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# Stochastic optimal control of homogenous systems

Ying Hu \*

Xiaomin Shi<sup>†</sup>

Zuo Quan Xu<sup>‡</sup>

April 18, 2024

## Abstract

This paper studies a new class of homogenous stochastic control problems subject to cone control constraints. It extends the classical homogenous stochastic linear-quadratic (LQ) problems to problems with nonlinear dynamics and non-quadratic cost functionals. Similar to LQ problems, the optimal controls and optimal values of the new class problems are closely related to some backward stochastic differential equations (BSDEs) which are highly non-linear and new in the literatures. We prove the existence and uniqueness of the solutions to these BSDEs under three different conditions using tools including truncation function technique, log transformation etc. Finally, we provide feedback optimal controls and optimal values in terms of the solutions to the aforementioned BSDEs by verification argument. Because our solvability conditions are fairly general, many existing results in homogenous LQ problems in both standard and singular cases are recovered by our results.

**Keywords:** Homogenous system; stochastic LQ problems; cone constraint; BSDE; existence and uniqueness; verification theorem.

## 1 Introduction

Consider a controlled stochastic differential equation (SDE):

$$dX_t = b_t(X_t, u_t) dt + \sigma_t(X_t, u_t)^\top dW_t, \quad X_0 = x, \quad (1.1)$$

and a related cost functional:

$$J(X, u) := \mathbb{E} \left[ \int_0^T f_t(X_t, u_t) dt + g(X_T) \right], \quad (1.2)$$

where  $u$  is the control variable,  $X$  is the corresponding state variable, and  $b, \sigma, f, g$  are given deterministic or stochastic maps. The stochastic control problem associated with (1.1) and (1.2)

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\*Univ Rennes, CNRS, IRMAR-UMR 6625, F-35000 Rennes, France. Partially supported by Lebesgue Center of Mathematics “Investissements d’avenir” program-ANR-11-LABX-0020-01. Email: [ying.hu@univ-rennes1.fr](mailto:ying.hu@univ-rennes1.fr)

<sup>†</sup>School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan 250100, China. Partially supported by NSFC (No. 11801315). Email: [shixm@mail.sdu.edu.cn](mailto:shixm@mail.sdu.edu.cn)

<sup>‡</sup>Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. Partially supported by NSFC (No. 11971409), The Hong Kong RGC (GRF 15204622 and 15203423), the PolyU-SDU Joint Research Center on Financial Mathematics, the CAS AMSS-PolyU Joint Laboratory of Applied Mathematics, the Research Centre for Quantitative Finance (1-CE03), and internal grants from The Hong Kong Polytechnic University. Email: [maxu@polyu.edu.hk](mailto:maxu@polyu.edu.hk)

is to minimize over control processes the cost functional  $J$ , and we introduce the associated value function:

$$V(x) := \inf_{u \in \mathcal{U}} J(X, u). \quad (1.3)$$

More precise formulation will be given in the next section.

A vast number of papers deal with the above stochastic control problem (1.3) subject to various conditions. Please refer the seminal books Yong and Zhou [13] and Pham [11] for a systematic account on stochastic optimal control. There are two well-known approaches to tackle the problem nowadays. The first one is called the dynamic programming. It leads to the study of the so-called Hamilton-Jacobi-Bellman equations when all the maps  $b, \sigma, f, g$  are deterministic. The other approach is called the (stochastic) maximum principle, which provides a necessary condition for optimal control; see, e.g., Peng [10], Hu, Ji and Xue [4]. It leads to the study of a system of forward and backward stochastic differential equations (FBSDEs) when some of the maps  $b, \sigma, f, g$  are stochastic. Generally, such FBSDEs are fully coupled so that they cannot be further reduced to simple forms.

But when the problem (1.3) is a linear-quadratic (LQ) problem, where the controlled dynamics is in linear form (i.e.  $b$  and  $\sigma$  are linear functions of  $x$  and  $u$ ) and the cost functional  $J$  in quadratic form (i.e.  $f$  and  $g$  are quadratic functions of  $x$  and  $u$ ), its FBSDEs system can reduce to a system of backward stochastic differential equations (BSDEs) only, which is called the stochastic Riccati equations. This will significantly reduce the difficulty of solving the problem since BSDE theory is well-established; see, e.g., [2, 12, 14].

Inspired by LQ problems, this paper proposes a new class of homogeneous stochastic control problems, which extend the classical homogeneous LQ problems to new ones with nonlinear controlled dynamics and non-quadratic cost functionals, but their FBSDEs can still reduce to certain BSDEs which are similar to Riccati equations for LQ problems.

To introduce our new class of problem, we first define homogeneous functions as follows.

**Definition 1.1 (Homogeneous function).** *Given a real constant  $p \neq 0$  and a closed cone  $\mathcal{Y} \subset \mathbb{R}^k$ , a function  $\varphi : \mathcal{Y} \rightarrow \mathbb{R}^d$  is called a homogeneous function of degree  $p$  in  $y \in \mathcal{Y}$  if*

$$\varphi(\lambda y) = \lambda^p \varphi(y)$$

holds for any scalar  $\lambda > 0$  and  $y \in \mathcal{Y}$ .

**Example 1.1.** *The following functions, possibly random, are homogeneous of degree  $p > 0$  in  $(x, u) \in \mathbb{R}^k \times \mathbb{R}^m$ :*

$$\begin{aligned} & |Ax + Bu|^p, \quad |Ax + B|u||^p, \quad |x|^p + |u|^p, \quad (|x|^2 + |u|^2)^{p/2}, \\ & A(|x||u|)^{p/2} + B|x|^{p/3}|u|^{2p/3}, \quad |\underline{A}x^+ + \overline{A}x^- + \underline{B}^\top u^+ - \overline{B}^\top u^-|^p, \\ & \inf_{(A,B)} |Ax + B^\top u|^p, \quad \sup_{(A,B)} |Ax + B^\top u|^p. \end{aligned}$$

As usual,  $x^+$  is the unique vector in  $\mathbb{R}_+^k$  determined by  $x^+ = \operatorname{argmin}_{y \in \mathbb{R}_+^k} |y - x|$  for  $x \in \mathbb{R}^k$ .  $x^- := (-x)^+$ . We remark that any homogeneous function in  $(x, u)$  multiplied by  $F\left(\frac{u}{|x|} \mathbf{1}_{x \neq 0}\right)$  is still a homogeneous function of the same degree, where  $F$  is any function.

In this paper, we only focus on scalar-valued state processes, namely  $X_t \in \mathbb{R}$  in the process (1.1) and the control constraint will belong to a closed cone  $\Gamma \subseteq \mathbb{R}^m$ . Every problem in the new class satisfies the following homogeneous assumption.

**Assumption 1.1 (Homogeneous system).** *For all  $t \in [0, T]$ , the dynamic coefficients  $b_t$  and  $\sigma_t$  are homogeneous functions of degree 1 in  $(x, u) \in \mathbb{R} \times \Gamma$ ; and the cost functional coefficients  $f_t$  and  $g$  are homogeneous functions of degree  $p \neq 0$  in  $(x, u) \in \mathbb{R} \times \Gamma$  and  $x \in \mathbb{R}$  respectively.*

By setting  $\lambda = 2$  and  $y = 0$  in Definition 1.1, we see  $\varphi(0) = 0$  if  $\varphi$  is homogeneous of degree  $p \neq 0$ . Therefore, all the coefficients  $b_t$ ,  $\sigma_t$ ,  $f_t$  and  $g$  are zero at the origin of coordinates under Assumption 1.1. We will use this fact without claim in the future.

**Definition 1.2 (Homogeneous stochastic control problem).** *If the coefficients in the problem (1.3) satisfy Assumption 1.1, it is called a homogeneous (stochastic control) problem of degree  $p \neq 0$ .*

Clearly, the classical homogeneous LQ control problems are special homogeneous problems of degree 2, therefore the new class of homogeneous problems, which allows non-linear state processes and non-quadratic cost functionals, is an extension of the homogeneous LQ problems.

This paper focuses on the solvability of the homogeneous stochastic control problems under three different conditions. Although this paper only studies the case  $p > 0$ , many results can be parallel established for the case  $p < 0$  without difficulties. We encourage the interested readers to give the details.

The idea to solve homogeneous stochastic control problems stems from Hu and Zhou [6] and Ji, Jin and Shi [7]. It is well known that for LQ stochastic control problems, the optimal controls take state feedback form through the famous stochastic Riccati equations, see, e.g., Tang [12]. By performing Tanaka's formula to  $X^+$  and  $X^-$  respectively, it is proved in [6] that the cone-constrained stochastic LQ problem admits a piecewise linear feedback optimal control expressed in terms of the solutions of two BSDEs. The above method (combined with some extra convex duality analysis) still works for mean-variance portfolio selection problems with a specific kind of non-linear wealth dynamic studied in [7]. In the models of [6, 7], the optimal (translated) state processes will never cross 0, i.e. it will remain positive (resp. negative) if the initial state is positive (resp. negative). This phenomenon actually results from the homogeneity of the control system. Therefore, our new homogeneous stochastic control problems will inherit this property.

In this paper, we extend the models of [6, 7] to a new class of homogeneous stochastic control problems with possibly non-linear state processes and non-quadratic cost functionals. We will show that the problem (1.3) admits an optimal control, in a state feedback form via two associated BSDEs. We firstly prove the existence and uniqueness of solutions to the two associated highly non-linear BSDEs based on truncation function technique, log transformation and some delicate analysis. The famous result of Kobylanski [9] on quadratic growth BSDEs plays a fundamental tool in our argument. Finally, the solutions of the associated BSDEs are used to construct a candidate control which is verified to be admissible, hence optimal to the original homogeneous stochastic control problem (1.3). We remark that the uniqueness of

solutions to the BSDEs in [6] was obtained as a by product of the verification theorem. Here we will provide a direct proof, based on the bounded mean oscillation (BMO) martingale theory, which is interesting in its own from the point of view of BSDE theory.

The rest part of this paper is organized as follows. In Section 2, we present some notations and the precise assumptions on the homogeneous stochastic control problem (1.3). Section 3 is devoted to establishing the solubility of two associated BSDEs under three different conditions. In Section 4, we determine the optimal controls and optimal values to the homogeneous stochastic control problems via the solutions to the two BSDEs through a straightforward and rigorous verification theorem. Some concluding remarks are given in Section 5.

## 2 Preliminaries

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a fixed complete probability space on which are defined a standard  $n$ -dimensional Brownian motion  $\{W_t\}_{t \geq 0}$ . Let  $\{\mathcal{F}_t\}_{t \geq 0}$  be the natural filtration of the Brownian motion  $W$  augmented by the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ . Let  $\mathcal{P}$  be the predictable  $\sigma$ -field of  $[0, T] \times \Omega$ . We stipulate that, in what follows, all the processes under consideration, unless otherwise stated, are  $\mathcal{P}$  measurable.

We denote by  $\mathbb{R}^m$  the set of  $m$ -dimensional column vectors, by  $\mathbb{R}_+^m$  the set of vectors in  $\mathbb{R}^m$  whose components are nonnegative, by  $\mathbb{R}^{m \times n}$  the set of  $m \times n$  real matrices, and by  $\mathbb{S}^n$  the set of symmetric  $n \times n$  real matrices. Therefore,  $\mathbb{R}^m \equiv \mathbb{R}^{m \times 1}$ . For any matrix  $M = (m_{ij}) \in \mathbb{R}^{m \times n}$ , we denote its transpose by  $M^\top$ , and its norm by  $|M| = \sqrt{\sum_{ij} m_{ij}^2}$ . If  $M \in \mathbb{S}^n$  is positive definite (resp. positive semidefinite), we write  $M >$  (reps.  $\geq$ )  $0$ . We write  $A >$  (resp.  $\geq$ )  $B$  if  $A, B \in \mathbb{S}^n$  and  $A - B >$  (resp.  $\geq$ )  $0$ .

We fix a constant  $T > 0$  throughout the paper to denote the control horizon. We use the following notations throughout the paper:

$$\begin{aligned} L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}) &= \left\{ \xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, and } \mathbb{E}(|\xi|^2) < \infty \right\}, \\ L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R}) &= \left\{ \xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, and essentially bounded} \right\}, \\ L_{\mathbb{F}}^0(0, T; \mathbb{R}) &= \left\{ \phi : [0, T] \times \Omega \rightarrow \mathbb{R} \mid \phi \text{ is } \mathcal{P}\text{-measurable} \right\}, \\ L_{\mathbb{F}}^2(0, T; \mathbb{R}) &= \left\{ \phi \in L_{\mathbb{F}}^0(0, T; \mathbb{R}) \mid \mathbb{E} \int_0^T |\phi_t|^2 dt < \infty \right\}, \\ L_{\mathbb{F}}^\infty(0, T; \mathbb{R}) &= \left\{ \phi \in L_{\mathbb{F}}^0(0, T; \mathbb{R}) \mid \phi \text{ is essentially bounded} \right\}, \\ L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n) &= \left\{ \Lambda \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^n) \mid \int_0^\cdot \Lambda_s^\top dW_s \text{ is a BMO martingale on } [0, T] \right\}. \end{aligned}$$

These definitions are generalized in the obvious way to the cases that  $\mathbb{R}$  is replaced by  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times m}$  or  $\mathbb{S}^n$ . For the definition of BMO martingales, we refer the reader to [8] or [5]. Here we will use the following fact about BMO martingales: the Doléans-Dade stochastic exponential  $\mathcal{E}(\int_0^\cdot \Lambda' dW)$  is a uniformly integrable martingale if  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .

In our argument,  $s$ ,  $t$ ,  $\omega$ , “ $\mathbb{P}$ -almost surely” and “almost everywhere”, will be suppressed for simplicity in many circumstances, when no confusion occurs. We shall use  $c$ , which can be different at each occurrence, to represent some positive constant that may depend on the

problem coefficients, but is independent of  $s, t, \omega, P, \Lambda$  and any stopping times. We will use the elementary inequality  $|xy| \leq \varepsilon|x|^2 + \frac{1}{2\varepsilon}|y|^2$  frequently without claim.

Our aim is to solve the homogeneous stochastic control problem (1.3) associated with the state process (1.1) and the cost functional (1.2), where the coefficients  $b : [0, T] \times \Omega \times \mathbb{R} \times \Gamma \rightarrow \mathbb{R}$ ,  $\sigma : [0, T] \times \Omega \times \mathbb{R} \times \Gamma \rightarrow \mathbb{R}^n$ ,  $f : [0, T] \times \Omega \times \mathbb{R} \times \Gamma \rightarrow \mathbb{R}$  are given  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Gamma)$ -measurable functions,  $g(x)$  is an  $\mathcal{F}_T$ -measurable random variable for any  $x \in \mathbb{R}$ , and they fulfill the homogeneous Assumption 1.1.

The class of admissible constrained controls is defined as the set

$$\mathcal{U} := \left\{ u : [0, T] \times \Omega \rightarrow \Gamma \mid \begin{array}{l} u \text{ is a } \mathcal{P} \text{ measurable process and the corresponding state} \\ \text{process (1.1) admits a unique continuous solution } X \text{ such that the cost} \\ \text{functional } J(X, u) \text{ in (1.2) is well-defined and finite and } \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t|^p \right] < \infty \end{array} \right\}.$$

For any homogeneous function  $\varphi$  of degree  $p$  in  $(x, u) \in \mathbb{R} \times \mathbb{R}^m$ , we have

$$\varphi(x, u) = |x|^p \left( \varphi\left(1, \frac{u}{|x|}\right) \mathbb{1}_{x>0} + \varphi\left(-1, \frac{u}{|x|}\right) \mathbb{1}_{x<0} \right) + \varphi(0, u) \mathbb{1}_{x=0}. \quad (2.1)$$

We may use this property throughout our analysis without claim.

### 3 The associated BSDEs

Let  $p \in \mathbb{R}$ . For  $(t, v, P, \Lambda) \times [0, T] \times \Gamma \times \mathbb{R} \times \mathbb{R}^n$ , define the following (random) mappings:

$$\begin{aligned} \mathcal{G}_{1,t}(v, P, \Lambda) &:= f_t(1, v) + \frac{1}{2}p(p-1)P|\sigma_t(1, v)|^2 + pPb_t(1, v) + p\Lambda^\top \sigma_t(1, v), \\ \mathcal{G}_{2,t}(v, P, \Lambda) &:= f_t(-1, v) + \frac{1}{2}p(p-1)P|\sigma_t(-1, v)|^2 - pPb_t(-1, v) - p\Lambda^\top \sigma_t(-1, v), \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_{1,t}^*(P, \Lambda) &:= \inf_{v \in \Gamma} \mathcal{G}_{1,t}(v, P, \Lambda), \\ \mathcal{G}_{2,t}^*(P, \Lambda) &:= \inf_{v \in \Gamma} \mathcal{G}_{2,t}(v, P, \Lambda). \end{aligned}$$

Note that  $\mathcal{G}_{1,t}^*$  and  $\mathcal{G}_{2,t}^*$  may take the value of  $-\infty$ .

Now we introduce the following two one-dimensional BSDEs:

$$\begin{cases} dP = -\mathcal{G}_1^*(P, \Lambda) dt + \Lambda^\top dW, & t \in [0, T], \\ P(T) = g(1), \end{cases} \quad (3.1)$$

and

$$\begin{cases} dP = -\mathcal{G}_2^*(P, \Lambda) dt + \Lambda^\top dW, & t \in [0, T], \\ P(T) = g(-1). \end{cases} \quad (3.2)$$

They will play the critical role in solving the control problem (1.3).

**Definition 3.1.** *A stochastic process  $(P, \Lambda)$  is called a solution to the BSDE (3.1) (resp. to (3.2)) if it satisfies (3.1) (resp. (3.2)), and  $(P, \Lambda) \in S_{\mathbb{F}}^\infty(0, T; \mathbb{R}) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$ . The solution*

is called nonnegative if  $P \geq 0$ ; and called uniformly positive if  $P \geq c$  for some deterministic constant  $c > 0$ .

Note in the above definition, it is implicitly required that  $\mathcal{G}_1^*(P, \Lambda)$  (resp.  $\mathcal{G}_2^*(P, \Lambda)$ ) is finite if  $(P, \Lambda)$  is a solution to the BSDE (3.1) (resp. (3.2)), for otherwise the SDEs are not well-defined.

When the coefficients in (3.1) or (3.2) satisfy some boundedness assumption, we have better estimates for their solutions shown as below.

**Lemma 3.1.** *Suppose*

$$f_t(\pm 1, 0), b_t(\pm 1, 0) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}), \int_0^T |\sigma_s(\pm 1, 0)|^2 ds \in L_{\mathcal{F}_T}^{\infty}(\Omega; \mathbb{R}).$$

If  $(P, \Lambda)$  is a solution to the BSDE (3.1) or (3.2), then  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .

*Proof.* We now assume  $(P, \Lambda)$  is a solution to the BSDE (3.1). The case for (3.2) can be dealt in the same way. Take a constant  $M > 0$  an upper bound of  $|P|$ . Using  $0 \in \Gamma$  and the boundedness of coefficients, we have

$$\begin{aligned} \mathcal{G}_{1,t}^*(P, \Lambda) &\leq \mathcal{G}_{1,t}(0, P, \Lambda) \\ &= f_t(1, 0) + \frac{1}{2}p(p-1)P|\sigma_t(1, 0)|^2 + pPb_t(1, 0) + p\Lambda^{\top}\sigma_t(1, 0) \\ &\leq c + c|\sigma_t(1, 0)|^2 + \frac{1}{5M}|\Lambda|^2, \end{aligned} \tag{3.3}$$

where  $c$  is a positive constant that may depend on  $M$ .

We now show  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ . For any stopping time  $\tau \leq T$ , define a sequence of stopping times:

$$\tau_k = \inf \left\{ t > \tau : \int_{\tau}^t |\Lambda_s|^2 ds > k \right\} \wedge T.$$

Since  $\Lambda \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$ , we see  $\tau_k$  increasingly approaches to  $T$  as  $k \rightarrow +\infty$ . Applying Itô's formula to  $(P - M)^2$  and using (3.3) and  $0 \leq M - P \leq 2M$ , we get

$$\begin{aligned} \mathbb{E} \left[ \int_{\tau}^{\tau_k} |\Lambda|^2 ds \mid \mathcal{F}_{\tau} \right] &= (P_{\tau_k} - M)^2 - (P_{\tau} - M)^2 + \mathbb{E} \left[ \int_{\tau}^{\tau_k} 2(M - P)\mathcal{G}_1^*(P, \Lambda) ds \mid \mathcal{F}_{\tau} \right] \\ &\leq (P_{\tau_k} - M)^2 + \mathbb{E} \left[ \int_{\tau}^{\tau_k} 2(M - P) \left( c + c|\sigma_s(1, 0)|^2 + \frac{1}{5M}|\Lambda|^2 \right) ds \mid \mathcal{F}_{\tau} \right] \\ &\leq 4M^2 + \mathbb{E} \left[ \int_{\tau}^{\tau_k} 4M \left( c + c|\sigma_s(1, 0)|^2 + \frac{1}{5M}|\Lambda|^2 \right) ds \mid \mathcal{F}_{\tau} \right]. \end{aligned}$$

Since  $\int_0^T |\sigma_s(1, 0)|^2 ds$  is bounded and  $\tau \leq \tau_k \leq T$ , it follows

$$\mathbb{E} \left[ \int_{\tau}^{\tau_k} |\Lambda|^2 ds \mid \mathcal{F}_{\tau} \right] \leq c + \frac{4}{5} \mathbb{E} \left[ \int_{\tau}^{\tau_k} |\Lambda|^2 ds \mid \mathcal{F}_{\tau} \right],$$

where  $c$  is a positive constant that does not depend on  $\tau$  or  $\tau_k$ . Thanks to the definition of  $\tau_k$ , the expectations in above are finite, so

$$\mathbb{E} \left[ \int_{\tau}^{\tau_k} |\Lambda|^2 ds \mid \mathcal{F}_{\tau} \right] \leq 5c.$$

By sending  $k \rightarrow \infty$  and using Fatou's lemma, we conclude  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .  $\square$

We now solve the BSDEs (3.1) and (3.2) under different conditions.

**Assumption 3.1.** *It holds that*

$$f_t(\pm 1, 0) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}^+), \quad g(\pm 1) \in L_{\mathcal{F}_T}^{\infty}(\Omega; \mathbb{R}^+),$$

and

$$f_t(0, v), \quad f_t(\pm 1, v) \in L_{\mathbb{F}}^0(0, T; \mathbb{R}^+),$$

for any  $v \in \Gamma$ . Also, there are constants  $L, \delta > 0$  such that

$$\max\{|b_t(\pm 1, v)|^2, |\sigma_t(\pm 1, v)|^2\} \leq \delta f_t(\pm 1, v) + L$$

holds for any  $v \in \Gamma$  and  $t \in [0, T]$ .

**Theorem 3.2.** *Suppose Assumptions 1.1 and 3.1 hold and  $p \geq 1$ . Then there exists a unique nonnegative solution  $(P, \Lambda)$  to (3.1) (resp. to (3.2)). Moreover,  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .*

*Proof.* We will focus on the solvability of BSDE (3.1), and the proof for (3.2) is similar.

For notation simplicity, we write  $w_{1,t}(v) = \max\{|b_t(1, v)|, |\sigma_t(1, v)|\}$ . Then Assumption 3.1 implies  $f_t(1, v) \geq \frac{1}{\delta}(w_{1,t}(v)^2 - L)^+$ . Thanks to  $p \geq 1$ , we have

$$\begin{aligned} \mathcal{G}_{1,t}^*(P^+, \Lambda) &\geq \inf_{v \in \Gamma} \left[ f_t(1, v) + pP^+ b_t(1, v) + p\Lambda^\top \sigma_t(1, v) \right] \\ &\geq \inf_{v \in \Gamma} \left[ \frac{1}{\delta}(w_{1,t}(v)^2 - L)^+ - c(P^+ + |\Lambda|)w_{1,t}(v) \right] \\ &\geq \inf_{y \in \mathbb{R}} \left[ \frac{1}{\delta}(y^2 - L)^+ - c(P^+ + |\Lambda|)y \right]. \end{aligned}$$

Clearly,

$$\begin{aligned} &\inf_{y^2 \leq 2L} \left[ \frac{1}{\delta}(y^2 - L)^+ - c(P^+ + |\Lambda|)y \right] \\ &\geq \inf_{y^2 \leq 2L} \left[ -c(P^+ + |\Lambda|)y \right] \geq -c(P^+ + |\Lambda|), \end{aligned}$$

and

$$\begin{aligned} &\inf_{y^2 > 2L} \left[ \frac{1}{\delta}(y^2 - L)^+ - c(P^+ + |\Lambda|)y \right] \\ &\geq \inf_{y^2 > 2L} \left[ \frac{1}{\delta} \left( y^2 - \frac{1}{2}y^2 \right) - c(P^+ + |\Lambda|)y \right] \geq -c(P^+ + |\Lambda|)^2, \end{aligned}$$

so

$$\mathcal{G}_{1,t}^*(P^+, \Lambda) \geq -c((P^+ + |\Lambda|)^2 + P^+ + |\Lambda|). \quad (3.4)$$

On the other hand,

$$\begin{aligned} \mathcal{G}_{1,t}^*(P^+, \Lambda) &\leq \mathcal{G}_{1,t}(0, P^+, \Lambda) \\ &= f_t(1, 0) + \frac{1}{2}p(p-1)P^+|\sigma_t(1, 0)|^2 + pP^+b_t(1, 0) + p\Lambda^\top \sigma_t(1, 0) \\ &\leq c(1 + P^+ + |\Lambda|). \end{aligned} \quad (3.5)$$

Combining the above two estimates, we have

$$|\mathcal{G}_{1,t}^*(P^+, \Lambda)| \leq c(1 + (P^+)^2 + |\Lambda|^2). \quad (3.6)$$

Let us consider the following linear BSDE with bounded coefficients:

$$\begin{cases} d\bar{P} = -\mathcal{G}_1(0, \bar{P}, \bar{\Lambda}) dt + \bar{\Lambda}^\top dW, \\ \bar{P}(T) = g(1). \end{cases} \quad (3.7)$$

Since  $f_t(1, 0) \geq 0$  and  $g(1) \geq 0$ , it admits a unique, nonnegative bounded solution  $(\bar{P}, \bar{\Lambda})$ .

Take a constant  $M > 0$  an upper bound of  $\bar{P}$ . Let  $h : \mathbb{R} \rightarrow [0, 1]$  be a smooth truncation function satisfying  $h(x) = 1$  for  $x \in [-M, M]$ , and  $h(x) = 0$  for  $x \in (-\infty, -2M] \cup [2M, \infty)$ . Since  $h$  is compactly supported and bounded, it follows from (3.6) that

$$|\mathcal{G}_{1,t}^*(P^+, \Lambda)|h(P) \leq c(h(P) + (P^+)^2h(P) + |\Lambda|^2h(P)) \leq c(1 + |\Lambda|^2).$$

According to [9, Theorem 2.3], there is a bounded, maximal solution  $(\hat{P}, \hat{\Lambda})$  to the following BSDE

$$\begin{cases} d\hat{P} = -\mathcal{G}_1^*(\hat{P}^+, \hat{\Lambda})h(\hat{P}) dt + \hat{\Lambda}^\top dW_t, \\ \hat{P}(T) = g(1). \end{cases} \quad (3.8)$$

Notice  $(\underline{P}, \underline{\Lambda}) = (0, 0)$  is a solution to the following BSDE

$$\underline{P}_t = - \int_t^T c((\underline{P}_s^+ + |\underline{\Lambda}_s|)^2 + \underline{P}_s^+ + |\underline{\Lambda}_s|)h(\underline{P}_s) ds - \int_t^T \underline{\Lambda}_s^\top dW_s,$$

where  $c$  is given in (3.4). Then the maximal solution argument of [9, Theorem 2.3], combined with (3.4), yields

$$\hat{P} \geq \underline{P} = 0. \quad (3.9)$$

On the other hand, since  $0 \leq \bar{P} \leq M$ ,  $h(\bar{P}) = 1$ . It follows from (3.7) that

$$\begin{cases} d\bar{P} = -\mathcal{G}_1(0, \bar{P}^+, \bar{\Lambda})h(\bar{P}) dt + \bar{\Lambda}^\top dW, \\ \bar{P}(T) = g(1). \end{cases}$$

Applying comparison theorem to (3.8) and the above BSDE with Lipschitz continuous driver yields

$$\hat{P} \leq \bar{P} \leq M. \quad (3.10)$$

Combining (3.10) and (3.9), we can assert that  $h(\hat{P}) = 1$ . Together with (3.8) and (3.9), we see  $(\hat{P}, \hat{\Lambda})$  is actually a solution to (3.1). By virtue of Lemma 3.1,  $\hat{\Lambda} \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .

We are now ready to prove the uniqueness. Suppose  $(P, \Lambda)$  and  $(\tilde{P}, \tilde{\Lambda})$  are two solutions of (3.1). Then  $0 \leq P, \tilde{P} \leq M$  for some constant  $M > 0$ , and  $\Lambda, \tilde{\Lambda} \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .

Define processes

$$(U, V) = \left( \ln(P + a), \frac{\Lambda}{P + a} \right), \quad (\tilde{U}, \tilde{V}) = \left( \ln(\tilde{P} + a), \frac{\tilde{\Lambda}}{\tilde{P} + a} \right),$$

where  $a > 0$  is a small constant to be determined later. Then  $(U, V)$  and  $(\tilde{U}, \tilde{V}) \in L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}) \times L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ . Furthermore, by Itô's formula,

$$\begin{cases} dU = -\left[\mathcal{H}^*(U, V) + \frac{1}{2}V'V\right] dt + V' dW, \\ U_T = \ln(G + a), \end{cases} \quad (3.11)$$

where

$$\mathcal{H}_t^*(U, V) = \inf_{v \in \Gamma} \mathcal{H}_t(v, U, V),$$

and

$$\begin{aligned} \mathcal{H}_t(v, U, V) &= f_t(1, v)e^{-U} + \frac{1}{2}p(p-1)(1 - ae^{-U})|\sigma_t(1, v)|^2 \\ &\quad + p(1 - ae^{-U})b_t(1, v) + pV^{\top}\sigma_t(1, v). \end{aligned}$$

Noticing  $U$  is bounded and using Assumption 3.1, we have

$$\mathcal{H}_t(0, U, V) \leq c(1 + |V|).$$

On the other hand, since  $0 \leq P \leq M$ ,  $e^{-U} = \frac{1}{P+a} \geq \frac{1}{M+a}$  and  $1 - ae^{-U} = \frac{P}{P+a} \in [0, 1)$ . Since  $p \geq 1$ , it follows

$$\frac{1}{2}p(p-1)(1 - ae^{-U})|\sigma_t(1, v)|^2 \geq 0.$$

Therefore, under Assumption 3.1,

$$\begin{aligned} \mathcal{H}_t(v, U, V) - \mathcal{H}_t(0, U, V) &\geq f_t(1, v)e^{-U} + p(1 - ae^{-U})b_t(1, v) + pV^{\top}\sigma_t(1, v) - c(1 + |V|) \\ &\geq \frac{w_{1,t}(v)^2 - L}{\delta(M + a)} - cw_{1,t}(v)(1 + |V|) - c(1 + |V|) \\ &\geq \frac{(w_{1,t}(v) + 1)^2}{\delta(M + a)} - c(1 + w_{1,t}(v))(1 + |V|) \\ &> 0, \end{aligned}$$

if  $w_{1,t}(v) > c'(1 + |V|)$  with a sufficiently large constant  $c'$ . So we conclude

$$\mathcal{H}_t^*(U, V) = \inf_{w_{1,t}(v) \leq c(1+|V|)} \mathcal{H}_t(v, U, V). \quad (3.12)$$

Similar estimates hold if  $(U, V)$  replaced by  $(\tilde{U}, \tilde{V})$ .

Set  $\bar{U} = U - \tilde{U}$ ,  $\bar{V} = V - \tilde{V}$ . Then  $(\bar{U}, \bar{V})$  is a solution to the following BSDE,

$$\begin{cases} d\bar{U} = -\left[\mathcal{H}^*(U, V) - \mathcal{H}^*(\tilde{U}, \tilde{V}) + \frac{1}{2}(V + \tilde{V})'\bar{V}\right] dt + \bar{V}' dW, \\ \bar{U}_T = 0. \end{cases}$$

Applying Itô's formula to  $\bar{U}^2$ , we deduce that

$$\bar{U}_t^2 = \int_t^T \left\{ 2\bar{U} \left[ \mathcal{H}^*(U, V) - \mathcal{H}^*(\tilde{U}, \tilde{V}) + \frac{1}{2}(V + \tilde{V})' \tilde{V} \right] - \bar{V}' \tilde{V} \right\} ds - \int_t^T 2\bar{U} \tilde{V}' dW. \quad (3.13)$$

We now estimate  $\bar{U} \left( \mathcal{H}^*(U, V) - \mathcal{H}^*(\tilde{U}, \tilde{V}) \right)$ . Let  $c_2$  be a large positive constant to be chosen. Set

$$\begin{aligned} H_t(v, \tilde{U}, U, V) &= (f_t(1, v) + c_2)e^{-\tilde{U}} + \frac{1}{2}p(p-1)(1 - ae^{-\tilde{U}})|\sigma_t(1, v)|^2 - c_2e^{-U} \\ &\quad + p(1 - ae^{-U})b_t(1, v) + pV^\top \sigma_t(1, v). \end{aligned}$$

Then we can rewrite  $\mathcal{H}$  in terms of  $H$ ,

$$\begin{aligned} \mathcal{H}_t^*(U, V) - \mathcal{H}_t^*(\tilde{U}, \tilde{V}) &= \inf_{w_{1,t}(v) \leq c(1+|V|)} \mathcal{H}_t(v, U, V) - \inf_{w_{1,t}(v) \leq c(1+|\tilde{V}|)} \mathcal{H}_t(v, \tilde{U}, \tilde{V}) \\ &= \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, U, U, V) - \inf_{w_{1,t}(v) \leq c(1+|\tilde{V}|)} H_t(v, \tilde{U}, \tilde{U}, \tilde{V}) \\ &= \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, U, U, V) - \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, \tilde{U}, U, V) \\ &\quad + \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, \tilde{U}, U, V) - \inf_{w_{1,t}(v) \leq c(1+|\tilde{V}|)} H_t(v, \tilde{U}, \tilde{U}, \tilde{V}). \end{aligned} \quad (3.14)$$

Choose  $a > 0$  and  $c_2$  such that

$$\frac{1}{2}p(p-1)a < \frac{1}{\delta}, \quad c_2 > \frac{L}{\delta}.$$

Then by Assumption 3.1,

$$f_t(1, v) + c_2 - \frac{1}{2}p(p-1)a|\sigma_t(1, v)|^2 \geq \frac{1}{\delta}(w_{1,t}(v)^2 - L) + c_2 - \frac{1}{2}p(p-1)aw_{1,t}(v)^2 > 0.$$

Consequently, for every given  $(v, U, V)$ , the map

$$\tilde{U} \mapsto H_t(v, \tilde{U}, U, V)$$

is decreasing, so

$$\bar{U} \left[ \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, U, U, V) - \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, \tilde{U}, U, V) \right] \leq 0. \quad (3.15)$$

On the other hand, noting that  $U$  and  $\tilde{U}$  are bounded and recalling Assumption 3.1, we

have

$$\begin{aligned}
& \left| \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, \tilde{U}, U, V) - \inf_{w_{1,t}(v) \leq c(1+|\tilde{V}|)} H_t(v, \tilde{U}, \tilde{U}, \tilde{V}) \right| \\
& \leq \sup_{w_{1,t}(v) \leq c(1+|V|+|\tilde{V}|)} |H_t(v, \tilde{U}, U, V) - H_t(v, \tilde{U}, \tilde{U}, \tilde{V})| \\
& = \sup_{w_{1,t}(v) \leq c(1+|V|+|\tilde{V}|)} | -c_2(e^{-U} - e^{-\tilde{U}}) - pa(e^{-U} - e^{-\tilde{U}})b_t(1, v) + p\bar{V}^\top \sigma_t(1, v) | \\
& \leq \sup_{w_{1,t}(v) \leq c(1+|V|+|\tilde{V}|)} c(|\bar{U}| + |\bar{V}|)(1 + w_{1,t}(v)) \\
& \leq c(1 + |V| + |\tilde{V}|)(|\bar{U}| + |\bar{V}|). \tag{3.16}
\end{aligned}$$

Therefore, we can define a process  $\beta$ , satisfying  $|\beta| \leq c(1+|V|+|\tilde{V}|)$ , hence  $\beta \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ , in an obvious way such that

$$2\bar{U} \left[ \inf_{w_{1,t}(v) \leq c(1+|V|)} H_t(v, \tilde{U}, U, V) - \inf_{w_{1,t}(v) \leq c(1+|\tilde{V}|)} H_t(v, \tilde{U}, \tilde{U}, \tilde{V}) \right] \leq |\beta| \bar{U}^2 + 2\beta^\top \bar{U} \bar{V}. \tag{3.17}$$

Moreover, notice that  $U$  and  $\tilde{U}$  are bounded and combining (3.14)-(3.17), we have

$$2\bar{U} \left[ \mathcal{H}_t^*(U, V) - \mathcal{H}_t^*(\tilde{U}, \tilde{V}) + \frac{1}{2}(V + \tilde{V})' \bar{V} \right] - \bar{V}' \bar{V} \leq |\beta| \bar{U}^2 + c\beta^\top \bar{U} \bar{V}. \tag{3.18}$$

Together with (3.13), we see

$$\bar{U}_t^2 \leq \int_t^T (|\beta| \bar{U}^2 + c\beta^\top \bar{U} \bar{V}) ds - \int_t^T 2\bar{U} \bar{V}' dW.$$

Now one can repeat the proof of [5, Theorem 3.5] to get  $\bar{U} = \bar{V} = 0$ . This completes the proof of uniqueness.  $\square$

**Assumption 3.2.** *It holds that*

$$f_t(\pm 1, 0) \in L_{\mathbb{F}}^\infty(0, T; \mathbb{R}^+), \quad g(\pm 1) \in L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R}^+),$$

and

$$f_t(0, v), \quad f_t(\pm 1, v) \in L_{\mathbb{F}}^0(0, T; \mathbb{R}^+),$$

for any  $v \in \Gamma$ . Also, there is a constant  $L \geq 0$  such that  $|\sigma_t(\pm 1, v)| \leq L$  holds for any  $v \in \Gamma$  and  $t \in [0, T]$ ; for any constant  $\varepsilon > 0$ , there is a constant  $L_\varepsilon \geq 0$  such that

$$|b_t(\pm 1, v)| \leq \varepsilon f_t(\pm 1, v) + L_\varepsilon$$

holds for any  $v \in \Gamma$  and  $t \in [0, T]$ .

This covers two important cases: (1) the volatility coefficient  $\sigma_t$  does not depend on the control; (2) both the drift coefficient  $b_t$  and volatility coefficient  $\sigma_t$  are bounded.

**Theorem 3.3.** *Suppose Assumptions 1.1 and 3.2 hold and  $p \geq 1$ . Then there exists a unique nonnegative solution  $(P, \Lambda)$  to (3.1) (resp. to (3.2)). Moreover,  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .*

*Proof.* The proof is similar to that of Theorem 3.2. We only point out the main differences. Under Assumption 3.2 and  $p \geq 1$ ,

$$\begin{aligned} \mathcal{G}_{1,t}^*(P^+, \Lambda) &\geq \inf_{v \in \Gamma} \left[ f_t(1, v) + pP^+ b_t(1, v) + p\Lambda^\top \sigma_t(1, v) \right] \\ &\geq \inf_{v \in \Gamma} \left[ f_t(1, v) - pP^+ |b_t(1, v)| \right] - \inf_{v \in \Gamma} f_t(1, v) - c|\Lambda| \\ &= m(P^+) - c|\Lambda|, \end{aligned} \tag{3.19}$$

where we used the fact that  $\inf_{v \in \Gamma} f_t(1, v) \geq 0$  and

$$m(x) := \inf_{v \in \Gamma} \left[ f_t(1, v) - px|b_t(1, v)| \right] - \inf_{v \in \Gamma} f_t(1, v), \quad x \in \mathbb{R}.$$

We now show  $m$  is a real-valued function on  $(-c, c)$  for any  $c > 0$ . Indeed, by taking  $\varepsilon = 1/(pc)$  in Assumption 3.2, one has, for  $x \in (-c, c)$ ,

$$\begin{aligned} m(x) &\geq \inf_{v \in \Gamma} \left[ f_t(1, v) - pc \left( f_t(1, v)/(pc) + L_{1/(pc)} \right) \right] - f_t(1, 0) \\ &= -pcL_{1/(pc)} - f_t(1, 0) > -\infty. \end{aligned}$$

Meanwhile,

$$m(x) \leq f_t(1, 0) - px|b_t(1, 0)| \leq f_t(1, 0) + p|x|(f_t(1, 0) + L_1) < \infty.$$

Therefore,  $m$  is a real-valued function on  $\mathbb{R}$ . It is also trivial to see that  $m$  is decreasing and concave (thus continuous) on  $\mathbb{R}$  with  $m(0) = 0$ . One can replace (3.4) by the above lower bound (3.19) to show the existence.

To establish the uniqueness, we choose constants  $a, \varepsilon > 0$  such that  $0 < \varepsilon p(1 - ae^{-U}) < \frac{1}{2}e^{-U}$ . Then

$$\begin{aligned} &\mathcal{H}_t(v, U, V) - \mathcal{H}_t(0, U, V) \\ &\geq f_t(1, v)e^{-U} + p(1 - ae^{-U})b_t(1, v) + pV^\top \sigma_t(1, v) - c(1 + |V|) \\ &\geq f_t(1, v)e^{-U} - p(1 - ae^{-U})(\varepsilon f_t(1, v) + L_\varepsilon) - c(1 + |V|) \\ &\geq \frac{1}{2}f_t(1, v)e^{-U} - c(1 + |V|) > 0 \end{aligned}$$

if  $f_t(1, v) \geq c'(1 + |V|)$  with  $c'$  being a sufficiently large constant. Hence,

$$\mathcal{H}_t^*(U, V) = \inf_{f_t(1, v) \leq c(1 + |V|)} \mathcal{H}_t(v, U, V).$$

Now we can replace (3.16) by

$$\begin{aligned} &\left| \inf_{f_t(1, v) \leq c(1 + |V|)} \mathcal{H}_t(v, \tilde{U}, U, V) - \inf_{f_t(1, v) \leq c(1 + |\tilde{V}|)} \mathcal{H}_t(v, \tilde{U}, \tilde{U}, \tilde{V}) \right| \\ &\leq \sup_{f_t(1, v) \leq c(1 + |V| + |\tilde{V}|)} | \mathcal{H}_t(v, \tilde{U}, U, V) - \mathcal{H}_t(v, \tilde{U}, \tilde{U}, \tilde{V}) | \\ &= \sup_{f_t(1, v) \leq c(1 + |V| + |\tilde{V}|)} \left| -c_2(e^{-U} - e^{-\tilde{U}}) - pa(e^{-U} - e^{-\tilde{U}})b_t(1, v) + p\bar{V}^\top \sigma_t(1, v) \right| \\ &\leq c(1 + |V| + |\tilde{V}|)(|\bar{U}| + |\bar{V}|), \end{aligned}$$

where we used  $|b_t(\pm 1, v)| \leq f_t(\pm 1, v) + L_1$  to get the last inequality.  $\square$

**Assumption 3.3.** *It holds that*

$$f_t(\pm 1, 0) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}^+), \quad \sigma_t(\pm 1, 0) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}^n), \quad g(\pm 1) \in L_{\mathcal{F}_T}^{\infty}(\Omega; \mathbb{R}^+),$$

and

$$f_t(0, v), f_t(\pm 1, v) \in L_{\mathbb{F}}^0(0, T; \mathbb{R}^+),$$

for any  $v \in \Gamma$ . Also, there are constants  $L, \delta \geq 0$  and  $\eta > 0$  such that

$$|b_t(\pm 1, v)| \leq \delta |\sigma_t(\pm 1, v)|^2 + L, \quad g(\pm 1) \geq \eta,$$

holds for any  $v \in \Gamma$  and  $t \in [0, T]$ .

**Theorem 3.4.** *Suppose Assumptions 1.1 and 3.3 hold and  $p > 1 + 2\delta$ . Then there exists a unique uniformly positive solution  $(P, \Lambda)$  to (3.1) (resp. to (3.2)). Moreover,  $\Lambda \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ .*

*Proof.* Since  $p > 1 + 2\delta$ , there is a constant  $\varepsilon > 0$  such that  $p - 1 - 2\delta - 2\varepsilon > 0$ . For  $P > 0$ , we have

$$\Lambda^\top \sigma_t(1, v) \leq \varepsilon P |\sigma_t(1, v)|^2 + \frac{|\Lambda|^2}{2\varepsilon P},$$

so it follows from Assumption 3.3 that

$$\begin{aligned} \mathcal{G}_{1,t}^*(P, \Lambda) &= \inf_{v \in \Gamma} \left[ \frac{1}{2} p(p-1) P |\sigma_t(1, v)|^2 + f_t(1, v) + p P b_t(1, v) + p \Lambda^\top \sigma_t(1, v) \right] \\ &\geq \inf_{v \in \Gamma} \left[ \frac{1}{2} p(p-1) P |\sigma_t(1, v)|^2 - p P (\delta |\sigma_t(1, v)|^2 + L) - \varepsilon p P |\sigma_t(1, v)|^2 - \frac{p |\Lambda|^2}{2\varepsilon P} \right] \\ &= \inf_{v \in \Gamma} \frac{1}{2} (p-1-2\delta-2\varepsilon) |\sigma_t(1, v)|^2 p P - L p P - \frac{p |\Lambda|^2}{2\varepsilon P} \\ &\geq -L p P - \frac{p |\Lambda|^2}{2\varepsilon P} \\ &\geq -c(P + |\Lambda|^2/P), \end{aligned} \tag{3.20}$$

where we used  $p - 1 - 2\delta - 2\varepsilon, p, P > 0$  to get the the second inequality.

Let  $c$  be the constant given in (3.20). Let  $h : \mathbb{R} \rightarrow [0, 1]$  be a smooth truncation function satisfying  $h(x) = 0$  for  $x \in (-\infty, \frac{1}{2}\eta e^{-cT}]$ , and  $h(x) = 1$  for  $x \geq \eta e^{-cT}$ . Noting (3.5) and (3.20), according to [9, Theorem 2.3], there exists a bounded maximal solution  $(\hat{P}, \hat{\Lambda})$  to the following BSDE

$$\begin{cases} d\hat{P} = -\mathcal{G}_1^*(\hat{P}, \hat{\Lambda}) h(\hat{P}) dt + \hat{\Lambda}' dW, \\ \hat{P}_T = g(1). \end{cases} \tag{3.21}$$

Notice that  $(\underline{P}, \underline{\Lambda}) = (\eta e^{-c(T-t)}, 0)$  satisfies the following BSDE:

$$\begin{cases} d\underline{P} = c(\underline{P} + |\underline{\Lambda}|^2/\underline{P}) h(\underline{P}) dt + \underline{\Lambda}' dW, \\ \underline{P}_T = \eta. \end{cases}$$

Then the maximal argument in [9, Theorem 2.3] applied to the above two BSDEs yields

$$\hat{P} \geq \underline{P} \geq \eta e^{-cT}.$$

It hence follows  $h(\hat{P}) = 1$ . Together with (3.21), we see  $(\hat{P}, \hat{\Lambda})$  is actually a uniformly positive solution to (3.1).

To establish the uniqueness, notice  $\hat{P}$  is uniformly positive, so we can take  $a = 0$  in the proof of Theorem 3.2. Under Assumption 3.3, we have

$$|b_t(\pm 1, v)| \leq |\sigma_t(\pm 1, v)|^2(\delta + \varepsilon)$$

when  $|\sigma_t(1, v)|^2 \geq L/\varepsilon$ . Therefore,

$$\begin{aligned} & \mathcal{H}_t(v, U, V) - \mathcal{H}_t(0, U, V) \\ & \geq f_t(1, v)e^{-U} + \frac{1}{2}p(p-1)|\sigma_t(1, v)|^2 + pb_t(1, v) + pV^\top \sigma_t(1, v) - c(1 + |V|) \\ & \geq \frac{1}{2}(p-1-2\delta-2\varepsilon)p|\sigma_t(1, v)|^2 - pL - p|V||\sigma_t(1, v)| - c(1 + |V|) > 0, \end{aligned}$$

if  $|\sigma_t(1, v)| > c'(1 + |V|)$  with  $c'$  being a sufficiently large constant. Hence,

$$\mathcal{H}_t^*(U, V) = \inf_{|\sigma_t(1, v)| \leq c(1+|V|)} \mathcal{H}_t(v, U, V).$$

Recall that  $a = 0$ , so we can replace (3.16) by

$$\begin{aligned} & \left| \inf_{|\sigma_t(1, v)| \leq c(1+|V|)} H_t(v, \tilde{U}, U, V) - \inf_{|\sigma_t(1, v)| \leq c(1+|\tilde{V}|)} H_t(v, \tilde{U}, \tilde{U}, \tilde{V}) \right| \\ & \leq \sup_{|\sigma_t(1, v)| \leq c(1+|V|+|\tilde{V}|)} |H_t(v, \tilde{U}, U, V) - H_t(v, \tilde{U}, \tilde{U}, \tilde{V})| \\ & = \sup_{|\sigma_t(1, v)| \leq c(1+|V|+|\tilde{V}|)} | -c_2(e^{-U} - e^{-\tilde{U}}) + p\tilde{V}^\top \sigma_t(1, v) | \\ & \leq c(1 + |V| + |\tilde{V}|)(|\bar{U}| + |\bar{V}|). \end{aligned}$$

Hence, the estimate (3.16) still holds and the reminder argument is similar as before.  $\square$

## 4 Verification: solution to the stochastic control problem (1.3)

In this section, we will solve the stochastic control problem (1.3) by providing an optimal control expressed by the solutions to BSDEs (3.1) and (3.2) obtained in the previous section.

**Theorem 4.1.** *Suppose  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are lower semi-continuous w.r.t.  $v$  and one of the following cases holds:*

**Case I:** *Assumptions 1.1 and 3.1 hold and  $p > 1$ ;*

**Case II:** *Assumptions 1.1 and 3.2 hold and  $p > 1$ ;*

**Case III:** *Assumptions 1.1 and 3.3 hold and  $p > 1 + 2\delta$ .*

*Then the problem (1.3) admits an optimal feedback control of time  $t$  and state  $X$ :*

$$u^*(t, X) = \hat{v}_{1,t}X^+ + \hat{v}_{2,t}X^-; \quad (4.1)$$

with the corresponding optimal value:

$$\inf_{u \in \mathcal{U}} J(x, u) = P_{1,0}(x^+)^p + P_{2,0}(x^-)^p,$$

where  $(P_1, \Lambda_1)$  and  $(P_2, \Lambda_2)$  are the unique nonnegative solutions (in Case I and Case II) and uniformly positive solutions (in Case III) to (3.1) and (3.2), respectively; and  $\hat{v}_1$  and  $\hat{v}_2$  are any  $\mathcal{P}$ -measurable processes such that

$$\begin{aligned} \hat{v}_{1,t} \in \Gamma, \quad \mathcal{G}_{1,t}(\hat{v}_{1,t}, P_{1,t}, \Lambda_{1,t}) &= \mathcal{G}_{1,t}^*(P_{1,t}, \Lambda_{1,t}), \\ \hat{v}_{2,t} \in \Gamma, \quad \mathcal{G}_{2,t}(\hat{v}_{2,t}, P_{2,t}, \Lambda_{2,t}) &= \mathcal{G}_{2,t}^*(P_{2,t}, \Lambda_{2,t}). \end{aligned}$$

*Proof.* In any case, we know the existence of  $(P_1, \Lambda_1)$  and  $(P_2, \Lambda_2)$  by Theorems 3.2, 3.3 or 3.4. In particular, we have the finiteness of  $\mathcal{G}_{1,t}^*(P_{1,t}, \Lambda_{1,t})$  and  $\mathcal{G}_{2,t}^*(P_{2,t}, \Lambda_{2,t})$ . Thanks to the homogeneity, we also have  $f_t \geq 0$  in all cases.

**Step 1.** We first show the existence of  $\hat{v}_1$  and  $\hat{v}_2$  and provide some estimates for them. In Case I, we have, recalling  $w_{1,t}(v) = \max\{|b_t(1, v)|, |\sigma_t(1, v)|\}$ ,

$$\begin{aligned} \mathcal{G}_1(v, P_1, \Lambda_1) - \mathcal{G}_1^*(P_1, \Lambda_1) &\geq \mathcal{G}_1(v, P_1, \Lambda_1) - \mathcal{G}_1(0, P_1, \Lambda_1) \\ &\geq f_t(1, v) + pP_1 b_t(1, v) + p\Lambda_1 \sigma_t(1, v) - c(1 + |\Lambda_1|) \\ &\geq \frac{1}{\delta}(w_{1,t}(v)^2 - L) - c(1 + |\Lambda_1|)(1 + w_{1,t}(v)) > 0, \end{aligned}$$

if  $w_{1,t}(v) > c'(1 + |\Lambda_1|)$  with  $c'$  being a sufficiently large constant. Similarly, we have  $\mathcal{G}_2(v, P_2, \Lambda_2) - \mathcal{G}_2^*(P_2, \Lambda_2) > 0$  if  $w_{2,t}(v) > c'(1 + |\Lambda_2|)$ . Thanks to the finiteness and lower semi-continuity of  $\mathcal{G}_1$  and  $\mathcal{G}_2$  w.r.t.  $v$ , we see that  $\hat{v}_1$  and  $\hat{v}_2$  exist and are real-valued and satisfy

$$w_{1,t}(\hat{v}_1) \leq c(1 + |\Lambda_1|), \quad w_{2,t}(\hat{v}_2) \leq c(1 + |\Lambda_2|),$$

so

$$|b_t(1, \hat{v}_1)| + |\sigma_t(1, \hat{v}_1)| \leq c(1 + |\Lambda_1|), \quad |b_t(2, \hat{v}_2)| + |\sigma_t(2, \hat{v}_2)| \leq c(1 + |\Lambda_2|). \quad (4.2)$$

The existence and finiteness of  $\hat{v}_1$  and  $\hat{v}_2$  in Cases II and III can be established similarly. In Case II, one can show that

$$f_t(1, \hat{v}_1) \leq c(1 + |\Lambda_1|), \quad f_t(1, \hat{v}_2) \leq c(1 + |\Lambda_2|), \quad (4.3)$$

which together with Assumption 3.2 implies

$$|b_t(1, \hat{v}_1)| \leq c(1 + |\Lambda_1|), \quad |b_t(2, \hat{v}_2)| \leq c(1 + |\Lambda_2|), \quad |\sigma_t(1, \hat{v}_1)| + |\sigma_t(2, \hat{v}_2)| \leq c. \quad (4.4)$$

In Case III, one can show that

$$|\sigma_t(1, \hat{v}_1)| \leq c(1 + |\Lambda_1|), \quad |\sigma_t(2, \hat{v}_2)| \leq c(1 + |\Lambda_2|), \quad (4.5)$$

which together with Assumption 3.3 implies

$$|b_t(1, \hat{v}_1)| \leq c(1 + |\Lambda_1|^2), \quad |b_t(2, \hat{v}_2)| \leq c(1 + |\Lambda_2|^2). \quad (4.6)$$

**Step 2.** Next we establish the following lower bound

$$J(X, u) = \mathbb{E} \left[ \int_0^T f_t(X_t, u_t) dt + g(X_T) \right] \geq P_{1,0}(x^+)^p + P_{2,0}(x^-)^p, \quad (4.7)$$

for any admissible control  $u \in \mathcal{U}$ .

Note we have  $p > 1$  in all cases, so for any admissible control  $u \in \mathcal{U}$ , by [1, Lemma 2.2],

$$\begin{aligned} d(X_t^+)^p &= p(X_t^+)^{p-1}(b_t(X_t, u_t) dt + \sigma_t(X_t, u_t)^\top dW_t) \\ &\quad + \frac{1}{2}p(p-1)(X_t^+)^{p-2} \mathbf{1}_{X_t > 0} |\sigma_t(X_t, u_t)|^2 dt, \\ -d(X_t^-)^p &= p(X_t^-)^{p-1}(b_t(X_t, u_t) dt + \sigma_t(X_t, u_t)^\top dW_t) \\ &\quad + \frac{1}{2}p(p-1)(X_t^-)^{p-2} \mathbf{1}_{X_t < 0} |\sigma_t(X_t, u_t)|^2 dt. \end{aligned}$$

Applying Itô's lemma to  $P_{1,t}(X_t^+)^p$  and  $P_{2,t}(X_t^-)^p$  respectively, we get

$$\begin{aligned} d(P_{1,t}(X_t^+)^p) &= \mathbf{1}_{X_t > 0} \left[ \frac{1}{2}p(p-1)P_{1,t}(X_t^+)^{p-2} |\sigma_t(X_t, u_t)|^2 - (X_t^+)^p \mathcal{G}_{1,t}^*(P_{1,t}, \Lambda_{1,t}) \right. \\ &\quad \left. + p(X_t^+)^{p-1}(P_{1,t}b_t(X_t, u_t) + \Lambda_{1,t}^\top \sigma_t(X_t, u_t)) \right] dt \\ &\quad + \left[ (X_t^+)^p \Lambda_{1,t} + pP_{1,t}(X_t^+)^{p-1} \sigma_t(X_t, u_t) \right]^\top dW_t, \end{aligned}$$

and

$$\begin{aligned} d(P_{2,t}(X_t^-)^p) &= \mathbf{1}_{X_t < 0} \left[ \frac{1}{2}p(p-1)P_{2,t}(X_t^-)^{p-2} |\sigma_t(X_t, u_t)|^2 - (X_t^-)^p \mathcal{G}_{2,t}^*(P_{2,t}, \Lambda_{2,t}) \right. \\ &\quad \left. - p(X_t^-)^{p-1}(P_{2,t}b_t(X_t, u_t) + \Lambda_{2,t}^\top \sigma_t(X_t, u_t)) \right] dt \\ &\quad + \left[ (X_t^-)^p \Lambda_{2,t} - pP_{2,t}(X_t^-)^{p-1} \sigma_t(X_t, u_t) \right]^\top dW_t. \end{aligned}$$

By standard construction, there is a sequence of increasing stopping times  $\{\tau_k\}$  with  $\lim_k \tau_k = T$  such that, after summing the above two equations,

$$\begin{aligned} &\mathbb{E} \left[ \int_0^{\tau_k} f_t(X_t, u_t) dt + P_{1,\tau_k}(X_{\tau_k}^-)^p + P_{2,\tau_k}(X_{\tau_k}^-)^p \right] \\ &= P_{1,0}(x^+)^p + P_{2,0}(x^-)^p + \mathbb{E} \int_0^{\tau_k} \varphi_t(X_t, u_t) dt, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} \varphi_t(x, u) &= f_t(x, u) + \mathbf{1}_{x > 0} \frac{1}{2}p(p-1)P_{1,t}(x^+)^{p-2} |\sigma_t(x, u)|^2 - (x^+)^p \mathcal{G}_{1,t}^*(P_{1,t}, \Lambda_{1,t}) \\ &\quad + p(x^+)^{p-1}(P_{1,t}b_t(x, u) + \Lambda_{1,t}^\top \sigma_t(x, u)) \\ &\quad + \mathbf{1}_{x < 0} \frac{1}{2}p(p-1)P_{2,t}(x^-)^{p-2} |\sigma_t(x, u)|^2 - (x^-)^p \mathcal{G}_{2,t}^*(P_{2,t}, \Lambda_{2,t}) \\ &\quad - p(x^-)^{p-1}(P_{2,t}b_t(x, u) + \Lambda_{2,t}^\top \sigma_t(x, u)). \end{aligned} \quad (4.9)$$

We now show that  $\varphi_t(x, u) \geq 0$  for any  $x \in \mathbb{R}$ ,  $u \in \Gamma$  and  $t \in [0, T]$ . Since  $u \in \Gamma$  and  $\Gamma$  is a cone,  $v := \frac{u}{|x|} \mathbf{1}_{x \neq 0} \in \Gamma$ . Recall that, under Assumption 1.1,  $f_t$  is homogeneous of degree  $p$  and

$b_t, \sigma_t$  are homogeneous of degree 1 in  $(x, u)$ , so

$$\begin{aligned}\varphi_t(x, u) &= \mathbb{1}_{x>0} |x|^p \left( \frac{1}{2} p(p-1) P_{1,t} |\sigma_t(1, v)|^2 - \mathcal{G}_{1,t}^*(P_{1,t}, \Lambda_{1,t}) \right. \\ &\quad \left. + p(P_{1,t} b_t(1, v) + \Lambda_{1,t}^\top \sigma_t(1, v)) + f_t(1, v) \right) \\ &\quad + \mathbb{1}_{x<0} |x|^p \left( \frac{1}{2} p(p-1) P_{2,t} |\sigma_t(-1, v)|^2 - \mathcal{G}_{2,t}^*(P_{2,t}, \Lambda_{2,t}) \right. \\ &\quad \left. - p(P_{2,t} b_t(-1, v) + \Lambda_{2,t}^\top \sigma_t(-1, v)) + f_t(-1, v) \right) \\ &\quad + \mathbb{1}_{x=0} f_t(0, u) \\ &\geq 0,\end{aligned}$$

by the definitions of  $\mathcal{G}_1^*$ ,  $\mathcal{G}_2^*$  and  $f_t(0, u) \geq 0$ . Now it follows from (4.8) that

$$\mathbb{E} \left[ \int_0^{\tau_k} f_t(X_t, u_t) dt + P_{1,\tau_k} (X_{\tau_k}^+)^p + P_{2,\tau_k} (X_{\tau_k}^-)^p \right] \geq P_{1,0} (x^+)^p + P_{2,0} (x^-)^p.$$

Since  $f_t \geq 0$ , we can apply the monotone convergence theorem to the integral in above; and since  $\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t|^p \right] < \infty$  and  $P_1$  and  $P_2$  are bounded, we can apply the dominated convergence theorem to the other two terms to get

$$\mathbb{E} \left[ \int_0^T f_t(X_t, u_t) dt + P_{1,T} (X_T^+)^p + P_{2,T} (X_T^-)^p \right] \geq P_{1,0} (x^+)^p + P_{2,0} (x^-)^p.$$

Since  $P_{1,T} = g(1)$ ,  $P_{2,T} = g(-1)$  and  $g$  is homogenous, the estimate (4.7) is established.

**Step 3.** We next show the following claims hold:

**Claim (i).** The state process (1.1) admits a unique continuous solution, denoted by  $X^*$ , under the feedback control  $u^*$  given in (4.1);

**Claim (ii).**  $J(X^*, u^*) = \mathbb{E} \left[ \int_0^T f_t(X_t^*, u_t^*) dt + g(X_T^*) \right] \leq P_{1,0} (x^+)^p + P_{2,0} (x^-)^p$ ;

**Claim (iii).**  $\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^*|^p \right] < \infty$ .

Claims (i)-(iii) show that  $u^*$  is an admissible control in  $\mathcal{U}$  to the problem (1.3). Furthermore, Claim (ii) and (4.7) indicate that  $u^*$  is indeed an optimal control to (1.3). Therefore, the proof of the theorem will be complete if Claims (i)-(iii) can be proved. We now prove them one by one.

**Proof of Claim (i).** Substituting the feedback control  $u^*$  into the state process (1.1), and using the homogeneous property of  $b_t$  and  $\sigma_t$ , we have

$$\begin{cases} dX_t = b_t(X_t, u^*(t, X_t)) dt + \sigma_t(X_t, u^*(t, X_t))^\top dW_t \\ \quad = \mathbb{1}_{X_t>0} X_t [b_t(1, \hat{v}_1) dt + \sigma_t(1, \hat{v}_1)^\top dW_t] \\ \quad \quad + \mathbb{1}_{X_t<0} X_t [b_t(-1, \hat{v}_2) dt + \sigma_t(-1, \hat{v}_2)^\top dW_t], \\ X_0 = x. \end{cases} \quad (4.10)$$

By the estimates (4.2)-(4.6) and  $\Lambda_1, \Lambda_2 \in L_{\mathbb{F}}^{2, \text{BMO}}(0, T; \mathbb{R}^n)$ , according to Gal'chuk [3, basic theorem on pp. 756-757], the SDE (4.10) admits a unique continuous solution, which is clearly

given by

$$X_t^* = \begin{cases} X_{1,t}^*, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ X_{2,t}^*, & \text{if } x < 0; \end{cases} \quad (4.11)$$

where

$$\begin{aligned} X_{1,t}^* &= x \exp \left\{ \int_0^t \left( b_s(1, \hat{v}_1) - \frac{1}{2} |\sigma_s(1, \hat{v}_1)|^2 \right) ds + \int_0^t \sigma_s(1, \hat{v}_1)^\top dW_s \right\}, \\ X_{2,t}^* &= x \exp \left\{ \int_0^t \left( b_s(-1, \hat{v}_2) - \frac{1}{2} |\sigma_s(-1, \hat{v}_2)|^2 \right) ds + \int_0^t \sigma_s(-1, \hat{v}_2)^\top dW_s \right\}. \end{aligned}$$

This completes the proof of Claim (i).

**Proof of Claim (ii).** Thanks to the continuity of  $X^*$ , the estimates (4.2)-(4.6) and  $\varphi_t(X_t^*, u_t^*) = 0$ , we know, after recalling (4.8), there is a sequence of increasing stopping times  $\{\tau_k\}$  with  $\lim_k \tau_k = T$  such that,

$$\mathbb{E} \left[ \int_0^{\iota \wedge \tau_k} f_s(X_s^*, u_s^*) ds + P_{1, \iota \wedge \tau_k} (X_{\iota \wedge \tau_k}^{*,+})^p + P_{2, \iota \wedge \tau_k} (X_{\iota \wedge \tau_k}^{*, -})^p \right] = P_{1,0} (x^+)^p + P_{2,0} (x^-)^p, \quad (4.12)$$

for any stopping time  $\iota$  valued in  $[0, T]$ . Now applying Fatou's lemma to the left hand side of above leads to

$$\mathbb{E} \left[ \int_0^\iota f_t(X_t^*, u_t^*) dt + P_{1,\iota} (X_{\iota}^{*,+})^p + P_{2,\iota} (X_{\iota}^{*, -})^p \right] \leq P_{1,0} (x^+)^p + P_{2,0} (x^-)^p. \quad (4.13)$$

Especially, we deduce Claim (ii) by taking  $\iota = T$  in above.

Furthermore, in Case III, we have  $f_t \geq 0$  and  $P_1$  and  $P_2$  are uniformly positive, so (4.13) implies

$$\sup_{\iota} \mathbb{E}[|X_{\iota}^*|^p] \leq K \quad \text{in Case III} \quad (4.14)$$

with some constant  $K$ , where the supreme is taken over all stopping times  $\iota$  valued in  $[0, T]$ .

**Proof of Claim (iii).** We now prove

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^*|^p \right] < \infty.$$

This is trivial by (4.11) when  $x = 0$ . We only deal with the case  $x > 0$  since the case  $x < 0$  can be dealt similarly.

Case I. Under Assumptions 1.1 and 3.1, we have

$$\begin{aligned} |X_s^*|^{p-1} |b_s(X_s^*, u_s^*)| &= |X_s^*|^p |b_s(1, \hat{v}_1)| \leq |X_s^*|^p (\delta f_s(1, \hat{v}_1) + L)^{1/2} \\ &\leq \frac{1}{2} |X_s^*|^p (\delta f_s(1, \hat{v}_1) + L + 1) = \frac{1}{2} (\delta f_s(X_s^*, u_s^*) + (L + 1) |X_s^*|^p), \end{aligned}$$

and

$$\begin{aligned} |X_s^*|^{p-2} \mathbf{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 &= |X_s^*|^p \mathbf{1}_{X_s^* \neq 0} |\sigma_s(1, \hat{v}_1)|^2 \\ &\leq |X_s^*|^p \mathbf{1}_{X_s^* \neq 0} (\delta f_s(1, \hat{v}_1) + L) = \delta f_s(X_s^*, u_s^*) + L |X_s^*|^p. \end{aligned}$$

Let  $a$  be any real constant. Applying Itô's lemma [1, Lemma 2.2] to  $e^{at}|X_t^*|^p$  gives

$$\begin{aligned} e^{at}|X_t^*|^p &= x^p + \int_0^t e^{as} \left( a |X_s^*|^p + p |X_s^*|^{p-1} \operatorname{sgn}(X_s^*) b_s(X_s^*, u_s^*) \right. \\ &\quad \left. + \frac{1}{2} p(p-1) |X_s^*|^{p-2} \mathbf{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 \right) ds \\ &\quad + \int_0^t e^{as} p |X_s^*|^{p-1} \operatorname{sgn}(X_s^*) \sigma_s(X_s^*, u_s^*)^\top dW_s \\ &\leq x^p + \int_0^t e^{as} \left( a |X_s^*|^p + \frac{1}{2} p (\delta f_s(X_s^*, u_s^*) + (L+1) |X_s^*|^p) \right. \\ &\quad \left. + \frac{1}{2} p(p-1) (\delta f_s(X_s^*, u_s^*) + L |X_s^*|^p) \right) ds \\ &\quad + \int_0^t e^{as} p |X_s^*|^{p-1} \operatorname{sgn}(X_s^*) \sigma_s(X_s^*, u_s^*)^\top dW_s. \end{aligned}$$

Construct a sequence of stopping times as follows:

$$\theta_k := \inf \left\{ t \geq 0 \mid |X_t^*| > k \right\} \wedge T.$$

Since  $X^*$  is continuous,  $\theta_k$  increasingly converges to  $T$  as  $k \rightarrow \infty$ . By the Burkholder-Davis-Gundy (BDG) inequality,

$$\begin{aligned} &\mathbb{E} \left[ \sup_{t \in [0, \theta_k]} \left| \int_0^t e^{as} p |X_s^*|^{p-1} \operatorname{sgn}(X_s^*) \sigma_s(X_s^*, u_s^*)^\top dW_s \right| \right] \\ &\leq M \mathbb{E} \left[ \left( \int_0^{\theta_k} e^{2as} |X_s^*|^{2(p-1)} |\sigma_s(X_s^*, u_s^*)|^2 ds \right)^{1/2} \right] \\ &\leq M \mathbb{E} \left[ \sup_{t \in [0, \theta_k]} e^{at/2} |X_t^*|^{p/2} \left( \int_0^{\theta_k} e^{as} |X_s^*|^{p-2} \mathbf{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 ds \right)^{1/2} \right] \\ &\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, \theta_k]} e^{at} |X_t^*|^p \right] + M^2 \mathbb{E} \left[ \int_0^{\theta_k} e^{as} |X_s^*|^{p-2} \mathbf{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 ds \right] \\ &\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, \theta_k]} e^{at} |X_t^*|^p \right] + M^2 \mathbb{E} \left[ \int_0^{\theta_k} e^{as} (\delta f_s(X_s^*, u_s^*) + L |X_s^*|^p) ds \right], \end{aligned} \quad (4.15)$$

where  $M$  is a constant independent of  $k$  and  $a$ . Combining the above two estimates, we conclude,

for  $a \leq -\frac{1}{2}p^2L - \frac{1}{2}p - M^2L$ ,

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\sup_{t \in [0, \theta_k]} e^{at} |X_t^*|^p\right] &\leq x^p + \mathbb{E}\left[\sup_{t \in [0, \theta_k]} \int_0^t e^{as} \left(a|X_s^*|^p + \frac{1}{2}p(\delta f_s(X_s^*, u_s^*) + (L+1)|X_s^*|^p) \right. \right. \\
&\quad \left. \left. + \frac{1}{2}p(p-1)(\delta f_s(X_s^*, u_s^*) + L|X_s^*|^p)\right) ds\right] \\
&\quad + M^2\mathbb{E}\left[\int_0^{\theta_k} e^{as} (\delta f_s(X_s^*, u_s^*) + L|X_s^*|^p) ds\right] \\
&\leq x^p + \mathbb{E}\left[\sup_{t \in [0, \theta_k]} \int_0^t e^{as} \left(a + \frac{1}{2}p^2L + \frac{1}{2}p + M^2L\right) |X_s^*|^p ds\right] \\
&\quad + \left(\frac{1}{2}p^2 + M^2\right) \delta \mathbb{E}\left[\sup_{t \in [0, \theta_k]} \int_0^t e^{as} f_s(X_s^*, u_s^*) ds\right] \\
&\leq x^p + \left(\frac{1}{2}p^2 + M^2\right) \delta e^{aT} \mathbb{E}\left[\int_0^T f_s(X_s^*, u_s^*) ds\right] \leq c,
\end{aligned}$$

where the last inequality is due to Claim (ii) and non-negativity of  $P_1$  and  $P_2$  and  $f_s$ . Now applying Fatou's lemma, the proof of Claim (iii) in Case I is complete.

Case II. In this case,  $\sigma_t$  is bounded. Recall that  $x > 0$  and (4.11), applying Hölder's inequality yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \in [0, T]} |X_t^*|^p\right] \\
&= x^p \mathbb{E}\left[\sup_{t \in [0, T]} \exp\left\{\int_0^t \left(pb_s(1, \hat{v}_1) - \frac{1}{2}p|\sigma_s(1, \hat{v}_1)|^2\right) ds + \int_0^t p\sigma_s(1, \hat{v}_1)^\top dW_s\right\}\right] \\
&\leq c \mathbb{E}\left[\sup_{t \in [0, T]} \exp\left\{\int_0^t \left(2pb_s(1, \hat{v}_1) + (2p^2 - p)|\sigma_s(1, \hat{v}_1)|^2\right) ds\right\}\right]^{1/2} \\
&\quad \times \mathbb{E}\left[\sup_{t \in [0, T]} \exp\left\{-\int_0^t 2p^2|\sigma_s(1, \hat{v}_1)|^2 ds + \int_0^t 2p\sigma_s(1, \hat{v}_1)^\top dW_s\right\}\right]^{1/2}.
\end{aligned}$$

The estimate (4.4) holds in Case II, so

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \in [0, T]} \exp\left\{\int_0^t \left(2pb_s(1, \hat{v}_1) + (2p^2 - p)|\sigma_s(1, \hat{v}_1)|^2\right) ds\right\}\right] \\
&\leq \mathbb{E}\left[\exp\left\{c \int_0^T (1 + |\Lambda_1|) ds\right\}\right] < \infty,
\end{aligned}$$

where the last inequality is due to standard estimate for BMO martingales. It is left to show

$$\mathbb{E}\left[\sup_{t \in [0, T]} Y_t\right] < \infty, \tag{4.16}$$

where

$$Y_t = \exp\left\{-\int_0^t 2p^2|\sigma_s(1, \hat{v}_1)|^2 ds + \int_0^t 2p\sigma_s(1, \hat{v}_1)^\top dW_s\right\}.$$

Since  $\sigma_t$  is bounded,  $Y$  is a martingale with  $\mathbb{E}[Y_T^k] < \infty$  for any  $k \geq 1$ . Then the estimate (4.16) comes from martingale inequality immediately. This complete the proof of Claim (iii) in Case II.

Case III. We now study Case III. Under Assumptions 1.1 and 3.3, we have

$$p|X_s^*|^{p-1}|b_s(X_s^*, u_s^*)| \leq \delta p|X_s^*|^{p-2} \mathbb{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 + L|X_s^*|^p. \quad (4.17)$$

Similar to Case I, applying Itô's lemma to  $e^{Lt}|X_t^*|^p$  gives

$$\begin{aligned} e^{Lt}|X_t^*|^p &= x^p + \int_0^t e^{Ls} \left( L|X_s^*|^p + p|X_s^*|^{p-1} \operatorname{sgn}(X_s^*) b_s(X_s^*, u_s^*) \right. \\ &\quad \left. + \frac{1}{2}p(p-1)|X_s^*|^{p-2} \mathbb{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 \right) ds \\ &\quad + \int_0^t e^{Ls} p|X_s^*|^{p-1} \operatorname{sgn}(X_s^*) \sigma_s(X_s^*, u_s^*)^\top dW_s \end{aligned} \quad (4.18)$$

$$\begin{aligned} &\geq x^p + \left( \frac{1}{2}(p-1) - \delta \right) p \int_0^t e^{Ls} |X_s^*|^{p-2} \mathbb{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 ds \\ &\quad + \int_0^t e^{Ls} p|X_s^*|^{p-1} \operatorname{sgn}(X_s^*) \sigma_s(X_s^*, u_s^*)^\top dW_s. \end{aligned} \quad (4.19)$$

Define

$$\theta_k := \inf \left\{ t \geq 0 \mid |X_t^*| + \int_0^t |\sigma_s(1, \hat{v}_1)|^2 ds > k \right\} \wedge T.$$

Since  $X^*$  is continuous, by (4.5) we see  $\theta_k$  is a sequence of stopping times that increasingly converges to  $T$  as  $k \rightarrow \infty$ . Noticing

$$|X_s^*|^{p-2} \mathbb{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 = |X_s^*|^p |\sigma_s(1, \hat{v}_1)|^2,$$

recalling  $\delta < \frac{1}{2}(p-1)$  and (4.14), it then follows from (4.19) that

$$\mathbb{E} \left[ \int_0^{\theta_k} e^{Ls} |X_s^*|^{p-2} \mathbb{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 ds \right] \leq \frac{\mathbb{E} \left[ e^{L\theta_k} |X_{\theta_k}^*|^p \right]}{\left( \frac{1}{2}(p-1) - \delta \right) p} \leq \frac{K e^{LT}}{\left( \frac{1}{2}(p-1) - \delta \right) p}.$$

We notice (4.15) holds except for the last inequality, so it follows from (4.14), (4.17), (4.18) and the above estimate that

$$\begin{aligned} &\mathbb{E} \left[ \sup_{t \in [0, \theta_k]} e^{Lt} |X_t^*|^p \right] \\ &\leq x^p + \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, \theta_k]} e^{Lt} |X_t^*|^p \right] + M^2 \mathbb{E} \left[ \int_0^{\theta_k} e^{Ls} \left( |X_s^*|^{p-2} \mathbb{1}_{X_s^* \neq 0} |\sigma_s(X_s^*, u_s^*)|^2 + 2L|X_s^*|^p \right) ds \right] \\ &\leq x^p + \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, \theta_k]} e^{Lt} |X_t^*|^p \right] + M^2 \left( \frac{K e^{LT}}{\left( \frac{1}{2}(p-1) - \delta \right) p} + 2KT e^{LT} \right). \end{aligned}$$

In this estimate the constants are independent of  $k$ , so Claim (iii) in Case III follows from Fatou's lemma.  $\square$

## 5 Concluding remarks

In this paper, we introduced and solved a new class of homogeneous stochastic control problems with cone control constraints in three different situations. The optimal value and optimal control in the state feedback form are provided in terms of solutions to two associated

BSDEs via rigorous verification procedure. The solvability of these BSDEs is interesting in its own from the point of view of BSDE theory. Further research along this line can be interesting as well; for instance: (1) Please note that the state process (1.1) is one-dimensional, so it is interesting to investigate the problem (1.3) with multi-dimensional state process. Does it admit the state feedback optimal control like (4.13)? (2) When the state process are discontinuous (e.g. with Poisson jumps), how to solve the problem (1.3)?

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