



ON ALMOST-FUCHSIAN SUBMANIFOLDS OF HADAMARD SPACES AND THE ASYMPTOTIC PLATEAU PROBLEM

Samuel Bronstein

► To cite this version:

Samuel Bronstein. ON ALMOST-FUCHSIAN SUBMANIFOLDS OF HADAMARD SPACES AND THE ASYMPTOTIC PLATEAU PROBLEM. 2024. hal-04549710

HAL Id: hal-04549710

<https://hal.science/hal-04549710>

Preprint submitted on 17 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

ON ALMOST-FUCHSIAN SUBMANIFOLDS OF HADAMARD SPACES AND THE ASYMPTOTIC PLATEAU PROBLEM

SAMUEL BRONSTEIN

ABSTRACT. We consider minimal submanifolds of negatively curved spaces with small curvature. We show that in a Hadamard space with negatively pinched curvature $-C \leq K \leq -1$, complete minimal submanifolds with second fundamental form less than 1 everywhere bound a class of spheres at infinity for which the asymptotic Plateau problem is uniquely solvable.

CONTENTS

1. Introduction	1
2. The convex hull barrier	2
3. Almost-fuchsian submanifolds	3
4. The asymptotic Plateau problem	7
References	10

1. INTRODUCTION

In 1983, Uhlenbeck [Uhl83] introduced the notion of *almost-fuchsian representation*. A representation of a surface group in $\mathrm{PSL}(2, \mathbb{C})$ is almost-fuchsian if it is discrete, faithful and there is an equivariant minimal disc in \mathbb{H}^3 whose principal values lie in a compact set of $(-1, 1)$. Since then, almost-fuchsian representations have been a very fruitful field of study, see for instance [Eps86, KS07, HW13, Sep16, San17, Jia21, EES22, BS23]. We consider here a generalization of almost-fuchsian submanifolds in negatively curved spaces: Let \mathbb{X} be a complete simply connected riemannian manifold whose sectional curvature is less or equal than -1 .

Definition 1.1. An immersion $f : Y \rightarrow \mathbb{X}$ is *almost-fuchsian* if it is minimal, proper, and it satisfies

$$(1.1) \quad \sup_{y \in Y} |\mathbb{I}_f| < 1$$

Remark that Y can always be endowed with the induced metric which will be complete by the properness assumption. By extension, a submanifold $Y \subset \mathbb{X}$ will be said to be almost-fuchsian if the inclusion map is almost-fuchsian.

In this paper, we prove the following embedding theorem:

Theorem 1.2. *Let $f : Y \rightarrow \mathbb{X}$ be an almost-fuchsian immersion. Then Y is contractible. Denote by NY the normal bundle to Y . The exponential map $\exp : NY \rightarrow \mathbb{X}$ is a diffeomorphism. Also, f extends at infinity to an embedding from a sphere $S \rightarrow \partial_\infty \mathbb{X}$.*

Date: 2023.

We also consider the following asymptotic Plateau problem: given a sphere $S \subset \partial_\infty \mathbb{X}$, can we count the complete minimal submanifolds of \mathbb{X} asymptotically bounding S ? Jiang [Jia21] and Huang–Lowe–Seppi [HLS23] considered this problem for spheres in $\partial_\infty \mathbb{H}^n$ bounding respectively an almost-fuchsian disc or an almost-fuchsian hypersurface. They proved that in that case, the unique minimal submanifold of the given dimension bounding S is the almost-fuchsian one, which is unique.

Here we prove that this statement holds in a broader generality, provided that \mathbb{X} has negatively pinched curvature between $-C$ and -1 .

Theorem 1.3. *Let \mathbb{X} be an n -dimensional negatively pinched Hadamard space with sectional curvature less or equal than -1 . Let $Y \subset \mathbb{X}$ be a k -dimensional almost-fuchsian submanifold bounding a $(k-1)$ -dimensional sphere $S \subset \partial_\infty \mathbb{X}$. Then Y is the unique k -dimensional complete minimal submanifold of \mathbb{X} asymptotically bounding S .*

We don't know whether this statement holds when the sectional curvature of \mathbb{X} has no lower bound, because we lack of convex sets to work with. Note that Huang–Lowe–Seppi [HLS23] also cover the case of weakly-almost-fuchsian discs in \mathbb{H}^3 , that is when the supremum of the principal values is allowed to be one.

This paper is divided in three parts. The first part is about a very standard fact, that minimal submanifolds of a space \mathbb{X} with negatively pinched sectional curvature remain in the convex hull of their asymptotic boundary. Surprisingly, we couldn't prove it without a lower bound on the curvature of \mathbb{X} . The second part is devoted to the proof of the embedding theorem, with some explicit estimates on the geometry of an almost-fuchsian submanifold. Finally, the third part is devoted to the proof of the uniqueness to the asymptotic Plateau problem for spheres bounding an almost-fuchsian submanifold.

The author thanks Andrea Seppi, Graham Smith and Nicolas Tholozan for their help, insights and discussions on these topics.

2. THE CONVEX HULL BARRIER

This section is devoted to the proof of the following theorem:

Theorem 2.1. *Let \mathbb{X} be a Hadamard space with negatively pinched curvature. Let Y be a minimal submanifold of \mathbb{X} , bounding a subset $F \subset \partial_\infty \mathbb{X}$. Denote C the convex hull of F . Then $Y \subset C$.*

While this theorem is well known when $\mathbb{X} = \mathbb{H}^n$, we didn't find a general proof of this fact in negative curvature. A very elegant proof in \mathbb{H}^n uses totally geodesic hypersurfaces as barriers, as the convex hull is the intersection of all half spaces containing C [Jia21]. While this proof can be word for word done in a symmetric space of a semisimple Lie group of rank 1, here we need to be more careful, as totally geodesic hypersurfaces might not exist.

In all this section \mathbb{X} will denote a negatively pinched Hadamard space, its boundary $\partial_\infty \mathbb{X}$ will be its ideal boundary which equals its Gromov boundary in that case [BGS85, Gro78, EO73]. The main idea is to use the following property:

Proposition 2.2. *Let F be a closed set of $\partial_\infty \mathbb{X}$, and denote by C its convex hull. Let $p \in X - C$. Then there is a closed domain $D \subset \mathbb{X} \cup \partial_\infty \mathbb{X}$ disjoint from $C \cup F$ such that $p \in D$, and D is foliated by smooth strictly convex hypersurfaces.*

Proof. We use a theorem of Bowditch [Bow94]. As \mathbb{X} has negatively pinched curvature, there is a metric on $\mathbb{X} \cup \partial_\infty \mathbb{X}$ compatible with the topologies such that the map $F \mapsto CH(F)$ is

continuous on the closed sets of $\mathbb{X} \cup \partial_\infty \mathbb{X}$ equipped with the Hausdorff distance. Therefore, there is a neighborhood V of F such that \tilde{C} the convex hull of V does not contain p . The complement of \tilde{C} is then foliated by the equidistant to \tilde{C} , which are strictly convex.

Now a priori the convex \tilde{C} has no smooth boundary, and so our foliation is only $C^{1,1}$. However, Parkkonen–Paulin [PP12] (Proposition 6) proved that a ε -neighborhood of \tilde{C} contains a smooth convex set, so up to replacing \tilde{C} by this smooth convex set we assume the foliation of the complement to be by smooth convex hypersurfaces. Denote D the closure of the complement of \tilde{C} . \square

Proof of theorem 2.1. Let Y be a minimal submanifold and let C denote the convex hull of $\partial_\infty Y$. Let $p \in \mathbb{X} - C$ and D denote the domain provided by proposition 2.2.

Suppose that Y intersects D . Because D does not bound $\partial_\infty Y$, the intersection $Y \cap D$ is compact. Then there must some point q at which Y is tangent to one of the strictly convex hypersurfaces foliating D . This contradicts the minimality of Y . Hence $Y \cap D$ is empty. Repeating this argument for any $p \in \mathbb{X} - C$ proves that $Y \subset C$. \square

Remark 2.1. The lower bound assumption on the curvature of \mathbb{X} is crucial here, otherwise there might not be enough convex sets to pursue our proof. See for instance Ancona [Anc94], there are spaces with curvature less than -1 such that the convex hull of any nontrivial open set at infinity is the full space \mathbb{X} .

3. ALMOST-FUCHSIAN SUBMANIFOLDS

This part is about the notion of almost-fuchsian submanifold. First introduced for surface group representations in $\mathrm{PSL}(2, \mathbb{C})$ by Uhlenbeck [Uhl83], almost-fuchsian discs and hypersurfaces have a rich literature [Eps86, KS93, Sep16, San17, Jia21, EES22, BS23, HLS23].

Throughout this section \mathbb{X} will be a Hadamard space with sectional curvature less or equal than -1 . Note that we do not require its curvature to admit a lower bound. We consider an immersion $f : Y \rightarrow \mathbb{X}$, where Y is connected. We assume f to be complete, i.e. the induced metric on Y is complete.

We will prove the following theorem.

Theorem 3.1. *Assume $f : Y \rightarrow \mathbb{X}$ is minimal and satisfies:*

$$(3.1) \quad \sup_{y \in Y} |II_f| < 1$$

Then Y is contractible. Denote by NY the normal bundle to Y . The exponential map $\exp : NY \rightarrow \mathbb{X}$ is a diffeomorphism. Also, f extends at infinity to an embedding from a sphere $S \rightarrow \partial_\infty \mathbb{X}$

The proof is divided in several steps. First, we show that the induced metric by \exp is nondegenerate on NY . This will imply that \exp is a covering map to \mathbb{X} , and so it is a diffeomorphism. Finally, we will see that f is a quasi-isometric embedding from Y to \mathbb{X} , so it extends at infinity to an embedding $S \rightarrow \partial_\infty \mathbb{X}$.

First note that because of the condition on f , the induced metric has negatively pinched curvature on Y , and it makes sense to talk about its boundary $\partial_\infty Y$, which is constructed in the works of Gromov [Gro78], Eberlein–O’Neill [EO73] and Ballmann–Gromov–Schroeder [BGS85].

Lemma 3.2. *Let $f : Y \rightarrow \mathbb{X}$ be an almost-fuchsian immersion. Let $\varepsilon > 0$ such that*

$$(3.2) \quad \sup_{y \in Y} |\mathbb{I}_f| \leq 1 - \varepsilon.$$

*Endow Y with the induced metric by f . Then its sectional curvature is less than $-\varepsilon(2 - \varepsilon)$. If the sectional curvature of \mathbb{X} is lower bounded by $-C$, then the sectional curvature of $f^*g_{\mathbb{X}}$ is bigger than $-C - 2(1 - \varepsilon)^2$.*

Remark 3.1. When Y is a surface, the minimality condition ensures that we have more precise bounds: The sectional curvature of an almost-fuchsian surface is then less than -1 and bigger than $-C - (1 - \varepsilon)^2$.

Proof. This is a consequence of the Gauss condition: For u, v an orthonormal basis of a tangent plane to Y , By the Gauss' equation, the curvature of $f^*g_{\mathbb{X}}$ is then

$$(3.3) \quad R_{f^*g_{\mathbb{X}}}(u, v, v, u) = R_{g_{\mathbb{X}}}(u, v, v, u) + g_{\mathbb{X}}(\mathbb{I}_f(u, u), \mathbb{I}_f(v, v)) - g_{\mathbb{X}}(\mathbb{I}_f(u, v), \mathbb{I}_f(u, v))$$

As $R_{g_{\mathbb{X}}}(u, v, v, u) \leq -1$ and $|\mathbb{I}_f(u, u)| \leq 1 - \varepsilon$, we directly get

$$(3.4) \quad R_{f^*g_{\mathbb{X}}}(u, v, v, u) \leq -1 + (1 - \varepsilon)^2 = -\varepsilon(2 - \varepsilon).$$

as desired. Moreover, if $R_{g_{\mathbb{X}}}(u, v, v, u) \geq -C$, we directly get that

$$(3.5) \quad R_{f^*g_{\mathbb{X}}}(u, v, v, u) \geq -C - 2(1 - \varepsilon)^2,$$

as claimed. \square

Proposition 3.3. *Let $f : Y \rightarrow \mathbb{X}$ be an almost-fuchsian immersion. Denote by NY the normal bundle to Y , and note G the induced metric by the exponential map on NY . For $t > 0$, denote by $N_t Y$ the tangent bundle to the distance t hypersurface to Y . Let $(x, tv) \in NY$ with v a unit normal vector. Let B_t be the parallel transport map from $T_x \mathbb{X} \cap (v)^\perp$ to $T_{\exp tv}(N_t Y)$. Let $w \in T_v(UY)$ and let f be the function:*

$$(3.6) \quad f(t) = |B_t w|^2$$

If \mathbb{X} is the hyperbolic space, f satisfies the following controls:

$$(3.7) \quad (\sqrt{f})'' = \sqrt{f}$$

If \mathbb{X} has sectional curvature strictly less than -1 , then

$$(3.8) \quad (\sqrt{f})'' > \sqrt{f}$$

Also, in both cases

$$(3.9) \quad (\ln f)'' + \frac{(\ln f)^2}{2} \geq 2$$

with the same equality and inequality conditions.

Proof. With the introduced notations, fix $\gamma : I \rightarrow UY$ a smooth path such that $\gamma(0) = v$ and $\gamma'(0) = w$. The family $c_s(t)$ is a smooth variation of geodesics, hence $J(t) = \partial_s c_s(t)|_{s=0}$ is a Jacobi field along c_0 . Evaluate the characteristic equation of Jacobi fields against J :

$$(3.10) \quad g_{\mathbb{X}}(\ddot{J}, J) + R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) = 0$$

In the second term, we recognize the sectional curvature of the tangent plane spanned by J and $\partial_t c$. Choose a local orthonormal chart of $T_x \mathbb{X}$ such that $B_0 = I_k \oplus O_{n-k-1}$ and

$\dot{B}_0 = B(v) \oplus I_{n-k-1}$, where $B(v)$ is the shape operator of f at v . Then equation 3.10 rewrites as

$$(3.11) \quad {}^\perp w \ddot{B}_t B_t w + R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) = 0$$

and as the sectional curvature is less than -1 , we replace the second term with the control

$$(3.12) \quad R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) \leq -{}^\perp w {}^\perp B_t B_t w$$

to get the following

$$(3.13) \quad {}^\perp w {}^\perp (\ddot{B}_t - B_t) B_t w \geq 0$$

Now remark that

$$(3.14) \quad \begin{cases} f'(t) &= 2{}^\perp w {}^\perp B_t \dot{B}_t w \\ f''(t) &= 2{}^\perp w {}^\perp B_t \ddot{B}_t w + 2{}^\perp w {}^\perp \dot{B}_t \dot{B}_t w \end{cases}$$

So the characteristic equation rewrites as

$$(3.15) \quad f''(t) \geq 2f(t) + \frac{f'(t)^2}{2f(t)}$$

with equality when \mathbb{X} is the hyperbolic space, and strict inequality if \mathbb{X} has sectional curvature strictly less than -1 . As a direct consequence,

$$(3.16) \quad (\sqrt{f})'' \geq \sqrt{f}$$

with the same equality and inequality conditions. Finally, introduce $g = \ln(f)$. As long as it is well defined, we deduce from 3.15 that

$$(3.17) \quad g'' + \frac{(g')^2}{2} \geq 2$$

□

Remark 3.2. When \mathbb{X} has varying curvature less or equal than -1 , up to rescaling the metric we can assume its sectional curvature to be strictly less than -1 .

Proof of theorem 3.1. With the notations introduced, Fix $w \in T_v(UY)$ and consider the function f defined in proposition 3.3. Denote w_1 the projection of w on TY , and assume it is nonzero. The initial conditions, by construction of f ,

$$(3.18) \quad \begin{cases} f(0) &= \alpha^2 = |w_1|^2 > 0 \\ f'(0) &= 2\alpha\beta = 2{}^\perp w_1 B(v)w_1 < 2\alpha \end{cases}$$

From proposition 3.3, we deduce that

$$(3.19) \quad \begin{cases} f(t) &\geq (\alpha \cosh(t) + \beta \sinh(t))^2 \\ f'(t) &\geq 2(\alpha \cosh(t) + \beta \sinh(t))(\alpha \sinh(t) + \beta \cosh(t)) \\ g'(t) &= \frac{f'(t)}{f(t)} \geq 2 \frac{\alpha \sinh(t) + \beta \cosh(t)}{\alpha \cosh(t) + \beta \sinh(t)} \end{cases}$$

with equality when \mathbb{X} is hyperbolic, and strict inequality when the curvature of \mathbb{X} is strictly less than -1 .

A first consequence is that f never vanishes. Hence (B_t) has trivial kernel, and the metric induced by \exp on NY is nondegenerate. So \exp is a local diffeomorphism from NY to \mathbb{X} .

Moreover, because the immersion $Y \rightarrow \mathbb{X}$ is complete, it is proper. Hence for any point of $x \in \mathbb{X}$ there is at least one point in Y which minimizes the distance to x . This implies that the map \exp is surjective. As a proper surjective local diffeomorphism, \exp satisfies the

path lifting property, so it is a covering map. But Y is connected and \mathbb{X} is contractible, so \exp is a global diffeomorphism, Y is embedded and contractible too.

It remains to prove that f is a quasi-isometric embedding. In order to prove we introduce a metric on NY for which the inverse map of the exponential map will be Lipschitz, and such that Y sits in NY totally geodesically with the induced metric by f :

Consider $(x, tv) \in NY$ and decompose its tangent space into $T_{(x,tv)}NY = T_t(x, \mathbb{R}v) \oplus T_xY \oplus T_v\mathbb{S}^{n-k-1}$.

Introduce the metric h on NY which in that decomposition is:

$$(3.20) \quad h_{(x,tv)} = 1 \oplus \cosh^2(t)f^*g_{\mathbb{X},x} \oplus \sinh^2(t)g_{\mathbb{S}^{n-k-1}}$$

By construction, Y sits inside NY totally geodesically with the metric induced by f , that we denote g_Y .

The equation on f 3.19 ensures that:

$$(3.21) \quad \exp^*g_{\mathbb{X}} \geq 1 \oplus (\cosh(t)I_k + B(v)\sinh(t))^2g_Y \oplus \sinh^2(t)g_{\mathbb{S}^{n-k-1}}$$

Denote $\delta = 1 - \sup |\mathbb{I}_f| > 0$. Then

$$(3.22) \quad \exp^*g_{\mathbb{X}} \geq \delta^2(1 \oplus \cosh^2(t)g_Y \oplus \sinh^2(t)g_{\mathbb{S}^{n-k-1}}) = \delta^2h$$

Hence the inverse map of the exponential map is $\frac{1}{\delta}$ -Lipschitz.

As a consequence, f is a quasi-isometric embedding, as it satisfies:

$$(3.23) \quad d_{\mathbb{X}}(f(x_1), f(x_2)) \leq d_Y(x_1, x_2) \leq \frac{1}{\delta}d_{\mathbb{X}}(f(x_1), f(x_2))$$

Combined with the fact that Y has negatively pinched sectional curvature proven in lemma 3.2, it ensures that f extends at infinity to an embedding from a sphere $\mathbb{S}^{k-1} \rightarrow \partial_{\infty}\mathbb{X}$. \square

As a corollary of the computations we made, we have some explicit estimates of the geometry of NY , endowed with the induced metric by \exp .

Corollary 3.4. *Let $f : Y \rightarrow \mathbb{X}$ be a complete almost-fuchsian immersion, with \mathbb{X} a Hadamard space. Denote by N_tY the distance t hypersurface to Y . At a point $\exp_x(tv) \in N_tY$, consider $\lambda_1 \leq \dots \leq \lambda_k$ the eigenvalues of $B(v)$ the shape operator of Y evaluated at (x, v) .*

Then the second fundamental form \mathbb{I}_t of N_tY has eigenvalues $\lambda_1^t \leq \dots \leq \lambda_{n-1}^t$ which satisfy

$$(3.24) \quad \forall i \leq k, \quad \sum_{j=1}^i \lambda_j^t \geq \sum_{j=1}^i \frac{\lambda_j + \tanh(t)}{1 + \lambda_j \tanh(t)}$$

$$(3.25) \quad \forall i > k, \quad \sum_{j=1}^i \lambda_j^t \geq \sum_{j=1}^k \frac{\lambda_j + \tanh(t)}{1 + \lambda_j \tanh(t)} + (i - k) \frac{1}{\tanh(t)}$$

In particular, N_tY is k -convex in the sense of Sha [Sha86].

Proof. This is a consequence of the control on $\frac{f'}{f}$ gotten in 3.19. The trace of \mathbb{I}_t on a k -plane P is the sum of $\frac{1}{2} \frac{f'(t)}{f(t)}$ for w spanning a basis of the plane P .

The almost-fuchsian condition ensures that the sum of the λ_i is 0, and that they live in a compact set of $(-1, 1)$, which ensures the k -convexity of \mathbb{I}_t . \square

4. THE ASYMPTOTIC PLATEAU PROBLEM

The Plateau problem was introduced in 1847, originally about a mathematical proof of the existence of soap films bounding a wire frame. It was solved by Douglas [Dou31] and Rado [Rad30]. Here we are interested in a noncompact analogous problem: the asymptotic Plateau problem. Fix \mathbb{X} a Hadamard space of sectional curvature less than -1 , the asymptotic Plateau problem can be formulated as

Problem 4.1 (Asymptotic Plateau Problem). Let S be a $(k-1)$ -sphere embedded in $\partial_\infty \mathbb{X}$. Is there a minimal k -dimensional submanifold M of \mathbb{X} asymptotically bounding S ? Is M unique?

The asymptotic Plateau problem has been studied by Anderson [And83]. The problem has since had a rich history, see the survey of Cozkunuzer [Cos09].

Recently, Huang–Lowe–Seppi [HLS23] considered the asymptotic Plateau problem for a class of almost-fuchsian submanifolds of \mathbb{H}^n . If a sphere in $\partial_\infty \mathbb{H}^n$ bounds an almost-fuchsian hypersurface, then it is the unique minimal hypersurface bounding it. Jiang [Jia21] also solved the asymptotic Plateau problem for Jordan curves bounding an almost-fuchsian disc in $\partial_\infty \mathbb{H}^n$. In this section, we generalize these results to any sphere bounding an almost-fuchsian submanifold in a space \mathbb{X} , under the assumption that \mathbb{X} has negatively pinched curvature.

Theorem 4.2 (Asymptotic Plateau Problem for almost-fuchsian spheres). *Let \mathbb{X} be a negatively pinched Hadamard space with sectional curvature less or equal than -1 . Let $Y \subset \mathbb{X}$ be an almost-fuchsian submanifold of dimension k , bounding at infinity a $(k-1)$ -dimensional sphere $S \subset \partial_\infty \mathbb{X}$. Then Y is the unique complete k -dimensional minimal submanifold of \mathbb{X} bounding S .*

In all the remainder of the article, \mathbb{X} is assumed to have pinched sectional curvature between $-C$ and -1 . $Y \subset \mathbb{X}$ is a complete almost-fuchsian k -dimensional submanifold, and S is its asymptotic boundary in $\partial_\infty \mathbb{X}$.

The first step in the proof is that eventual other minimal submanifolds bounding S remain at bounded distance from Y :

Proposition 4.3. *Let Z be a minimal submanifold of \mathbb{X} bounding S . Then $Z \subset \mathcal{N}_r Y$, where $\mathcal{N}_r Y$ is the r -uniform neighborhood of Y , and $r = \tanh^{-1}(1 - \sup |\mathbb{I}_f|)$.*

Proof. Thanks to theorem 2.1, we know that Z is included in the convex hull of S . But the explicit bounds shown in corollary 3.4 show that for $r \geq \tanh^{-1}(1 - \sup |\mathbb{I}_f|)$, the uniform neighborhoods $\mathcal{N}_r(Y)$ are convex, and contain Y so bound S too. As a consequence, $\mathcal{N}_r(Y)$ contains the convex hull of Y which contains Z . \square

Remark 4.1. The same corollary 3.4 shows that all uniform neighborhoods are k -convex, which directly implies that the function $z \in Z \mapsto d(z, Y)$ cannot have a local maximum. When the induced metric of Z has lower bounded sectional curvature, we could apply the Omori–Yau [Omo67, Yau75] maximum principle to conclude that $Z = Y$. Here we don't prove that Z has lower bounded sectional curvature, but we prove that the maximum principle is still applicable. This property is called *stochastic completeness*, cf Pigola–Rigoli–Setti [PRS05].

We want to apply a maximum principle to the distance function to Y restricted on a minimal submanifold Z , so we first prove it satisfies a strong subharmonicity condition.

Proposition 4.4. *Let $Z \subset \mathbb{X}$ be a minimal k -dimensional submanifold. Consider the function $u : z \in Z \mapsto d(z, Y)^2$. Then there is $C > 0$ such that*

$$(4.1) \quad \Delta u \geq Cu$$

Proof. Consider d the distance function to Y , defined on the whole space X . Because Y is almost-fuchsian, for any point of $\mathbb{X} - Y$, the distance is attained at a unique point of \mathbb{X} , so the function d is smooth on $X - Y$.

Furthermore, at a point x where $d(x) = t > 0$, we have the formula

$$(4.2) \quad \nabla^2 d = 0 \oplus \mathbb{I}_{N_t(Y)}$$

the 0 is simply because d is linear in the direction of the minimizing geodesic to Y . Using the formula

$$(4.3) \quad \nabla^2(d^2) = 2\nabla d \cdot \nabla d + 2d\nabla^2 d$$

We deduce that

$$(4.4) \quad \nabla^2(u) \geq 2 \oplus 2d\mathbb{I}_{N_t(Y)}$$

Now, because Z is a minimal submanifold of X , the laplacian of the restriction of u equals the trace of the restriction of its Hessian to the tangent space of Z

$$(4.5) \quad \Delta(u|_Z) = \text{Tr}((\nabla^2 u)|_{TZ})$$

Hence the bounds on the eigenvalues of $\mathbb{I}_{N_t(Y)}$ computed in corollary 3.4 and the control

$$(4.6) \quad \frac{\tanh(t) + \lambda_i}{1 + \lambda_i \tanh(t)} \leq 1$$

allow us to get the control:

$$(4.7) \quad \Delta(u|_Z) \geq 2d \inf_{\lambda_1 + \dots + \lambda_k = 0, |\lambda_i| \leq 1 - \varepsilon} \sum_{i=1, \dots, k} \frac{\tanh(d) + \lambda_i}{1 + \tanh(d)\lambda_i} = 2d\Phi(d)$$

Remark that $\Phi(d)$ is continuous, positive, vanishes only when $d = 0$, and satisfies

$$(4.8) \quad \Phi'(0) \geq k(1 - (1 - \sup |\mathbb{I}_f|)^2) > 0$$

As d is bounded by a constant $r = \tanh^{-1}(1 - \sup |\mathbb{I}_f|)$, there is a constant $C > 0$ such that on $[0, r]$, $\Phi(d) \geq Cd$. We deduce that the restriction of u to Z satisfies

$$(4.9) \quad \Delta u \geq 2Cu$$

, as claimed. □

Now we want to prove that we can apply the maximum principle to u . In order to do so, we will use the Khas'minskii test [Km60], as stated in Theorem 3.1 and Proposition 3.2. of [PRS05].

Theorem 4.5. *Let M, g be a Riemannian manifold, and assume that M supports a C^2 function γ , which tends to infinity at infinity, and satisfies*

$$(4.10) \quad \Delta \gamma \leq \lambda \gamma \text{ off a compact set}$$

for some $\lambda > 0$. Then M, g is stochastically complete, or equivalently for every $\lambda > 0$ the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on M is zero.

Note that the equivalence between stochastical completeness and the applicability of the maximum principle is due to Grigor'Yan [Gri99].

The following proposition will ensure that we can apply our maximum principle on Z :

Proposition 4.6. *Let $Z \subset \mathbb{X}$ be a proper submanifold of \mathbb{X} complete contractible space with sectional curvature pinched between $-b$ and $-a$, $b \geq a \geq 0$. Fix $p \in \mathbb{X}$ and denote f the distance function to the point p . Then out of a compact set K , there is a constant $C > 0$ such that the restriction of f to Z satisfies*

$$(4.11) \quad \Delta(f|_Z) \leq C$$

Proof. As \mathbb{X} is negatively pinched, there are $b \geq a \geq 0$ such that $-b \leq K_{\mathbb{X}} \leq -a$. By the Hessian comparison Theorem, the Hessian of f is bounded between the hessian of the distance functions in the spaceforms of sectional curvature $-b$ and $-a$. Explicitely, this means

$$(4.12) \quad a \coth(af)g_{\mathbb{X}} \leq \nabla^2 f \leq b \coth(bf)g_{\mathbb{X}}$$

If a is zero, one has to replace $a \coth(af)$ by $\frac{1}{f}$, but it doesn't change anything to the proof. In particular, out of a compact set K containing p , the hessian of f is bounded by a constant C . Now because Z is proper, the intersection $K \cap Z$ is a compact subset of Z . Also, because Z is minimal, the laplacian of the restriction of f satisfies

$$(4.13) \quad \Delta(f|_Z) = \text{Tr}(\nabla^2 f|_{TZ}) \leq C \dim Z.$$

as desired. \square

Applying theorem 4.5, we deduce that the maximum principle is applicable on a proper submanifold of such a space \mathbb{X} .

Corollary 4.7. *Let \mathbb{X} be a complete contractible space whose sectional curvature is bounded between $-b$ and 0 for some $b > 0$. Let $Z \subset \mathbb{X}$ a proper minimal submanifold. Then for any $u \in C^2(Z)$ such that $\sup u < \infty$, for every $c < \sup u$,*

$$(4.14) \quad \inf_{z \in Z: u(z) > c} \Delta u \leq 0$$

Proof. Thanks to proposition 4.6, the function $f|_Z$ tends to ∞ at ∞ , and out of a compact set K , it satisfies

$$(4.15) \quad \Delta f \leq C \leq C' f$$

for some constant $C' > 0$. In particular, we can apply theorem 4.5 to deduce that Z is stochastically complete, which is equivalent to the statement of our corollary by Theorem 3.1. of [PRS05]. \square

We now have everything needed to prove our main theorem.

Proof of theorem 4.2. Introduce $u : Z \rightarrow \mathbb{R}$, such that $u(z) = d(z, Y)^2$. Thanks to proposition 4.3, u is bounded. But thanks to proposition 4.4, there is $C > 0$ such that

$$(4.16) \quad \Delta u \geq Cu$$

Now because of corollary 4.7, we can apply the maximum principle on u to get that $u = 0$. Hence $Z \subset Y$, and by completeness $Z = Y$, as desired. \square

REFERENCES

- [Anc94] Alano Ancona. Convexity at infinity and brownian motion on manifolds with unbounded negative curvature. *Revista Matemática Iberoamericana*, 10(1):189–220, 1994.
- [And83] Michael T Anderson. Complete minimal hypersurfaces in hyperbolic n-manifolds. *Commentarii Mathematici Helvetici*, 58:264–290, 1983.
- [BGS85] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder. *Manifolds of nonpositive curvature*, volume 61. Progress in Mathematics, 1985.
- [Bow94] Brian H Bowditch. Some results on the geometry of convex hulls in manifolds of pinched negative curvature. *Commentarii Mathematici Helvetici*, 69:49–81, 1994.
- [BS23] Samuel Bronstein and Graham Andrew Smith. A geometric description of the set of almost-fuchsian representations. *arXiv preprint arXiv:2301.00715*, 2023.
- [Cos09] Baris Coskunuzer. Asymptotic plateau problem. *arXiv preprint arXiv:0907.0552*, 2009.
- [Dou31] Jesse Douglas. Solution of the problem of Plateau. *Transactions of the American Mathematical Society*, 33(1):263–321, 1931.
- [EES22] Christian El Emam and Andrea Seppi. On the Gauss map of equivariant immersions in hyperbolic space. *Journal of Topology*, 15(1):238–301, 2022.
- [EO73] Patrick Eberlein and Barrett O’Neill. Visibility manifolds. *Pacific Journal of Mathematics*, 46(1):45–109, 1973.
- [Eps86] Charles L Epstein. The hyperbolic gauss map and quasiconformal reflections. 1986.
- [Gri99] Alexander Grigor’Yan. Analytic and geometric background of recurrence and non-explosion of the brownian motion on riemannian manifolds. *Bulletin of the American Mathematical Society*, 36(2):135–249, 1999.
- [Gro78] M Gromov. Hyperbolic manifolds, groups and actions in riemann surfaces and related topics (stony brook, ny, 1978). *Ann. of Math. Stud*, 97, 1978.
- [HLS23] Zheng Huang, Ben Lowe, and Andrea Seppi. Uniqueness and non-uniqueness for the asymptotic plateau problem in hyperbolic space. *arXiv preprint arXiv:2309.00599*, 2023.
- [HW13] Zheng Huang and Biao Wang. On almost-fuchsian manifolds. *Transactions of the American Mathematical Society*, 365(9):4679–4698, 2013.
- [Jia21] Ruojing Jiang. Counting essential minimal surfaces in closed negatively curved n-manifolds. *arXiv preprint arXiv:2108.01796*, 2021.
- [Km60] Rafail Z Khas’minskii. Ergodic properties of recurrent diffusion processes and stabilization of the solution to the cauchy problem for parabolic equations. *Theory of Probability & Its Applications*, 5(2):179–196, 1960.
- [KS93] Nicholas J Korevaar and Richard M Schoen. Sobolev spaces and harmonic maps for metric space targets. *Communications in Analysis and Geometry*, 1(4):561–659, 1993.
- [KS07] Kirill Krasnov and Jean-Marc Schlenker. Minimal surfaces and particles in 3-manifolds. *Geometriae Dedicata*, 126(1):187–254, 2007.
- [Omo67] Hideki Omori. Isometric immersions of riemannian manifolds. *Journal of the Mathematical Society of Japan*, 19(2):205–214, 1967.
- [PP12] Jouni Parkkonen and Frédéric Paulin. On strictly convex subsets in negatively curved manifolds. *Journal of Geometric Analysis*, 22:621–632, 2012.
- [PRS05] Stefano Pigola, Marco Rigoli, and Alberto Giulio Setti. *Maximum principles on Riemannian manifolds and applications*. American Mathematical Soc., 2005.
- [Rad30] Tibor Radó. On Plateau’s problem. *Annals of Mathematics*, pages 457–469, 1930.
- [San17] Andrew Sanders. Domains of discontinuity for almost-fuchsian groups. *Transactions of the American Mathematical Society*, 369(2):1291–1308, 2017.
- [Sep16] Andrea Seppi. Minimal discs in hyperbolic space bounded by a quasicircle at infinity. *Commentarii Mathematici Helvetici*, 91(4):807–839, 2016.
- [Sha86] Ji-Ping Sha. p-convex riemannian manifolds. *Inventiones mathematicae*, 83(3):437–447, 1986.
- [Uhl83] Karen K Uhlenbeck. Closed minimal surfaces in hyperbolic 3-manifolds. In *Seminar on minimal submanifolds*, volume 103, pages 147–168, 1983.
- [Yau75] Shing-Tung Yau. Harmonic functions on complete riemannian manifolds. *Communications on Pure and Applied Mathematics*, 28(2):201–228, 1975.