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► **To cite this version:**

Kaïs Ammari, Fathi Hassine, Souleymane Kadri Harouna, Louis Tebou. STABILIZATION OF TWO COUPLED WAVE EQUATIONS WITH A LOCALIZED SINGULAR KELVIN-VOIGT DAMPING. 2024. hal-04551132

**HAL Id: hal-04551132**

**<https://hal.science/hal-04551132>**

Preprint submitted on 18 Apr 2024

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# STABILIZATION OF TWO COUPLED WAVE EQUATIONS WITH A LOCALIZED SINGULAR KELVIN-VOIGT DAMPING

KAÏS AMMARI, FATHI HASSINE, SOULEYMANE KADRI HAROUNA, AND LOUIS TEBOU

ABSTRACT. We consider two wave equations coupled through a singular Kelvin-Voigt damping mechanism in a bounded domain. We are interested in investigating stability issues for this system. We prove the polynomial stability of the semigroup if the damping region is big enough, and logarithmic stability of the semigroup if the damping region is an arbitrarily small nonempty open subset of the domain under consideration. The main features of our proofs: i) frequency domain approach and, ii) flow multipliers combined with extra auxiliary elliptic systems in the case of polynomial stability, or iii) Carleman estimate in the case of logarithmic stability. A numerical analysis of the spectrum of the one dimensional space semi-discretized system using mixed finite element method indicates that uniform (with respect to the mesh size) exponential decay is not to be expected. This latter result leads us to conjecture that our first polynomial stability result cannot be improved to an exponential stability one.

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## 1. Introduction and main results

There are several mathematical models representing physical damping. The most often encountered type of damping in vibration studies are linear viscous damping, e.g. [1, 10, 27, 29] and Kelvin-Voigt damping, e.g. [2, 3, 4, 5, 20, 22, 32, 33, 34, 37, 38] which are special cases of proportional damping. Viscous damping usually models external friction forces such as air resistance acting on the vibrating structures and is thus called "external damping", while Kelvin-Voigt damping originates from the internal friction of the material of the vibrating structures and thus called "internal damping" or "material damping". This type of material is encountered in real life when one uses patches to suppress vibrations, the modeling aspect of which may be found in [8]. This type of question was examined in the one-dimensional setting in [32] where

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2020 *Mathematics Subject Classification.* 35A01, 35A02, 35L05, 35M33, 93D20.

*Key words and phrases.* Carleman estimate, simultaneous stabilization, coupled wave equations, Singular Kelvin-Voigt damping.

it was shown that the longitudinal motion of an Euler-Bernoulli beam modeled by a locally damped wave equation with Kelvin-Voigt damping is not exponentially stable when the junction between the elastic part and the viscoelastic part of the beam is not smooth enough. Later on, the wave equation with Kelvin-Voigt damping in the multidimensional setting was examined in [34]; in particular, those authors showed the exponential decay of the energy by assuming that the damping region is a neighborhood of the whole boundary. Later on, it was shown that the exponential decay of the energy could be obtained for a more general damping region e.g. [17, 37].

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a connected bounded domain with a sufficiently smooth boundary  $\Gamma = \partial\Omega$ .

Consider the damped wave system

$$(1.1) \quad \partial_t^2 u - d \Delta u - \operatorname{div}(a(x) \nabla(\partial_t u + \partial_t v)) = 0 \text{ in } \Omega \times (0, +\infty),$$

$$(1.2) \quad \partial_t^2 v - c \Delta v - \operatorname{div}(a(x) \nabla(\partial_t u + \partial_t v)) = 0 \text{ in } \Omega \times (0, +\infty),$$

$$(1.3) \quad u = v = 0 \text{ on } \partial\Omega \times (0, +\infty),$$

$$(1.4) \quad u(x, 0) = u^0(x), v(x, 0) = v^0(x), \partial_t u(x, 0) = u^1(x), \partial_t v(x, 0) = v^1(x) \text{ in } \Omega,$$

where  $c \neq d > 0$  are constants and  $a \in L^\infty(\Omega)$ , is nonnegative in  $\Omega$  and positive in  $\omega$ , where  $\omega$  is an arbitrary nonempty open subset of  $\Omega$

We define the natural energy of the solution of (1.1)-(1.4) at instant  $t$  by

$$E(u, v, t) = \frac{1}{2} \int_{\Omega} \left( |\partial_t u(x)|^2 + |\partial_t v(x)|^2 + d |\nabla u(x)|^2 + c |\nabla v(x)|^2 \right) dx, \forall t \geq 0.$$

Simple formal calculations give

$$E(u, v, t) - E(u, v, 0) = - \int_0^t \int_{\Omega} a(x) |\nabla(\partial_t u(x, s) + \partial_t v(x, s))|^2 dx ds, \forall t \geq 0,$$

and therefore, the energy is a non-increasing function of the time variable  $t$ .

We define the energy space by  $\mathcal{H} = (H_0^1(\Omega))^2 \times (L^2(\Omega))^2$  which is endowed with the usual inner product

$$\begin{aligned} & \langle (u_1, u_2, u_3, u_4); (v_1, v_2, v_3, v_4) \rangle = \\ & d \int_{\Omega} \nabla u_1(x) \cdot \nabla \bar{v}_1(x) dx + c \int_{\Omega} \nabla u_2(x) \cdot \nabla \bar{v}_2(x) dx + \int_{\Omega} u_3(x) \bar{v}_3(x) dx + \int_{\Omega} u_4(x) \bar{v}_4(x) dx. \end{aligned}$$

Next, we define the linear unbounded operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{ (u_1, u_2, v_1, v_2) \in \mathcal{H} : v_1, v_2 \in H_0^1(\Omega), d \Delta u_1 + \operatorname{div}(a \nabla(v_1 + v_2)) \in L^2(\Omega), \\ c \Delta u_2 + \operatorname{div}(a \nabla(v_1 + v_2)) \in L^2(\Omega) \} \end{aligned}$$

and

$$\mathcal{A}(u_1, u_2, v_1, v_2)^t = (v_1, v_2, d \Delta u_1 + \operatorname{div}(a \nabla(v_1 + v_2)), c \Delta u_2 + \operatorname{div}(a \nabla(v_1 + v_2)))^t.$$

Before stating our stability results, we recall the following geometric condition that will be useful in the proofs of exponential and polynomial stability results.

Introduce a geometric constraint (GC) on the subset  $\omega$  where the dissipation is effective; we proceed as in [31], (see also [25, 30] for a special case).

**(GC).** There exist open sets  $\Omega_j \subset \Omega$  with piecewise smooth boundary  $\partial\Omega_j$ , and points  $x_0^j \in \mathbb{R}^n$ ,  $j = 1, 2, \dots, J$ , such that  $\Omega_i \cap \Omega_j = \emptyset$ , for any  $1 \leq i < j \leq J$ , and:

$$\Omega \cap \mathcal{N}_{\delta} \left[ \left( \bigcup_{j=1}^J \Gamma_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right) \right] \subset \omega,$$

for some  $\delta > 0$ , where  $\mathcal{N}_\delta(S) = \bigcup_{x \in S} \{y \in \mathbb{R}^n; |x - y| < \delta\}$ , for  $S \subset \mathbb{R}^n$ ,

$\Gamma_j = \left\{x \in \partial\Omega_j; (x - x_0^j) \cdot \nu^j(x) > 0\right\}$ ,  $\nu^j$  being the unit normal vector pointing into the exterior of  $\Omega_j$ .

Let  $0 < \delta_1 < \delta$ . Set  $Q_1 = \mathcal{N}_{\delta_1}(S)$  and  $\omega_1 = \Omega \cap Q_1$ .

**Theorem 1.1.** *Assume that  $a \in C^2(\bar{\Omega})$  and further satisfies the following constraints*

$$\exists C > 0 : |\nabla a(x)| \leq Ca^{1/2}(x), \text{ and } |\Delta a(x)| \leq Ca^{1/2}(x), \text{ for every } x \text{ in } \Omega.$$

*Suppose that the damping region  $\omega$  satisfies the geometric condition above, and the damping coefficient  $a$  further satisfies:*

$$\exists a_0 > 0 : a(x) \geq a_0, \forall x \in \omega_1.$$

*Then for any  $k \in \mathbb{N}^*$  there exists  $C > 0$  such that for any initial data  $(u^0, v^0, u^1, v^1) \in \mathcal{D}(\mathcal{A}^k)$  the solution  $u(x, t)$  of (1.1)-(1.4) satisfies*

$$E(u, v, t) \leq \frac{C}{(1+t)^{2k}} \|(u^0, v^0, u^1, v^1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \quad \forall t > 0,$$

**Remark 1.1.** *Before stating our next result, we find it useful to comment on the fact that we have no exponential stability. In the case of a single wave equation or elasticity equations with localized Kelvin-Voigt damping, it is well known that under a geometric constraint on the damping region, and a less stringent structural constraint on the damping coefficient (in those cases, the condition on the Laplacian is not needed), uniform exponential decay of the energy holds, e.g. [17, 34, 37, 38]. This exponential decay is obtained by exploiting the fact that the energy localized in the damping region is more regular. In fact, estimates of the localized energy are exactly those that would lead to the analyticity of the semigroup if the damping were distributed everywhere in the domain under consideration. Those estimates of the localized energy play a critical role in the proof of the exponential stability of the semigroup. In the case of the system at hand, it is well known that when the damping is distributed everywhere in the domain under consideration, the corresponding semigroup is not differentiable, but it is exponentially stable, [6, 26]. Given this lack of regularity at the global level, it is reasonable to conjecture that exponential stability fails when the Kelvin-Voigt damping is singular and localized. Indeed, the numerical analysis of the spectrum of the one dimensional space semi-discretized system, using mixed finite element method, indicates that uniform (with respect to the mesh size) exponential decay is not to be expected; at high frequencies, the real parts of the eigenvalues approach the imaginary axis. Now, it is well-known that the mixed finite element method preserves the behavior of eigenvalues, so that the stability properties of the continuous and the space semi-discretized systems are identical. This latter observation comforts us in our conjecture that this polynomial stability result cannot be improved to an exponential stability one.*

**Theorem 1.2.** *Let  $\omega$  satisfy the geometric constraint above. Assume that the damping coefficient  $a$  is bounded measurable satisfies:*

$$(1.5) \quad \exists a_0 > 0 : a(x) \geq a_0, \forall x \in \omega.$$

*Then for any  $k \in \mathbb{N}^*$  there exists  $C > 0$  such that for any initial data  $(u^0, v^0, u^1, v^1) \in \mathcal{D}(\mathcal{A}^k)$  the solution  $u(x, t)$  of (1.1)-(1.4) satisfies*

$$E(u, v, t) \leq \frac{C}{(1+t)^k} \|(u^0, v^0, u^1, v^1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \quad \forall t > 0.$$

**Theorem 1.3.** *Let  $\omega$  be a nonempty open subset of  $\Omega$ . Assume that the damping coefficient satisfies (1.5). Then for any  $k \in \mathbb{N}^*$  there exists  $C > 0$  such that for any initial data  $(u^0, v^0, u^1, v^1) \in \mathcal{D}(\mathcal{A}^k)$  the solution  $u(x, t)$  of (1.1) starting from  $(u^0, v^0, u^1, v^1)$  satisfying*

$$E(u, v, t) \leq \frac{C}{(\ln(2+t))^{2k}} \|(u^0, v^0, u^1, v^1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \quad \forall t > 0.$$

The remainder of this paper is organized as follows. In Section 2, we prove the well-posedness of the system (1.1)-(1.4) as well as its strong stability. In section 3, we show the polynomial stability of the system where we prove Theorem 1.1 and Theorem 1.1. In Section 4, we prove the logarithm stability of the system given by Theorem 1.3. Finally, in section 5 we present numerical results of the spectrum of the operator claiming that the exponential stability of the system can not hold.

## 2. Well-posedness and strong stability

We can write (1.1)-(1.4) as the following Cauchy problem

$$\begin{cases} \frac{d}{dt}(u(t), v(t), \partial_t u(t), \partial_t v(t))^t = \mathcal{A}(u(t), v(t), \partial_t u(t), \partial_t v(t))^t, \\ (u(0), v(0), \partial_t u(0), \partial_t v(0)) = (u^0, u^1, v^0, v^1). \end{cases}$$

**Theorem 2.1.** *The operator  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions on the energy space  $\mathcal{H}$ .*

*Proof.* Firstly, it is easy to see that for all  $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ , we have

$$\operatorname{Re} \langle \mathcal{A}(u_1, u_2, v_1, v_2); (u_1, u_2, v_1, v_2) \rangle = - \int_{\Omega} a(x) |\nabla(v_1(x) + v_2(x))|^2 dx,$$

which show that the operator  $\mathcal{A}$  is dissipative.

Next, for any given  $(f_1, f_2, g_1, g_2) \in \mathcal{H}$ , we solve the equation  $\mathcal{A}(u_1, u_2, v_1, v_2) = (f_1, f_2, g_1, g_2)$ , which is recast on the following way

$$(2.1) \quad \begin{cases} v_1 = f_1, \\ v_2 = f_2, \\ d\Delta u_1 + \operatorname{div}(a\nabla(f_1 + f_2)) = g_1, \\ c\Delta u_2 + \operatorname{div}(a\nabla(f_1 + f_2)) = g_2. \end{cases}$$

It is well known that by Lax-Milgram's theorem the system (2.1) admits a unique solution  $u \in H_0^1(\Omega)$ . Moreover by multiplying the second line of (2.1) by  $\bar{u}$  and integrating over  $\Omega$  and using Poincaré inequality and Cauchy-Schwarz inequality we find that there exists a constant  $C > 0$  such that

$$\begin{aligned} & d \int_{\Omega} |\nabla u_1(x)|^2 dx + c \int_{\Omega} |\nabla u_2(x)|^2 dx \\ & \leq C \left( \int_{\Omega} |\nabla f_1(x)|^2 dx + \int_{\Omega} |\nabla f_2(x)|^2 dx + \int_{\Omega} |\nabla g_1(x)|^2 dx + \int_{\Omega} |g_2(x)|^2 dx \right). \end{aligned}$$

It follows that for all  $(u, v) \in \mathcal{D}(\mathcal{A})$  we have

$$\|(u_1, u_2, v_1, v_2)\|_{\mathcal{H}} \leq C \|(f_1, f_2, g_1, g_2)\|_{\mathcal{H}}.$$

This implies that  $0 \in \rho(\mathcal{A})$  and by contraction principle, we easily get  $R(\lambda I - \mathcal{A}) = \mathcal{H}$  for sufficient small  $\lambda > 0$ . The density of the domain of  $\mathcal{A}$  follows from [35, Theorem 1.4.6]. Then thanks to Lumer-Phillips Theorem (see [35, Theorem 1.4.3]), the operator  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions on the Hilbert  $\mathcal{H}$ .  $\square$

**Theorem 2.2.** *The semigroup  $e^{t\mathcal{A}}$  is strongly stable in the energy space  $\mathcal{H}$ , i.e.,*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}}(u^0, v^0, u^1, v^1)^t\|_{\mathcal{H}} = 0, \quad \forall (u^0, v^0, u^1, v^1) \in \mathcal{H}.$$

*Proof.* To show that the semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable we only have to prove that the intersection of  $\sigma(\mathcal{A})$  with  $i\mathbb{R}$  is an empty set. Since the resolvent of the operator  $\mathcal{A}$  is not compact (see [32, 34]) but  $0 \in \rho(\mathcal{A})$  we only need to prove that  $(i\mu I - \mathcal{A})$  is a one-to-one correspondence in the energy space  $\mathcal{H}$  for all  $\mu \in \mathbb{R}^*$ . The proof will be carried out in two steps: in the first step we will prove the injective property of  $(i\mu I - \mathcal{A})$  and in the second step we will prove the surjective property of the same operator.

i) Let  $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$  such that

$$(2.2) \quad \mathcal{A}(u_1, u_2, v_1, v_2)^t = i\mu(u_1, u_2, v_1, v_2)^t.$$

Then taking the real part of the scalar product of (2.2) with  $(u, v)$  we get

$$\operatorname{Re}(i\mu \|(u_1, u_2, v_1, v_2)\|_{\mathcal{H}}^2) = \operatorname{Re} \langle \mathcal{A}(u_1, u_2, v_1, v_2), (u_1, u_2, v_1, v_2) \rangle = - \int_{\Omega} a |\nabla(v_1 + v_2)|^2 dx = 0.$$

which implies that

$$(2.3) \quad a \nabla(v_1 + v_2) = 0 \quad \text{in } \Omega.$$

Inserting (2.3) into (2.2), we obtain

$$(2.4) \quad \begin{cases} \mu^2 u_1 + d\Delta u_1 = 0 & \text{in } \Omega, \\ \mu^2 u_2 + c\Delta u_2 = 0 & \text{in } \Omega, \\ \nabla(u_1 + u_2) = 0 & \text{in } \omega \\ u_1 = u_2 = 0 & \text{on } \Gamma, \end{cases}$$

We denote by  $w_j^i = \partial_{x_j} u_i, i = 1, 2$  and we derive the equations of (2.4), one gets

$$\begin{cases} \mu^2 w_j^1 + d\Delta w_j^1 = 0 & \text{in } \Omega, \\ \mu^2 w_j^2 + c\Delta w_j^2 = 0 & \text{in } \Omega, \\ w_j^1 + w_j^2 = 0 & \text{in } \omega. \end{cases}$$

Which implies that

$$d\Delta w_j^1 + c\Delta w_j^2 = 0 \text{ in } \omega \implies (d - c)\Delta w_j^i = 0, \text{ in } \omega, i = 1, 2 \implies \Delta w_j^i = 0 \text{ in } \omega, i = 1, 2.$$

According to the above system we have that  $w_j^i = 0$  in  $\omega, i = 1, 2$  and

$$\begin{cases} \mu^2 w_j^1 + d\Delta w_j^1 = 0 & \text{in } \Omega, \\ \mu^2 w_j^2 + c\Delta w_j^2 = 0 & \text{in } \Omega, \\ w_j^1 = w_j^2 = 0 & \text{in } \omega. \end{cases}$$

Hence, from the unique continuation theorem we deduce that  $w_j^i = 0$  in  $\Omega, i = 1, 2$  and therefore  $u_1, u_2$  are constants in  $\Omega$  and since  $u_i|_{\Gamma} = 0, i = 1, 2$  we follow that  $u_i \equiv 0, i = 1, 2$ . We have thus proved that  $\operatorname{Ker}(i\mu I - \mathcal{A}) = \{0\}$ .

ii) Now given  $(f_1, f_2, g_1, g_2) \in \mathcal{H}$ , we solve the equation

$$(i\mu I - \mathcal{A})(u_1, u_2, v_1, v_2) = (f_1, f_2, g_1, g_2)$$

Or equivalently,

$$(2.5) \quad \begin{cases} v_1 = i\mu u_1 - f_1 \\ v_2 = i\mu u_2 - f_2 \\ \mu^2 u_1 + d\Delta u_1 + i\mu \operatorname{div}(a\nabla(u_1 + u_2)) = \operatorname{div}(a\nabla f_1) - i\mu f_1 - g_1, \\ \mu^2 u_2 + c\Delta u_2 + i\mu \operatorname{div}(a\nabla(u_1 + u_2)) = \operatorname{div}(a\nabla f_2) - i\mu f_2 - g_2. \end{cases}$$

Let us define the operator

$$A \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} d\Delta u + i\mu \operatorname{div}(a\nabla(u + v)) \\ c\Delta v + i\mu \operatorname{div}(a\nabla(u + v)) \end{pmatrix}, \quad \forall (u, v) \in (H_0^1(\Omega))^2.$$

It is easy to show that  $A$  is an isomorphism from  $(H_0^1(\Omega))^2$  onto  $(H^{-1}(\Omega))^2$ . Then the second line of (2.5) can be written as follows

$$(2.6) \quad \begin{pmatrix} u \\ v \end{pmatrix} - \mu^2 A^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} g_1 + i\mu f_1 - \operatorname{div}(a\nabla f_1) \\ g_2 + i\mu f_2 - \operatorname{div}(a\nabla f_2) \end{pmatrix}.$$

If  $\begin{pmatrix} u \\ v \end{pmatrix} \in \operatorname{Ker}(I - \mu^2 A^{-1})$ , then  $(\mu^2 I - A) \begin{pmatrix} u \\ v \end{pmatrix} = 0$ . It follows that

$$(2.7) \quad \begin{pmatrix} \mu^2 u + d\Delta u + i\mu \operatorname{div}(a\nabla(u+v)) \\ \mu^2 v + c\Delta v + i\mu \operatorname{div}(a\nabla(u+v)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Multiplying (2.7) by  $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$ , then by Green's formula we obtain

$$\begin{aligned} & \mu^2 \int_{\Omega} (|u(x)|^2 + |v(x)|^2) dx - d \int_{\Omega} |\nabla u(x)|^2 dx - c \int_{\Omega} |\nabla v(x)|^2 dx \\ & - i\mu \int_{\Omega} a(x) |\nabla(u(x) + v(x))|^2 dx = 0. \end{aligned}$$

This shows that

$$\int_{\Omega} a(x) |\nabla(u(x) + v(x))|^2 dx = 0,$$

which imply that  $a\nabla(u+v) = 0$  in  $\Omega$ .

Inserting this last equation into (2.7) we get

$$\mu^2 u + d\Delta u = 0, \quad \mu^2 v + c\Delta v = 0 \quad \text{in } \Omega.$$

Once again, using the unique continuation theorem as in the first step where we recall that  $u|_{\Gamma} = v|_{\Gamma} = 0$ , we get  $u = v = 0$  in  $\Omega$ . This imply that  $\operatorname{Ker}(I - \mu^2 A^{-1}) = \{0\}$ . On the other hand thanks to the compact embeddings  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  and  $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$  we see that  $A^{-1}$  is a compact operator in  $(H_0^1(\Omega))^2$ . Now thanks to the Fredholm's alternative, the operator  $(I - \mu^2 A^{-1})$  is bijective in  $(H_0^1(\Omega))^2$ , hence the equation (2.6) has a unique solution in  $(H_0^1(\Omega))^2$ , which shows that the operator  $(i\mu I - \mathcal{A})$  is surjective in the energy space  $\mathcal{H}$ . The proof is thus complete.  $\square$

### 3. Proofs of Theorems 1.1 and 1.2

In this section we shall provide the proofs of Theorem 1.1 and Theorem 1.2. The two wave equations are coupled through the damping mechanism. The main idea of the proof is to treat each equation separately, then suitably combine the estimates to derive the claimed energy estimate in each of Theorem 1.1 or Theorem 1.2. For both proofs, we will rely on resolvent estimates.

**3.1. Proof of Theorem 1.1.** We recall that we have just established that the semigroup is strongly stable. Thanks to [14, Theorem 2.4] and [11, Proposition 3.1], the proof of Theorem 1.1 will be completed once we show the resolvent estimate

$$(3.1) \quad \|(ib\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(|b|) \text{ as } |b| \nearrow +\infty.$$

To this end, let  $U \in \mathcal{H}$ , and let  $b$  be a real number with  $|b| > 1$ . Since the range of  $ib\mathcal{I} - \mathcal{A}$  is  $\mathcal{H}$ , there exists  $Z \in D(\mathcal{A})$  such that

$$(3.2) \quad ibZ - \mathcal{A}Z = U.$$

The desired resolvent estimate will be established once we prove

$$(3.3) \quad \|Z\|_{\mathcal{H}} \leq K_0 |b| \|U\|_{\mathcal{H}},$$

where here and in the sequel,  $K_0$  is a generic positive constant that may eventually depend on  $\Omega$ ,  $\omega$ , and  $a$ , but never on  $b$ .

To establish (3.3), first, we note that if  $Z = (u, w, v, z)$ , and  $U = (f, h, g, \ell)$ , then (3.2) may be recast as

$$(3.4) \quad \begin{aligned} ibu - v &= f \\ ibv - d\Delta u - \operatorname{div}(a\nabla(v + z)) &= g \\ ibw - z &= h \\ ibz - c\Delta w - \operatorname{div}(a\nabla(v + z)) &= \ell. \end{aligned}$$

First taking the inner product with  $Z$  on both sides of (3.2), then taking the real parts, we immediately get

$$(3.5) \quad |\sqrt{a}(\nabla v + \nabla z)|_2^2 \leq \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}}.$$

It follows from (3.5), and the first and third equations in (3.4):

$$(3.6) \quad \begin{aligned} b^2|\sqrt{a}(\nabla u + \nabla w)|_2^2 &\leq 2|\sqrt{a}(\nabla v + \nabla z)|_2^2 + 2|\sqrt{a}(\nabla f + \nabla h)|_2^2 \\ &\leq 2\|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + K_0 (\|f\|_1^2 + \|h\|_1^2) \leq 2\|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + K_0 \|U\|_{\mathcal{H}}^2. \end{aligned}$$

Henceforth,  $|\cdot|_2$  stands for  $\|\cdot\|_{L^2(\Omega)}$  or  $\|\cdot\|_{[L^2(\Omega)]^n}$ , while  $\|\cdot\|_s$  stands for  $\|\cdot\|_{H^s(\Omega)}$  for every nonzero  $s$  in  $\mathbb{R}$ .

**Methodology of the proof.** Due to the fact that the damping is localized, we will need first order or flow multipliers to propagate the effect of the damping mechanism from the damped region  $\omega$  to the whole domain  $\Omega$ . This is a challenging task for the following reasons:

- the damping is unbounded,
- the operator  $\mathcal{A}$  does not have a compact resolvent; so solutions are not smooth enough to apply the first order multipliers directly to either the second or fourth equation in (3.4). We shall introduce new variables that have the requisite smoothness,
- the matrix defining the damping is singular, making it tricky to combine the estimates established for  $(u, v)$  and  $(w, z)$ -systems; this is why we need the speeds of propagation of the two waves to be distinct. Indeed, if  $c = d$ , then the wave equation corresponding to  $q = u - v$  in (1.1)-(1.4) is conservative and the coupled system is then unstable.

The rest of the proof will go along the following steps:

Step 1: Smoothing of solutions. In this step, we shall introduce new variables that will enable us to propagate the localized effect of the damping to the whole domain using flow multipliers.

Step 2: Preliminary estimates for the  $(u, v)$ -system.

Step 3: Geometric propagation of the damping effect. In this step, we shall use appropriate flow multipliers to propagate the damping effect localized in  $\omega$  to the whole domain  $\Omega$ .

Step 4: Estimating the localized kinetic energy. Since the dissipation law involves the sum of velocities, the idea is to express the localized kinetic energy using the sum of velocities.

Step 5: Estimating the localized potential energy. Once the localized kinetic energy is estimated, we just rely on the local equipartition of energy identity to estimate the localized potential energy.

Step 6: Final estimate and completion of the proof.

**STEP 1: Smoothing of solutions.** Let  $\phi$  in  $H_0^1(\Omega)$  be the solutions of

$$(3.7) \quad -\Delta\phi = \operatorname{div}(a\nabla(v + z)).$$

Thanks to (3.5), one readily checks that

$$(3.8) \quad \|\phi\|_1 \leq K_0 \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}}.$$



Now, set  $u_1 = du - \phi$ , and  $w_1 = cw - \phi$ , then the second and fourth equation in (3.4) may be written

$$(3.9) \quad \begin{aligned} ibv - \Delta u_1 &= g \\ ibz - \Delta w_1 &= \ell. \end{aligned}$$

Elliptic regularity theory then shows that both  $u_1$  and  $w_1$  lie in  $H^2(\Omega) \cap H_0^1(\Omega)$ . One readily checks that the following estimates hold

$$(3.10) \quad \begin{aligned} \|u_1\|_1 &= \|du - \phi\|_1 \leq K_0 \|Z\|_{\mathcal{H}} + K_0 \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}}, \text{ and} \\ \|w_1\|_1 &= \|cw - \phi\|_1 \leq K_0 \|Z\|_{\mathcal{H}} + K_0 \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}}. \end{aligned}$$

**STEP 2: Preliminary estimates for the  $(u, v)$ -system.** In this step, we are going to estimate  $\|v\|_{-1}$  and discuss the equipartition of energy inequality.

Taking the  $H^{-1}$ -norm in the first equation in (3.9) and using (3.10), we derive:

$$(3.11) \quad |b| \|v\|_{-1} \leq K_0 (\|u_1\|_1 + |g|_2) \leq K_0 (\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}}).$$

Multiply the first equation in (3.9) by  $\beta \bar{u}_1$ , integrate on  $\Omega$ , and take real parts to get

$$(3.12) \quad \beta \Re \int_{\Omega} g \bar{u}_1 \, dx = \beta \Re \int_{\Omega} (ibv - \Delta u_1) \bar{u}_1 \, dx = \beta \|u_1\|_1^2 + \beta \Re \int_{\Omega} v (ib\bar{u} - ib\bar{\phi}) \, dx.$$

Using the first equation in (3.4) it follows:

$$\Re \int_{\Omega} v (ib\bar{u} - ib\bar{\phi}) \, dx = \Re \int_{\Omega} v (-\bar{v} - \bar{f} - ib\bar{\phi}) \, dx.$$

Now, let  $\beta$  be a real parameter to be specified later on. Then we have

$$(3.13) \quad \beta \Re \int_{\Omega} g \bar{u}_1 \, dx = \beta \|u_1\|_1^2 - \beta |v|_2^2 - \beta \Re \int_{\Omega} v (\bar{f} + ib\bar{w}) \, dx.$$

Using Cauchy-Schwarz inequality as well as Poincaré inequality and the duality product between  $H_0^1(\Omega)$  and  $H^{-1}(\Omega)$ , we obtain

$$(3.14) \quad \begin{aligned} \left| \beta \Re \int_{\Omega} \{g \bar{u}_1 + v (\bar{f} + ib\bar{\phi})\} \, dx \right| &\leq K_0 (|g|_2 \|u\|_1 + \|f\|_1 |v|_2 + |b| \|v\|_{-1} \|\phi\|_1) \\ &\leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} \right). \end{aligned}$$

Whence

$$(3.15) \quad K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} \right) \geq \beta \|u_1\|_1^2 - \beta |v|_2^2.$$

Now we are going to introduce some further notations that will be useful in the next step. For each  $j = 1, \dots, J$ , where  $J$  appears in the geometric constraint (GC) stated above, set  $m^j(x) = x - x_0^j$ . Let  $0 < \delta_0 < \delta_1 < \delta$ , where  $\delta$  is the one given in (GC). Set

$$S = \left( \bigcup_{j=1}^J \Gamma_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right), \quad Q_0 = \mathcal{N}_{\delta_0}(S), \quad Q_1 = \mathcal{N}_{\delta_1}(S), \quad \omega_1 = \Omega \cap Q_1,$$

and for each  $j$ , let  $\varphi_j$  be a function satisfying

$$\varphi_j \in W^{1,\infty}(\Omega), \quad 0 \leq \varphi_j \leq 1, \quad \varphi_j = 1 \text{ in } \bar{\Omega}_j \setminus Q_1, \quad \varphi_j = 0 \text{ in } \Omega \cap Q_0.$$

Let  $\alpha > 0$  be another parameter such that  $\alpha(n-2) < \beta < \alpha n$ .

**STEP 3: Geometric propagation of the damping effect.** Multiply the first equation in

(3.9) by  $2\alpha\varphi_j m^j \cdot \nabla \bar{u}_1$ , integrate on  $\Omega_j$ , and take real parts to get

$$(3.16) \quad 2\alpha\Re \int_{\Omega_j} g(\varphi_j m^j \cdot \nabla \bar{u}_1) dx = 2\alpha\Re \int_{\Omega_j} v\varphi_j m^j \cdot \nabla(-\bar{v} - \bar{f} - ib\bar{\phi}) dx \\ - 2\alpha\Re \int_{\Omega_j} \Delta u_1(\varphi_j m^j \cdot \nabla \bar{u}_1) dx.$$

The application of Green's formula shows

$$(3.17) \quad -2\alpha\Re \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{v} dx = \alpha n \int_{\Omega_j} \varphi_j |v|^2 dx + \alpha \int_{\Omega_j} (m^j \cdot \nabla \varphi_j) |v|^2 - 2\alpha \int_{\partial\Omega_j} \varphi_j (m^j \cdot \nu^j) |v|^2 d\Gamma,$$

and

$$(3.18) \quad -2\alpha\Re \int_{\Omega_j} \Delta u_1(\varphi_j m^j \cdot \nabla \bar{u}_1) dx = 2\alpha\Re \int_{\Omega_j} (\nabla u_1 \cdot \nabla \varphi_j) m^j \cdot \nabla \bar{u}_1 dx - (n-2)\alpha \int_{\Omega_j} \varphi_j |\nabla u_1|^2 dx \\ - \alpha \int_{\Omega_j} (m^j \cdot \nabla \varphi_j) |\nabla u_1|^2 dx + \alpha \int_{\partial\Omega_j} \varphi_j (m^j \cdot \nu^j) |\nabla u_1|^2 d\Gamma \\ - 2\alpha\Re \int_{\partial\Omega_j} \left(\frac{\partial u_1}{\partial \nu^j}\right) \varphi_j m^j \cdot \nabla \bar{u}_1 d\Gamma.$$

It can be checked that the boundary integral in (3.17) vanishes while

$$\alpha \int_{\partial\Omega_j} \varphi_j (m^j \cdot \nu^j) |\nabla u_1|^2 d\Gamma - 2\alpha\Re \int_{\partial\Omega_j} \left(\frac{\partial u_1}{\partial \nu^j}\right) \varphi_j m^j \cdot \nabla \bar{u}_1 d\Gamma \geq 0.$$

Hence, taking the sum over  $j$ , one derives from (3.16)-(3.18):

$$(3.19) \quad K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\frac{3}{2}\mathcal{H}}\|Z\|_{\frac{1}{2}\mathcal{H}}) \geq \alpha n \int_{\Omega} |v|^2 dx - (n-2)\alpha d \|u_1\|_1^2 - K_0 \int_{\omega_1} |v|^2 dx \\ - K_0 \int_{\Omega} a |\nabla u_1|^2 dx - 2\alpha\Re ib \sum_{j=1}^J \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{\phi} dx.$$

Gathering (3.15) and (3.19), we find

$$(3.20) \quad K_0 \left( \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\frac{1}{2}\mathcal{H}}\|Z\|_{\frac{3}{2}\mathcal{H}} + \|U\|_{\frac{3}{2}\mathcal{H}}\|Z\|_{\frac{1}{2}\mathcal{H}} \right) \geq (\alpha n - \beta) |v|_2^2 + (\beta - \alpha(n-2)) \|u_1\|_1^2 \\ - K_0 \int_{\omega_1} |v|^2 dx - K_0 \int_{\omega_1} |\nabla u_1|^2 dx \\ - 2\alpha\Re ib \sum_{j=1}^J \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{\phi} dx.$$

Thanks to the definition of  $u_1$  and (3.8), we derive from (3.20):

$$(3.21) \quad K_0 \left( \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\frac{1}{2}\mathcal{H}}\|Z\|_{\frac{3}{2}\mathcal{H}} + \|U\|_{\frac{3}{2}\mathcal{H}}\|Z\|_{\frac{1}{2}\mathcal{H}} \right) \geq (\alpha n - \beta) |v|_2^2 + \frac{(\beta - \alpha(n-2))}{2} \|u\|_1^2 \\ - K_0 \int_{\omega_1} |v|^2 dx - K_0 \int_{\omega_1} |\nabla u|^2 dx \\ - 2\alpha\Re ib \sum_{j=1}^J \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{\phi} dx.$$

Proceeding in a similar way, we derive for the  $(w, z)$ -system the following estimate

$$\begin{aligned}
(3.22) \quad K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\frac{1}{2}\mathcal{H}} \|Z\|_{\frac{3}{2}\mathcal{H}} + \|U\|_{\frac{3}{2}\mathcal{H}} \|Z\|_{\frac{1}{2}\mathcal{H}} \right) &\geq (\alpha n - \beta) |z|_2^2 + \frac{(\beta - \alpha(n-2))}{2} \|w\|_1^2 \\
&- K_0 \int_{\omega_1} |z|^2 dx - K_0 \int_{\omega_1} |\nabla w|^2 dx \\
&- 2\alpha \Re i b \sum_{j=1}^J \int_{\Omega_j} z \varphi_j m^j \cdot \nabla \bar{\phi} dx.
\end{aligned}$$

Adding (3.21) and (3.22) side by side leads to

$$\begin{aligned}
(3.23) \quad \|Z\|_{\mathcal{H}}^2 &\leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\frac{1}{2}\mathcal{H}} \|Z\|_{\frac{3}{2}\mathcal{H}} + \|U\|_{\frac{3}{2}\mathcal{H}} \|Z\|_{\frac{1}{2}\mathcal{H}} \right) \\
&+ K_0 \int_{\omega_1} (|v|^2 + |z|^2) dx + K_0 \int_{\omega_1} (|\nabla u|^2 + |\nabla w|^2) dx \\
&+ K_0 \left| \Re i b \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} dx \right|,
\end{aligned}$$

where we have set  $\psi = v + z$ .

**STEP 4: Estimating the localized kinetic energy.** In this step, we are going to estimate the first integral term in the second line of (3.23). To this end, introduce the cutoff function  $\chi$  in  $C^1(\bar{\Omega})$  with  $\chi(x) \geq d_0$  in  $\omega_1$ , for some constant  $d_0 > 0$ , and  $\chi(x) = 0$  in  $\Omega \setminus \omega$ .

Notice that

$$(3.24) \quad \int_{\omega_1} (|v|^2 + |z|^2) dx \leq \int_{\Omega} \chi^2 (|v|^2 + |z|^2) dx = \int_{\Omega} \chi^2 |\psi|^2 dx - 2\Re \int_{\Omega} \chi^2 v \bar{z} dx.$$

We can now use interpolation to estimate the first integral after the equality sign as follows

$$(3.25) \quad \int_{\Omega} \chi^2 |v + z|^2 \leq K_0 \|\chi \psi\|_{-1} \|\chi \psi\|_1 \leq K_0 (\|v\|_{-1} + \|z\|_{-1}) (|v|_2 + |z|_2 + |\chi \nabla \psi|_2).$$

If we choose  $\chi \leq a$  and use both (3.11) and (3.5), we get

$$\begin{aligned}
(3.26) \quad \int_{\Omega} \chi^2 |v + z|^2 &\leq K_0 |b|^{-1} \left( \|Z\|_{\mathcal{H}} + \|U\|_{\frac{1}{2}\mathcal{H}} \|Z\|_{\frac{3}{2}\mathcal{H}} + \|U\|_{\mathcal{H}} \right) \left( \|Z\|_{\mathcal{H}} + \|U\|_{\frac{1}{2}\mathcal{H}} \|Z\|_{\frac{3}{2}\mathcal{H}} \right) \\
&\leq K_0 |b|^{-1} \left( \|Z\|_{\mathcal{H}}^2 + \|U\|_{\frac{1}{2}\mathcal{H}} \|Z\|_{\frac{3}{2}\mathcal{H}} + \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\frac{3}{2}\mathcal{H}} \|Z\|_{\frac{1}{2}\mathcal{H}} \right).
\end{aligned}$$

We find it useful to stress that the assumption  $\chi \leq a$  enables us to roughly replace  $|\chi \nabla \psi|_2$  with  $|a \nabla \psi|_2$ , then invoke the dissipation law (3.5).

To estimate the second integral, we shall rely on the multipliers technique. Multiply the second equation in (3.4) by  $c\chi^2 \bar{w}$  and its fourth equation by  $d\chi^2 \bar{u}$ , then use Green's formula and the first and third equations in (3.4) to find

$$\begin{aligned}
(3.27) \quad &c \int_{\Omega} \chi^2 v (-\bar{z} - \bar{h}) dx + cd \int_{\Omega} \{ \chi^2 (\nabla u \cdot \nabla \bar{w}) + 2\chi \bar{w} (\nabla u \cdot \nabla \chi) \} dx \\
&+ c \int_{\Omega} a \{ \chi^2 (\nabla \psi \cdot \nabla \bar{w}) + 2\chi \bar{w} (\nabla \psi \cdot \nabla \chi) \} dx = c \int_{\Omega} g \chi^2 \bar{w} dx,
\end{aligned}$$

and

$$\begin{aligned}
(3.28) \quad &d \int_{\Omega} \chi^2 z (-\bar{v} - \bar{f}) dx + cd \int_{\Omega} \{ \chi^2 (\nabla w \cdot \nabla \bar{u}) + 2\chi \bar{u} (\nabla w \cdot \nabla \chi) \} dx \\
&+ d \int_{\Omega} a \{ \chi^2 (\nabla \psi \cdot \nabla \bar{u}) + 2\chi \bar{u} (\nabla \psi \cdot \nabla \chi) \} dx = d \int_{\Omega} \ell \chi^2 \bar{u} dx,
\end{aligned}$$

Subtracting (3.28) from (3.27) and taking real parts lead to

$$(3.29) \quad \begin{aligned} (d-c)\Re \int_{\Omega} \chi^2 v \bar{z} \, dx &= \Re \int_{\Omega} \chi^2 (cv\bar{h} - dz\bar{f}) \, dx - 2cd\Re \int_{\Omega} \chi(\bar{w}(\nabla u \cdot \nabla \chi) - \bar{u}(\nabla w \cdot \nabla \chi)) \, dx \\ &- \Re \int_{\Omega} a\{\chi^2 \nabla \psi \cdot \nabla (c\bar{w} - d\bar{u}) + 2\chi(\nabla \psi \cdot \nabla \chi)(c\bar{w} - d\bar{u})\} \, dx + \Re \int_{\Omega} \chi^2 (cg\bar{w} - d\ell\bar{u}) \, dx, \end{aligned}$$

The application of Cauchy-Schwarz and Poincaré inequalities yield

$$(3.30) \quad \begin{aligned} \left| \Re \int_{\Omega} \chi^2 \{(cv\bar{h} - dz\bar{f}) + (cg\bar{w} - d\ell\bar{u})\} \, dx \right| &\leq K_0 \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} \\ 2cd \left| \Re \int_{\Omega} \chi(\bar{w}(\nabla u \cdot \nabla \chi) - \bar{u}(\nabla w \cdot \nabla \chi)) \, dx \right| &\leq K_0 (|w|_2 \|u\|_1 + |u|_2 \|w\|_1) \\ &\leq K_0 |b|^{-1} (\|Z\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}}) \\ \left| \Re \int_{\Omega} a\{\chi^2 \nabla \psi \cdot \nabla (c\bar{w} - d\bar{u}) + 2\chi(\nabla \psi \cdot \nabla \chi)(c\bar{w} - d\bar{u})\} \, dx \right| &\leq K_0 a \|\nabla \psi\|_2 (\|u\|_1 + \|w\|_1) \\ &\leq K_0 \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}}. \end{aligned}$$

Hence

$$(3.31) \quad \left| \Re \int_{\Omega} \chi^2 v \bar{z} \, dx \right| \leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + |b|^{-1} \|Z\|_{\mathcal{H}}^2 \right).$$

Gathering (3.24), (3.26) and (3.31), we derive

$$(3.32) \quad \int_{\Omega} \chi^2 (|v|^2 + |z|^2) \, dx \leq K_0 \left( \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + |b|^{-1} \|Z\|_{\mathcal{H}}^2 \right).$$

**STEP 5: Estimating the localized potential energy.** Let  $\chi$  be the cutoff function introduced in Step 4.

Multiply the second equation in (3.4) by  $\chi^2 \bar{u}$  and its fourth equation by  $\chi^2 \bar{w}$  and apply Green's formula to derive

$$(3.33) \quad \begin{aligned} \int_{\Omega} \chi^2 (d|\nabla u|^2 + c|\nabla w|^2) \, dx &= \Re \int_{\Omega} \chi^2 (|v|^2 + |z|^2 + v\bar{f} + z\bar{h}) \, dx + \Re \int_{\Omega} \chi^2 (g\bar{u} + \ell\bar{w}) \, dx \\ &- 2\Re \int_{\Omega} \chi(d\bar{u}(\nabla u \cdot \nabla \chi) + c\bar{w}(\nabla w \cdot \nabla \chi)) \, dx \\ &- \Re \int_{\Omega} a\{\chi^2 \nabla \psi \cdot \nabla (\bar{w} + \bar{u}) + 2\chi(\nabla \psi \cdot \nabla \chi)(\bar{w} + \bar{u})\} \, dx. \end{aligned}$$

Estimating the integrals in the right of (3.33) as we did in Step 4, and use (3.32), we get

$$(3.34) \quad \begin{aligned} &\int_{\Omega} \chi^2 (d|\nabla u|^2 + c|\nabla w|^2) \, dx \\ &\leq K_0 \left( \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + |b|^{-1} \|Z\|_{\mathcal{H}}^2 \right). \end{aligned}$$

At this stage, if we combine (3.23), (3.32) and (3.34), and choose  $b$  large enough, we should be able to absorb the term involving  $|b|^{-1}$  to the left, thereby finding

$$(3.35) \quad \begin{aligned} \|Z\|_{\mathcal{H}}^2 &\leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}}^2 \right) \\ &+ K_0 \left| \Re i b \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} \, dx \right|. \end{aligned}$$

It remains to estimate the integral term. This will be done in the next step.

**STEP 6: Final estimate and completion of the proof.** In this step, we are going to estimate the integral term in (3.35). To this end introduce for each  $j = 1, \dots, J$  the function  $q_j$  lying in  $H_0^1(\Omega)$  that satisfies:

$$(3.36) \quad \Delta q_j = \operatorname{div}(\mathbf{1}_{\Omega_j} \psi \varphi_j m^j),$$

where  $\mathbf{1}_{\Omega_j}$  denotes the characteristic function of  $\Omega_j$ .

On the one hand, multiplying (3.36) by  $\bar{\phi}$  and applying Green's formula, one readily derives

$$(3.37) \quad \Re ib \sum_{j=1}^J \int_{\Omega} \nabla q_j \cdot \nabla \bar{\phi} \, dx = \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} \, dx.$$

On the other hand, first taking the conjugates in (3.7), then multiplying the resulting equation by  $q_j$  and finally applying Green's formula, one finds

$$(3.38) \quad \int_{\Omega} \nabla q_j \cdot \nabla \bar{\phi} \, dx = - \int_{\Omega} a \nabla q_j \cdot \nabla \bar{\psi} \, dx.$$

Applying Green's formula to the right side of the last equation leads to

$$(3.39) \quad - \int_{\Omega} a \nabla q_j \cdot \nabla \bar{\psi} \, dx = \int_{\Omega} \bar{\psi} \nabla q_j \cdot \nabla a + a \bar{\psi} \Delta q_j \, dx.$$

Now, using (3.36) and differentiation yield

$$(3.40) \quad \begin{aligned} \int_{\Omega} a \bar{\psi} \Delta q_j \, dx &= \int_{\Omega} a \bar{\psi} \operatorname{div}(\mathbf{1}_{\Omega_j} \psi \varphi_j m^j) \, dx \\ &= \int_{\Omega} \{a |\psi|^2 \operatorname{div}(\mathbf{1}_{\Omega_j} \varphi_j m^j) + a \bar{\psi} \mathbf{1}_{\Omega_j} \varphi_j \nabla \psi \cdot m^j\} \, dx. \end{aligned}$$

Therefore, it follows from (3.37)-(3.40):

$$(3.41) \quad \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} \, dx = \Re ib \sum_{j=1}^J \int_{\Omega} \{\bar{\psi} \nabla q_j \cdot \nabla a + a \bar{\psi} \mathbf{1}_{\Omega_j} \varphi_j \nabla \psi \cdot m^j\} \, dx.$$

Thanks to Cauchy-Schwarz inequality one gets from (3.41) and the structural constraint on the damping coefficient  $a$ , one gets

$$(3.42) \quad \left| \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} \, dx \right| \leq K_0 |b| |\sqrt{a} \psi|_2 \left( \sum_{j=1}^J \|q_j\|_1 + |\sqrt{a} \nabla \psi|_2 \right).$$

Using the variational principle, one readily checks that for each  $j$ , one has

$$(3.43) \quad \|q_j\|_1 \leq K_0 (|v|_2 + |z|_2) \leq K_0 \|Z\|_{\mathcal{H}},$$

so that (3.42) becomes

$$(3.44) \quad \left| \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} \, dx \right| \leq K_0 |b| |\sqrt{a} \psi|_2 (\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}}).$$

Therefore, it remains to estimate  $|\sqrt{a} \psi|_2$ .

Before doing so, we find it useful to indicate that for exponential decay to hold, we should be able to suitably bound from above the term  $|b| |\sqrt{a} \psi|_2$  uniformly with respect to  $b$ ; but this bound fails even in the case where the damping is distributed everywhere in  $\Omega$ , that is to say, when  $a \equiv 1$  in  $\Omega$ , see [6, 26]. This failure is due to the fact that the matrix defining the damping is singular. So, the singularity and local nature of the damping mechanism both seem to preclude the exponential stability of the coupled system.

Back to estimating  $|\sqrt{a} \psi|_2$ . We start by adding the second and fourth equations in (3.4), thereby obtaining

$$ib\psi = d\Delta u + c\Delta w + 2\operatorname{div}(a\nabla\psi) + g + \ell.$$

Now multiply that equation by  $a\bar{\psi}$ , then apply Green's formula to get

$$\begin{aligned}
(3.45) \quad ib \int_{\Omega} a|\psi|^2 dx &= - \int_{\Omega} \{\bar{\psi} \nabla a \cdot (d \nabla u + c \nabla w) + a \nabla \bar{\psi} \cdot (d \nabla u + c \nabla w)\} dx \\
&\quad - 2 \int_{\Omega} \{a^2 |\nabla \psi|^2 + a \bar{\psi} \nabla a \cdot \nabla \psi\} dx + \int_{\Omega} a(g + \ell) \bar{\psi} dx \\
&= \int_{\Omega} \{(\nabla \bar{\psi} \cdot \nabla a + \bar{\psi} \Delta a)(du + cw) - a \nabla \bar{\psi} \cdot (d \nabla u + c \nabla w)\} dx \\
&\quad - 2 \int_{\Omega} \{a^2 |\nabla \psi|^2 + a \bar{\psi} \nabla a \cdot \nabla \psi\} dx + \int_{\Omega} a(g + \ell) \bar{\psi} dx
\end{aligned}$$

Taking the imaginary parts in (3.45) and using Cauchy-Schwarz inequality, one derives

$$\begin{aligned}
(3.46) \quad |b| \int_{\Omega} a|\psi|^2 dx &\leq K_0 |\sqrt{a}\psi|_2 (|u|_2 + |w|_2 + |\sqrt{a}\nabla\psi|_2 + |g + \ell|_2) \\
&\quad + K_0 |\sqrt{a}\nabla\psi|_2 (|\sqrt{a}\nabla u|_2 + |\sqrt{a}\nabla w|_2) + K_0 |\sqrt{a}\nabla\psi|_2 (|u|_2 + |w|_2) \\
&\leq \frac{|b|}{2} \int_{\Omega} a|\psi|^2 dx + K_0 (|b|^{-1} \|U\|_{\mathcal{H}}^2 + |b|^{-2} \|Z\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}}) \\
&\quad + |\sqrt{a}\nabla u|_2^2 + |\sqrt{a}\nabla w|_2^2.
\end{aligned}$$

To complete estimating  $|\sqrt{a}\psi|_2$ , it remains to estimate the terms in the last line of (3.46). To this end, we notice that one may rewrite the sum of those two terms as

$$(3.47) \quad |\sqrt{a}\nabla u|_2^2 + |\sqrt{a}\nabla w|_2^2 = |\sqrt{a}(\nabla u + \nabla w)|_2^2 - 2\Re \int_{\Omega} a \nabla u \nabla \bar{w} dx.$$

Thanks to (3.6), one readily estimates the first term in (3.47). To estimate the integral term in (3.47), first we are going to express it differently using a combination of multipliers technique, Green's formula and algebra. For this purpose, multiply the second (3.4) by  $a\bar{w}$  and its fourth equation by  $a\bar{u}$ , then use Green's formula to derive

$$ib \int_{\Omega} a\bar{w} dx + d \int_{\Omega} \{a \nabla u \cdot \nabla \bar{w} + \bar{w} \nabla u \cdot \nabla a\} dx + \int_{\Omega} \{a^2 \nabla \psi \cdot \nabla \bar{w} + a \bar{w} \nabla \psi \cdot \nabla a\} dx = \int_{\Omega} g a \bar{w} dx,$$

and

$$ib \int_{\Omega} a\bar{u} dx + c \int_{\Omega} \{a \nabla w \cdot \nabla \bar{u} + \bar{u} \nabla w \cdot \nabla a\} dx + \int_{\Omega} \{a^2 \nabla \psi \cdot \nabla \bar{u} + a \bar{u} \nabla \psi \cdot \nabla a\} dx = \int_{\Omega} \ell a \bar{u} dx.$$

Subtracting the latter equation from the former, then using the first and third equation on (3.4), we get

$$\begin{aligned}
(3.48) \quad (d - c) \Re \int_{\Omega} a \nabla u \cdot \nabla \bar{w} dx &= \Re \int_{\Omega} \{av(\bar{z} + \bar{h}) - az(\bar{p} + \bar{f})\} dx + \Re \int_{\Omega} (c\bar{u}\nabla w - d\bar{w}\nabla u) \cdot \nabla a dx \\
&\quad + \Re \int_{\Omega} \{a^2 \nabla \psi \cdot \nabla (u - w) + a(\nabla a \cdot \nabla \psi)(\bar{u} - \bar{w}) + a(g\bar{w} - \ell\bar{u})\} dx.
\end{aligned}$$

Using the Cauchy Schwarz inequality in (3.48) leads to

$$\begin{aligned}
(3.49) \quad 2 \left| \Re \int_{\Omega} a \nabla u \cdot \nabla \bar{w} dx \right| &\leq K_0 (|v|_2 |h|_2 + |z|_2 |f|_2 + |u|_2 |\sqrt{a}\nabla w|_2 + |w|_2 |\sqrt{a}\nabla u|_2) + |g|_2 |w|_2 \\
&\quad + (|\sqrt{a}\nabla u|_2 + |\sqrt{a}\nabla w|_2) |\sqrt{a}\nabla \psi|_2 + (|u|_2 + |w|_2) |\sqrt{a}\nabla \psi|_2 + |\ell|_2 |u|_2.
\end{aligned}$$

Using Poincaré and Young inequalities as needed, and it follows from (3.49):

$$\begin{aligned}
(3.50) \quad 2 \left| \Re \int_{\Omega} a \nabla u \cdot \nabla \bar{w} dx \right| &\leq K_0 \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \frac{1}{2} (|\sqrt{a}\nabla u|_2^2 + |\sqrt{a}\nabla w|_2^2) \\
&\quad + K_0 b^{-2} (\|U\|_{\mathcal{H}}^2 + \|Z\|_{\mathcal{H}}^2),
\end{aligned}$$

where the terms in the last line is obtained by using the first and third equation in (3.4). Gathering (3.47) and (3.50), and using (3.6), we find

$$(3.51) \quad |\sqrt{a}\nabla u|_2^2 + |\sqrt{a}\nabla w|_2^2 \leq K_0 \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + K_0 b^{-2} (\|U\|_{\mathcal{H}}^2 + \|Z\|_{\mathcal{H}}^2).$$

The combination of (3.46) and (3.51) then shows that

$$(3.52) \quad |b| \int_{\Omega} a |\psi|^2 dx \leq K_0 (|b|^{-1} \|U\|_{\mathcal{H}}^2 + |b|^{-2} \|Z\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}}).$$

Using (3.52) in (3.44), we deduce

$$(3.53) \quad \left| \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} dx \right| \leq K_0 \left( |b|^{-\frac{1}{2}} \|Z\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + |b|^{\frac{1}{2}} \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + |b|^{\frac{1}{2}} \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} \right).$$

Collecting (3.35) and (3.53), then choosing  $|b|$  large enough, we obtain

$$(3.54) \quad \|Z\|_{\mathcal{H}}^2 \leq K_0 \left( |b|^{\frac{1}{2}} \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + |b|^{\frac{1}{2}} \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}}^2 \right).$$

Finally, the application of Young inequality yields

$$(3.55) \quad \|Z\|_{\mathcal{H}}^2 \leq K_0 b^2 \|U\|_{\mathcal{H}}^2 \text{ or } \|Z\|_{\mathcal{H}} \leq K_0 |b| \|U\|_{\mathcal{H}},$$

which is the desired estimate, thereby completing the proof of Theorem 1.1.  $\square$

**3.2. Proof of Theorem 1.2.** Thanks to [14, Theorem 2.4] and [11, Proposition 3.1], the proof of Theorem 1.2 will be completed once we show the resolvent estimate

$$(3.56) \quad \|(ib\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(|b|^2) \text{ as } |b| \nearrow +\infty.$$

To prove this estimate, using the notations introduced above amounts to showing that we have

$$(3.57) \quad \|Z\|_{\mathcal{H}} \leq K_0 b^2 \|U\|_{\mathcal{H}}.$$

The proof of that estimate goes along the proof provided in the section above. Therefore, we already have:

$$(3.58) \quad \|Z\|_{\mathcal{H}}^2 \leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}}^2 \right) + K_0 \left| \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} dx \right|.$$

Notice that, that inequality was established without any smoothness assumption on the damping coefficient. To estimate the integral term now, we just apply the Cauchy-Schwarz inequality, getting

$$(3.59) \quad \left| \Re ib \sum_{j=1}^J \int_{\Omega_j} \psi \varphi_j m^j \cdot \nabla \bar{\phi} dx \right| \leq K_0 |b| (|v|_2 + |z|_2) \|\phi\|_1 \leq K_0 |b| \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}},$$

where the last inequality is obtained by invoking (3.8).

Combining (3.58) and (3.59), we find

$$(3.60) \quad \|Z\|_{\mathcal{H}}^2 \leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + |b| \|U\|_{\mathcal{H}}^{\frac{1}{2}} \|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^{\frac{3}{2}} \|Z\|_{\mathcal{H}}^{\frac{1}{2}} + \|U\|_{\mathcal{H}}^2 \right).$$

The application of Young inequality leads to (3.57), thereby completing the proof.  $\square$

#### 4. Proof of logarithmic stability

In this section, we will prove the logarithmic stability of the system (1.1). To this end, we establish a particular resolvent estimate. Precisely we will show that for some constant  $C > 0$  we have

$$(4.1) \quad \|(i\mu \text{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{C|\mu|}, \quad \forall |\mu| \gg 1,$$

and then by Burq's result [15] and the remark of Duyckaerts [19, section 7] (see also [12, 14]) we obtain the expected decay rate of the energy given in Theorem 1.3.

Reasoning by contraposition, assume that the resolvent estimate (4.1) does not hold. Then by the continuity of the resolvent and the resonance theorem there exist  $K_m > 0$ ,  $\mu_m \in \mathbb{R}$  and a two sequences  $(u_{1,m}, u_{2,m}, v_{1,m}, v_{2,m}) \in \mathcal{D}(\mathcal{A})$  and  $(f_{1,m}, f_{2,m}, g_{1,m}, g_{2,m}) \in \mathcal{H}$ ,  $m = 1, 2, \dots$ , such that

$$(4.2) \quad |\mu_m| \xrightarrow{m \rightarrow \infty} +\infty, \quad K_m \xrightarrow{m \rightarrow \infty} +\infty, \quad \|(u_{1,m}, u_{2,m}, v_{1,m}, v_{2,m})\|_{\mathcal{H}} = 1 \quad \forall m \in \mathbb{N},$$

and

$$(4.3) \quad e^{K_m|\mu_m|}(i\mu_m \text{Id} - \mathcal{A}) \begin{pmatrix} u_{1,m} \\ u_{2,m} \\ v_{1,m} \\ v_{2,m} \end{pmatrix} = \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ g_{1,m} \\ g_{2,m} \end{pmatrix} \xrightarrow{m \rightarrow \infty} 0 \text{ in } \mathcal{H}.$$

This implies that

$$(4.4) \quad e^{K_m|\mu_m|}(i\mu_m u_{1,m} - v_{1,m}) = f_{1,m} \xrightarrow{m \rightarrow \infty} 0 \text{ in } H^1(\Omega),$$

$$(4.5) \quad e^{K_m|\mu_m|}(i\mu_m u_{2,m} - v_{2,m}) = f_{2,m} \xrightarrow{m \rightarrow \infty} 0 \text{ in } H^1(\Omega),$$

$$(4.6) \quad e^{K_m|\mu_m|}(i\mu_m v_{1,m} - a_1 \Delta u_{1,m} - \text{div}(a(x)\nabla(v_{1,m} + v_{2,m}))) = g_{1,m} \xrightarrow{m \rightarrow \infty} 0 \text{ in } L^2(\Omega),$$

$$(4.7) \quad e^{K_m|\mu_m|}(i\mu_m v_{2,m} - a_2 \Delta u_{2,m} - \text{div}(a(x)\nabla(v_{1,m} + v_{2,m}))) = g_{2,m} \xrightarrow{m \rightarrow \infty} 0 \text{ in } L^2(\Omega).$$

where we have denoted by  $a_1 = d$  and  $a_2 = c$ . From (4.2) and (4.3), we get

$$(4.8) \quad \Re \left\langle \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ g_{1,m} \\ g_{2,m} \end{pmatrix}, \begin{pmatrix} u_{1,m} \\ u_{2,m} \\ v_{1,m} \\ v_{2,m} \end{pmatrix} \right\rangle_{\mathcal{H}} = -e^{K_m|\mu_m|} \int_{\Omega} a(x) |\nabla v_{1,m} + \nabla v_{2,m}|^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

Taking the gradient of (4.4) and (4.5), summing up and taking the  $L^2$  norm over  $\omega$  then following to (4.8) one gets

$$(4.9) \quad |\mu_m|^2 e^{K_m|\mu_m|} \int_{\omega} |\nabla u_{1,m} + \nabla u_{2,m}|^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

From (4.8) and (4.9) we have

$$(4.10) \quad e^{K_m|\mu_m|} \left( \int_{\omega} |\nabla u_{1,m} + \nabla u_{2,m}|^2 dx + \int_{\omega} |\nabla v_{1,m} + \nabla v_{2,m}|^2 dx \right) \xrightarrow{m \rightarrow \infty} 0.$$

Let  $\chi \in \mathcal{C}_c^\infty(\omega)$  then from (4.2), (4.4) and (4.5) we have

$$(4.11) \quad \frac{1}{|\mu_m|^2} \left( \|\nabla(\chi \cdot v_{1,m})\|_{L^2(\omega)}^2 + \|\nabla(\chi \cdot v_{2,m})\|_{L^2(\omega)}^2 \right) = O(1).$$

Multiplying (4.6) by  $\frac{\mu_m^{-1}}{a_1} \chi \cdot \bar{v}_{1,m}$  and (4.7) by  $\frac{\mu_m^{-1}}{a_2} \chi \cdot \bar{v}_{2,m}$ , integrating over  $\Omega$ , summing up and taking the imaginary part then by (4.2), (4.10) and (4.11) we find

$$(4.12) \quad e^{K_m|\mu_m|} \frac{1}{a_1} \int_{\omega} \chi |v_{1,m}|^2 dx + e^{K_m|\mu_m|} \frac{1}{a_2} \Re \int_{\omega} \chi v_{1,m} \bar{v}_{2,m} dx \xrightarrow{m \rightarrow \infty} 0.$$



Arguing the same as in (4.12) by just replacing  $\bar{v}_{1,m}$  with  $\bar{v}_{2,m}$  in the previous operations then one obtains

$$(4.13) \quad e^{K_m|\mu_m|} \frac{1}{a_2} \int_{\omega} \chi |v_{2,m}|^2 dx + e^{K_m|\mu_m|} \frac{1}{a_1} \Re \int_{\omega} \chi v_{1,m} \bar{v}_{2,m} dx \xrightarrow{m \rightarrow \infty} 0.$$

Multiplying (4.12) by  $\frac{1}{a_1}$  and (4.13) by  $\frac{1}{a_2}$  and summing up, we follow

$$(4.14) \quad e^{K_m|\mu_m|} \int_{\omega} \chi \left| \frac{1}{a_1} v_{1,m} + \frac{1}{a_2} v_{2,m} \right|^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

Multiplying (4.6) by  $-i\mu_m^{-1}\chi^2 \cdot \left( \frac{1}{a_1} \bar{v}_{1,m} + \frac{1}{a_2} \bar{v}_{2,m} \right)$  and integrating by parts

$$(4.15) \quad \begin{aligned} & \frac{e^{K_m|\mu_m|}}{a_1} \int_{\omega} |v_{1,m}|^2 \chi^2 dx + \frac{e^{K_m|\mu_m|}}{a_2} \int_{\omega} v_{1,m} \bar{v}_{2,m} \chi^2 dx \\ & + 2a_1 \frac{e^{K_m|\mu_m|}}{i\mu_m} \int_{\omega} \nabla u_{1,m} \left( \frac{1}{a_1} \bar{v}_{1,m} + \frac{1}{a_2} \bar{v}_{2,m} \right) \chi \nabla \chi dx \\ & + a_1 \frac{e^{K_m|\mu_m|}}{i\mu_m} \int_{\omega} \nabla u_{1,m} \left( \frac{1}{a_1} \nabla \bar{v}_{1,m} + \frac{1}{a_2} \nabla \bar{v}_{2,m} \right) \chi^2 dx \\ & + \frac{e^{K_m|\mu_m|}}{i\mu_m} \int_{\omega} a(x) (\nabla v_{1,m} + \nabla v_{2,m}) \nabla \left( \left( \frac{1}{a_1} \bar{v}_{1,m} + \frac{1}{a_2} \bar{v}_{2,m} \right) \chi^2 \right) dx \\ & = \frac{1}{i\mu_m} \int_{\omega} g_{1,m} \left( \frac{1}{a_1} \bar{v}_{1,m} + \frac{1}{a_2} \bar{v}_{2,m} \right) \chi^2 dx. \end{aligned}$$

Due to (4.2), (4.10), (4.11) and (4.14), the third and the fifth left hand side of (4.15) go to zero as  $m$  goes to the infinity. Moreover, by taking its real part and using (4.14) we get

$$(4.16) \quad \frac{e^{K_m|\mu_m|}}{\mu_m} \Im \int_{\omega} \nabla u_{1,m} \left( \frac{1}{a_1} \nabla \bar{v}_{1,m} + \frac{1}{a_2} \nabla \bar{v}_{2,m} \right) \chi^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

Following to (4.4) we have

$$(4.17) \quad \begin{aligned} \frac{e^{K_m|\mu_m|}}{\mu_m} \int_{\omega} \nabla u_{1,m} \left( \frac{1}{a_1} \nabla \bar{v}_{1,m} + \frac{1}{a_2} \nabla \bar{v}_{2,m} \right) \chi^2 dx &= \frac{e^{K_m|\mu_m|}}{a_2 \mu_m} \int_{\omega} \nabla u_{1,m} (\nabla \bar{v}_{1,m} + \nabla \bar{v}_{2,m}) \chi^2 dx \\ &+ \frac{e^{K_m|\mu_m|}}{\mu_m} \left( \frac{1}{a_1} - \frac{1}{a_2} \right) \int_{\omega} \nabla u_{1,m} \nabla \bar{v}_{1,m} \chi^2 dx \\ &= \frac{e^{K_m|\mu_m|}}{a_2 \mu_m} \int_{\omega} \nabla u_{1,m} (\nabla \bar{v}_{1,m} + \nabla \bar{v}_{2,m}) \chi^2 dx \\ &- i e^{K_m|\mu_m|} \left( \frac{1}{a_1} - \frac{1}{a_2} \right) \int_{\omega} |\nabla u_{1,m}|^2 \chi^2 dx \\ &+ \frac{1}{\mu_m} \left( \frac{1}{a_1} - \frac{1}{a_2} \right) \int_{\omega} \nabla u_{1,m} \nabla \bar{f}_{1,m} \chi^2 dx. \end{aligned}$$

In view of (4.2), (4.3) and (4.14) the first and the third terms of the right hand side of (4.17) go to zero as  $m$  goes to the infinity. As a result we deduce from (4.16) and (4.17) that

$$(4.18) \quad e^{K_m|\mu_m|} \int_{\omega} |\nabla u_{1,m}|^2 \chi^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

Multiplying (4.6) by  $\frac{1}{i\mu_m} \bar{v}_1 \chi^2$ , and integrating over  $\Omega$  by integration by parts

$$\begin{aligned} e^{K_m|\mu_m|} \int_{\omega} |v_{1,m}|^2 \chi^2 dx &= \frac{1}{i\mu_m} \int_{\omega} g_{1,m} \bar{v}_{1,m} dx - \frac{e^{K_m|\mu_m|}}{i\mu_m} \int_{\omega} a(x) (\nabla v_{1,m} + \nabla v_{2,m}) \nabla (\bar{v}_{1,m} \chi^2) dx \\ &- \frac{a_1 e^{K_m|\mu_m|}}{i\mu_m} \left( \int_{\omega} \nabla u_{1,m} \cdot \nabla \bar{v}_{1,m} \chi^2 dx + 2 \int_{\omega} \nabla u_{1,m} \cdot \nabla \chi \bar{v}_{1,m} \chi dx \right), \end{aligned}$$

then by virtue of (4.2), (4.4), (4.10), (4.11) and (4.18) it derives

$$(4.19) \quad e^{K_m|\mu_m|} \int_{\omega} |v_{1,m}|^2 \chi^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

We can arguing the same as in (4.19) then we can prove also that

$$(4.20) \quad e^{K_m|\mu_m|} \int_{\omega} |v_{2,m}|^2 \chi^2 dx \xrightarrow{m \rightarrow \infty} 0.$$

By setting  $B_{5r}$  a ball of radius  $5r > 0$ , such that  $B_{5r} \subset \omega$  then we can deduce from (4.19) and (4.20) that

$$e^{K_m|\mu_m|} \left( \int_{B_{5r}} |v_{1,m}|^2 dx + \int_{B_{5r}} |v_{2,m}|^2 dx \right) \xrightarrow{m \rightarrow \infty} 0$$

and due to (4.4) and (4.5) we thus have

$$(4.21) \quad |\mu_m|^2 e^{K_m|\mu_m|} \left( \int_{B_{5r}} |u_{1,m}|^2 dx + \int_{B_{5r}} |u_{2,m}|^2 dx \right) \xrightarrow{m \rightarrow \infty} 0.$$

At this stage, our focus lies in establishing a stability estimates for the following Helmholtz equation

$$(4.22) \quad \mu^2 u + \alpha \Delta u = f \quad \text{in } \Omega$$

for large wave numbers  $\mu$  where  $\alpha$  is positive constant. To do so the Carleman estimates will be our main ingredient in this part. So first let's recall the classical Carleman observation inequality: Consider  $\mathcal{O}$  an open set of  $\mathbb{R}^n$  with a boundary  $\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$  such that  $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$ . Let  $\varphi \in \mathcal{C}^\infty(\mathcal{O})$  be a real valued function and define the operator  $P = -\alpha\Delta - \mu^2$  and its adjoint operator  $P_\varphi = e^{\mu\varphi} P e^{-\mu\varphi}$  whose principal symbol is given by  $p(x, \xi, \mu) = \alpha|\xi + i\nabla\varphi|^2 - \mu^2$ . Now we are ready to state the classical Carleman estimate (see [15, 28, 29])

**Proposition 4.1.** *Assume that the weight function  $\varphi$  satisfying the following assumptions:  $|\nabla\varphi(x)| > 0$  for every  $x \in \bar{\mathcal{O}}$  and  $\partial_\nu\varphi|_{\partial\mathcal{O}} \neq 0$  and  $\partial_\nu\varphi|_{\Gamma_1} < 0$  where we denoted by  $\nu$  the unit outward normal vector of  $\mathcal{O}$ . We suppose also that the sub-ellipticity condition: For some  $c > 0$  and  $\mu_0 > 0$*

$$\forall(x, \xi) \in \bar{\mathcal{O}} \times \mathbb{R}^n, \mu \geq \mu_0; \quad p_\varphi(x, \xi, \mu) \implies \{\Re(p_\varphi), \Im(p_\varphi)\}(x, \xi, \mu) \geq c\langle \xi, \mu \rangle^3,$$

where we have denoted by  $\langle \xi, \mu \rangle = (|\xi|^2 + \mu^2)^{\frac{1}{2}}$  and  $\{., .\}$  is the Poisson bracket.

Then there exists  $C > 0$  such that

$$\mu^3 \|e^{\mu\varphi} u\|_{L^2(\mathcal{O})}^2 + \mu \|e^{\mu\varphi} \nabla u\|_{L^2(\mathcal{O})}^2 \leq C \left( \|e^{\mu\varphi} P u\|_{L^2(\mathcal{O})}^2 + \mu^3 \|e^{\mu\varphi} u\|_{L^2(\Gamma_2)}^2 + \mu \|e^{\mu\varphi} \partial_\nu u\|_{L^2(\Gamma_2)}^2 \right)$$

for all  $\mu \geq \mu_0$  and  $u \in H^2(\mathcal{O})$  such that  $u|_{\Gamma_1} = 0$ .

Now we are ready to come back to the Helmholtz equation (4.22) and establishing the following estimate

**Proposition 4.2.** *There exists  $C > 0$  such that*

$$(4.23) \quad \|u\|_{H^1(\Omega)}^2 \leq C e^{C|\mu|} \left( \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(B_{4r})}^2 \right)$$

for every  $u \in H^2(\Omega)$  solution of (4.22) with  $u = 0$  in  $\partial\Omega$ .

*Proof.* Let  $\chi \in \mathcal{C}^\infty(\Omega)$  be a cutt-off function defined by

$$\chi(x) = \begin{cases} 1 & \text{in } B_{3r}^c \\ 0 & \text{in } B_{2r}, \end{cases}$$

and denoting by  $\tilde{u} = \chi u$ . Then according to (4.22), we have

$$(4.24) \quad -\alpha \Delta \tilde{u} - \mu^2 \tilde{u} = \tilde{f} \quad \text{in } \Omega,$$

where

$$\tilde{f} = -\chi f - [\alpha\Delta, \chi]u.$$

Now we recall that according to [15, 20] there exist  $\varepsilon > 0$ , a finite numbers of points  $x_1^1, \dots, x_{N_1}^1$  and  $x_1^2, \dots, x_{N_2}^2$  in  $\tilde{\Omega} = \Omega \setminus \overline{B_r}$  and two weight functions  $\varphi_1$  and  $\varphi_2$  that satisfy the assumptions of Proposition 4.1 in  $\mathcal{U}_1 = \tilde{\Omega} \cap \left( \bigcup_{i=1}^{N_1} B(x_i^1, \varepsilon) \right)^c$  and  $\mathcal{U}_2 = \tilde{\Omega} \cap \left( \bigcup_{i=1}^{N_2} B(x_i^2, \varepsilon) \right)^c$  respectively. Moreover, we can also suppose that  $\varphi_1 < \varphi_2$  in  $B(x_i^1, 2\varepsilon)$  for all  $i = 1, \dots, N_1$  and  $\varphi_2 < \varphi_1$  in  $B(x_i^2, 2\varepsilon)$  for all  $i = 1, \dots, N_2$ . Let  $\chi_1$  and  $\chi_2$  be two cut-off functions equal to one respectively in  $\left( \bigcup_{i=1}^{N_1} B(x_i^1, 2\varepsilon) \right)^c$  and  $\left( \bigcup_{i=1}^{N_2} B(x_i^2, 2\varepsilon) \right)^c$  and supported respectively in  $\left( \bigcup_{i=1}^{N_1} B(x_i^1, \varepsilon) \right)^c$  and  $\left( \bigcup_{i=1}^{N_2} B(x_i^2, \varepsilon) \right)^c$  (in order to eliminate the critical points of the phases functions  $\varphi_1$  and  $\varphi_2$ ).

We set now  $w_1 = \chi_1 \tilde{u}$  and  $w_2 = \chi_2 \tilde{u}$ . Then from (4.24) and for  $k = 1, 2$  we have

$$(4.25) \quad -\alpha\Delta w_k - \mu^2 w_k = \Phi_k \quad \text{in } \mathcal{U}_k,$$

where

$$\Phi_k = \chi_k \tilde{f} - [\alpha\Delta, \chi_k] \tilde{u}.$$

Applying now the Carleman estimate of Proposition 4.1 for  $k = 1, 2$  we obtain

$$\mu^3 \|e^{\mu\varphi_k} w_k\|_{L^2(\mathcal{U}_k)}^2 + \mu \|e^{\mu\varphi_k} \nabla w_k\|_{L^2(\mathcal{U}_k)}^2 \leq C \|e^{\mu\varphi_k} \Phi_k\|_{L^2(\mathcal{U}_k)}^2,$$

which yields to

$$(4.26) \quad \mu^3 \|e^{\mu\varphi_k} w_k\|_{L^2(\mathcal{U}_k)}^2 + \mu \|e^{\mu\varphi_k} \nabla w_k\|_{L^2(\mathcal{U}_k)}^2 \leq C \left( \|e^{\mu\varphi_k} \tilde{f}\|_{L^2(\mathcal{U}_k)}^2 + \|e^{\mu\varphi_k} [\Delta, \chi_k] \tilde{u}\|_{L^2(\mathcal{U}_k)}^2 \right).$$

Taking the sum for  $k = 1, 2$  in (4.26) and then the fact that  $\varphi_1 < \varphi_2$  in  $\left( \bigcup_{i=1}^{N_1} B(x_i^1, 2\varepsilon) \right)$  and  $\varphi_2 < \varphi_1$  in  $\left( \bigcup_{i=1}^{N_2} B(x_i^2, 2\varepsilon) \right)$  allow us to absorb the terms  $[\Delta, \chi_k] \tilde{u}$ , supported in  $\left( \bigcup_{i=1}^{N_2} B(x_i^k, 2\varepsilon) \right)$ , into the left hand side for  $\mu > 0$  large enough

$$\mu^3 \int_{\tilde{\Omega}} (e^{2\mu\varphi_1} + e^{2\mu\varphi_2}) |\tilde{u}|^2 dx + \mu \int_{\tilde{\Omega}} (e^{2\mu\varphi_1} + e^{2\mu\varphi_2}) |\nabla \tilde{u}|^2 dx \leq C \int_{\tilde{\Omega}} (e^{2\mu\varphi_1} + e^{2\mu\varphi_2}) |\tilde{f}|^2 dx,$$

and consequently we end up with

$$(4.27) \quad \int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx \leq C e^{C\mu} \left( \int_{\Omega} |f|^2 dx + \int_{B_{2r}} |u|^2 dx + \int_{\Omega} |[\alpha\Delta, \chi]u|^2 dx + \int_{\tilde{\Omega}} |\nabla((1-\chi)u)|^2 dx \right)$$

for some constant  $C > 0$ . Let  $\tilde{\chi}$  be a cut-off function equal to 1 in a neighborhood of  $B_{3r}$  and supported in  $B_{4r}$  then from (4.22) we have

$$\alpha\Delta(\tilde{\chi}u) = [\alpha\Delta, \tilde{\chi}]u + \alpha\tilde{\chi}\Delta u = [\alpha\Delta, \tilde{\chi}]u - \mu^2 \tilde{\chi}u + \tilde{\chi}f.$$

Elliptic estimates imply

$$(4.28) \quad \begin{aligned} \|u\|_{H^1(B_{3r})}^2 &\leq C \left( \|\Delta(\tilde{\chi}u)\|_{H^{-1}(B_{4r})}^2 + \|u\|_{L^2(B_{4r})}^2 \right) \\ &\leq C \left( (1 + \mu^4) \|u\|_{L^2(B_{4r})}^2 + \|f\|_{L^2(B_{4r})}^2 \right) \end{aligned}$$

Since both operators  $\nabla((1-\chi))$  and  $[\alpha\Delta, \chi]$  are of order one and supported in  $B_{3r}$ , from (4.28) we can deduce

$$(4.29) \quad \int_{\Omega} |[\alpha\Delta, \chi]u|^2 dx + \int_{\bar{\Omega}} |\nabla((1-\chi)u)|^2 \leq \|u\|_{H^1(B_{3r})}^2 \\ \leq C \left( (1+\mu^4) \|u\|_{L^2(B_{4r})}^2 + \|f\|_{L^2(B_{4r})}^2 \right).$$

Now estimate (4.23) fulfilled from (4.27) and (4.29).  $\square$

The following result is a refinement of the previous proposition (see [16]).

**Proposition 4.3.** *There exists  $C > 0$  such that*

$$\|u\|_{H^1(\Omega)}^2 \leq C e^{C|\mu|} \left( \|f\|_{H^{-1}(\Omega)}^2 + \|u\|_{L^2(B_{5r})}^2 \right)$$

for every  $u \in H_0^1(\Omega)$  solution of (4.22) where  $f \in H^{-1}(\Omega)$ .

*Proof.* Let  $\chi \in C_0^\infty(\Omega)$  and equal to 1 in  $B_{4r}$  with support in  $B_{5r}$ . Then estimate (4.23) implies

$$(4.30) \quad \|u\|_{H^1(\Omega)}^2 \leq C e^{C|\mu|} \left( \|f\|_{L^2(\Omega)}^2 + \|\chi u\|_{L^2(B_{5r})}^2 \right),$$

when  $f \in L^2(\Omega)$ . We consider now the following equation

$$(4.31) \quad \alpha\Delta v + \mu^2 v + i b^2 v = g,$$

where  $g \in L^2(\Omega)$  and  $v \in H_0^1(\Omega)$ . Multiplying (4.31) by  $\bar{v}$ , integrating over  $\Omega$  and taking the imaginary part, by integrating by parts we have

$$(4.32) \quad \|\chi v\|_{L^2(\Omega)}^2 = \Re \left( \int_{\Omega} g \bar{v} dx \right) \leq \|g\|_{H^{-1}(\Omega)} \|v\|_{H^1(\Omega)}.$$

Using (4.30) and (4.32) we deduce

$$\|v\|_{H^1(\Omega)} \leq C e^{C|\mu|} \left( \|g - i\chi^2 v\|_{L^2(\Omega)} + \|\chi v\|_{L^2(B_{5r})} \right) \\ \leq C e^{C|\mu|} \left( \|g\|_{L^2(\Omega)} + \|g\|_{H^{-1}(\Omega)}^{\frac{1}{2}} \|v\|_{H^1(\Omega)}^{\frac{1}{2}} \right).$$

Then by the Young inequality

$$\|v\|_{H^1(\Omega)} \leq C e^{C|\mu|} \|g\|_{L^2(\Omega)},$$

which shows that the operator  $(\alpha\Delta + \mu^2 + i\chi^2)^{-1}$  mapping from  $L^2(\Omega)$  to  $H_0^1(\Omega)$  is bounded and we have

$$\|(\alpha\Delta + \mu^2 + i\chi^2)^{-1}\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))} \leq C e^{C|\mu|},$$

for some positive constant  $C$ . By duality  $(\alpha\Delta + \mu^2 + i\chi^2)^{-1}$  mapping from  $H^{-1}(\Omega)$  to  $H_0^1(\Omega)$  is bounded and we have also

$$\|(\alpha\Delta + \mu^2 + i\chi^2)^{-1}\|_{\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))} \leq C e^{C|\mu|}.$$

Now since

$$\alpha\Delta u + \mu^2 u + i\chi^2 u = f + i\chi^2 u,$$

we have

$$\|u\|_{H^1(\Omega)} \leq C e^{C|\mu|} \|f + i\chi^2 u\|_{H^{-1}(\Omega)} \\ \leq C e^{C|\mu|} \left( \|f\|_{H^{-1}(\Omega)} + \|\chi^2 u\|_{H^{-1}(\Omega)} \right) \\ \leq C e^{C|\mu|} \left( \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(B_{5r})} \right),$$

and this completes the proof.  $\square$

Let's come back to the original problem. Then by combining (4.6) with (4.4) and (4.7) with (4.5), we obtain

$$(4.33) \quad -\mu_m^2 v_{1,m} - a_1 \Delta u_{1,m} = e^{-K_m |\mu_m|} g_{1,m} + i \mu_m e^{-K_m |\mu_m|} f_{1,m} + \operatorname{div} (d(x) \nabla (v_{1,m} + v_{2,m})),$$

$$(4.34) \quad -\mu_m^2 v_{2,m} - a_2 \Delta u_{2,m} = e^{-K_m |\mu_m|} g_{2,m} + i \mu_m e^{-K_m |\mu_m|} f_{2,m} + \operatorname{div} (d(x) \nabla (v_{1,m} + v_{2,m})).$$

Applying Proposition 4.3 to the Helmholtz equations (4.33) and (4.34) then we follow

$$(4.35) \quad \|u_{1,m}\|_{H^1(\Omega)}^2 \leq C e^{C|\mu_m|} \left( |\mu_m|^2 e^{-2K_m |\mu_m|} \|f_{1,m}\|_{L^2(\Omega)}^2 + e^{-2K_m |\mu_m|} \|g_{1,m}\|_{L^2(\Omega)}^2 + \int_{\Omega} d(x) |\nabla (v_{1,m} + v_{2,m})|^2 dx + \|u_{1,m}\|_{L^2(B_{5r})}^2 \right)$$

and

$$(4.36) \quad \|u_{2,m}\|_{H^1(\Omega)}^2 \leq C e^{C|\mu_m|} \left( |\mu_m|^2 e^{-2K_m |\mu_m|} \|f_{2,m}\|_{L^2(\Omega)}^2 + e^{-2K_m |\mu_m|} \|g_{2,m}\|_{L^2(\Omega)}^2 + \int_{\Omega} d(x) |\nabla (v_{1,m} + v_{2,m})|^2 dx + \|u_{2,m}\|_{L^2(B_{5r})}^2 \right).$$

Due to (4.2), (4.3) (4.7) and (4.21) then the right hand side of (4.35) and (4.36) goes to 0 and consequently

$$(4.37) \quad |\mu_m|^2 \left( \|u_{1,m}\|_{H^1(\Omega)}^2 + \|u_{2,m}\|_{H^1(\Omega)}^2 \right) \xrightarrow{m \rightarrow \infty} 0.$$

Through (4.4), (4.5) and (4.37) we can deduce

$$(4.38) \quad \|v_{1,m}\|_{L^2(\Omega)}^2 + \|v_{2,m}\|_{L^2(\Omega)}^2 \xrightarrow{m \rightarrow \infty} 0.$$

Estimates (4.37) and (4.38) leads to a contradiction with (4.2) and this prove that the resolvent estimate (4.1) is true and consequently a logarithmic decay rate follows.

## 5. Numerical study of the approximate operator

In this section we present numerical results to study the spectrum of the operator resulting from the discretization of (1.1) using the finite element method [9]. This study is inspired by the work of [7] done in a framework without transmission conditions. Specifically, the operator considered here is

$$(5.1) \quad \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} u_2 \\ 2\partial_x^2 u_1 \\ u_4 \\ 2\partial_x^2 u_3 + \partial_x [x^4 (\partial_x u_4 + \partial_x u_8)] \\ u_6 \\ 3\partial_x^2 u_5 \\ u_8 \\ 3\partial_x^2 u_7 + \partial_x [x^4 (\partial_x u_4 + \partial_x u_8)] \end{pmatrix}.$$

In (5.1), the damping function is taken as  $a(x) = x^4$  and due to the transmission conditions, the domain of  $\mathcal{A}$  is

$$\mathcal{D}(\mathcal{A}) = V_-^2 \times V_+^2 \times V_-^2 \times V_+^2,$$

where

$$V_- = \{v \in H^1(-1, 0) : v(-1) = 0\} \quad \text{and} \quad V_+ = \{v \in H^1(0, 1) : v(1) = 0\},$$

supplemented with the following conditions:  $2\partial_x^2 u_3 + \partial_x [x^4 (\partial_x u_4 + \partial_x u_8)] \in L^2(0, 1)$ ,  $3\partial_x^2 u_7 + \partial_x [x^4 (\partial_x u_4 + \partial_x u_8)] \in L^2(0, 1)$ ,  $u_1(0) = u_3(0)$ ,  $u_5(0) = u_7(0)$ ,  $2\partial_x u_1(0) = [2\partial_x u_3 + x^4 (\partial_x u_4 + \partial_x u_8)](0)$  and  $3\partial_x u_5(0) = [3\partial_x u_7 + x^4 (\partial_x u_4 + \partial_x u_8)](0)$ .

To study the spectrum of the numerical approximation of  $\mathcal{A}$ , we first project it onto the finite element basis of  $\mathcal{D}(\mathcal{A})$ . This is done in practice by using the `FreeFem++` software [24]. Then, as in the classical transfer problems [24], to compute the stiffness matrix associated to  $\mathcal{A}$ , we rely on the use of the indicator function  $h = \mathbb{1}_{[-1,0[} + \mathbb{1}_{]0,1]}$  to define the corresponding weak bi-linear form:

$$(5.2) \quad \forall u, v \in V, a(u, v) := \langle \mathcal{A}u, v \rangle = \int_{-1}^1 h(x) \mathcal{A}_1 u(x) \cdot \mathcal{A}_2 v(x) dx,$$

where the operators  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are defined by

$$(5.3) \quad \mathcal{A}_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} u_2 \\ -2\partial_x u_1 \\ u_4 \\ -2\partial_x u_3 - x^4(\partial_x u_4 + \partial_x u_8) \\ u_6 \\ -3\partial_x u_5 \\ u_8 \\ -3\partial_x u_7 - x^4(\partial_x u_4 + \partial_x u_8) \end{pmatrix} \quad \text{and} \quad \mathcal{A}_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} u_1 \\ \partial_x u_2 \\ u_3 \\ \partial_x u_4 \\ u_5 \\ \partial_x u_6 \\ u_7 \\ \partial_x u_8 \end{pmatrix},$$

with  $V = H_0^1(-1, 1)^d$  and  $d = 8$ .

When the damping term is acting on the boundary of the spatial domain, in the case of wave equation, since the speed and the position have different smoothness, it is well known that the use of mixed finite elements basis is necessary to have stability of the discrete system [7]. Here we have an internal damping and, conversely, this implies more regularity for the components  $u_4$  and  $u_8$  of  $u$ . We have therefore studied the eigenvalues of the matrix of the discretization of (5.2) when two different method are used: a standard discretization and a mixed discretization, with piecewise linear continuous finite elements denoted by  $P1$  and  $P2$ , respectively.

In the case of standard discretization with  $P1$ , according to the number of elements of the used basis, Figure 1, Figure 3 and Figure 5 show the plot of the stiffness matrices of (5.2) and the location of their eigenvalues. We used  $N = 5$ ,  $N = 10$  and  $N = 20$ , respectively. As can be observed, the structure of the stiffness matrix is somewhat unusual. Especially, `FreeFem++` stores the coefficients that correspond to the boundary elements on the diagonal and our code has been implemented in such a way that these coefficients are 1. Thus, it is easy to rearrange the stiffness matrix coefficients before to compute the eigenvalues. The code is available on request and the figures were produced using `MATLAB`.

For the mixed finite element discretization, the used order of the elements according to the components of  $u$  is:  $[P2, P1, P2, P1, P2, P1, P2, P1]$ . Likewise, Figure 2, Figure 4 and Figure 6 show the plot of the stiffness matrices corresponding to such a discretization and the location of their eigenvalues.

To see the location of the eigenvalues, for the sake of presentation, the minimum value on the x-axis has been set to  $-10$  for all figures. In all cases we can see that we have more eigenvalues on the imaginary axis or close to it. This prove that for the discrete operator of the form (5.2) we do not have an exponential stability and let us to conjecture that polynomial stability result cannot be improved to an exponential stability one, even in the continuous case.

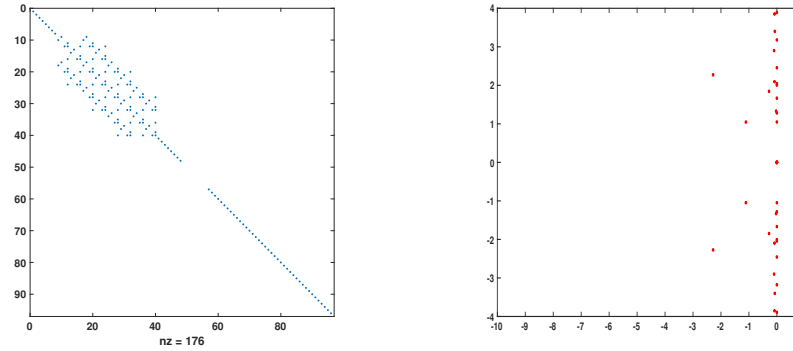


FIGURE 1. Plot of the stiffness matrix of the form  $a(.,.)$  (left) and the location of its eigenvalues (right): standard discretization with  $N = 5$  elements  $P1$  per direction and order  $[P1, P1, P1, P1, P1, P1, P1, P1]$ .

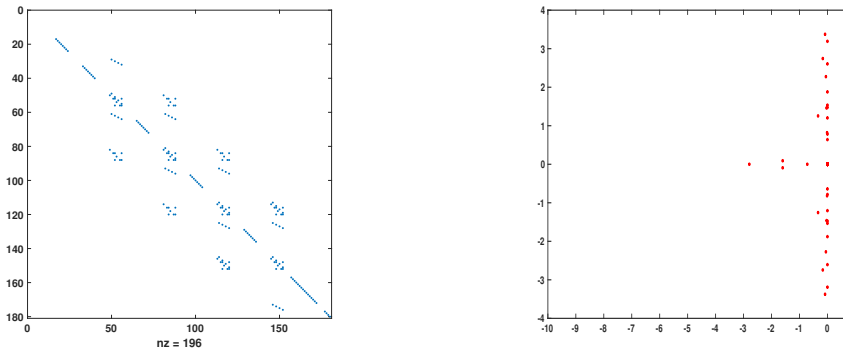


FIGURE 2. Plot of the stiffness matrix of the form  $a(.,.)$  (left) and the location of its eigenvalues (right): mixed discretization with  $N = 5$  elements in the following order  $[P2, P1, P2, P1, P2, P1, P2, P1]$ .

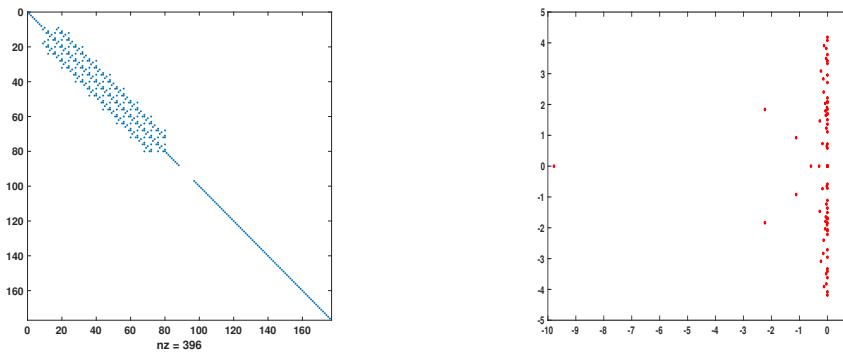


FIGURE 3. Plot of the stiffness matrix of the form  $a(.,.)$  (left) and the location of its eigenvalues (right): standard discretization with  $N = 10$  elements  $P1$  per direction and order  $[P1, P1, P1, P1, P1, P1, P1, P1]$ .

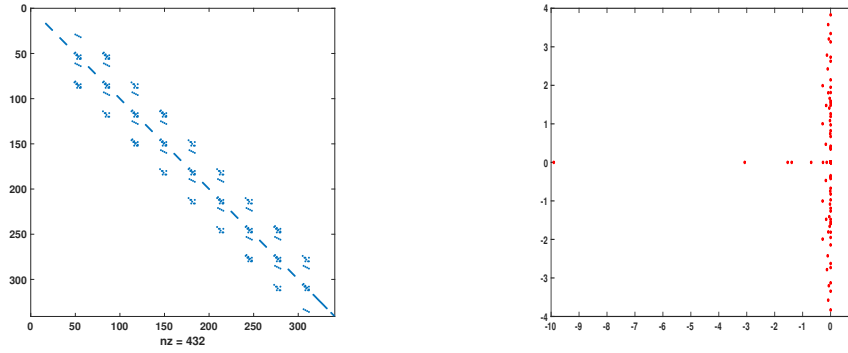


FIGURE 4. Plot of the stiffness matrix of the form  $a(.,.)$  (left) and the location of its eigenvalues (right): mixed discretization with  $N = 10$  elements in the following order  $[P2, P1, P2, P1, P2, P1, P2, P1]$ .

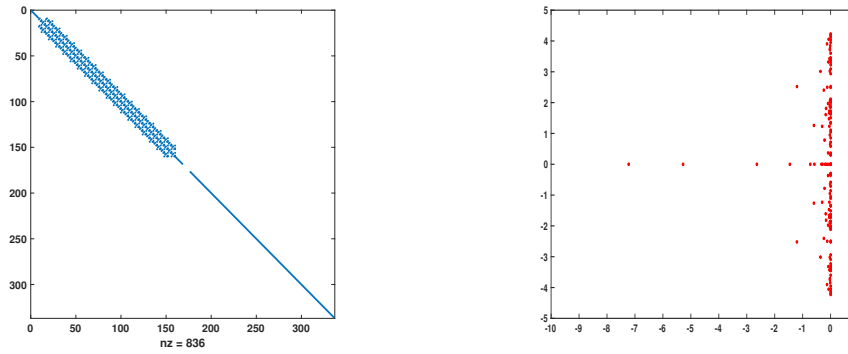


FIGURE 5. Plot of the stiffness matrix of the form  $a(.,.)$  (left) and the location of its eigenvalues (right): standard discretization with  $N = 20$  elements  $P1$  per direction and order  $[P1, P1, P1, P1, P1, P1, P1, P1]$ .

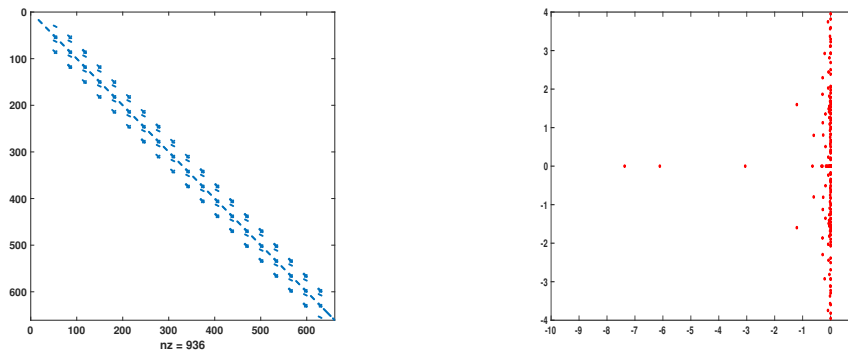


FIGURE 6. Plot of the stiffness matrix of the form  $a(.,.)$  (left) and the location of its eigenvalues (right): mixed discretization with  $N = 20$  elements in the following order  $[P2, P1, P2, P1, P2, P1, P2, P1]$ .



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