $n \geq 0$ , which gives the stability estimate (3.33) and concludes the proof.  $\square$

A corollary from the stability theorem 3.9 and the *a priori estimates* from the theoretical section is the boundedness result:

**Lemma 3.10.** *There exists  $C > 0$  depending only on the data such that*

$$(3.34) \quad \|\mathbf{u}_h\|_{W_h} \leq C \text{ and } \|\mathbf{u}_h^n\|_{W_h} \leq C,$$

for all  $n \geq 0$ .

We define the errors between the Galerkin theoretical velocities and the Galerkin approximated velocities obtained from the Oseen scheme  $\mathbf{e}_h^n = \mathbf{u}_h^n - \mathbf{u}_h$  and the pressure errors  $\eta_h^n = p_h^n - p_h$ .

This leads to the error convergence result:

**Theorem 3.11.** *Provided the condition on the data*

$$(3.35) \quad \min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D \geq 0 \text{ and } \frac{C(C_b + C_D)}{\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D} < 1,$$

the method is converging and we have the error estimate:

$$\|\mathbf{e}_h^n\|_{W_h} \leq \frac{2}{\nu} \left( \frac{C(C_b + C_D)}{\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D} \right)^{n+1} \|\mathbf{f}\|_{-1}.$$

*Proof.* We consider the difference at the  $n$ -th step, between the equations satisfied by  $(\mathbf{u}_h, p_h)$  and by  $(\mathbf{u}_h^n, p_h^n)$ , which gives the relation

(3.36)

$$(3.37) \quad a(\mathbf{e}_h^n, \mathbf{v}_h) + b(\mathbf{e}_h^{n-1}, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h^{n-1}, \mathbf{e}_h^n, \mathbf{v}_h) + g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + h(\mathbf{e}_h^n, \mathbf{v}_h) - d(\mathbf{v}_h, \eta_h^n) + d(\mathbf{e}_h^n, q_h) = 0,$$

for  $n \geq 1$  and  $(\mathbf{v}_h, q_h) \in W_h \times Q_h$ .

Consider the difference  $g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h)$ , we have

$$\begin{aligned} & g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ &= C_D \int_{\Gamma_I} \left( (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n) |\mathbf{u}_{1,h}^{n-1} - \mathbf{u}_{2,h}^{n-1}| - (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) |\mathbf{u}_{1,h} - \mathbf{u}_{2,h}| \right) (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}), \end{aligned}$$

and denoting  $\mathbf{U}_h^n := \mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n$  and  $\mathbf{U}_h := \mathbf{u}_{1,h} - \mathbf{u}_{2,h}$ ,  $\mathbf{V}_h = \mathbf{v}_{1,h} - \mathbf{v}_{2,h}$ , we get

$$\begin{aligned} g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) &= C_D \int_{\Gamma_I} (\mathbf{U}_h^n |\mathbf{U}_h^{n-1}| - \mathbf{U}_h |\mathbf{U}_h|) \mathbf{V}_h \\ &= C_D \int_{\Gamma_I} (\mathbf{U}_h^n |\mathbf{U}_h^{n-1}| - \mathbf{U}_h^n |\mathbf{U}_h| + \mathbf{U}_h^n |\mathbf{U}_h| - \mathbf{U}_h |\mathbf{U}_h|) \mathbf{V}_h \\ &= C_D \int_{\Gamma_I} (\mathbf{U}_h^n (|\mathbf{U}_h^{n-1}| - |\mathbf{U}_h|) + |\mathbf{U}_h| (\mathbf{U}_h^n - \mathbf{U}_h)) \mathbf{V}_h. \end{aligned}$$

We get by the triangular inequality

$$\begin{aligned} \|g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h)\|_{W_h} &\leq C_D (\|\mathbf{U}_h^n\|_{W_h} \|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{U}_h\|_{W_h} \|\mathbf{e}_h^n\|_{W_h}) \|\mathbf{V}_h\|_{W_h} \\ &= C_D (\|\mathbf{u}_h^n\|_{W_h} \|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{u}_h\|_{W_h} \|\mathbf{e}_h^n\|_{W_h}) \|\mathbf{v}_h\|_{W_h} \\ &\leq CC_D (\|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{e}_h^n\|_{W_h}) \|\mathbf{v}_h\|_{W_h}, \end{aligned}$$

using the stability result 3.10.

Choosing  $\mathbf{v}_h = \mathbf{e}_h^n$  and  $q_h = \eta_h^n$  in (3.36), we get

$$a(\mathbf{e}_h^n, \mathbf{e}_h^n) + b(\mathbf{e}_h^{n-1}, \mathbf{u}_h, \mathbf{e}_h^n) + b(\mathbf{u}_h^{n-1}, \mathbf{e}_h^n, \mathbf{e}_h^n) + g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{e}_h^n) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}_h^n) + h(\mathbf{e}_h^n, \mathbf{e}_h^n) = 0.$$

Using the properties of the different operations, this yields to

$$\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) \|\mathbf{e}_h^n\|_{W_h}^2 \leq C_b \|\mathbf{u}_h\|_{W_h} \|\mathbf{e}_h^n\|_{W_h} \|\mathbf{e}_h^{n-1}\|_{W_h} + CC_D (\|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{e}_h^n\|_{W_h}) \|\mathbf{e}_h\|_{W_h},$$

hence

$$(\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D) \|\mathbf{e}_h^n\|_{W_h} \leq (CC_b + CC_D) \|\mathbf{e}_h^{n-1}\|_{W_h}.$$

Using again the stability result 3.10, we conclude thanks to the recursivity principle.  $\square$

**Remark 3.12.** *The major issue of this convergence result is the fact that it is difficult to have access to the constants  $C$  and  $C_b$  involved. Nevertheless, choosing small values of  $C_D$  relatively to the range of values of  $\nu_1, \nu_2, c_{d,1}$  and  $c_{d,2}$  guarantees both conditions. We will see in the numerical simulations an example of this criteria being not fulfilled.*

**3.5. Numerical simulations.** Finally, we are numerically simulating with Freefem++ the problem (3.28) thanks to the Oseen linearisation method.

We consider  $P2$  finite elements for the velocities and  $P1$  finite elements for the pressure. The figure 6 shows the mesh we have used in Freefem++.

We also decide to consider different viscosities:  $\nu_1 = 0.4$  and  $\nu_2 = 0.08$  and horizontal periodic boundary conditions, where  $\Omega_1 = 1 \times 0.6$  and  $\Omega_2 = 1 \times 0.4$ . The convergence criteria we consider is

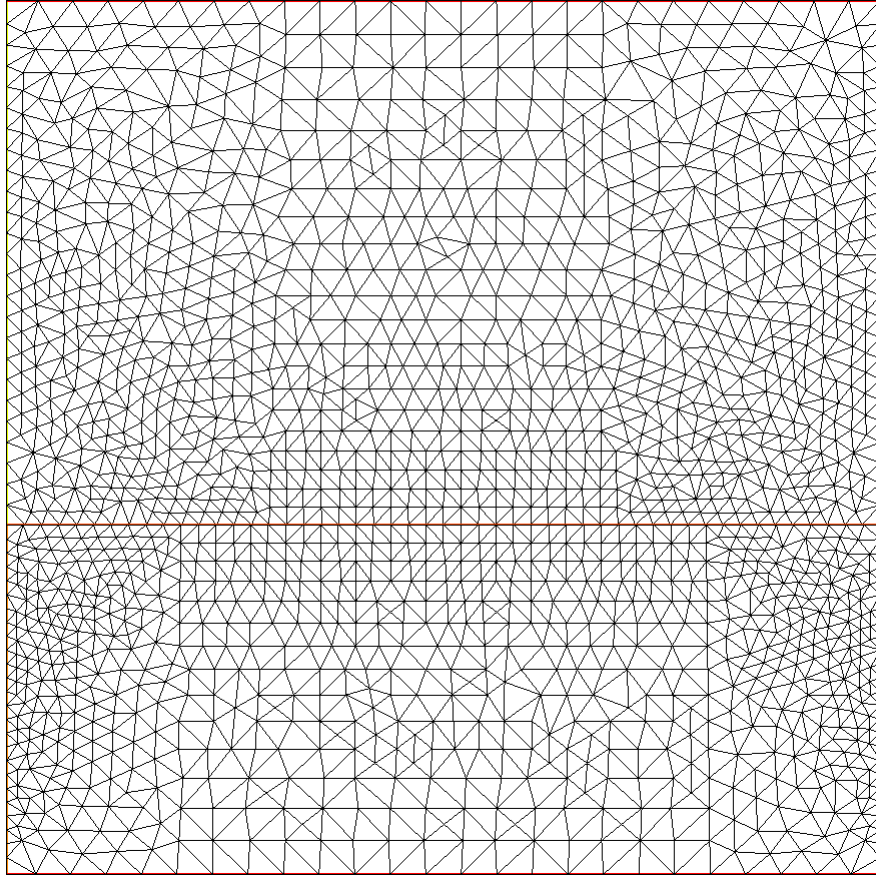
$$(3.38) \quad \frac{\|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{H^1}}{\|\mathbf{u}^n\|_{H^1}} < \varepsilon_{tol},$$

where the tolerance error  $\varepsilon_{tol} = 10^{-3}$ . Note that the stopping criteria is based on the relative error. The source  $\mathbf{f}$  has been chosen constant equal to 1 everywhere in the domains and the constants  $c_{d,1}$  and  $c_{d,2}$  equal to 1. The velocity  $\mathbf{V}$  has been chosen such that the air velocity at the top  $\mathbf{V}_1 = 1$  and the water velocity at the bottom  $\mathbf{V} = 0.1$ .

The table 1 shows the number of iterations until the stopping criteria (3.38) is fulfilled, for different values of  $C_D$ .

We have proved that there is convergence if the condition (3.35) is fulfilled; we don't know what is happening when it is not fulfilled.

In this configuration, we have seen that there was quite a fast convergence when  $C_D < 2$ , but when  $C_D \rightarrow 2.1$ , the simulation is "exploding" and the numerical results are not what we expect. The figure 7 shows the simulation of the flows after 144 iterations for  $C_D = 1$  and the figure 8, the case  $C_D = 2.5$  after only 3 iterations.

FIGURE 6. Mesh for the domain  $\Omega = \Omega_1 \times \Omega_2$ 

Value of $C_D$	First value of $n$	$\ \mathbf{u}_1^n\ _{L^2}$	$\ \mathbf{u}_2^n\ _{L^2}$	$\ \mathbf{u}_1^n\ _{H^1}$	$\ \mathbf{u}_2^n\ _{H^1}$
0.1	160	5.48849	16.1656	4.42758	15.9186
0.2	152	5.48479	16.1702	4.38152	16.0235
0.5	146	5.47764	16.1937	4.32994	16.3304
1	144	5.46818	16.2515	4.29199	16.7047
2	461	5.45242	16.4263	4.25692	17.2604
2.01	558	5.45227	16.4292	4.2564	17.2688
2.02	722	5.45214	16.4313	4.25593	17.2738
2.03	1325	5.45208	16.4322	4.25607	17.2753
2.05	8250	5.45192	16.4341	4.25618	17.2771
2.1	$+\infty$				

TABLE 1. Convergence rate for different values of  $C_D$ 

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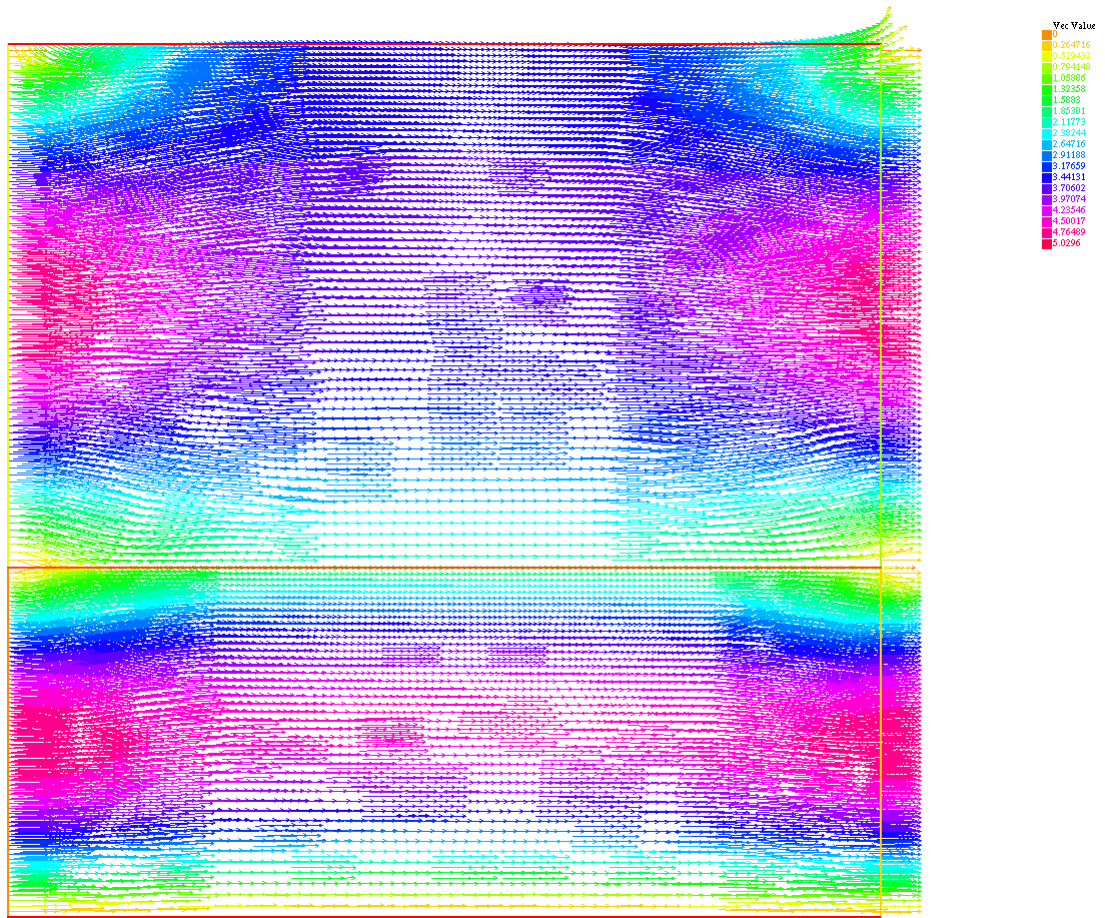


FIGURE 7. Two fluids with  $\nu_1 = 0.4$ ,  $\nu_2 = 0.08$ ,  $C_D = 1$ ,  $c_{d,1} = c_{d,2} = 1$

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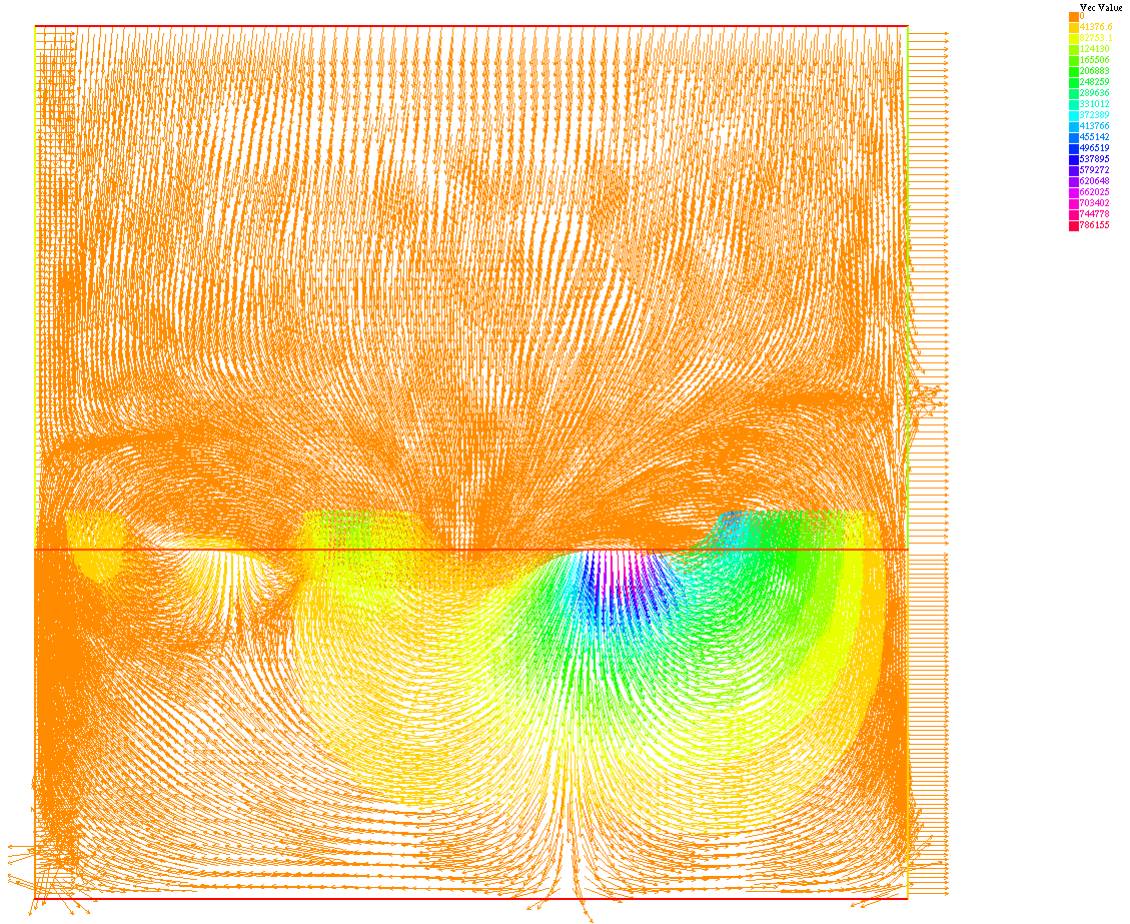


FIGURE 8. Two fluids with  $\nu_1 = 0.4$ ,  $\nu_2 = 0.08$ ,  $C_D = 2.5$ ,  $c_{d,1} = c_{d,2} = 1$

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