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ON ALMOST-FUCHSIAN SUBMANIFOLDS OF HADAMARD SPACES AND THE ASYMPTOTIC PLATEAU PROBLEM

SAMUEL BRONSTEIN

ABSTRACT. We consider minimal submanifolds of negatively curved spaces with small curvature. We show that in a Hadamard space with negatively pinched curvature $-C \leq K \leq -1$, complete minimal submanifolds with second fundamental form less than 1 everywhere bound a class of spheres at infinity for which the asymptotic Plateau problem is uniquely solvable.

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1. INTRODUCTION

In 1983, Uhlenbeck [Uhl83] introduced the notion of *almost-fuchsian representation*. A representation of a surface group in $\mathrm{PSL}(2, \mathbb{C})$ is almost-fuchsian if it is discrete, faithful and there is an equivariant minimal disc in \mathbb{H}^3 whose principal values lie in a compact set of $(-1, 1)$. Since then, almost-fuchsian representations have been a very fruitful field of study, see for instance [Eps86, KS07, HW13, Sep16, San17, Jia21, EES22, BS23]. We consider here a generalization of almost-fuchsian submanifolds in negatively curved spaces: Let \mathbb{X} be a complete simply connected riemannian manifold whose sectional curvature is less or equal than -1 .

Definition 1.1. An immersion $f : Y \rightarrow \mathbb{X}$ is *almost-fuchsian* if it is minimal, proper, and it satisfies

$$(1.1) \quad \sup_{y \in Y} |\mathbb{I}_f| < 1$$

Remark that Y can always be endowed with the induced metric which will be complete by the properness assumption. By extension, a submanifold $Y \subset \mathbb{X}$ will be said to be almost-fuchsian if the inclusion map is almost-fuchsian.

In this paper, we prove the following embedding theorem:

Theorem 1.2. *Let $f : Y \rightarrow \mathbb{X}$ be an almost-fuchsian immersion. Then Y is contractible. Denote by NY the normal bundle to Y . The exponential map $\exp : NY \rightarrow \mathbb{X}$ is a diffeomorphism. Also, f extends at infinity to an embedding from a sphere $S \rightarrow \partial_\infty \mathbb{X}$.*

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We also consider the following asymptotic Plateau problem: given a sphere $S \subset \partial_\infty \mathbb{X}$, can we count the complete minimal submanifolds of \mathbb{X} asymptotically bounding S ? Jiang [Jia21] and Huang–Lowe–Seppi [HLS23] considered this problem for spheres in $\partial_\infty \mathbb{H}^n$ bounding respectively an almost-fuchsian disc or an almost-fuchsian hypersurface. They proved that in that case, the unique minimal submanifold of the given dimension bounding S is the almost-fuchsian one, which is unique.

Here we prove that this statement holds in a broader generality, provided that \mathbb{X} has negatively pinched curvature between $-C$ and -1 .

Theorem 1.3. *Let \mathbb{X} be an n -dimensional negatively pinched Hadamard space with sectional curvature less or equal than -1 . Let $Y \subset \mathbb{X}$ be a k -dimensional almost-fuchsian submanifold bounding a $(k-1)$ -dimensional sphere $S \subset \partial_\infty \mathbb{X}$. Then Y is the unique k -dimensional complete minimal submanifold of \mathbb{X} asymptotically bounding S .*

We don't know whether this statement holds when the sectional curvature of \mathbb{X} has no lower bound, because we lack of convex sets to work with. Note that Huang–Lowe–Seppi [HLS23] also cover the case of weakly-almost-fuchsian discs in \mathbb{H}^3 , that is when the supremum of the principal values is allowed to be one.

This paper is divided in three parts. The first part is about a very standard fact, that minimal submanifolds of a space \mathbb{X} with negatively pinched sectional curvature remain in the convex hull of their asymptotic boundary. Surprisingly, we couldn't prove it without a lower bound on the curvature of \mathbb{X} . The second part is devoted to the proof of the embedding theorem, with some explicit estimates on the geometry of an almost-fuchsian submanifold. Finally, the third part is devoted to the proof of the uniqueness to the asymptotic Plateau problem for spheres bounding an almost-fuchsian submanifold.

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2. THE CONVEX HULL BARRIER

This section is devoted to the proof of the following theorem:

Theorem 2.1. *Let \mathbb{X} be a Hadamard space with negatively pinched curvature. Let Y be a minimal submanifold of \mathbb{X} , bounding a subset $F \subset \partial_\infty \mathbb{X}$. Denote C the convex hull of F . Then $Y \subset C$.*

While this theorem is well known when $\mathbb{X} = \mathbb{H}^n$, we didn't find a general proof of this fact in negative curvature. A very elegant proof in \mathbb{H}^n uses totally geodesic hypersurfaces as barriers, as the convex hull is the intersection of all half spaces containing C [Jia21]. While this proof can be word for word done in a symmetric space of a semisimple Lie group of rank 1, here we need to be more careful, as totally geodesic hypersurfaces might not exist.

In all this section \mathbb{X} will denote a negatively pinched Hadamard space, its boundary $\partial_\infty \mathbb{X}$ will be its ideal boundary which equals its Gromov boundary in that case [BGS85, Gro78, EO73]. The main idea is to use the following property:

Proposition 2.2. *Let F be a closed set of $\partial_\infty \mathbb{X}$, and denote by C its convex hull. Let $p \in X - C$. Then there is a closed domain $D \subset \mathbb{X} \cup \partial_\infty \mathbb{X}$ disjoint from $C \cup F$ such that $p \in D$, and D is foliated by smooth strictly convex hypersurfaces.*

Proof. We use a theorem of Bowditch [Bow94]. As \mathbb{X} has negatively pinched curvature, there is a metric on $\mathbb{X} \cup \partial_\infty \mathbb{X}$ compatible with the topologies such that the map $F \mapsto CH(F)$ is

continuous on the closed sets of $\mathbb{X} \cup \partial_\infty \mathbb{X}$ equipped with the Hausdorff distance. Therefore, there is a neighborhood V of F such that \tilde{C} the convex hull of V does not contain p . The complement of \tilde{C} is then foliated by the equidistant to \tilde{C} , which are strictly convex.

Now a priori the convex \tilde{C} has no smooth boundary, and so our foliation is only $C^{1,1}$. However, Parkkonen–Paulin [PP12] (Proposition 6) proved that a ε -neighborhood of \tilde{C} contains a smooth convex set, so up to replacing \tilde{C} by this smooth convex set we assume the foliation of the complement to be by smooth convex hypersurfaces. Denote D the closure of the complement of \tilde{C} . \square

Proof of theorem 2.1. Let Y be a minimal submanifold and let C denote the convex hull of $\partial_\infty Y$. Let $p \in \mathbb{X} - C$ and D denote the domain provided by proposition 2.2.

Suppose that Y intersects D . Because D does not bound $\partial_\infty Y$, the intersection $Y \cap D$ is compact. Then there must some point q at which Y is tangent to one of the strictly convex hypersurfaces foliating D . This contradicts the minimality of Y . Hence $Y \cap D$ is empty. Repeating this argument for any $p \in \mathbb{X} - C$ proves that $Y \subset C$. \square

Remark 2.1. The lower bound assumption on the curvature of \mathbb{X} is crucial here, otherwise there might not be enough convex sets to pursue our proof. See for instance Ancona [Anc94], there are spaces with curvature less than -1 such that the convex hull of any nontrivial open set at infinity is the full space \mathbb{X} .

3. ALMOST-FUCHSIAN SUBMANIFOLDS

This part is about the notion of almost-fuchsian submanifold. First introduced for surface group representations in $\mathrm{PSL}(2, \mathbb{C})$ by Uhlenbeck [Uhl83], almost-fuchsian discs and hypersurfaces have a rich literature [Eps86, KS93, Sep16, San17, Jia21, EES22, BS23, HLS23].

Throughout this section \mathbb{X} will be a Hadamard space with sectional curvature less or equal than -1 . Note that we do not require its curvature to admit a lower bound. We consider an immersion $f : Y \rightarrow \mathbb{X}$, where Y is connected. We assume f to be complete, i.e. the induced metric on Y is complete.

We will prove the following theorem.

Theorem 3.1. *Assume $f : Y \rightarrow \mathbb{X}$ is minimal and satisfies:*

$$(3.1) \quad \sup_{y \in Y} |II_f| < 1$$

Then Y is contractible. Denote by NY the normal bundle to Y . The exponential map $\exp : NY \rightarrow \mathbb{X}$ is a diffeomorphism. Also, f extends at infinity to an embedding from a sphere $S \rightarrow \partial_\infty \mathbb{X}$

The proof is divided in several steps. First, we show that the induced metric by \exp is nondegenerate on NY . This will imply that \exp is a covering map to \mathbb{X} , and so it is a diffeomorphism. Finally, we will see that f is a quasi-isometric embedding from Y to \mathbb{X} , so it extends at infinity to an embedding $S \rightarrow \partial_\infty \mathbb{X}$.

First note that because of the condition on f , the induced metric has negatively pinched curvature on Y , and it makes sense to talk about its boundary $\partial_\infty Y$, which is constructed in the works of Gromov [Gro78], Eberlein–O’Neill [EO73] and Ballmann–Gromov–Schroeder [BGS85].

Lemma 3.2. *Let $f : Y \rightarrow \mathbb{X}$ be an almost-fuchsian immersion. Let $\varepsilon > 0$ such that*

$$(3.2) \quad \sup_{y \in Y} |\mathbb{I}_f| \leq 1 - \varepsilon.$$

*Endow Y with the induced metric by f . Then its sectional curvature is less than $-\varepsilon(2 - \varepsilon)$. If the sectional curvature of \mathbb{X} is lower bounded by $-C$, then the sectional curvature of $f^*g_{\mathbb{X}}$ is bigger than $-C - 2(1 - \varepsilon)^2$.*

Remark 3.1. When Y is a surface, the minimality condition ensures that we have more precise bounds: The sectional curvature of an almost-fuchsian surface is then less than -1 and bigger than $-C - (1 - \varepsilon)^2$.

Proof. This is a consequence of the Gauss condition: For u, v an orthonormal basis of a tangent plane to Y , By the Gauss' equation, the curvature of $f^*g_{\mathbb{X}}$ is then

$$(3.3) \quad R_{f^*g_{\mathbb{X}}}(u, v, v, u) = R_{g_{\mathbb{X}}}(u, v, v, u) + g_{\mathbb{X}}(\mathbb{I}_f(u, u), \mathbb{I}_f(v, v)) - g_{\mathbb{X}}(\mathbb{I}_f(u, v), \mathbb{I}_f(u, v))$$

As $R_{g_{\mathbb{X}}}(u, v, v, u) \leq -1$ and $|\mathbb{I}_f(u, u)| \leq 1 - \varepsilon$, we directly get

$$(3.4) \quad R_{f^*g_{\mathbb{X}}}(u, v, v, u) \leq -1 + (1 - \varepsilon)^2 = -\varepsilon(2 - \varepsilon).$$

as desired. Moreover, if $R_{g_{\mathbb{X}}}(u, v, v, u) \geq -C$, we directly get that

$$(3.5) \quad R_{f^*g_{\mathbb{X}}}(u, v, v, u) \geq -C - 2(1 - \varepsilon)^2,$$

as claimed. □

Proposition 3.3. *Let $f : Y \rightarrow \mathbb{X}$ be an almost-fuchsian immersion. Denote by NY the normal bundle to Y , and note G the induced metric by the exponential map on NY . For $t > 0$, denote by $N_t Y$ the tangent bundle to the distance t hypersurface to Y . Let $(x, tv) \in NY$ with v a unit normal vector. Let B_t be the parallel transport map from $T_x \mathbb{X} \cap (v)^\perp$ to $T_{\exp tv}(N_t Y)$. Let $w \in T_v(UY)$ and let f be the function:*

$$(3.6) \quad f(t) = |B_t w|^2$$

If \mathbb{X} is the hyperbolic space, f satisfies the following controls:

$$(3.7) \quad (\sqrt{f})'' = \sqrt{f}$$

If \mathbb{X} has sectional curvature strictly less than -1 , then

$$(3.8) \quad (\sqrt{f})'' > \sqrt{f}$$

Also, in both cases

$$(3.9) \quad (\ln f)'' + \frac{(\ln f)^2}{2} \geq 2$$

with the same equality and inequality conditions.

Proof. With the introduced notations, fix $\gamma : I \rightarrow UY$ a smooth path such that $\gamma(0) = v$ and $\gamma'(0) = w$. The family $c_s(t)$ is a smooth variation of geodesics, hence $J(t) = \partial_s c_s(t)|_{s=0}$ is a Jacobi field along c_0 . Evaluate the characteristic equation of Jacobi fields against J :

$$(3.10) \quad g_{\mathbb{X}}(\ddot{J}, J) + R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) = 0$$

In the second term, we recognize the sectional curvature of the tangent plane spanned by J and $\partial_t c$. Choose a local orthonormal chart of $T_x \mathbb{X}$ such that $B_0 = I_k \oplus O_{n-k-1}$ and

$\dot{B}_0 = B(v) \oplus I_{n-k-1}$, where $B(v)$ is the shape operator of f at v . Then equation 3.10 rewrites as

$$(3.11) \quad {}^\perp w \ddot{B}_t B_t w + R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) = 0$$

and as the sectional curvature is less than -1 , we replace the second term with the control

$$(3.12) \quad R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) \leq -{}^\perp w {}^\perp B_t B_t w$$

to get the following

$$(3.13) \quad {}^\perp w {}^\perp (\ddot{B}_t - B_t) B_t w \geq 0$$

Now remark that

$$(3.14) \quad \begin{cases} f'(t) &= 2{}^\perp w {}^\perp B_t \dot{B}_t w \\ f''(t) &= 2{}^\perp w {}^\perp B_t \ddot{B}_t w + 2{}^\perp w {}^\perp \dot{B}_t \dot{B}_t w \end{cases}$$

So the characteristic equation rewrites as

$$(3.15) \quad f''(t) \geq 2f(t) + \frac{f'(t)^2}{2f(t)}$$

with equality when \mathbb{X} is the hyperbolic space, and strict inequality if \mathbb{X} has sectional curvature strictly less than -1 . As a direct consequence,

$$(3.16) \quad (\sqrt{f})'' \geq \sqrt{f}$$

with the same equality and inequality conditions. Finally, introduce $g = \ln(f)$. As long as it is well defined, we deduce from 3.15 that

$$(3.17) \quad g'' + \frac{(g')^2}{2} \geq 2$$

□

Remark 3.2. When \mathbb{X} has varying curvature less or equal than -1 , up to rescaling the metric we can assume its sectional curvature to be strictly less than -1 .

Proof of theorem 3.1. With the notations introduced, Fix $w \in T_v(UY)$ and consider the function f defined in proposition 3.3. Denote w_1 the projection of w on TY , and assume it is nonzero. The initial conditions, by construction of f ,

$$(3.18) \quad \begin{cases} f(0) &= \alpha^2 = |w_1|^2 > 0 \\ f'(0) &= 2\alpha\beta = 2{}^\perp w_1 B(v) w_1 < 2\alpha \end{cases}$$

From proposition 3.3, we deduce that

$$(3.19) \quad \begin{cases} f(t) &\geq (\alpha \cosh(t) + \beta \sinh(t))^2 \\ f'(t) &\geq 2(\alpha \cosh(t) + \beta \sinh(t))(\alpha \sinh(t) + \beta \cosh(t)) \\ g'(t) &= \frac{f'(t)}{f(t)} \geq 2 \frac{\alpha \sinh(t) + \beta \cosh(t)}{\alpha \cosh(t) + \beta \sinh(t)} \end{cases}$$

with equality when \mathbb{X} is hyperbolic, and strict inequality when the curvature of \mathbb{X} is strictly less than -1 .

A first consequence is that f never vanishes. Hence (B_t) has trivial kernel, and the metric induced by \exp on NY is nondegenerate. So \exp is a local diffeomorphism from NY to \mathbb{X} .

Moreover, because the immersion $Y \rightarrow \mathbb{X}$ is complete, it is proper. Hence for any point of $x \in \mathbb{X}$ there is at least one point in Y which minimizes the distance to x . This implies that the map \exp is surjective. As a proper surjective local diffeomorphism, \exp satisfies the

path lifting property, so it is a covering map. But Y is connected and \mathbb{X} is contractible, so \exp is a global diffeomorphism, Y is embedded and contractible too.

It remains to prove that f is a quasi-isometric embedding. In order to prove we introduce a metric on NY for which the inverse map of the exponential map will be Lipschitz, and such that Y sits in NY totally geodesically with the induced metric by f :

Consider $(x, tv) \in NY$ and decompose its tangent space into $T_{(x,tv)}NY = T_t(x, \mathbb{R}v) \oplus T_xY \oplus T_v\mathbb{S}^{n-k-1}$.

Introduce the metric h on NY which in that decomposition is:

$$(3.20) \quad h_{(x,tv)} = 1 \oplus \cosh^2(t)f^*g_{\mathbb{X},x} \oplus \sinh^2(t)g_{\mathbb{S}^{n-k-1}}$$

By construction, Y sits inside NY totally geodesically with the metric induced by f , that we denote g_Y .

The equation on f 3.19 ensures that:

$$(3.21) \quad \exp^*g_{\mathbb{X}} \geq 1 \oplus (\cosh(t)I_k + B(v)\sinh(t))^2g_Y \oplus \sinh^2(t)g_{\mathbb{S}^{n-k-1}}$$

Denote $\delta = 1 - \sup |\mathbb{I}_f| > 0$. Then

$$(3.22) \quad \exp^*g_{\mathbb{X}} \geq \delta^2(1 \oplus \cosh^2(t)g_Y \oplus \sinh^2(t)g_{\mathbb{S}^{n-k-1}}) = \delta^2h$$

Hence the inverse map of the exponential map is $\frac{1}{\delta}$ -Lipschitz.

As a consequence, f is a quasi-isometric embedding, as it satisfies:

$$(3.23) \quad d_{\mathbb{X}}(f(x_1), f(x_2)) \leq d_Y(x_1, x_2) \leq \frac{1}{\delta}d_{\mathbb{X}}(f(x_1), f(x_2))$$

Combined with the fact that Y has negatively pinched sectional curvature proven in lemma 3.2, it ensures that f extends at infinity to an embedding from a sphere $\mathbb{S}^{k-1} \rightarrow \partial_{\infty}\mathbb{X}$. \square

As a corollary of the computations we made, we have some explicit estimates of the geometry of NY , endowed with the induced metric by \exp .

Corollary 3.4. *Let $f : Y \rightarrow \mathbb{X}$ be a complete almost-fuchsian immersion, with \mathbb{X} a Hadamard space. Denote by N_tY the distance t hypersurface to Y . At a point $\exp_x(tv) \in N_tY$, consider $\lambda_1 \leq \dots \leq \lambda_k$ the eigenvalues of $B(v)$ the shape operator of Y evaluated at (x, v) .*

Then the second fundamental form \mathbb{I}_t of N_tY has eigenvalues $\lambda_1^t \leq \dots \leq \lambda_{n-1}^t$ which satisfy

$$(3.24) \quad \forall i \leq k, \quad \sum_{j=1}^i \lambda_j^t \geq \sum_{j=1}^i \frac{\lambda_j + \tanh(t)}{1 + \lambda_j \tanh(t)}$$

$$(3.25) \quad \forall i > k, \quad \sum_{j=1}^i \lambda_j^t \geq \sum_{j=1}^k \frac{\lambda_j + \tanh(t)}{1 + \lambda_j \tanh(t)} + (i - k) \frac{1}{\tanh(t)}$$

In particular, N_tY is k -convex in the sense of Sha [Sha86].

Proof. This is a consequence of the control on $\frac{f'_t}{f}$ gotten in 3.19. The trace of \mathbb{I}_t on a k -plane P is the sum of $\frac{1}{2} \frac{f'_t(t)}{f(t)}$ for w spanning a basis of the plane P .

The almost-fuchsian condition ensures that the sum of the λ_i is 0, and that they live in a compact set of $(-1, 1)$, which ensures the k -convexity of \mathbb{I}_t . \square

4. THE ASYMPTOTIC PLATEAU PROBLEM

The Plateau problem was introduced in 1847, originally about a mathematical proof of the existence of soap films bounding a wire frame. It was solved by Douglas [Dou31] and Rado [Rad30]. Here we are interested in a noncompact analogous problem: the asymptotic Plateau problem. Fix \mathbb{X} a Hadamard space of sectional curvature less than -1 , the asymptotic Plateau problem can be formulated as

Problem 4.1 (Asymptotic Plateau Problem). Let S be a $(k-1)$ -sphere embedded in $\partial_\infty \mathbb{X}$. Is there a minimal k -dimensional submanifold M of \mathbb{X} asymptotically bounding S ? Is M unique?

The asymptotic Plateau problem has been studied by Anderson [And83]. The problem has since had a rich history, see the survey of Cozkunuzer [Cos09].

Recently, Huang–Lowe–Seppi [HLS23] considered the asymptotic Plateau problem for a class of almost-fuchsian submanifolds of \mathbb{H}^n . If a sphere in $\partial_\infty \mathbb{H}^n$ bounds an almost-fuchsian hypersurface, then it is the unique minimal hypersurface bounding it. Jiang [Jia21] also solved the asymptotic Plateau problem for Jordan curves bounding an almost-fuchsian disc in $\partial_\infty \mathbb{H}^n$. In this section, we generalize these results to any sphere bounding an almost-fuchsian submanifold in a space \mathbb{X} , under the assumption that \mathbb{X} has negatively pinched curvature.

Theorem 4.2 (Asymptotic Plateau Problem for almost-fuchsian spheres). *Let \mathbb{X} be a negatively pinched Hadamard space with sectional curvature less or equal than -1 . Let $Y \subset \mathbb{X}$ be an almost-fuchsian submanifold of dimension k , bounding at infinity a $(k-1)$ -dimensional sphere $S \subset \partial_\infty \mathbb{X}$. Then Y is the unique complete k -dimensional minimal submanifold of \mathbb{X} bounding S .*

In all the remainder of the article, \mathbb{X} is assumed to have pinched sectional curvature between $-C$ and -1 . $Y \subset \mathbb{X}$ is a complete almost-fuchsian k -dimensional submanifold, and S is its asymptotic boundary in $\partial_\infty \mathbb{X}$.

The first step in the proof is that eventual other minimal submanifolds bounding S remain at bounded distance from Y :

Proposition 4.3. *Let Z be a minimal submanifold of \mathbb{X} bounding S . Then $Z \subset \mathcal{N}_r Y$, where $\mathcal{N}_r Y$ is the r -uniform neighborhood of Y , and $r = \tanh^{-1}(1 - \sup |\mathbb{I}_f|)$.*

Proof. Thanks to theorem 2.1, we know that Z is included in the convex hull of S . But the explicit bounds shown in corollary 3.4 show that for $r \geq \tanh^{-1}(1 - \sup |\mathbb{I}_f|)$, the uniform neighborhoods $\mathcal{N}_r(Y)$ are convex, and contain Y so bound S too. As a consequence, $\mathcal{N}_r(Y)$ contains the convex hull of Y which contains Z . \square

Remark 4.1. The same corollary 3.4 shows that all uniform neighborhoods are k -convex, which directly implies that the function $z \in Z \mapsto d(z, Y)$ cannot have a local maximum. When the induced metric of Z has lower bounded sectional curvature, we could apply the Omori–Yau [Omo67, Yau75] maximum principle to conclude that $Z = Y$. Here we don't prove that Z has lower bounded sectional curvature, but we prove that the maximum principle is still applicable. This property is called *stochastic completeness*, cf Pigola–Rigoli–Setti [PRS05].

We want to apply a maximum principle to the distance function to Y restricted on a minimal submanifold Z , so we first prove it satisfies a strong subharmonicity condition.

Proposition 4.4. *Let $Z \subset \mathbb{X}$ be a minimal k -dimensional submanifold. Consider the function $u : z \in Z \mapsto d(z, Y)^2$. Then there is $C > 0$ such that*

$$(4.1) \quad \Delta u \geq Cu$$

Proof. Consider d the distance function to Y , defined on the whole space X . Because Y is almost-fuchsian, for any point of $\mathbb{X} - Y$, the distance is attained at a unique point of \mathbb{X} , so the function d is smooth on $X - Y$.

Furthermore, at a point x where $d(x) = t > 0$, we have the formula

$$(4.2) \quad \nabla^2 d = 0 \oplus \mathbb{I}_{N_t(Y)}$$

the 0 is simply because d is linear in the direction of the minimizing geodesic to Y . Using the formula

$$(4.3) \quad \nabla^2(d^2) = 2\nabla d \cdot \nabla d + 2d\nabla^2 d$$

We deduce that

$$(4.4) \quad \nabla^2(u) \geq 2 \oplus 2d\mathbb{I}_{N_t(Y)}$$

Now, because Z is a minimal submanifold of X , the laplacian of the restriction of u equals the trace of the restriction of its Hessian to the tangent space of Z

$$(4.5) \quad \Delta(u|_Z) = \text{Tr}((\nabla^2 u)|_{TZ})$$

Hence the bounds on the eigenvalues of $\mathbb{I}_{N_t(Y)}$ computed in corollary 3.4 and the control

$$(4.6) \quad \frac{\tanh(t) + \lambda_i}{1 + \lambda_i \tanh(t)} \leq 1$$

allow us to get the control:

$$(4.7) \quad \Delta(u|_Z) \geq 2d \inf_{\lambda_1 + \dots + \lambda_k = 0, |\lambda_i| \leq 1 - \varepsilon} \sum_{i=1, \dots, k} \frac{\tanh(d) + \lambda_i}{1 + \tanh(d)\lambda_i} = 2d\Phi(d)$$

Remark that $\Phi(d)$ is continuous, positive, vanishes only when $d = 0$, and satisfies

$$(4.8) \quad \Phi'(0) \geq k(1 - (1 - \sup |\mathbb{I}_f|)^2) > 0$$

As d is bounded by a constant $r = \tanh^{-1}(1 - \sup |\mathbb{I}_f|)$, there is a constant $C > 0$ such that on $[0, r]$, $\Phi(d) \geq Cd$. We deduce that the restriction of u to Z satisfies

$$(4.9) \quad \Delta u \geq 2Cu$$

, as claimed. □

Now we want to prove that we can apply the maximum principle to u . In order to do so, we will use the Khas'minskii test [Km60], as stated in Theorem 3.1 and Proposition 3.2. of [PRS05].

Theorem 4.5. *Let M, g be a Riemannian manifold, and assume that M supports a C^2 function γ , which tends to infinity at infinity, and satisfies*

$$(4.10) \quad \Delta \gamma \leq \lambda \gamma \text{ off a compact set}$$

for some $\lambda > 0$. Then M, g is stochastically complete, or equivalently for every $\lambda > 0$ the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on M is zero.

Note that the equivalence between stochastical completeness and the applicability of the maximum principle is due to Grigor'Yan [Gri99].

The following proposition will ensure that we can apply our maximum principle on Z :

Proposition 4.6. *Let $Z \subset \mathbb{X}$ be a proper submanifold of \mathbb{X} complete contractible space with sectional curvature pinched between $-b$ and $-a$, $b \geq a \geq 0$. Fix $p \in \mathbb{X}$ and denote f the distance function to the point p . Then out of a compact set K , there is a constant $C > 0$ such that the restriction of f to Z satisfies*

$$(4.11) \quad \Delta(f|_Z) \leq C$$

Proof. As \mathbb{X} is negatively pinched, there are $b \geq a \geq 0$ such that $-b \leq K_{\mathbb{X}} \leq -a$. By the Hessian comparison Theorem, the Hessian of f is bounded between the hessian of the distance functions in the spaceforms of sectional curvature $-b$ and $-a$. Explicitely, this means

$$(4.12) \quad a \coth(af)g_{\mathbb{X}} \leq \nabla^2 f \leq b \coth(bf)g_{\mathbb{X}}$$

If a is zero, one has to replace $a \coth(af)$ by $\frac{1}{f}$, but it doesn't change anything to the proof. In particular, out of a compact set K containing p , the hessian of f is bounded by a constant C . Now because Z is proper, the intersection $K \cap Z$ is a compact subset of Z . Also, because Z is minimal, the laplacian of the restriction of f satisfies

$$(4.13) \quad \Delta(f|_Z) = \text{Tr}(\nabla^2 f|_{TZ}) \leq C \dim Z.$$

as desired. □

Applying theorem 4.5, we deduce that the maximum principle is applicable on a proper submanifold of such a space \mathbb{X} .

Corollary 4.7. *Let \mathbb{X} be a complete contractible space whose sectional curvature is bounded between $-b$ and 0 for some $b > 0$. Let $Z \subset \mathbb{X}$ a proper minimal submanifold. Then for any $u \in C^2(Z)$ such that $\sup u < \infty$, for every $c < \sup u$,*

$$(4.14) \quad \inf_{z \in Z: u(z) > c} \Delta u \leq 0$$

Proof. Thanks to proposition 4.6, the function $f|_Z$ tends to ∞ at ∞ , and out of a compact set K , it satisfies

$$(4.15) \quad \Delta f \leq C \leq C' f$$

for some constant $C' > 0$. In particular, we can apply theorem 4.5 to deduce that Z is stochastically complete, which is equivalent to the statement of our corollary by Theorem 3.1. of [PRS05]. □

We now have everything needed to prove our main theorem.

Proof of theorem 4.2. Introduce $u : Z \rightarrow \mathbb{R}$, such that $u(z) = d(z, Y)^2$. Thanks to proposition 4.3, u is bounded. But thanks to proposition 4.4, there is $C > 0$ such that

$$(4.16) \quad \Delta u \geq Cu$$

Now because of corollary 4.7, we can apply the maximum principle on u to get that $u = 0$. Hence $Z \subset Y$, and by completeness $Z = Y$, as desired. □

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