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# A logarithm law for nonautonomous systems fastly converging to equilibrium and mean field coupled systems.

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## Abstract

We prove that if a nonautonomous system has in a certain sense a fast convergence to equilibrium (faster than any power law behavior) then the time  $\tau_r(x, y)$  needed for a typical point  $x$  to enter for the first time in a ball  $B(y, r)$  centered in  $y$ , with small radius  $r$  scales as the local dimension of the equilibrium measure  $\mu$  at  $y$ , i.e.

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} = d_\mu(y).$$

We then apply the general result to concrete systems of different kind, showing such a logarithm law for asymptotically autonomous solenoidal maps and mean field coupled expanding maps.

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# 1 introduction

One way to express the rarity of an event in some evolving system is to estimate the time scale in which the event is likely to occur, given the current situation of the system, thus given some information on its initial condition.

In the context of dynamical systems this naturally leads to the study of waiting times or hitting times indicators and to the study of the hitting time distribution, which is in turn connected to the classical theory of extreme events (see [1] for a survey on these topics with a particular focus on dynamical systems theory).

Most of the results already established in this direction are related to autonomous dynamical systems or stationary processes. Many important natural and social phenomena are characterized by the fact that the parameters describing the dynamics of interest may evolve with time and the systems associated are then not autonomous. This is particularly relevant in the study of climate models, and in particular in connection with the study of climate change. Due to their deep impact on the society, the study of extreme events is also particularly important in the context of climate and meteorological studies. In that context the models considered are naturally nonautonomous. In this case the theoretical study is still at its infancy and it is not clear under which assumptions, results similar to the ones currently used in the autonomous case can be established. We stress that the kind of non autonomous systems we are interested to study having in mind the application to climate dynamics (and to the mean field dynamics) is the so called *sequential* one, where the parameters change in time in a certain deterministic way and not the *random* one, in which the parameters vary randomly according to some stationary law.

In this article we focus on one of the most basic results, linking the time scale in which some rare event is likely to occur with the *fractal* dimension of the system in a neighborhood of the event itself, expressed by the so called *local dimension*. Let  $X$  be the phase space in which our dynamics occur. We will always suppose that  $(X, d)$  is a compact metric space. Let  $x_0, x_1, \dots \in X$  be a trajectory of our system with initial condition  $x_0$ , let  $y \in X$  be a target point. Let

$$\tau_r(x_0, y) = \min\{n \in \mathbb{N} : d(x_n, y) < r\}$$

be the time needed for the trajectory starting from  $x_0$  to enter a target of radius  $r$  centered in  $y$ . In the context of autonomous dynamics, supposing

the system generating the trajectory has an invariant measure  $\mu$ , in many cases of having fast speed of mixing the following result can be proved: for  $\mu$  almost all initial conditions  $x_0$

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x_0, y)}{-\log r} = d_\mu(y) \quad (1)$$

where

$$d_\mu(y) := \lim_{r \rightarrow 0} \frac{\log \mu(B_r(y))}{\log r} \quad (2)$$

is the local dimension of  $\mu$  at  $y$  and  $B_r(y)$  denotes the ball with center  $y$  and radius  $r$ . This kind of result was also called a logarithm law and relates the scaling behavior of the hitting time on small targets, with the one of the measure of the targets themselves, given by the local dimension. A logarithm law is a weaker result with respect to the ones on distribution of hitting times and extreme values theory. This result is also somewhat weaker with respect to the so called dynamical Borel Cantelli results (see [2]). In the autonomous case Logarithm laws were established for the geodesic flows and similar systems (see e.g. [3, 4, 5]), similar results have been established for Lorenz-like flows ([6], [7], [8]) or infinite systems ([9]). Generally speaking, these types of statements hold true for systems having superpolynomial decay of correlations ([10]) even for targets which are not balls ([11]). Logarithm laws however also hold for systems which are not chaotic like rotations and interval exchanges. In this case their behavior is related to arithmetical properties of the system ([4], [12], [13], [14]). Deep relations have indeed been shown with diophantine approximation (see e.g. [15], [13], [16]). It is worth to remark that (relatively slowly) mixing systems are known for which a logarithm law does not hold at all, and the time needed for a typical orbit to hit a small target is much larger than the inverse of the measure of the target (see [13], [16]).

In the paper [17] Extreme Values Theory results are established with the aim of application to non autonomous dynamical systems. In [17] a previous approach of [18] is adapted, by weakening the uniform mixing condition that was previously used to a non uniform condition which can be verified in the context of dynamical systems. The paper [17] establishes Extreme Values Laws and exponential distribution of the hitting times for a class of sequential dynamical systems whose transfer operators satisfy uniformly a list of assumption which usually are used to establish the spectral gap for those operators on a Banach space of absolutely continuous measures.

This lead to application to a sequential composition of (multidimensional) expanding maps. The result is hence particularly interesting in the context of non autonomous dynamics, but cannot be applied to the case of systems having fractal attractor, whose dimension plays an important role in the study of the event's rarity, which is the main goal of this paper.

The link between the scaling behavior of the occurrence of the hitting time and the local dimension already established in the autonomous case was successfully used in climate science to estimate the rarity of given events. Logarithm laws and the the results coming from extreme value theory were used as theoretical tools to interpret empirical data and validate the use of certain statistical estimators [19, 20, 21, 22, 23].

In non-autonomous systems, where the governing equations evolve with time, this can lead to a time-varying hitting time statistics and traditional methods for analyzing hitting time distributions may not directly apply. Moreover, the presence of external forcing or environmental perturbations further complicates the analysis, potentially leading to deviations from expected hitting time behaviors, as highlighted numerically in [24]. In the context of climate change applications, where understanding the timing and occurrence of extreme events is crucial, these extensions are particularly pertinent. This paper contributes to this endeavor by demonstrating the existence of a logarithm law for hitting times in certain non-autonomous systems, shedding light on the dynamics of rare events in evolving environments.

In the main result of the paper (see Theorem 4) we consider a sequential nonautonomous deterministic dynamical system  $(X, T_i)$  where  $i \in \mathbb{N}^*$  and  $T_i : X \rightarrow X$ . Supposing that the system has a fast convergence to equilibrium to some measure  $\mu$  we show that typical trajectories satisfy a logarithm law in the sense of (1).

The convergence to equilibrium notion which we consider is based on the convergence of the iterates  $L_{T_n} \circ \dots \circ L_{T_1}(\mu_0)$  of some initial reference measures  $\mu_0$  in a certain space, to some *equilibrium* measure  $\mu$  by the sequential composition of transfer operators  $L_{T_i}$  associated to the maps  $T_i$ . Some important class of systems where one is led to consider a sequential composition of maps behave like this. We then indeed apply our main general theorem to a class of *asymptotically autonomous* solenoidal maps which can have fractal attractors of different dimensions (See Section 3) and to a class of mean field coupled systems having exponential convergence to equilibrium (See Section 4). We remark that asymptotically autonomous systems, in which the maps considered have a certain limit map  $T_i \rightarrow T_0$  have been proposed in [25] and

[26] as natural kind of models to study in order to understand tipping points and the statistical properties of climate change.

## 2 A logarithm law in the nonautonomous case

Let us introduce some notation and terminology that will be used in the following: let us consider two compact metric spaces  $X, Y$ . Without loss of generality we will suppose that the diameter of  $X$  and  $Y$  is 1. Let us consider the spaces of Borel probability measures  $PM(X)$ ,  $PM(Y)$  on  $X$  and  $Y$ , and a Borel measurable  $F : X \rightarrow Y$ . We denote the pushforward of  $F$  as  $L_F : PM(X) \rightarrow PM(Y)$ , defined by the relation

$$[L_F(\mu)](A) = \mu(F^{-1}(A))$$

for all  $\mu \in PM(X)$  and measurable set  $A \subseteq Y$ . With the same definition, the pushforward can be extended as a linear function  $L_F : SM(X) \rightarrow SM(Y)$  from the vector space of Borel signed measures on  $X$  to the same space on  $Y$ . In this case  $L_F$  is linear and will be also called as the transfer operator associated with the function  $F$ .

Let us consider on  $(X, d)$ , a family of maps  $T_i : X \rightarrow X$  with  $i \in \mathbb{N}^*$  and a sequential non autonomous system  $(X, T_i)$ .

Consider two points  $x, y \in X$ . The orbit of  $x$  is the sequence

$$x, T_1(x), T_2(T_1(x)), \dots$$

We denote the sequential composition of the maps by  $T^{(0)}(x) = x$  and inductively for  $k \in \mathbb{N}^*$ ,  $T^{(k)}(x) := T_k(T^{(k-1)}(x))$ . Let  $B_r(y)$  be the ball of radius  $r$  centered in  $y$ , we denote the hitting time of  $B_r(y)$  for the orbit of  $x$  as

$$\tau_r(x, y) = \min\{n \in \mathbb{N} : T^{(n)}(x) \in B_r(y)\}.$$

Typically  $\tau_r(x, y) \rightarrow +\infty$  as  $r \rightarrow 0$ . To give an estimate on how rare is the hitting of such small targets as an event on our system, in the following we will estimate the speed, asymptotically  $\tau_r(x, y)$  goes to  $+\infty$ .

In a system which is not autonomous there is not an invariant measure, we will replace it with a kind of asymptotically invariant one, which we will call the equilibrium measure.

We will suppose that in our phase space  $X$  a starting "reference" Borel probability measure  $\mu_0$  is considered (it can be for example the normalized volume measure when  $X$  is a Riemannian manifold), and that the iterates of the pushforward of  $\mu_0$  through the dynamics converge to a certain measure  $\mu$ . We will suppose that there is a certain  $\mu \in PM(X)$  such that as  $n \rightarrow \infty$

$$L_{T^{(n)}}\mu_0 \rightarrow \mu$$

with convergence in a certain topology, and with a certain superpolynomial speed.

To formalize the assumptions, let us define a certain weak norm and distance to be considered in spaces of measures on metric spaces. Let  $(X, d)$  be a compact metric space and let  $g : X \rightarrow \mathbb{R}$  be a Lipschitz function and let  $Lip(g)$  be its best Lipschitz constant, i.e.

$$Lip(g) = \sup_{x, y \in X} \left\{ \frac{|g(x) - g(y)|}{d(x, y)} \right\}.$$

We also define the Lipschitz norm of  $g$  as

$$\|g\|_{Lip} = \max(Lip(g), \sup_{x \in X} |g(x)|).$$

**Definition 1** *Given a Borel signed measure  $\mu$  on  $X$ , we define a **Wasserstein-Kantorovich Like** norm of  $\mu$  by*

$$\|\mu\|_W = \sup_{\|g\|_{Lip} \leq 1} \left| \int g d\mu \right|. \quad (3)$$

*To this norm one can associate the distance*

$$W(\mu, \nu) = \|\mu - \nu\|_W \quad (4)$$

*for  $\mu, \nu \in SM(X)$ .*

Let us denote the sequential composition of transfer operators  $L_{T_k} : SM(X) \rightarrow SM(X)$  associated to the maps  $T_k$  as

$$L^{(j,k)} := L_{T_k} \circ L_{T_{k-1}} \circ \dots \circ L_{T_j}$$

for  $k > j$  and

$$L^{(k)} := L_{T_k} \circ \dots \circ L_{T_1}$$

for  $k > 1$ . Coherently we denote  $L^{(k,k)} := L_{T_k}$ ,  $L^{(0)} := Id$  and  $L^{(1)} := L_{T_1}$ .

Now we can formalize the general framework in which our abstract result is stated. As usual in the study of transfer operators we consider the action of the operator itself on a suitable normed vector space of measures or distributions. We suppose that the space considered, which we will denote by  $B_s$  has a topology which is stronger than the one induced by the  $W$  distance above defined.

**Definition 2** *Let  $(B_s, || \cdot ||_s) \subseteq SM(X)$  be a normed vector subspaces of the space of Borel signed measures on  $X$ . Suppose there is  $C \geq 0$  such that  $|| \cdot ||_W \leq C || \cdot ||_s$ . Suppose that for each  $i$ ,  $L_{T_i}$  preserves  $B_s$ . We say that the nonautonomous system  $(X, T_i)$  has weak convergence to equilibrium with superpolynomial speed if there is a probability measure  $\mu \in B_s$  and  $\Phi$  superpolynomially decreasing <sup>1</sup> such that  $\forall k, j \in \mathbb{N}$  with  $k \geq j$  and each probability measure  $\mu_0 \in B_s$*

$$||\mu - L^{(j,k)}\mu_0||_W \leq \Phi(k-j) \max(1, ||\mu - \mu_0||_s).$$

**Definition 3** *We say that a set  $A \subseteq B_s$  has uniformly bounded Lipschitz multipliers if there is  $C_A \geq 0$  depending on  $A$  such that for each  $\mu_0 \in A$  and  $\phi \in Lip(X)$  we have  $\phi\mu_0 \in B_s$  and*

$$||\phi\mu_0||_s \leq C_A ||\phi||_{Lip}.$$

With the above definitions we can state the main general result of the paper, linking the scaling behavior of the hitting time of typical orbits and the local dimension of  $\mu$ .

**Theorem 4** *Let us consider a probability measure  $\mu_0 \in PM(X)$ , suppose that the set*

$$A := \{\mu_k := L^{(k)}\mu_0, k \in \mathbb{N}\}$$

*is bounded in  $B_s$  and has uniformly bounded Lipschitz multipliers. Suppose furthermore that  $(X, T_i)$  has convergence to equilibrium with superpolynomial speed as in Definition 2. Suppose  $y \in X$  is such that the local dimension  $d_\mu(y)$  of  $\mu$  at  $y$  exists in the sense of (2) and also suppose that the preimages of  $y$  have zero  $\mu_0$  measure: more precisely let us suppose that  $\forall i \in \mathbb{N}$*

$$\mu_0(\{x \text{ s.t. } T^{(i)}(x) = y\}) = 0. \quad (5)$$

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<sup>1</sup>We say  $\Phi$  is superpolynomially decreasing if the function  $\Phi : \mathbb{N} \rightarrow \mathbb{R}$  is decreasing and for each  $\alpha > 0$ ,  $\lim_{n \rightarrow \infty} n^\alpha \Phi(n) = 0$ .



Then we have

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} = d_\mu(y)$$

for  $\mu_0$  almost every  $x$ .

**Remark 5** We remark that the assumption (5) is automatically satisfied if the maps considered have countable degree (that is  $\forall x \in X, i \in \mathbb{N}^*$  the set  $T^{-1}(x)$  is countable) and  $\mu_0$  is nonatomic.

In order to prove the main result we need some preliminary result.

The first one is a kind of dynamical Borel-Cantelli Lemma adapted to our case.

**Lemma 6** Let  $(X, T_i)$  be a sequential nonautonomous system, let  $\mu_0 \in PM(X)$  and  $\mu_k := L^{(k)}\mu_0$  as above.

Suppose there is  $\mu \in PM(X)$  and a superpolynomially decreasing  $\Phi$  such that for each  $g \in Lip(X)$ ,  $j, k \in \mathbb{N}$ , the measure  $g\mu_j$  converges to  $\mu \int g d\mu_j$  at a uniform superpolynomial speed in the  $W$  distance: more precisely for each such  $g, j, k$

$$\|L_{T_{j+k}} \circ \dots \circ L_{T_{j+1}}[g\mu_j] - \mu \int g d\mu_j\|_W \leq \max(1, \|g\|_{Lip})\Phi(k). \quad (6)$$

Let  $g_k$  be a sequence of positive Lipschitz observables such that

$$\sup_{x \in X, k \in \mathbb{N}} |g_k(x)| \leq 1.$$

Suppose that  $\exists B \geq 1, \beta > 0$  such that  $\|g_k\|_{Lip} \leq Bk^\beta$  and suppose that  $\exists \gamma, C > 0$  such that

$$\sum_{j \leq n} \int g_j(T^{(j)}(x)) d\mu_0 \geq Cn^\gamma. \quad (7)$$

Then

$$\frac{\sum_{j \leq n} g_j(T^{(j)}(x))}{\sum_{j \leq n} \int g_j(T^{(j)}(x)) d\mu_0} \rightarrow 1$$

$\mu_0$  almost everywhere.

**Proof.** First let us remark that for the Lipschitz observables  $g_j$ , by the fast convergence to equilibrium (6) we get that

$$|\int g_j(T^{(j)}(x))d\mu_0 - \int g_j d\mu| = |\int g_j d\mu_j - \int g_j d\mu| \quad (8)$$

$$\leq \|g_j\|_{Lip} \|\mu_j - \mu\|_W \leq B j^\beta \Phi(j) \quad (9)$$

and since  $B j^\beta \Phi(j)$  is summable we get that there is  $C_2 > 0$  such that

$$\sum_{j \leq n} \int g_j d\mu \geq C_2 n^\gamma.$$

Let  $\gamma$  as above, consider  $\alpha < \frac{\gamma}{2}$

$$\begin{aligned} \int \left( \sum_{1 \leq k \leq n} g_k(T^{(k)}(x)) \right)^2 d\mu_0 &= \sum_{1 \leq j \leq n} \int (g_j(T^{(j)}(x)))^2 d\mu_0 \\ &+ 2 \sum_{\substack{k, j \leq n, k > j \\ k < j + n^\alpha}} \int g_j(T^{(j)}(x)) g_k(T^{(k)}(x)) d\mu_0 \\ &+ 2 \sum_{\substack{k, j \leq n \\ k \geq j + n^\alpha}} \int g_j(T^{(j)}(x)) g_k(T^{(k)}(x)) d\mu_0. \end{aligned}$$

Since  $\forall i, 0 \leq g_i \leq 1$  this implies  $g_j(T^{(j)}(x)) g_k(T^{(k)}(x)) \leq g_k(T^{(k)}(x))$  and

$$\begin{aligned} &\sum_{1 \leq j \leq n} \int (g_j(T^{(j)}(x)))^2 d\mu_0 + 2 \sum_{\substack{k, j \leq n, k > j \\ k < j + n^\alpha}} \int g_j(T^{(j)}(x)) g_k(T^{(k)}(x)) d\mu_0 \\ &\leq 2n^\alpha \sum_{j \leq n} \int g_j(T^{(j)}(x)) d\mu_0. \end{aligned} \quad (11)$$

Now let us estimate

$$\sum_{k, j \leq n, k \geq j + n^\alpha} \int g_j(T^{(j)}(x)) g_k(T^{(k)}(x)) d\mu_0.$$

We have

$$|\int g_j(T^{(j)}(x)) g_k(T^{(k)}(x)) d\mu_0| \leq |\int g_k(x) dL^{(j+1, k)}[g_j d\mu_j]|$$

where  $L^{(j+1,k)} := L_{T^k} \circ \dots \circ L_{T^{j+1}}$ . By (6)

$$\|L^{(j+1,k)}[g_j \mu_j] - [\int g_j(x) d\mu_j] \mu\|_W \leq \max(1, \|g_j\|_{Lip}) \Phi(k-j-1)$$

and then

$$\begin{aligned} |\int g_k(x) dL^{(j+1,k)}[g_j \mu_j]| &\leq \int g_k(x) d\mu \int g_j(x) d\mu_j + B^2[k]^\beta [j+1]^\beta \Phi(k-j-1) \\ &\leq [\int g_k \circ T^{(k)} d\mu_0 + B[k]^\beta \Phi(k)] \int g_j \circ T^{(j)} d\mu_0 \\ &\quad + B^2[k]^\beta [j+1]^\beta \Phi(k-j-1) \\ &\leq \int g_k \circ T^{(k)} d\mu_0 \int g_j \circ T^{(j)} d\mu_0 \\ &\quad + B[k]^\beta \Phi(k) + B^2[k]^\beta [j+1]^\beta \Phi(k-j-1) \end{aligned}$$

using again (8). Hence

$$\begin{aligned} \sum_{\substack{k,j \leq n \\ k \geq j+n^\alpha}} \int g_j \circ T^{(j)} g_k \circ T^{(k)} d\mu_0 &\leq \sum_{\substack{k,j \leq n \\ k \geq j+n^\alpha}} [\int g_j \circ T^{(j)} d\mu_0 \int g_k \circ T^{(k)} d\mu_0 \quad (12) \\ &\quad + B[k]^\beta \Phi(k) + B^2[k]^\beta [j+1]^\beta \Phi(k-j-1)] \\ &\leq \sum_{\substack{k,j \leq n \\ k \geq j+n^\alpha}} [\int g_j \circ T^{(j)} d\mu_0 \int g_k \circ T^{(k)} d\mu_0] \quad (14) \\ &\quad + 2B^2 n^{2\beta+2} \Phi(n^\alpha) + B n^{\beta+2} \Phi(n^\alpha). \quad (15) \end{aligned}$$

Now consider the sequence of random variables  $Z_n(x) := \sum_{1 \leq j \leq n} g_j(T^{(j)}(x))$  and denote by  $E(Z_n) := \int \sum_{1 \leq j \leq n} g_j(T^{(j)}(x)) d\mu_0(x)$  let us consider the additional sequence of random variables

$$Y_n = \frac{Z_n}{E(Z_n)} - 1 = \frac{Z_n - E(Z_n)}{E(Z_n)}.$$

And since  $\int (Z_n - E(Z_n))^2 d\mu_0 = \int (Z_n)^2 d\mu_0 - (E(Z_n))^2$  we get

$$\begin{aligned} E((Y_n)^2) &= \frac{\int Z_n^2 d\mu_0 - E(Z_n)^2}{E(Z_n)^2} \\ &= \frac{\int (\sum_{1 \leq k \leq n} g_k(T^{(k)}(x)))^2 d\mu_0 - (\sum_{1 \leq k \leq n} \int g_k(T^{(k)}(x)) d\mu_0)^2}{(\sum_{1 \leq k \leq n} \int g_k(T^{(k)}(x)) d\mu_0)^2} \\ &\leq \frac{2n^\alpha \sum_{j \leq n} \int g_j(T^{(j)}(x)) d\mu_0 + B^2 4n^{2\beta+2} \Phi(n^\alpha) + 2Bn^{\beta+2} \Phi(n^\alpha)}{(\sum_{1 \leq k \leq n} \int g_k(T^{(k)}(x)) d\mu_0)^2} \end{aligned}$$

where in the last line we used (12) and (10). By this and (7), since  $\alpha < \frac{\gamma}{2}$  we establish  $E((Y_n)^2) \rightarrow 0$ . Now consider

$$n_k = \inf \{n : \sum_{1 \leq j \leq n} \int g_j(T^{(j)}(x)) d\mu_0 \geq k^2\}. \quad (16)$$

$$\begin{aligned} E((Y_{n_k})^2) &\leq \frac{2n_k^\alpha \sum_{j \leq n_k} \int g_j(T^{(j)}(x)) d\mu_0 + 2B^2 n_k^{2\beta+2} \Phi(n_k^\alpha) + 2Bn_k^{\beta+2} \Phi(n_k^\alpha)}{(\sum_{1 \leq j \leq n_k} \int g_j(T^{(j)}(x)) d\mu_0)^2} \\ &\leq \frac{2n_k^\alpha}{\sum_{j \leq n_k} \int g_j(T^{(j)}(x)) d\mu_0} + \frac{4B^2 n_k^{2\beta+2} \Phi(n_k^\alpha)}{(\sum_{1 \leq j \leq n_k} \int g_j(T^{(j)}(x)) d\mu_0)^2} \\ &\quad + \frac{2Bn_k^{\beta+2} \Phi(n_k^\alpha)}{(\sum_{1 \leq j \leq n_k} \int g_j(T^{(j)}(x)) d\mu_0)^2} \end{aligned}$$

Since  $\forall \epsilon > 0$ , for  $n$  big enough,  $\sum_{j \leq n} \int g_j(T^{(j)}(x)) d\mu_0 \geq n^{\gamma-\epsilon}$  then  $n_k \leq (k+1)^{\frac{2}{\gamma-\epsilon}} \leq (2k)^{\frac{2}{\gamma-\epsilon}}$  and

$$\frac{2n_k^\alpha}{\sum_{j \leq n_k} \int g_j(T^{(j)}(x)) d\mu_0} \leq \frac{2(2k)^{\frac{2\alpha}{\gamma-\epsilon}}}{k^2}$$

and since  $\alpha < \frac{\gamma}{2}$ , and  $\epsilon$  can be taken small as wanted, we have that

$$\sum_{k \geq 0} E((Y_{n_k})^2) < \infty$$

then by the Borel Cantelli Lemma,  $Y_{n_k} \rightarrow 0$  a.e. Now we prove that the whole  $Y_n \rightarrow 0$  a.e. Indeed if  $n_k \leq n \leq n_{k+1}$

$$\frac{Z_n}{E(Z_n)} \leq \frac{Z_{n_{k+1}}}{E(Z_{n_k})} = \frac{Z_{n_{k+1}}}{E(Z_{n_{k+1}})} \frac{E(Z_{n_{k+1}})}{E(Z_{n_k})} \leq \frac{Z_{n_{k+1}}}{E(Z_{n_{k+1}})} \frac{(k+2)^2}{k^2}$$

and

$$\frac{Z_n}{E(Z_n)} \geq \frac{Z_{n_k}}{E(Z_{n_{k+1}})} = \frac{Z_{n_k}}{E(Z_{n_k})} \frac{E(Z_{n_k})}{E(Z_{n_{k+1}})} \geq \frac{Z_{n_k}}{E(Z_{n_k})} \frac{k^2}{(k+2)^2}.$$

then we have  $\lim_{n \rightarrow \infty} \frac{Z_n}{E(Z_n)} = 1$ ,  $\mu$ -almost everywhere. ■

We will use the last Lemma to prove a proposition which will be an intermediate step in proving Theorem 4.

**Proposition 7** *Let  $(X, T_i)$  be a sequential nonautonomous system, let  $\mu_0 \in PM(X)$  as above and suppose there is  $\mu \in PM(X)$  towards  $g\mu_i$  converges at a superpolynomial speed as in (6). Let us consider a target point  $y \in X$  such that assumption (5) is satisfied. Then the equality*

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} = d_\mu(y)$$

*holds for  $\mu_0$  almost every  $x$ .*

This proposition is of independent interest since directly establishes the logarithm law. However the assumption required (see (6)) may look quite technical and difficult to verify. For this we decided to state the main result in the form shown at Theorem 4, whose assumptions are more similar to what is expected to be established in concrete examples, where the maps involved are already known to satisfy some regularization (Lasota Yorke) inequalities on certain functional spaces, and some convergence to equilibrium properties.

**Proof of Proposition 7.** Let  $r_k \rightarrow 0$  be a decreasing sequence, let  $B(y, r_k)$  be a sequence of balls with decreasing radius centered at  $y$ , let  $\phi_k$  be a Lipschitz function such that  $\phi_k(x) = 1$  for all  $x \in B(y, r_k)$ ,  $\phi_k(x) = 0$  if  $x \notin B(y, r_{k-1})$  and  $\|\phi_k\|_{Lip} \leq \frac{1}{r_{k-1} - r_k}$  (such functions can be constructed as  $\phi_k(x) = h(d(y, x))$  where  $h$  is a piecewise linear Lipschitz function  $\mathbb{R} \rightarrow [0, 1]$ ).

First we prove that  $\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{r} \geq d_\mu(y)$ ,  $\mu_0$  almost everywhere. This follows by a classical Borel-Cantelli argument.

Let us suppose that for some  $\epsilon > 0$ ,  $\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} \leq d_\mu(y) - \epsilon$  on a certain set  $A \subseteq X$ . Let us consider the sequence  $r_k = k^{-(d_\mu(y) - \epsilon)^{-1}}$ . From the properties of logarithms, it is standard to get (see the beginning of the proof of Theorem 4 of [27])  $\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} = \liminf_{k \rightarrow \infty} \frac{\log \tau_{r_k}(x, y)}{-\log r_k}$ . Hence  $\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} \leq d_\mu(y) - \epsilon$  implies that  $\frac{\log \tau_{r_k}(x, y)}{-\log r_k} \leq d_\mu(y) - \epsilon$  for infinitely many  $k$ 's.

We have that for each  $\epsilon' > 0$  eventually when  $k$  is large enough

$$\int \phi_k d\mu \leq (k-1)^{-(d_\mu(y)-\epsilon)^{-1}d_\mu(y)-\epsilon'}.$$

If  $\epsilon'$  is so small that  $(d_\mu(y)-\epsilon)^{-1}d_\mu(y)-\epsilon' > 1$ , then  $\sum_k \int \phi_k d\mu < \infty$ . Let us now consider the sequence  $\phi_k \circ T^{(k)}$  and let us estimate  $\int \phi_k \circ T^{(k)} d\mu_0$ . We have eventually as  $k \rightarrow \infty$  that

$$\begin{aligned} \left| \int \phi_k \circ T^{(k)} d\mu_0 - \int \phi_k d\mu \right| &= \left| \int \phi_k d\mu_k - \int \phi_k d\mu \right| \\ &\leq \|\phi_k\|_{Lip} \|\mu_k - \mu\|_W \\ &\leq k^\beta \Phi(k) \end{aligned}$$

where  $\beta > \lim_{k \rightarrow \infty} \frac{\log \frac{1}{r_k-1-r_k}}{\log k}$ , and since  $k^\beta \Phi(k)$  is summable we get that for each such  $\epsilon > 0$

$$\sum_{k \leq n} \int \phi_k \circ T^{(k)} d\mu_0 < \infty.$$

This means that the set of  $x \in X$  for which  $\sum_k \phi_k(T^{(k)}(x)) = \infty$  is a zero  $\mu_0$ -measure set. This set includes the set of  $x$  such that  $d(T^{(k)}(x), y) \leq r_k$  infinitely many times and the set of  $x \in X$  such that  $\forall i T^{(i)}(x) \neq y$  and for infinitely many  $k$ ,  $\tau_{r_k}(x, y) \leq k = r_k^{-(d_\mu(y)-\epsilon)}$  proving that  $A$  is a zero  $\mu_0$ -measure one <sup>2</sup>.

Now we prove that

$$\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{r} \leq d_\mu(y) \quad (17)$$

$\mu_0$  almost everywhere. Let us consider some small  $\epsilon' > 0$  and the set of  $x$  such that  $\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{r} \geq d_\mu(y) + \epsilon'$ . In order to estimate the measure of such set, let us consider some  $0 < \beta < \frac{1}{d_\mu(y)}$  (implying  $\beta d_\mu(y) < 1$ ) such that  $\beta(d_\mu(y) + \epsilon') > 1$ . Consider then the sequence of radii  $r_k = k^{-\beta}$ . We remark that as before, if (17) is proved for such a sequence, then it holds for

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<sup>2</sup>If  $\forall i T^{(i)}(x) \neq y$  and  $\tau_{r_k}(x, y) \leq k$  for infinitely many  $k$ , we have infinitely many  $k$  for which  $d(T^{(k)}(x), y) \leq r_k$ . Indeed assuming the opposite. Let us consider  $k$  to be the last index for which  $d(T^{(k)}(x), y) \leq r_k$ . Since  $\min_{i \leq k} d(T^{(i)}(x), y) > 0$  we can consider  $k' > k$  such that  $0 < r_{k'} < \min_{i \leq k} d(T^{(i)}(x), y)$  since still we have  $\tau_{r_{k'}}(x, y) \leq k'$  we have a new close approach to the target, negating the assumption.

all sequences. Now remark that for each small  $\epsilon < \beta^{-1} - d_\mu(y)$ , eventually as  $k \rightarrow \infty$ ,  $\int \phi_k d\mu \geq (r_k)^{d_\mu(y)+\epsilon} = k^{-\beta(d_\mu(y)+\epsilon)}$  and there is  $C > 0$  such that

$$\sum_0^k \int \phi_k d\mu \geq C k^{1-\beta(d_\mu(y)+\epsilon)} \quad (18)$$

for each  $k \geq 0$ . Now let us estimate the sequence  $\int \phi_k \circ T^{(k)} d\mu_0$ . We have as before that eventually, as  $k \rightarrow \infty$

$$\begin{aligned} \left| \int \phi_k \circ T^{(k)} d\mu_0 - \int \phi_k d\mu \right| &= \left| \int \phi_k d\mu_k - \int \phi_k d\mu \right| \\ &\leq \|\phi_k\|_{Lip} \|\mu_k - \mu\|_W \\ &\leq k^{\beta'} \Phi(n) \end{aligned}$$

for some  $\beta' > \lim_{k \rightarrow \infty} \frac{\log \frac{1}{r_{k-1}-r_k}}{\log k}$  since  $k^{\beta'} \Phi(n)$  is summable we get by (18) that for each  $\epsilon > 0$ , eventually

$$\sum_{k \leq n} \int \phi_k \circ T^{(k)} d\mu_0 \geq n^{1-\beta d_\mu(y)-\beta\epsilon}.$$

We can then apply Lemma 6 and obtain that setting  $Z_n(x) = \sum_{j \leq n} \phi_j(T^{(j)}(x))$  and  $E(Z_n) = \int \sum_{j \leq n} \phi_j(T^{(j)}(x)) d\mu_0$ , for such a sequence,  $\lim_{n \rightarrow \infty} \frac{Z_n}{E(Z_n)} = 1$   $\mu_0$ -almost everywhere. We are now going to use this to complete the proof. Let us hence still consider  $\beta$  as above, near but below  $\frac{1}{d_\mu(y)}$  and  $\epsilon' > 0$  such that  $\beta(d_\mu(y) + \epsilon') > 1$ . Let us consider  $x$  such that  $\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{r} \geq d_\mu(y) + \epsilon'$  then, for infinitely many  $n$ ,  $\tau_{(n-1)^{-\beta}}(x, y) \geq (n-1)^{\beta(d_\mu(y)+\epsilon')}$ , then  $T^{(i)}(x) \notin B(y, (n-1)^{-\beta})$  for each  $0 \leq i \leq (n-1)^{\beta(d_\mu(y)+\epsilon')}$  and in particular  $T^{(i)}(x) \notin B(y, (i-1)^{-\beta})$  for  $n \leq i \leq (n-1)^{\beta(d_\mu(y)+\epsilon')}$  which implies  $Z_n(x) = Z_{n^{\beta(d_\mu(y)+\epsilon')}}(x)$  for infinitely many  $n$ . But

$$\frac{\sum_{i=0}^{n^{\beta(d_\mu(y)+\epsilon')}} \int \phi_j d\mu}{\sum_{i=0}^n \int \phi_j d\mu} \geq \frac{\sum_{i=0}^{n^{\beta(d_\mu(y)+\epsilon')}} \mu(B(y, i^{-\beta}))}{\sum_{i=0}^n \mu(B(y, (i-1)^{-\beta}))} \rightarrow \infty$$

eventually as  $n \rightarrow \infty$  because  $\beta d_\mu(y) < 1$ , implying that the above sums go to  $\infty$  and because  $\beta(d_\mu(y) + \epsilon') > 1$ , implying that the numerator's sum goes to  $\infty$  faster than the denominator's one. Then as shown before

$$\frac{E(Z_{n^{\beta(d_\mu(y)+\epsilon')}})}{E(Z_n)} = \frac{\sum_{j=0}^{n^{\beta(d_\mu(y)+\epsilon')}} \int \phi_j(T^{(j)}(x)) d\mu_0}{\sum_{j=0}^n \int \phi_j(T^{(j)}(x)) d\mu_0} \rightarrow \infty$$

hence in order to get  $\lim_{n \rightarrow \infty} \frac{Z_n}{E(Z_n)} = 1$   $\mu_0$ -almost everywhere one must have  $\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{r} \geq d_\mu(y) + \epsilon'$  on a zero measure set. Since  $\beta$  can be chosen as near as we want to  $\frac{1}{d_\mu(y)}$ ,  $\epsilon'$  can be chosen to be arbitrary small we have the statement. ■

Now we see that the assumptions of Theorem 4 implies the ones of Proposition 7 and then we can get our main result applying the proposition.

**Lemma 8** *Given a probability measure  $\mu_0 \in B_s$ , let us suppose that the set  $A := \{\mu_k := L^{(k)}\mu_0, k \in \mathbb{N}\}$  is bounded in  $B_s$  and has uniformly bounded Lipschitz multipliers in the sense of Definition 2. Suppose  $(X, T_i)$  has convergence to equilibrium with superpolynomial speed in the sense of Definition 3 and there is  $C \geq 0$  such that  $\|\mu_j\|_s \leq C$ , then for each  $g \in Lip(X)$ ,  $i \in \mathbb{N}$  the measure  $g\mu_i$  converges to  $\mu \int g d\mu_i$  at a superpolynomial speed as expressed in (6).*

**Proof.** Since  $A$  is bounded there is a  $C_2 \geq 0$  s.t.  $\|\mu_j\|_s \leq C_2 \forall j$  and  $\|\mu\|_s \leq C_2$ . In order to prove (6), from the convergence to equilibrium we have

$$\|L_{T_{j+k}} \circ \dots \circ L_{T_{j+1}}[g\mu_j] - \mu \int g d\mu_j\|_W \leq \Phi(k) \max(1, \|g\mu_j - \mu \int g d\mu_j\|_s).$$

By the bounded Lipschitz multiplier property

$$\|g\mu_j - \mu \int g d\mu_j\|_s \leq C_A \|g\|_{Lip} + C_2 \|g\|_{Lip}$$

since  $\|g\|_\infty \leq \|g\|_{Lip}$  and  $\mu_j$  is a probability measure. We have then

$$\|L_{T_{j+k}} \circ \dots \circ L_{T_{j+1}}[g\mu_j] - \mu \int g d\mu_j\|_W \leq \Phi'(k) \max(1, \|g\|_{Lip})$$

for a superpolynomially decreasing  $\Phi'$  as required by (6). ■

Having collected the necessary results now we can prove the main theorem.

**Proof of Theorem 4.** By Lemma 8 we see that the assumptions of Theorem 4 imply the assumptions of Proposition 7. The application of this proposition directly lead to the result. ■



### 3 Application to asymptotically autonomous systems

In this section we show an example of application of Theorem 4 to a family of solenoidal maps forming a nonautonomous system. Such family is also an eventually autonomous system in the sense of [25]. Solenoidal maps are known to have a fractal attractor whose dimension can vary, depending on the map's contraction and expansion rates. The same can be said for the local dimension of the unique physical invariant measure. To keep the treatment short and avoid technicalities, we choose a relatively simple family of such maps where the maps vary in time only on the second coordinate. We hence consider a family  $F_i$  of solenoidal maps. Each element of  $F_i$  is a  $C^2$  map  $F_i : X \rightarrow X$  where  $X = \mathbb{S}^1 \times D^2$  the filled torus, and  $F_i$  is a skew product

$$F_i(x, y) = (T(x), G_i(x, y)), \quad (19)$$

where  $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  and  $G_i : X \rightarrow D^2$  are smooth maps. We suppose the map  $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  to be  $C^3$ , expanding<sup>3</sup> of degree  $q$ , giving rise to a map  $[0, 1] \rightarrow [0, 1]$ , which we denote by  $\tilde{T}$  and whose branches will be denoted by  $\tilde{T}_i$ ,  $i \in [1, \dots, q]$ . We make the following assumptions on the  $G_i$ :

- (a) Consider the  $F$ -invariant foliation  $\mathcal{F}^s := \{\{x\} \times D^2\}_{x \in \mathbb{S}^1}$ . We suppose that  $\mathcal{F}^s$  is contracted: there exists  $0 < \alpha < 1$  such that for all  $x \in \mathbb{S}^1, i \in \mathbb{N}$  holds

$$|G_i(x, y_1) - G_i(x, y_2)| \leq \alpha |y_1 - y_2| \text{ for all } y_1, y_2 \in D^2. \quad (20)$$

- (b)  $\sup_{x \in \mathbb{S}^1, i \in \mathbb{N}} \left| \frac{\partial G_i}{\partial x}(x) \right| < \infty$ .

- (c)  $\sup_{x, y} |G_i(x, y) - G_0(x, y)| \leq \Phi(i)$  with  $\Phi$  being decreasing and having superpolynomial decay.

In the following, applying the theory we have shown in the previous sections we will prove the logarithm law for this system:

**Proposition 9** *Let  $(X, F_i)$  be a sequential family of solenoidal maps satisfying the above assumptions (a), (b), (c), let  $\mu_0$  be the Lebesgue measure on  $X$  and  $\mu$  the unique physical measure of  $F_0$ .*

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<sup>3</sup>There is  $\alpha < 1$  such that  $\forall x \in \mathbb{S}^1, |T'_0(x)| \geq \alpha^{-1} > 1$ .

Suppose  $y \in X$  is such that the local dimension  $d_\mu(y)$  of  $\mu$  at  $y$  exists, then the equality

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x, y)}{-\log r} = d_\mu(y)$$

holds for  $\mu_0$  almost every  $x$ .

We will prove Proposition 9 by applying Theorem 4. In order to do this we construct some functional spaces which are suitable for the system we consider. We follow the simple construction of anisotropic spaces suitable for skew products which can be found in [28] and [29]. The idea is to consider spaces of measures with sign, with suitable norms constructed by disintegrating the measures along the stable, preserved foliation.

Given  $\mu \in SM(X)$  denote by  $\mu^+$  and  $\mu^-$  the positive and the negative parts of it ( $\mu = \mu^+ - \mu^-$ ).

Let  $\pi_x : X \rightarrow \mathbb{S}^1$  be the projection defined by  $\pi(x, y) = x$  and let  $\pi_x^*$  be the associated pushforward map.

Denote by  $\mathcal{AB}$  the set of measures  $\mu \in SM(X)$  such that its associated marginal measures,  $\mu_x^+ := \pi_x^* \mu^+$ ,  $\mu_x^- := \pi_x^* \mu^-$  are absolutely continuous with respect to the Lebesgue measure  $m$  on  $\mathbb{S}^1$  i.e.

$$\mathcal{AB} = \{\mu \in SM(X) : \pi_x^* \mu^+ \ll m \text{ and } \pi_x^* \mu^- \ll m\}. \quad (21)$$

Let us consider a finite positive measure  $\mu \in \mathcal{AB}$  on the space  $X$  foliated by the contracting leaves  $\mathcal{F}^s = \{\gamma_l\}_{l \in \mathbb{S}^1}$  such that  $\gamma_l = \pi_x^{-1}(l)$ . The Rokhlin Disintegration Theorem describes a disintegration  $(\{\mu_{\gamma_l}\}_{\gamma_l \in \mathcal{F}^s}, \mu_x =: \phi_\mu m)$  by a family  $\{\mu_\gamma\}$  of probability measures on the stable leaves and a non negative marginal density  $\phi_\mu : \mathbb{S}^1 \rightarrow \mathbb{R}$  with  $\|\phi_\mu\|_1 = \mu(X)$ . By this disintegration, for each measurable set  $E \subset X$ , with the above notations it holds

$$\mu(E) = \int_{\mathbb{S}^1} \mu_{\gamma_l}(E \cap \gamma_l) d\mu_x(l). \quad (22)$$

**Definition 10** Let  $\pi_y : X \rightarrow D^2$  be the projection defined by  $\pi_y(x, y) = y$  and  $\gamma \in \mathcal{F}^s$ . Given a positive measure  $\mu \in \mathcal{AB}$  and its disintegration along the stable leaves  $\mathcal{F}^s$ ,  $(\{\mu_{\gamma_l}\}_{\gamma_l}, \mu_x = \phi_\mu m)$  we define the **restriction of  $\mu$  on  $\gamma_l$**  as the positive measure  $\mu|_{\gamma_l}$  on  $D^2$  (not on the leaf  $\gamma_l$ ) defined, for all measurable set  $A \subset D^2$ , by

$$\mu|_{\gamma_l} := \pi_y^*(\phi_\mu(l)\mu_{\gamma_l}).$$

For a given signed measure  $\mu \in \mathcal{AB}$  and its decomposition  $\mu = \mu^+ - \mu^-$ , define the **restriction of  $\mu$  on  $\gamma_l$**  by

$$\mu|_{\gamma_l} := \mu^+|_{\gamma_l} - \mu^-|_{\gamma_l}. \quad (23)$$

Similarly we define the marginal density of  $\mu$  as

$$\phi_\mu := \phi_{\mu^+} - \phi_{\mu^-}.$$

Now we define a  $L^1$  like space of disintegrated measures.

**Definition 11** Let  $\mathcal{L}^1 \subseteq \mathcal{AB}$  be defined as

$$\mathcal{L}^1 := \left\{ \mu \in \mathcal{AB} : \int_{\mathbb{S}^1} W(\mu^+|_{\gamma_l}, \mu^-|_{\gamma_l}) dm(l) < \infty \right\} \quad (24)$$

and define a norm on this space,  $\|\cdot\|_1 : \mathcal{L}^1 \rightarrow \mathbb{R}$ , by

$$\|\mu\|_1 = \int_{\mathbb{S}^1} W(\mu^+|_{\gamma_l}, \mu^-|_{\gamma_l}) dm_1(l). \quad (25)$$

Let us now consider the transfer operator  $L_F$  associated with  $F$ . There is a nice characterization of the transfer operator in our case, showing that this operator works similarly to a one dimensional transfer operator. For the proof see [29] .

**Proposition 12** For each leaf  $\gamma \in \mathcal{F}^s$ , let us define the map  $F_\gamma : D_2 \rightarrow D_2$  by

$$F_\gamma = \pi_y \circ F|_\gamma \circ [\pi_y|_\gamma]^{-1}.$$

For all  $\mu \in \mathcal{L}^1$  and for almost all  $l \in \mathbb{S}^1$  the following holds

$$(L_F \mu)|_{\gamma_l} = \sum_{i=1}^q \frac{F_{\gamma_{T_i^{-1}(l)}}^* (\mu|_{\gamma_{T_i^{-1}(l)}})}{|T'_i \circ T_i^{-1}(l)|}. \quad (26)$$

Here, again  $F_\gamma^*$  stands for the pushforward of  $F_\gamma$ .

In [30], Section 12, for a solenoidal map  $F$  as defined in this section the following elementary facts are proved.

**Proposition 13** (The weak norm is weakly contracted by  $L_F$ ) *If  $\mu \in \mathcal{L}^1$  then*

$$\|L_F \mu\|_{1^*} \leq \|\mu\|_{1^*}. \quad (27)$$

**Proposition 14** *For all  $\mu \in \mathcal{L}^1$  it holds*

$$\|L_F \mu\|_{1^*} \leq \alpha \|\mu\|_{1^*} + (\alpha + 1) \|\phi_\mu\|_1. \quad (28)$$

We denote by  $\mathcal{V} \subseteq \mathcal{L}^1$  the set of measures having 0 average, i.e.

$$\mathcal{V} := \{\mu \in \mathcal{L}^1 | \mu(X) = 0\}.$$

**Proposition 15** (Exponential convergence to equilibrium) *There exist  $D \in \mathbb{R}$  and  $0 < \beta_1 < 1$  such that, for every signed measure  $\mu \in \mathcal{V}$ , it holds*

$$\|L_F^n \mu\|_{1^*} \leq D_2 \beta_1^n (\|\mu\|_{1^*} + \|\phi_\mu\|_{W^{1,1}})$$

for all  $n \geq 1$ .

In the previous proposition  $\|\cdot\|_{W^{1,1}}$  stands for the 1,1 Sobolev norm. Furthermore the system has an unique invariant measure in  $\mathcal{L}^1$ .

**Proposition 16** *There is a unique  $\mu \in \mathcal{L}^1$  such that  $L_F \mu = \mu$ .*

Let us now consider the following stronger norm

$$\|\mu\|_s = \|\mu\|_{1^*} + \|\phi_\mu\|_{W^{1,1}}.$$

We can then define

$$B_s := \{\mu \in \mathcal{L}^1, \text{ s.t. } \|\mu\|_s < +\infty\}. \quad (29)$$

Considering  $\|\cdot\|_s$  as a strong norm and  $\|\cdot\|_{1^*}$  as a weak norm we can easily prove a Lasota Yorke inequality, showing that the system has a kind of regularization for these two norms.

**Lemma 17** *For each  $\mu \in B_s$*

$$\|L_F \mu\|_s \leq \max(\alpha, \lambda) \|\mu\|_s + [(\alpha + 1) + b] \|\phi_\mu\|_1. \quad (30)$$

**Proof.** By Proposition 14 and the Lasota Yorke inequality for expanding maps

$$\begin{aligned}
\|L_F \mu\|_s &\leq \|L_F \mu\|_{1^*} + \|L_T \phi_\mu\|_{W^{1,1}} \\
&\leq \alpha \|\mu\|_{1^*} + (\alpha + 1) \|\phi_\mu\|_1 \\
&\quad + \lambda \|\phi_\mu\|_{W^{1,1}} + B \|\phi_\mu\|_1 \\
&\leq \max(\alpha, \lambda) \|\mu\|_s + [(\alpha + 1) + b] \|\phi_\mu\|_1.
\end{aligned}$$

■

**Remark 18** From (30), since  $\|\phi_\mu\|_1 \leq \|\mu\|_{1^*}$  one also can deduce

$$\|L_F \mu\|_s \leq \max(\alpha, \lambda) \|\mu\|_s + [(\alpha + 1) + b] \|\mu\|_{1^*}. \quad (31)$$

We are then going to apply Theorem 4 considering  $(B_s, \|\cdot\|_s)$  as a strong space, as just defined, we will also use  $(\mathcal{L}^1, \|\cdot\|_{1^*})$  as a weak space. To apply Theorem 4 we have to verify that the iterates of the Lebesgue measure have bounded Lipschitz multipliers.

In order to achieve this we need we need to recall some further results on the regularity of the iterates of measures by solenoidal maps.

Given  $\mu \in \mathcal{L}^1$  and its marginal density  $\phi_\mu$ . Let us consider the following stronger space of measures

$${}^*W^{1,1} = \left\{ \begin{array}{l} \mu \in \mathcal{L}^1 \text{ s.t. } \phi_\mu \in W^{1,1} \text{ and } \forall l_1 \lim_{l \rightarrow l_1} \|\mu|_{\gamma_l} - \mu|_{\gamma_{l_1}}\|_W = 0 \text{ and} \\ \text{for almost all } l_1, D(\mu, l_1) := \limsup_{l \rightarrow l_1} \left\| \frac{\mu|_{\gamma_{l_1}} - \mu|_{\gamma_l}}{l_1 - l} \right\|_W < \infty \text{ and} \\ \|\mu\|_{1^*} + \int |D(\mu, \gamma_l)| dl < \infty \end{array} \right\}.$$

**Definition 19** Let us consider the norm

$$\|\mu\|_{{}^*W^{1,1}} := \|\mu\|_{1^*} + \int |D(\mu, \gamma)| d\gamma.$$

The following is proved in [30], Section 12.

**Proposition 20** Let  $F$  be a solenoidal map satisfying (a), (b), (c), then  $L_F({}^*W^{1,1}) \subseteq {}^*W^{1,1}$  and there are  $\lambda < 1, B > 0$  s.t  $\forall \mu \in {}^*W^{1,1}$  with  $\mu \geq 0$

$$\|L_F \mu\|_{{}^*W^{1,1}} \leq \lambda(\alpha \|\mu\|_{{}^*W^{1,1}} + \|\phi'_\mu\|_1) + B \|\mu\|_{1^*}.$$

Iterating the inequality, one gets

**Corollary 21** *There are  $B > 0, \lambda < 1$  such that*

$$\|L^{(n)}\mu\|_{W^{1,1}} \leq \lambda^n(\|\mu\|_{W^{1,1}} + \|\phi'_\mu\|_1) + B\|\mu\|_1. \quad (32)$$

Where  $L^{(n)}$  stands for the sequential composition of the operators  $L_{F_i}$  as defined in Section 2.

**Proof.** By Propositions 13 and 20 the operators  $L_i := L_{F_i}$  satisfy a common Lasota Yorke inequality. Denoting  $\|\mu\|_s := \|\mu\|_{W^{1,1}} + \|\phi'_\mu\|_1$ , there are constants  $B, \lambda_1 \geq 0$  with  $\lambda_1 < 1$  such that for all  $f \in B_s, \mu \in P_w, i \in \mathbb{N}$

$$\begin{aligned} \|L_i\mu\|_{W^{1,1}} &\leq \|\mu\|_{W^{1,1}} \\ \|L_i\mu\|_s &\leq \lambda_1\|\mu\|_s + B\|\mu\|_{W^{1,1}}. \end{aligned} \quad (33)$$

First we remark that obviously

$$\|L^{(n)}\mu\|_{W^{1,1}} \leq \|\mu\|_{W^{1,1}}. \quad (34)$$

For the stronger norm  $\|\cdot\|_s$ , given some  $j \in \mathbb{N}$ , composing the operators we have

$$\|L_j f\|_s \leq \lambda_1\|f\|_s + B\|f\|_{W^{1,1}}$$

thus

$$\begin{aligned} \|L_j \circ L_{j+1}(f)\|_s &\leq \lambda_1\|L_j f\|_s + B\|L_j f\|_{W^{1,1}} \\ &\leq \lambda_1^2\|f\|_s + \lambda_1 B\|f\|_{W^{1,1}} + B\|f\|_{W^{1,1}} \\ &\leq \lambda_1^2\|f\|_s + (1 + \lambda_1)B\|f\|_{W^{1,1}}. \end{aligned}$$

Continuing the composition, noting that the second coefficient keeps being bounded by a geometric sum we get (42).

■

Now we are ready to prove that the iterates of the Lebesgue measure have bounded Lipschitz multipliers.

**Lemma 22** *Let  $\mu_0$  be the Lebesgue measure on  $X$ . The set*

$$A := \{L^{(0,k)}\mu_0, k \in \mathbb{N}\}$$

*has bounded Lipschitz multipliers. There is  $C_A$  such that for each  $i$  and  $g \in \text{Lip}(\mathbb{S}^1)$*

$$\|g\mu_i\|_s \leq C_A\|g\|_{\text{Lip}}.$$

**Proof.** We have that  $\|g\mu\|_s = \|g\mu\|_{1^*} + \|\phi_{g\mu}\|_{W^{1,1}}$ . We first show that for each  $\mu \in B_s$  the weak norm has bounded Lipschitz multipliers:

$$\|g\mu\|_{1^*} = \int_{S^1} W(g\mu^+|_\gamma, g\mu^-|_\gamma) dm(l) \leq 2\|g\|_{Lip}\|\mu\|_{1^*}. \quad (35)$$

In order to prove this it is sufficient to show that for each leaf  $\gamma$  considering  $g\mu|_\gamma$  we have  $W(g\mu^+|_\gamma, g\mu^-|_\gamma) \leq 2\|g\|_{Lip}W(\mu^+|_\gamma, \mu^-|_\gamma)$ . Indeed consider  $f$  such that  $Lip(f) \leq 1$ ,  $\|f\|_\infty \leq 1$ . We have that also  $Lip(f \frac{g}{\|g\|_{Lip}}) \leq 2$ ,  $\|f \frac{g}{\|g\|_{Lip}}\|_\infty \leq 1$  then

$$\begin{aligned} W(g\mu^+|_\gamma, g\mu^-|_\gamma) &= \sup_{f \text{ s.t. } Lip(f) \leq 1, \|f\|_\infty \leq 1} \left| \int f d[g\mu^-|_\gamma] - \int f d[g\mu^+|_\gamma] \right| \\ &= \|g\|_{Lip} \sup_{f \text{ s.t. } Lip(f) \leq 1, \|f\|_\infty \leq 1} \left| \int f \frac{g|_\gamma}{\|g\|_{Lip}} d[\mu^-|_\gamma] \right. \\ &\quad \left. - \int f \frac{g|_\gamma}{\|g\|_{Lip}} d[\mu^+|_\gamma] \right| \\ &\leq 2\|g\|_{Lip}W(\mu^+|_\gamma, \mu^-|_\gamma). \end{aligned}$$

From this, integrating we obtain 35. Since by Proposition 13,  $\|L^{(0,k)}\mu_0\|_{1^*}$  is uniformly bounded as  $k \rightarrow \infty$  there is  $C_{1,A}$  such that

$$\|g\mu_i\|_{1^*} \leq C_{1,A}\|g\|_{Lip}$$

for each  $i$ .

Now we prove that there is  $C \geq 0$  such that for all  $i$ ,

$$\|\phi_{g\mu_i}\|_{W^{1,1}} \leq \|g\|_{Lip}C. \quad (36)$$

Let  $l \in \mathbb{S}^1$ . We have

$$\begin{aligned} \left| \limsup_{\delta \rightarrow 0} \frac{\phi_{g\mu_i}(l + \delta) - \phi_{g\mu_i}(l)}{\delta} \right| &= \left| \limsup_{\delta \rightarrow 0} \frac{\int_D 1d[g\mu_i|_{\gamma_{l+\delta}}] - \int_D 1d[g\mu_i|_\gamma]}{\delta} \right| \\ &= \left| \limsup_{\delta \rightarrow 0} \frac{\int_D g(l + \delta, y) d\mu_i|_{\gamma_{l+\delta}}(y) - \int_D g(l, y) d\mu_i|_\gamma(y)}{\delta} \right| \\ &\leq \left| \limsup_{\delta \rightarrow 0} \frac{\int_D g(l + \delta, y) d\mu_i|_{\gamma_{l+\delta}}(y) - \int_D g(l + \delta, y) d\mu_i|_\gamma(y)}{\delta} \right| \\ &\quad + \left| \limsup_{\delta \rightarrow 0} \frac{\int_D g(l + \delta, y) d\mu_i|_\gamma(y) - \int_D g(l, y) d\mu_i|_\gamma(y)}{\delta} \right| \\ &\leq \|g\|_{Lip}|D(\mu_i, l)| + \|g\|_{Lip}\|\mu_i|_\gamma\|_W. \end{aligned}$$

This shows that  $\phi_{g\mu_i}$  is absolutely continuous and then in the Sobolev space  $W^{1,1}$  furthermore, integrating over  $\mathbb{S}^1$ , (36) is satisfied with  $C = \|\mu_i\|_{W^{1,1}} + \|\mu_i\|_s$ . Since by Corollary 21 we have that  $\|\mu_i\|_{W^{1,1}}$  is uniformly bounded as  $i$  vary we establish the Lemma. ■

### 3.1 Exponential loss of memory for sequential composition of operators

In this section we show a relatively simple and general argument establishing exponential loss of memory for a sequential composition of Markov operators converging to a limit. Since the approach is general, we will work in an abstract framework, stating a result which holds for a sequence of Markov operators acting on suitable spaces of measures. Let  $B_w$  and  $B_s$  be a normed vector subspaces of signed measures on  $X$ . Suppose  $(B_s, \|\cdot\|_s) \subseteq (B_w, \|\cdot\|_w)$  and  $\|\cdot\|_s \geq \|\cdot\|_w$ . Let us consider a sequence of Markov operators  $\{L_i\}_{i \in \mathbb{N}} : B_s \rightarrow B_s$ . We will suppose furthermore that the following assumptions are satisfied by the  $L_i$ :

*ML1* The operators  $L_i$  satisfy a common "one step" Lasota Yorke inequality. There are constants  $B, \lambda_1 \geq 0$  with  $\lambda_1 < 1$  such that for all  $f \in B_s$ ,  $\mu \in P_w$ ,  $i \in \mathbb{N}$

$$\begin{cases} \|L_i f\|_w \leq \|f\|_w \\ \|L_i f\|_s \leq \lambda_1 \|f\|_s + B \|f\|_w. \end{cases} \quad (37)$$

*ML2* There is a Markov operator  $L_0 : B_s \rightarrow B_s$  having an invariant probability measure  $\mu \in B_s$  such that the family of operators satisfy:  $\lim_{i \rightarrow +\infty} L_i = L_0$  in the  $B_s \rightarrow B_w$  topology.<sup>4</sup>

*ML3* There exists  $a_n \geq 0$  with  $a_n \rightarrow 0$  such that for all  $n \in \mathbb{N}$  and  $v \in V_s$

$$\|L_0^n(v)\|_w \leq a_n \|v\|_s \quad (39)$$

where

$$V_s = \{\mu \in B_s \mid \mu(X) = 0\}.$$

---

<sup>4</sup>In particular the family of operators satisfy:  $\forall \epsilon > 0 \exists N$  s.t.  $\forall i, j \geq N$

$$\|(L_i - L_j)\|_{B_s \rightarrow B_w} \leq \epsilon. \quad (38)$$



We recall that since  $\mu \rightarrow \mu(X)$  is continuous,  $V_s$  is closed. Furthermore  $\forall i, L_i(V_s) \subseteq V_s$ .

We remark that the assumption (ML1) implies that the family of operators  $L_i$  is uniformly bounded when acting on  $B_s$  and on  $B_w$ .

First we establish a Lasota Yorke inequality for a sequential composition of operators satisfying (ML1). The proof of the Lemma is essentially the same as the proof of Corollary 21.

**Lemma 23** *Let  $L_i$  be a family of Markov operators satisfying (ML1) and let*

$$L^{(j,j+n-1)} := L_j \circ L_{j+1} \circ \dots \circ L_{j+n-1} \quad (40)$$

*be a sequential composition of operators in such family, then  $\forall n, j$*

$$\|L^{(j,j+n-1)}f\|_w \leq \|f\|_w \quad (41)$$

*and*

$$\|L^{(j,j+n-1)}f\|_s \leq \lambda_1^n \|f\|_s + \frac{B}{1-\lambda_1} \|f\|_w. \quad (42)$$

The following Lemma is an estimate for the distance of the sequential composition of operators from the iterates of  $L_0$ .

**Lemma 24** *Let  $\delta \geq 0$  and let  $L^{(j,j+n-1)}$  be a sequential composition of operators  $\{L_i\}_{i \in \mathbb{N}}$  as in (40) satisfying the above assumptions. Let  $L_0$  as above such that  $\|L_i - L_0\|_{s \rightarrow w} \leq \delta$ . Then there is  $C \geq 0$  such that  $\forall g \in B_s, \forall j, n \geq 1$*

$$\|L^{(j,j+n-1)}g - L_0^n g\|_w \leq \delta(C\|g\|_s + n \frac{B}{1-\lambda} \|g\|_w). \quad (43)$$

*where  $B$  is the second coefficient of the Lasota Yorke inequality (37).*

**Proof.** By the assumptions we get

$$\|L_0 g - L_j g\|_w \leq \delta \|g\|_s$$

hence the case  $n = 1$  of (43) is trivial. Let us now suppose inductively

$$\|L^{(j,j+n-2)}g - L_0^{n-1}g\|_w \leq \delta(C_{n-1}\|g\|_s + (n-1) \frac{B}{1-\lambda_1} \|g\|_w)$$

then

$$\begin{aligned}
\|L_{j+n-1}L^{(j,j+n-2)}g - L_0^n g\|_w &\leq \|L_{j+n-1}L^{(j,j+n-2)}g - L_{j+n-1}L_0^{n-1}g + L_{j+n-1}L_0^{n-1}g - L_0^n g\|_w \\
&\leq \|L_{j+n-1}L^{(j,j+n-2)}g - L_{j+n-1}L_0^{n-1}g\|_w + \|L_{j+n-1}L_0^{n-1}g - L_0^n g\|_w \\
&\leq \delta(C_{n-1}\|g\|_s + (n-1)\frac{B}{1-\lambda_1}\|g\|_w) + \|[L_{j+n-1} - L_0](L_0^{n-1}g)\|_w \\
&\leq \delta(C_{n-1}\|g\|_s + (n-1)\frac{B}{1-\lambda_1}\|g\|_w) + \delta\|L_0^{n-1}g\|_s \\
&\leq \delta(C_{n-1}\|g\|_s + (n-1)\frac{B}{1-\lambda_1}\|g\|_w) \\
&\quad + \delta(\lambda_1^{n-1}\|g\|_s + \frac{B}{1-\lambda_1}\|g\|_w) \\
&\leq \delta[(C_{n-1} + \lambda_1^{n-1})\|g\|_s] + n\frac{B}{1-\lambda_1}\|g\|_w.
\end{aligned}$$

The statement follows from the observation that continuing the composition,  $C_n$  remains being bounded by the sum of a geometric series. ■

**Lemma 25** *Let  $L_i$  be a sequence of operators satisfying (ML1), ..., (ML3).*

*Then the sequence  $L_i$  has strong exponential loss of memory in the following sense. There are  $C, \lambda \geq 0$  such that  $\forall j, n \in \mathbb{N}, g \in V_s$*

$$\|L^{(j,j+n-1)}g\|_s \leq Ce^{-\lambda n}\|g\|_s.$$

**Proof.** It is standard to deduce from the assumptions that  $\mu$  is the unique invariant probability measure of  $L_0$  in  $B_s$ . Now let us consider  $\mu_0 \in B_s$ . Remark that because of the Lasota-Yorke inequality,  $\forall j, i \geq 1, g \in B_s$

$$\|L^{(j,j+i)}(g)\|_s \leq (\frac{B}{1-\lambda_1} + 1)\|g\|_s.$$

Now let us consider  $N_0$  such that  $\lambda_1^{N_0} \leq \frac{1}{100(\frac{B}{1-\lambda_1} + 1)}$  and by (ML3),  $N_2$  such that  $\forall i \geq N_2, g \in V_s$

$$\|L_0^{N_2}g\|_w \leq \frac{1}{100B}\|g\|_s.$$

Let  $M := \max(N_0, N_2)$ . Let  $N_1$  such that

$$\|L_i - L_0\|_{s \rightarrow w} \leq \frac{(1-\lambda_1)}{100MB(C+B)}$$

for all  $i \geq N_1$ . By (43),  $\forall j \geq N_1, i \geq M$

$$\begin{aligned} \|L^{(j,j+i-1)}g - L_0^i g\|_w &\leq \frac{(1-\lambda_1)}{100MB(C+B)}(C\|g\|_s + i\frac{B\|g\|_w}{(1-\lambda_1)}) \\ &\leq \frac{i}{100MB}\|g\|_s. \end{aligned}$$

Hence

$$\begin{aligned} \|L^{(j,j+i-1)}g\|_w &\leq \|L_0^i g\|_w + \frac{i}{100MB}\|g\|_s \\ &\leq \frac{1}{100B}\|g\|_s + \frac{i}{100MB}\|g\|_s. \end{aligned} \tag{44}$$

Applying now the Lasota-Yorke inequality we get, for any  $j \geq N_1$

$$\begin{aligned} \|L^{(j,j+2M-1)}g\|_s &\leq \lambda_1^{-M}\|L^{(j,j+M-1)}g\|_s + B\|L^{(j,j+M-1)}g\|_w \\ &\leq \frac{1}{100}\|g\|_s + B\frac{1}{100B}\|g\|_s + \frac{BM}{100MB}\|g\|_s \\ &\leq \frac{3}{100}\|g\|_s \end{aligned} \tag{45}$$

and

$$\|L^{(j,j+2kM-1)}g\|_s \leq \frac{3^k}{100}\|g\|_s$$

for each  $j \geq N_1$  and  $k \geq 1$ ,  $g \in V_s$  establishing the result. ■

### 3.2 Superpolynomial convergence to equilibrium for the family of Solenoidal maps and the proof of Proposition 9.

Now we apply the results of the previous section to a family of solenoidal maps satisfying the assumptions (a), (b), (c) stated at the beginning of Section 3.

**Proposition 26** *Let  $F_i$  be a sequence of maps satisfying the assumptions (a), (b), (c). Let  $B_s$  be the space defined in (29). Let  $\mu \in B_s$  be the invariant probability measure of the limit map  $F_0$ . Let  $L_{F_i}$  the sequence of transfer*

operators associated to  $F_i$ . Then the sequence  $L_{F_i}$  has superpolynomial convergence to equilibrium to  $\mu$  in the following strong sense. Denoting as before  $L^{(j,j+n-1)} := L_{F_{j+n-1}} \circ \dots \circ L_{F_j}$ , there are  $C, \lambda \geq 0$  such that  $\forall j, n \in \mathbb{N}$ ,  $\mu_0 \in B_s$

$$\|\mu - L^{(j,j+n-1)}\mu_0\|_s \leq \Phi(n) \max(1, \|\mu - \mu_0\|_s). \quad (46)$$

**Proof.** We will apply Lemma 25 to the family of transfer operators  $L_{F_i}$  using as strong space  $B_s$  the one defined in (29) and as a weak space  $B_w$  the one defined in (11). By Lemma 17 and (13) the action of the transfer operators on these two spaces satisfy the assumption (ML1). By assumption (c) (ML2) is satisfied. By Proposition 15 we have that (ML3) is satisfied. We can then apply Lemma 25 and get that there are  $N, M \geq 0$  such that for any  $j \geq N$

$$\|L^{(j,j+M-1)}(\mu - \mu_0)\|_s \leq \frac{1}{2} \|\mu - \mu_0\|_s.$$

Since

$$L^{(j,j+M-1)}(\mu - \mu_0) = L^{(j,j+M-1)}(\mu) - L^{(j,j+M-1)}(\mu_0)$$

by (c), considering that the map only changes on the leaves, where the Wasserstein like distance is considered on positive measures, by (c) we have

$$\|L^{(j,j+M-1)}(\mu) - \mu\|_s \leq M\Phi(j)$$

and then

$$\|\mu - L^{(j,j+M-1)}(\mu_0)\|_s \leq \frac{1}{2} \|\mu - \mu_0\|_s + M\Phi(j).$$

Denoting  $d_k := \|L^{(j,j+kM-1)}(\mu_0) - \mu\|_s$ , for  $k \geq 0$  the above computation shows that  $d_0 := \|\mu_0 - \mu\|_s$ ,  $d_{k+1} \leq \frac{1}{2}d_k + M\Phi(j + Mk) \leq \frac{1}{2}d_k + M\Phi(j + Mk)[\max(1, d_0)]$  showing that  $d_k$  decreases superpolynomially fast, satisfying (46)<sup>5</sup>.

■

**Proof (of Proposition 9).** The proof of the statement directly follows from the application of Theorem 4.

The boundedness of the set  $A = \{L^{(0,k)}\mu_0, k \in \mathbb{N}\}$  in  $B_s$  and of the Lipschitz multipliers is verified in Lemma 22, the superpolynomial strong convergence to equilibrium for the family of maps we consider is verified

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<sup>5</sup>If we have  $d_{k+1} \leq \frac{1}{2}d_k + a_n$  with  $a_n$  decreasing superpolynomially, then one can rewrite the relation as  $d_{k+1} - 2a_n \leq \frac{1}{2}(d_k - 2a_n)$  showing that  $d_{k+1}$  converges to  $2a_n$  exponentially fast.

in Lemma 26. The assumption (5) is trivially verified. This provides the assumptions necessary to apply Theorem 4, establishing the result.

■

**Remark 27** *We remark that in order to get a logarithm law as in Theorem 4 for an eventually autonomous system like the ones considered in this section, a quantitative bound on the speed of convergence of the sequence of the sequence of maps  $F_i \rightarrow F_0$  is necessary. Let us indeed consider  $i \geq 1$  and a family with a slow convergence like  $F_i(x, y) = (2x \bmod (1), [\frac{1}{\sqrt{i}}])$ . for this family of maps the limit map is  $F_0$  with  $F_0(x, y) = (2x \bmod (1), [0])$  whose physical measure is the one dimensional lebesgue measure on  $S := \mathbb{S}^1 \times \{0\}$ . So  $d_\mu(x) = 1$  for each  $x \in S$ . Let us fix  $y \in S$  as a target point. Let us consider  $i \in \mathbb{N}$  and an initial condition  $x_0$  such that  $d(x_0, x) > 0$ . We remark that because of the slow convergence to  $F_i$  to  $F_0$  we have that for  $i, j \geq 1$ ,  $d(F^{(0,j)}(x_0), x) \leq \frac{1}{i}$  implies that  $j \geq i^2$ . This implies that for this system  $\liminf_{r \rightarrow 0} \frac{\log(\tau_r(x_0, y))}{-\log(r)} \geq 2 > d_\mu(y)$ . Showing that a logarithm law as in Theorem 4 cannot hold in this case.*

## 4 Application to mean field coupled maps

In this section we show how to apply our main results to a system of mean field coupled expanding maps, obtaining a logarithm law for this kind of systems.

To this goal, we will use known results on the convergence to equilibrium of mean field coupled systems. Those results allow to treat the dynamics of a typical subsystem of the mean field coupled system as a nonautonomous system having fast convergence to equilibrium (see Definition 2). This general idea is hence applied in this section to a particularly simple class of chaotic systems.

### 4.1 A model for infinitely many mean field coupled maps

We now define a model for the dynamics of an infinite family of expanding maps interacting in the mean field. The mean field system will be composed by infinitely many interacting subsystems, where the dynamics is given by

some expanding map, perturbed deterministically by the state of all the other systems in a way which we are going to describe in this subsection.

The phase space for each interacting subsystem is the unit circle  $\mathbb{S}^1$ , we will equip  $\mathbb{S}^1$  with the Borel  $\sigma$ -algebra.

Let us consider an additional metric space  $M$  equipped with the Borel  $\sigma$ -algebra and a probability measure  $p \in PM(M)$ . Let us consider a collection of identical  $C^6$  expanding maps  $(\mathbb{S}^1, T)_i$ , with  $i \in M$ . An admissible global state for the dynamics of this extended system at some time  $t$  is given by a measurable function  $\mathbf{x}_t : M \rightarrow \mathbb{S}^1$  associating to every  $i \in M$  the state  $x_0(i)$  of the subsystem  $(\mathbb{S}^1, T)_i$ .

We say that the global state  $\mathbf{x}_t$  of the system is represented by a probability measure  $\mu_{\mathbf{x}_t} \in PM(\mathbb{S}^1)$  if

$$\mu_{\mathbf{x}_t} = L_{\mathbf{x}_t}(p)$$

(the pushforward of  $p$  by the function  $\mathbf{x}_t$ ). Let  $\mathcal{X}$  be the set of such measurable functions  $M \rightarrow \mathbb{S}^1$  defining the admissible global states of the system. We now define the dynamics of the interacting systems by defining a global map  $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$  and global trajectory of the system by

$$\mathbf{x}_{t+1} := \mathcal{T}(\mathbf{x}_t)$$

where  $\mathbf{x}_{t+1}$  is defined on every coordinate by applying at each step the common local dynamics  $T$ , plus a perturbation given by the mean field interaction with the other systems, by

$$x_{t+1}(i) = \Phi_{\delta, \mathbf{x}_t} \circ T(x_t(i)) \quad (47)$$

for all  $i \in M$ , where  $\Phi_{\delta, \mathbf{x}_t} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a diffeomorphism near to the identity when  $\delta$  is small and represents the perturbation provided by the global mean field coupling. Let us consider a coupling function  $h \in C^6(\mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R})$ . The function  $h(x, y)$  represents the way in which the presence of some subsystem in the state  $y \in \mathbb{S}^1$  perturbs a certain subsystems in the state  $x \in \mathbb{S}^1$ . The mean field perturbation  $\Phi_{\delta, \mathbf{x}_t}$  with strength  $\delta \geq 0$  is defined in the following way: let  $\pi_{\mathbb{S}^1} : \mathbb{R} \rightarrow \mathbb{S}^1$  be the universal covering projection, ; we define  $\Phi_{\delta, \mathbf{x}_t}$  as

$$\Phi_{\delta, \mathbf{x}_t}(x) := x + \pi_{\mathbb{S}^1}(\delta \int_{\mathbb{S}^1} h(x, y) d\mu_{\mathbf{x}_t}(y)). \quad (48)$$

We remark that in this definition the parameter  $\delta$  plays the role of the strength of the coupling. Since (47) is clearly a measurable map we see

that the measure representing the current state of the system fully determines the measure which represents the next state of the system, defining a function between measures  $\mu_{\mathbf{x}_t} \rightarrow \mu_{\mathbf{x}_{t+1}}$  defined as

$$\mu_{\mathbf{x}_{t+1}} = L_{\Phi_{\delta, \mu_{\mathbf{x}_t}} \circ T}(\mu_{\mathbf{x}_t}) := \mathcal{L}_{\delta}(\mu_{\mathbf{x}_t}).$$

Now, let us consider  $\delta \geq 0$  and denote by  $(\mathbb{S}^1, T, \delta, h)$  the extended system in which these maps are coupled by  $h$  as explained above. The function  $\mathcal{L}_{\delta}$  is also called to be the Self Consistent Transfer Operator associated to the mean field coupled system  $(\mathbb{S}^1, T, \delta, h)$ .

Since in every subsystem and coordinate, at each iteration, the map  $\Phi_{\delta, \mu_{\mathbf{x}_t}} \circ T$  is applied, if we observe the evolution of a single coordinate, we see the result of the application of a nonautonomous dynamical system  $(\mathbb{S}^1, T_n)$  where  $T_n = \Phi_{\delta, \mu_{\mathbf{x}_n}} \circ T$ . In the case where  $T$  is an expanding map and  $x_0$  is distributed smoothly we are in the framework of our main result and we can establish a logarithm law for the dynamics of each coordinate.

For this kind of extended system we prove:

**Proposition 28** *Let us fix  $i \in M$  and let  $x_t(i)$  the evolution of the  $i$ -th coordinate of the mean field coupled system  $(\mathbb{S}^1, T, \delta, h)$  as defined above. Let us suppose that the global initial condition of the system is distributed in a smooth way, that is  $\mu_{x_0}$  is an absolutely continuous measure having density in  $W^{1,1}$ .<sup>6</sup> We will also suppose that the coupling is small. In the sense that there is  $\hat{\delta} > 0$  such that for each  $0 \leq \delta \leq \hat{\delta}$  the following result will hold. We define the hitting time of a small target centered at  $y$  for the  $i$ -th subsystem with initial condition  $x_0(i)$  as*

$$\tau_r(x_0(i), y) = \sup(\{t \geq 0 | d(x_t(i), y) \geq r\}).$$

*Let  $m$  be the Lebesgue measure on  $\mathbb{S}^1$ . Then for each  $y \in \mathbb{S}^1$  and  $m$  almost each  $x_0(i)$  it holds*

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x_0(i), y)}{-\log r} = 1. \quad (49)$$

To prove Proposition 28 we need some preliminary results we will take from [31], Section 7. The following statement shows that our mean field coupled system has a unique regular invariant measure when  $\delta$  is small enough.

---

<sup>6</sup>We remark that the global initial distribution  $\mu_{x_0}$  and its evolution in time does not depend on the single  $i$ -th subsystem initial condition  $x_0(i)$ .

**Proposition 29 (Existence and uniqueness of the invariant measure)**

Let  $(\mathbb{S}^1, T, \delta, h)$  as above and let  $\mathcal{L}_\delta$  be the associated self consistent transfer operator. If  $\delta > 0$  is small enough then there is a unique probability measure  $\mu_\delta$  having density  $f_\delta \in L^1$  such that

$$\mathcal{L}_\delta(\mu_\delta) = \mu_\delta.$$

Furthermore  $f_\delta \in W^{5,1}$ .

The following statement is an estimate for the speed of convergence to equilibrium of mean field coupled expanding maps (see [32], or [33] for similar statements).

**Proposition 30 (Exponential convergence to equilibrium)** Let  $\mathcal{L}_\delta$  be the family of self-consistent transfer operators arising  $(\mathbb{S}^1, T, \delta, h)$  as above. Let  $\mu_\delta$  be the absolutely continuous invariant probability measure of  $\mathcal{L}_\delta$ . Let us denote by  $f_\delta \in W^{1,1}$  the density of  $\mu_\delta$  with respect to the Lebesgue measure  $m$ . Then there exists  $\bar{\delta} > 0$  and  $C, \gamma \geq 0$  such that for all  $0 < \delta < \bar{\delta}$ , and each probability measure  $\nu$  having density  $f_\nu \in W^{1,1}$  we have

$$\left\| \frac{d}{dm} \mathcal{L}_\delta^n(\nu) - f_\delta \right\|_{W^{1,1}} \leq C e^{-\gamma n} \|f_\nu - f_{\mu_\delta}\|_{W^{1,1}}.$$

We can now apply Theorem 4 to get a logarithm law result for mean field coupled maps.

**Proof of Proposition 28.** We will get the result by a direct application of Theorem 4 to the nonautonomous system  $(\mathbb{S}^1, T_i)$  where  $T_i = \Phi_{\delta, \mu_{x_i}} \circ T$  and  $\mu_{x_i}$  is the measure representing the global state at time  $i$ , satisfying  $\mu_{x_i} = \mathcal{L}^i(\mu_{x_o})$ . We will consider as a strong space  $B_s$  the space of signed measures having a density in  $W^{1,1}$  with the topology induced by the one on  $W^{1,1}$ . We remark that this topology is stronger than then one induced by the  $W$  distance. Furthermore we remark that by Proposition 30 the set

$$A = \{\mathcal{L}^i(\mu_{x_o})\}_{i \in \mathbb{N}}$$

is bounded in  $B_s$  and has obviously bounded Lipschitz multipliers. By Proposition 30 we also have that  $\mu_n := \mathcal{L}^i(\mu_{x_o})$  converge exponentially fast to the invariant measure  $\mu_\delta \in W^{1,1}$ , then Theorem 4 can be applied. We remark that since  $\mu_\delta$  has density  $f_\delta \in W^{1,1}$ ,  $d_\mu(y) = 1 \ \forall y \in \mathbb{S}^1$  establishing 49. ■



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