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# A HIGHER-DIMENSIONAL VERSION OF FÁRY'S THEOREM

KARIM ADIPRASITO AND ZUZANA PATÁKOVÁ

ABSTRACT. We prove a generalization of Istvan Fáry's celebrated theorem to higher dimension.

Fáry's theorem is simple:

**Theorem 1.** *Given a drawing of a graph in the plane, there exists a drawing with straight lines.*

That is simpler and more beautiful than rarely possible. So, could this be true in greater generality. Initially, the answer is no. There are many complexes that have an embedding into a euclidean space, but not one that is affine on every simplex [AKM23, BS92]: When embedding  $k$ -dimensional complexes into  $\mathbb{R}^d$ , and  $d \leq 2k$ , it can happen that the complex embeds, but does not embed linearly. In other dimensions, as one says, the linear and piecewise linear picture coincide, but only for a trivial reason: Every simplicial complex of dimension  $k$  embeds into  $\mathbb{R}^d$  if  $d$  exceeds  $2k$ .

So, paper done, right? There is no positive theorem, apparently.

Except, there is: We prove the following:

**Theorem 2 (Kind-of Fáry's theorem).** *Consider  $X$  a simplicial complex, and a piecewise linear embedding  $\varphi : X \rightarrow M$ , where  $M$  is a PL  $d$ -manifold. Then there is a triangulation  $M'$  of  $M$  that contains  $X$  as a subcomplex.*

This theorem is folklore, but as knowledge of PL topology is gradually lost, we feel it is useful to record it here. We note that it is necessary to modify the embedding map  $\varphi : X \rightarrow M$  to achieve this.

**Example 3.** Consider a 2-dimensional simplex  $\sigma$ , that we wish to embed into the 4-sphere  $S^4$  as follows:

Assume  $S^4$  is the suspension of a triangulated  $S^3$ , with apex points  $n$  and  $s$ . Consider  $\gamma$  a knot in  $S^3$ , and the 2-dimensional disk  $n * \gamma$ .

Choose now a "bad" homeomorphism  $\varphi$  sending  $(\sigma, \partial\sigma)$  to the pair  $(n * \gamma, \gamma)$  so that in particular some interior point of  $\sigma$  is mapped to  $\gamma$ . Obviously, every point of the image  $\varphi(\sigma^\circ)$  apart from  $n$  has a flat neighborhood with respect to  $\sigma$ .

Let us first note a lemma of Bing that almost does the trick.

**Lemma 4 (Section I.2, [Bing83]).** *Consider  $\tilde{X}$  a simplicial complex, and a simplexwise linear embedding  $\varphi : \tilde{X} \rightarrow M$ , where  $M$  is a PL  $d$ -manifold. Then there is a triangulation  $\tilde{M}$  of  $M$  that contains  $\tilde{X}$  as a subcomplex.*

Bing only proved this in the case of  $M$  being a 3-manifold, but his proof extends naturally to the case of arbitrary dimension.

Now, to reduce to Bing's lemma, let us note that because  $X$  embeds piecewise linearly, there is a refinement  $\tilde{X}$  of  $X$  that embeds simplexwise linearly. Let  $\tilde{M}$  denote the simplicial triangulation of  $M$  output by the lemma of Bing.

Recall that a *stellar subdivision* of a simplicial complex  $\Gamma$  at a simplex  $\sigma$  is the result of removing all simplices of  $N_\sigma\Gamma$ , the collection of simplices in  $\Gamma$  containing  $\sigma$ , and attaching the cone  $v_\sigma * \partial N_\sigma\Gamma$  to it along the natural identification map.

Recall that the derived subdivision of a simplicial complex is the simplicial complex of all order chains of simplices (or equivalently, applying stellar subdivisions at all faces in reverse order of inclusion), and that the  $n$ -th derived subdivision is the derived subdivision of the  $n - 1$ st derived subdivision.

If  $\Gamma$  is a subcomplex of a simplicial complex  $\Delta$ , then the derived subdivision of  $\Gamma$  is an *induced subcomplex* of the derived subdivision of  $\Delta$ : A simplex in the latter with vertices in the former is a simplex in the former itself.

Consider now the second derived subdivision  $\text{sd } \tilde{M}$ , in which  $\text{sd } \tilde{X}$  sits as a *strongly induced subcomplex*, since it is the derived subdivision of an induced pair: If  $\sigma$  is a simplex of  $\text{sd } \tilde{M}$  not in  $\text{sd } \tilde{X}$ , and  $\text{st}_\sigma \text{sd } \tilde{M}$  is its closed neighborhood, that is, the simplicial closure of all faces containing  $\sigma$ , then  $\text{sd } \tilde{X} \cap \text{st}_\sigma \text{sd } \tilde{M}$  consists of a single simplex: there is a simplex  $\tau$  of  $\text{sd } \tilde{X}$  such that

$$\tau = \text{sd } \tilde{X} \cap \text{st}_\sigma \text{sd } \tilde{M}$$

An *edge contraction* is the operation of removing the neighborhood  $N_e\Gamma$  of the edge  $e = \{0, 1\}$  and identifying  $N_0\partial N_e\Gamma$  and  $N_1\partial N_e\Gamma$ . It is valid if the result is a simplicial complex. Notice that it is valid if and only if the edge  $e$  is not contained in a minimal nonsimplex of  $\Gamma$ .

Recall that by results of Alexander and Newman [Ale30, New31], two PL homeomorphic complexes are related by edge subdivisions (that is, stellar subdivisions at edges) and valid edge contractions.

Note penultimately that if  $\Gamma \subset \Delta$  is an inclusion of simplicial complexes such that  $\Gamma$  is strongly induced in  $\Delta$ , then edge subdivisions and valid edge contractions applied to  $\Gamma$  naturally extend to  $\Delta$  (because minimal nonsimplices of  $\Delta$  intersecting  $\Gamma$  in more than a vertex are minimal nonsimplices of  $\Gamma$ ). Note also that strong inducedness is preserved under (valid) edge contraction that is carried out in the strongly induced subcomplex. And do not forget that any edge subdivision of  $\Gamma$  preserves strong inducedness provided we apply a stellar subdivision at every face of  $\Delta$  not in  $\Gamma$  in order of reverse inclusion immediately after the edge subdivision.

Finally, apply a suitable sequence of edge subdivisions and edge contractions to transform  $\text{sd } \tilde{X} \subset \text{sd } \tilde{M}$  into  $X$ ; the resulting triangulation  $M'$  of  $M$  is the desired one.  $\square$

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